Six-Critical Graphs on the Klein bottle
Extended Abstract

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Abstract

We exhibit an explicit list of nine graphs such that a graph drawn in the Klein bottle is 5-colorable if and only if it has no subgraph isomorphic to a member of the list. This answers a question of Thomassen [J. Comb. Theory Ser. B 70 (1997), 67–100] and implies an earlier result of Král', Mohar, Nakamoto, Pangrác and Suzuki that an Eulerian triangulation of the Klein bottle is 5-colorable if and only if it has no complete subgraph on six vertices.

1 Introduction

All graphs here are finite, undirected and simple. We study a specific instance of the following more general question: Given a surface Σ and an integer $t \geq 0$, which graphs drawn in Σ are t-colorable? Heawood proved that if Σ is not the sphere, then every graph in Σ is t-colorable as long as $t \geq H(\Sigma) := \lfloor (7 + \sqrt{24\gamma + 1})/2 \rfloor$, where γ is the *Euler genus of* Σ, defined as twice the genus if Σ is orientable and the cross-cap number otherwise. Ringel and Youngs proved that the bound is best possible for all surfaces except the Klein bottle. Dirac [6] and Albertson and Hutchinson [1] improved Heawood's result by showing that every graph in Σ is actually $(H(\Sigma) - 1)$-colorable, unless it has a subgraph isomorphic to the complete graph on $H(\Sigma)$ vertices.

We say that a graph is $(t + 1)$-*critical* if it is not t-colorable, but every proper subgraph is. Dirac [7] proved that for every $t \geq 8$ and every surface Σ there are only finitely many t-critical graphs on Σ. Using a result of Gallai [10] this can be extended to $t = 7$. In fact, the result extends to $t = 6$ by a deep theorem of Thomassen [20]. Thus for every $t \geq 5$ and every surface Σ there exists a polynomial-time algorithm to test whether a graph in Σ is t-colorable.

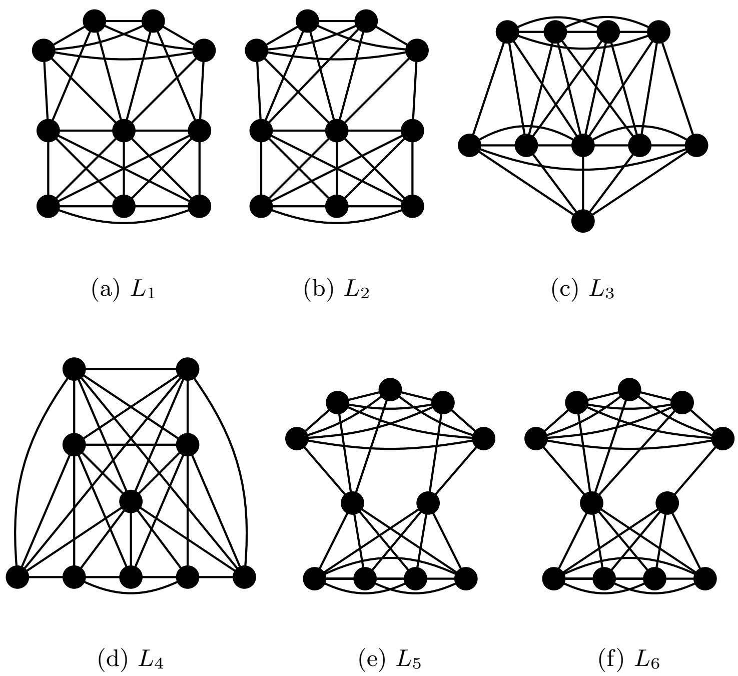
What about $t = 3$ and $t = 4$? The 3-coloring decision problem is NP-hard even when Σ is the sphere [11], and therefore we do not expect to be able to say much. By the Four-Color Theorem [2, 3, 4, 17] the 4-coloring decision problem is trivial when Σ is the sphere, but it is open for all other surfaces. A result of Fisk [9] can be used to construct infinitely many 5-critical graphs on any surface other than the sphere, and the structure of such graphs appears to be complicated [16, Section 8.4].

Thus the most interesting value of t for the t-colorability problem on a fixed surface seems to be $t = 5$. Albertson and Hutchinson [1] proved that a graph in the projective plane is 5-colorable if and only if it has no subgraph isomorphic to K_6, the complete graph on six vertices. Thomassen [19] proved the analogous (and much harder) result for the torus, which we now state. If K, L are graphs, then by $K + L$ we denote the graph obtained from the union of a copy of K with a disjoint copy of L by adding all edges between K and L. The graph T_{11} is obtained from a cycle of length 11 by adding edges joining all pairs of vertices at distance two or three. The graph H_7 is the Hajós' sum of two copies of K_4 and can be described as follows. Take two disjoint copies of K_4, and for $i = 1, 2$ let x_i, y_i be distinct vertices in the i^{th} copy. To obtain H_7 delete the edges $x_i y_i$, identify x_1 and x_2 and add the edge $y_1 y_2$. Now we can state Thomassen's theorem [19].

Theorem 1. *A graph in the torus is* 5*-colorable if and only if it has no subgraph isomorphic to* K_6*,* $C_3 + C_5$*,* $K_2 + H_7$*, or* T_{11}*.*

Our main theorem is the analogous result for the Klein bottle. The graphs $L_1, L_2, \ldots, L_6$ are defined in Figure 1.

Theorem 2. *A graph in the Klein bottle is* 5*-colorable if and only if it has no subgraph isomorphic to* K_6*,* $C_3 + C_5$*,* $K_2 + H_7$*, or any of the graphs* $L_1, L_2, \ldots, L_6$*.*



(a) L_1 (b) L_2 (c) L_3

(d) L_4 (e) L_5 (f) L_6

Figure 1: The graphs L_1 through L_6.

Thus in order to test 5-colorability of a graph G drawn in the Klein bottle it suffices to test subgraph isomorphism to one of the graphs listed in Theorem 2. Using the algorithms of [8] and [15] we obtain the following corollary.

Corollary 3. *There exists an explicit linear-time algorithm to decide whether an input graph embeddable in the Klein bottle is* 5*-colorable.*

It is not hard to see that with the sole exception of K_6, none of the graphs listed in Theorem 2 can be a subgraph of an Eulerian triangulation of the Klein bottle. Thus we deduce the following theorem of Král', Mohar, Nakamoto, Pangrác and Suzuki [13].

Corollary 4. *An Eulerian triangulation of the Klein bottle is* 5*-colorable if and only if it has no subgraph isomorphic to* K_6*.*

2 A lemma

We need a lemma about extensions of 5-colorings of facial cycles in a planar graph, an extension of [19, Lemma 4.1]. Let G be a planar graph, and let C be a cycle bounding a face of G. We say that G is C*-minimal* if there exists a proper 5-coloring $f : V(C) \to \{1, 2, \ldots, 5\}$ such that f does not extend to a 5-coloring of G, but extends to a 5-coloring of $G \backslash e$ for every edge $e \in E(G) - E(C)$. The following is shown in [12].

Lemma 5. For every cycle C there are only finitely many C-minimal graphs.

Thomassen [19, Lemma 4.1] found all C-minimal graphs for cycles of length at most six, and all those for cycles of length at most seven were found in [5]. The authors of [12] give an explicit algorithm to generate all C-minimal graphs for every fixed cycle C.

Here is why we need C-minimal graphs. Let G be a 6-critical graph drawn in the Klein bottle, let C be a cycle in G bounding a closed disk Δ in the Klein bottle, and let H be the subgraph of G consisting of all vertices and edges of G drawn in Δ. Then H is C-minimal. In the proof of Theorem 2 we will construct an explicit subgraph J of G such that every face of J is homeomorphic to an open disk (a "2-cell embedding"). The above observation will allow us to deduce what G looks like, by filling in each face f of J by a C-minimal graph, where C is the face boundary of f (making the obvious adjustment if C fails to be a cycle).

3 First proof

We have obtained Theorem 2 as two independent research groups [5, 12] using different, but related arguments. In particular, the proof [12] is computer-assisted, whereas the other one is not. The following observations are common to both proofs. Sasasuma [18] proved that every 6-regular graph in the Klein bottle is 5-colorable. Let G_0 be a 6-critical graph in the Klein bottle; then G_0 has a vertex v_0 of degree exactly five. We may assume that G_0 is not K_6, and hence it has no K_6 subgraph. It follows that v_0 has a pair of non-adjacent neighbors, say x and y. Let G_{xy} be the graph obtained from G_0 by deleting all edges incident with v_0 except xv_0 and yv_0, contracting the edges xv_0 and yv_0, and deleting all resulting parallel edges. This also defines a drawing of G_{xy} in the Klein bottle. If G_{xy} is 5-colorable, then so is G_0, as is easily seen. Thus G_{xy} has a 6-critical subgraph, say J. Let w be a vertex of J, and let $W = (W_1, W_2)$ be a partition of the neighbors of w into two non-empty disjoint sets. Let J^w_W be obtained from J by splitting w into two non-adjacent vertices w_1 and w_2 such that w_i has neighbors W_i, and then adding a new vertex joined to w_1 and w_2 only. It follows that J^w_W is isomorphic to a subgraph of G_0 for some choice of $w \in V(J)$ and some partition W of the neighbors of w. If every face of J^w_W is an open disk, then, as explained in the previous section, G can be regarded as being obtained from J^w_W by inserting a C-minimal graph into each face bounded by C.

The authors of [12] generate, for each 6-critical Klein bottle graph J and for each 2-cell embedding of J in the Klein bottle, all graphs J^w_W, and then fill their faces in all possible ways with C-minimal graphs. They discard graphs that are not 6-critical, and repeat the process. Thus they need C-minimal graphs for all cycles of length at most 10. Finally, they show, using a computer-free argument, that embeddings of J that are not 2-cell do not produce any additional 6-critical graphs. Their computer code is available for inspection [14].

4 Second proof

The proof of [5] uses the same basic idea, but instead of filling all faces of J^w_W by C-minimal graphs it takes advantage of different possible choices of the vertices x, y, whenever such choice is possible. More precisely, let G_0 be a graph drawn in the Klein bottle that is not 5-colorable and let a vertex $v_0 \in V(G_0)$ of degree exactly five be chosen so that $|V(G_0)|$ is minimum, and subject to that, several other parameters are optimized. An unordered pair of vertices $\{x, y\}$ is called an *identifiable pair* if x and y are not adjacent and $x, y \in N(v_0)$. Let (G_0, v_0) be as stated, let $\{x, y\}$ be an identifiable pair, and let G_{xy} be as in the previous section. By the minimality of G_0 the graph G_{xy} has a subgraph J isomorphic to one of the graphs from Theorem 2, and hence J^w_W is isomorphic to a subgraph of G_0 for some choice of $w \in V(J)$ and some partition W of the neighbors of w. If $J = C_3 + C_5$ or $J = K_2 + H_7$, then we conclude the proof using a minor modification of the corresponding argument in [19]. If J is one of the graphs L_i, then we need to examine possible drawings of those graphs in the Klein bottle. Luckily, in all cases the graph J^w_W has all faces of size at most seven, and so we can use our explicit version of Lemma 5 for cycles of length at most seven. Finally, the hardest case is when J is K_6, but even then we get by with the same version of Lemma 5, making use of all possible identifiable pairs. We refer to [5] for more details.

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