A New Proof Of The Independence Ratio Of Triangle-Free Cubic Graphs

Christopher Carl Heckman

and

Robin Thomas

School of Mathematics, Georgia Institute of Technology,
Atlanta, Georgia, 30332-0160

Abstract: Staton proved that every triangle-free graph on n vertices with maximum degree three has an independent set of size at least $5n/14$. A simpler proof was found by Jones. We give a yet simpler proof, and use it to design a linear-time algorithm to find such an independent set.

Let G be a triangle-free graph on n vertices with maximum degree three. By Brooks' theorem [1] G is 3-colorable; considering the largest color class, it follows that G has an independent set of size at least $n/3$. The coloring result is clearly best possible, but can we do better in terms of independent sets? Staton [6] proved that, in fact, the bound can be improved to $5n/14$, and this is best possible because, as noticed by Fajtlowicz [2], the generalized Petersen graph $P(7,2)$ has fourteen vertices and no independent set of size six. Jones [5] found a simpler proof of Staton's result. Griggs and Murphy [3] designed a linear-time algorithm to find an independent set in G of size at least $5(n-k)/14$, where k is the number of components of G that are 3-regular. The objective of this paper is to give a yet simpler proof of Staton's result, and to design a linear-time algorithm to find an independent set in G of size at least $5n/14$.

Graphs are finite and simple (that is, they have no loops or parallel edges). A *block* is either a 2-connected graph, or a complete graph on at most two vertices. A *block of a graph* G is a maximal subgraph of G that is a block. A *pentagon* is a cycle of length five. (*Paths* and *cycles* have no "repeated" vertices.) A block B is said to be *difficult* if it is isomorphic to a pentagon or L, where L is the graph obtained by subdividing both edges of a perfect matching of K_4 twice; it has eight vertices, ten edges and independent sets of size three. A graph G is said to be *difficult* if every block of G is either difficult or is an edge between two difficult blocks. We define $\lambda(G)$ to be the number of components of G that are difficult. Our main result reads as follows.

Theorem. *Every triangle-free graph G with maximum degree at most three has an independent set of size at least* $\frac{1}{7}\left(4\,|V(G)| - |E(G)| - \lambda(G)\right)$.

Since every difficult component has at least two vertices of degree two, we deduce Staton's theorem.

Corollary. *Every triangle-free graph on n vertices with maximum degree at most three has an independent set of size at least* $5n/14$.

We offer the following conjecture, which would also imply the corollary. The *fractional chromatic number* of a graph G is the infimum of all a/b such that to every vertex of G one can assign a subset of $\{1,2,\ldots,a\}$ of size b in such a way that adjacent vertices are assigned disjoint sets. It

Published in *Discr. Math.* **233** (2001), 233–237.

follows that the infimum is attained, because it is the optimum value of a certain linear program with rational data. The linear program is the linear programming relaxation of a certain integer program whose optimum is the chromatic number. It appears that the fractional chromatic number was first introduced in [4]. We conjecture the following.

Conjecture. *Every triangle-free graph with maximum degree at most three has fractional chromatic number at most* $14/5$.

Proof of the theorem. To show the theorem holds, suppose for a contradiction that it does not, and let G be a counterexample with $|V(G)|$ minimum. We proceed in a series of claims.

(1) Let $X \subseteq V(G)$ be nonempty, and let G' be obtained from $G \setminus X$ by (possibly) adding edges so that no triangles or vertices of degree more than 3 are created. If every independent set I' in G' can be extended to an independent set in G of size at least $|I'| + A$, then $7A + E - 4N < \Lambda$, where $E = |E(G)| - |E(G')|$, $N = |X|$, and $\Lambda = \lambda(G') - \lambda(G)$.

Proof. The graph G' satisfies the conclusion of the theorem by the minimality of $|V(G)|$. If $7A + E - 4N \geq \Lambda$, then so does G, a contradiction. □

(2) The minimum degree of G is at least two.

Proof. Suppose for a contradiction that v is a vertex of G of degree at most one. If v has degree zero, then let $X = \{v\}$; otherwise let X consist of v and its neighbor. Then the set X contradicts (1). □

A block B of a graph H is an *end-block* of H if B contains at most one vertex whose deletion disconnects H. If H is a subgraph of G we denote by $\Phi(H)$ the number of edges with one end in $V(H)$ and the other in $V(G) - V(H)$.

(3) Let H be an induced subgraph of G such that H is a difficult graph. Then $\Phi(H) \geq 3$. (In particular, $\lambda(G) = 0$.)

Proof. Let H be as stated, and suppose for a contradiction that $\Phi(H) \leq 2$. Then $\Phi(B) \leq 2$ for some end-block B of H. Let $X = V(B)$, let $G' = G \setminus X$, and let the notation be as in (1). Then $A = 2$, $E = \Phi(B) + 5$, $N = 5$ if B is a pentagon, and $A = 3$, $E = \Phi(B) + 10$, $N = 8$ if B is isomorphic to L. In either case $\Lambda \geq \Phi(B)$, which is impossible: if $\Phi(B) = 0$, then $\lambda(G') = \lambda(G) - 1$; if $\Phi(B) = 1$, then $\lambda(G') = \lambda(G)$; and if $\Phi(B) = 2$, then $\lambda(G') \leq \lambda(G) + 1$. □

(4) No vertex of G of degree two is adjacent to two vertices of degree three.

Proof. Suppose for a contradiction that v is vertex of G of degree two with both neighbors of degree three. Let X consist of v and its neighbors and $G' = G \setminus X$, and let us assume the notation of (1). Then $A = 1$, $E = 6$, $N = 3$, and so $\Lambda > 1$ by (1). By (3), $\Phi(G') \geq 6$, a contradiction, because there are only four edges between X and $V(G')$. □

(5) Every vertex of G has degree three.

Proof. Suppose for a contradiction that v is vertex of G of degree less than three. From (2) and (4) we deduce that v has degree two, and that it has a neighbor of degree two, say u. Let y and x be the other neighbors of u and v, respectively. Let $X = \{u, v\}$. If x is adjacent to y, then let $G' = G \setminus X$; otherwise let G' be obtained from $G \setminus X$ by adding an edge with ends x and y. If G' is triangle-free, then, using the notation of (1), we have $A = 1$, $E = 2$, $N = 2$, and hence $\Lambda > 1$ by (1), a contradiction, because all but one of the components of G' are components of G. So G' must contain a triangle and thus G has a vertex z adjacent to x and y.

By (3), applied to the graph induced by $\{u, v, x, y, z\}$, the vertices x, y, z all have degree three in G. Let $X = \{u, v, x\}$, and let $G' = G \setminus X$. Then, using the notation of (1), we have $A = 1$, $E = 5$, $N = 3$, and hence $\Lambda > 0$. Thus G' has a difficult component, but $\Phi(G') = 3$, and so (3) implies that G' is a difficult block. Now if G' is a pentagon, and x is adjacent to the neighbor of y other than u or z, then G is isomorphic to L. Otherwise, it is easy to see that G has an independent set of size at least $\frac{1}{7}(4|V(G)| - |E(G)|)$. In either case G is not a counterexample, a contradiction. This proves (5). □

(6) There is a vertex $v \in V(G)$ such that the graph obtained from G by deleting v and all its neighbors has no difficult component.

Proof. Let $v \in V(G)$ be an arbitrary vertex, and let G' be the graph obtained from G by deleting v and all its neighbors. We may assume that G' has a difficult component J, for otherwise (6) holds. From (5) we deduce that J is isomorphic to a pentagon, or L, or the graph obtained by adding an edge between two copies of L. In each case it is straightforward to select a vertex in $V(J)$ that satisfies the conclusion of (6). In fact, we can find such a vertex which is at distance at most 2 from v. □

Let $v \in V(G)$ be as in (6), let X consist of v and its neighbors, and let $G' = G \setminus X$. Using the notation of (1) and the fact that G is triangle-free, we have $A = 1$, $E = 9$, and $N = 4$, and so $\Lambda > 0$ by (1), contrary to (6). This completes the proof of the theorem. □

We now turn to the algorithm. We need the following simple data structure. A vertex in a graph G is called *special* if it has degree two, and if it belongs to a subgraph H of G such that $\Phi(H) \leq 2$ and H is a difficult block (that is, is isomorphic to a pentagon or L). Let G be a graph, and let S_1, S_2 be two multisets such that S_1 includes all vertices of G of degree at most one and all special vertices of G, and S_2 includes all vertices of G of degree two that are not special. We say that the pair (S_1, S_2) is a *signature* of G. It does not matter how signatures are implemented, as long as elements can be added and removed in constant time. Since a signature of a graph of maximum degree three can be found in linear time, it suffices to describe the following algorithm.

Algorithm. *There is an algorithm with the following specifications.*
Input: *A triangle-free cubic graph G with maximum degree at most three, and a signature (S_1, S_2) of G.*
Output: *An independent set in G of size at least $\frac{1}{7}(4|V(G)| - |E(G)| - \lambda(G))$.*
Running time: $O(|V(G)| + |S_1| + |S_2|)$.

Description. We will need to apply the algorithm recursively to smaller graphs. Each of the smaller graphs will be obtained by deleting a bounded number of vertices, and possibly adding a new edge. In those circumstances it is easy to see that a signature of G can be modified by adding a bounded number of vertices to become a signature of the smaller graph. Moreover, this can be done in constant time, because it suffices to examine vertices at bounded distance from the vertices being deleted, and because G has bounded degree.

We remove and examine members of S_1 (one by one, in arbitrary order), until either we find a vertex v of G of degree at most one, or we find a special vertex v of G, or S_1 becomes empty. If we find a vertex v of degree at most one, then let X be as in (2), and we apply the algorithm recursively to $G \setminus X$. If we find a special vertex we find a difficult block B with $\Phi(B) \leq 2$ containing that vertex, and apply the algorithm recursively to $G \setminus V(B)$. By the argument of (2) and (3) this results in an independent set of adequate size.

We may therefore assume that S_1 does not include any vertices as above, and so G satisfies (2) and (3). We remove and examine members of S_2 until either we find a vertex v of G of degree two, or S_2 becomes empty. If we find a vertex of degree two we proceed as in (4) or (5); otherwise it follows that every vertex of G has degree three, and we pick a vertex v satisfying (6). This can be done in constant time by mimicking the proof of (6). We then complete the algorithm as in the proof of the theorem.

This completes the description of the algorithm. Its correctness follows from the proof of the theorem, and the bound on the running time follows immediately, based on two observations. First, every iteration takes time proportional to the number of vertices removed from $S_1 \cup S_2$ (or constant time, if this set is empty). Second, for each iteration, at least one vertex is removed from the graph, and a bounded number of vertices are added to $S_1 \cup S_2$; hence the total number of vertices the algorithm puts into the signature (to update it) is linear in the number of vertices of the input graph. □

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