The Extremal Function for K_9 Minors

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Abstract

We prove that every (simple) graph on $n \geq 9$ vertices and at least $7n - 27$ edges either has a K_9 minor, or is isomorphic to $K_{2,2,2,3,3}$, or is isomorphic to a graph obtained from disjoint copies of $K_{1,2,2,2,2,2}$ by identifying cliques of size six. The proof of one of our lemmas is computer-assisted.

1 Introduction

All graphs in this paper are finite and simple. Our work is motivated by the following theorem of Mader [18].

Theorem 1.1 *For every integer $p = 1, 2, \ldots, 7$, a graph on $n \geq p$ vertices and at least $(p-2)n - \binom{p-1}{2} + 1$ edges has a K_p minor.*

For $p \leq 5$, this was first proved by Dirac [5]. Some years later but independently of Mader, Györi [7] proved Theorem 1.1 for $p \leq 6$.

Mader pointed out that Theorem 1.1 does not hold for $p = 8$: the graph $K_{2,2,2,2,2}$ is a counterexample. However, one can construct further counterexamples by repeatedly identifying cliques of size five. So for graphs H_1, H_2 and an integer k, let us define an (H_1, H_2, k)-*cockade* recursively as follows. Any graph isomorphic to H_1 or H_2 is an (H_1, H_2, k)-cockade. Now let G_1, G_2 be (H_1, H_2, k)-cockades and let G be obtained from the disjoint union of G_1 and G_2 by identifying a clique of size k in G_1 with a clique of the same size in G_2. Then the graph G

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is also an (H_1, H_2, k)-cockade, and every (H_1, H_2, k)-cockade can be constructed this way. If $H_1 = H_2 = H$, then G is called an (H, k)-cockade. Jørgensen [13] generalized Theorem 1.1 as follows.

Theorem 1.2 *Every graph on $n \geq 8$ vertices and at least $6n - 20$ edges either has a K_8 minor, or is a $(K_{2,2,2,2,2}, 5)$-cockade.*

To see that Theorem 1.2 implies Theorem 1.1, let G and p be as in Theorem 1.1, and apply Theorem 1.2 to the graph obtained from G by adding $8 - p$ vertices, each adjacent to every other vertex of the graph.

Our main result is the following next step. Note that every $(K_{2,2,2,3,3}, 6)$-cockade is isomorphic to $K_{2,2,2,3,3}$.

Theorem 1.3 *Every graph on $n \geq 9$ vertices and at least $7n - 27$ edges either has a K_9 minor, or is a $(K_{1,2,2,2,2,2}, 6)$-cockade, or is isomorphic to $K_{2,2,2,3,3}$.*

Our motivation was threefold. First, the bound in Theorem 1.1 is related to Hadwiger's conjecture [8], the following.

Conjecture 1.4 *For every integer $t \geq 1$, every graph with no K_{t+1} minor is t-colorable.*

Hadwiger's conjecture is trivially true for $t \leq 2$, and reasonably easy for $t = 3$, as shown by Dirac [4]. However, for $t \geq 4$, Hadwiger's conjecture implies the Four Color Theorem. (To see that, let H be a planar graph, and let G be obtained from H by adding $t-4$ vertices, each joined to every other vertex of the graph. Then G has no K_{t+1} minor, and hence is t-colorable by Hadwiger's conjecture, and hence H is 4-colorable.) Wagner [26] proved that the case $t = 4$ of Hadwiger's conjecture is, in fact, equivalent to the Four Color Theorem, and the same was shown for $t = 5$ by Robertson, Seymour, and the second author [19]. Their proof made use of Theorem 1.1 for $p = 6$. Hadwiger's conjecture remains open for $t \geq 6$.

In [3] Chen, Gould, Kawarabayashi, Pfender and Wei proved that every graph on n vertices and at least $9n - 45$ edges has a K_9 minor, and used that to deduce that if, in addition, G is 6-connected, then it is 3-linked. It turns out [22] that the latter conclusion can be obtained from a weaker bound on the number of edges by a more direct argument, but the work of Chen, Gould, Kawarabayashi, Pfender and Wei suggested that there may be interest in the extremal problem for K_9 minors.

Theorem 1.1 is such a nice result that it raises the question of whether it can be generalized to all values of p. But there are more depressing news than Mader's example above: for large p a graph must have at least $\Omega(p\sqrt{\log p}n)$ edges in order to guarantee a K_p minor, because, as noted by several people (Kostochka [15, 16], and Fernandez de la Vega [6] based on Bollobás, Catlin and Erdös [2]), a random graph with no K_p minor may have average degree of order $p\sqrt{\log p}$. Kostochka [15, 16] and Thomason [23] proved that this is indeed the correct order of magnitude, and in a remarkable recent result [24] Thomason was able to determine the constant of proportionality. Thus it may seem that an effort to generalize Theorem 1.1 will be in vain, but there is still the following possibility. The random graph examples provide only finitely many counterexamples for any given value of p. Of course, more counterexamples can be obtained by taking disjoint unions or even gluing counterexamples along small cutsets, but we know of no construction of highly connected infinite families of counterexamples. More specifically, Seymour and the second author conjecture the following.

Conjecture 1.5 *For every $p \geq 1$ there exists a constant $N = N(p)$ such that every $(p-2)$-connected graph on $n \geq N$ vertices and at least $(p-2)n - \binom{p-1}{2} + 1$ edges has a K_p minor.*

It was this conjecture that was the third motivating factor for our research. Thus our main result implies that Conjecture 1.5 holds for $p \leq 9$. Recently, Böhme, Kawarabayashi, Maharry and Mohar [1] showed that Conjecture 1.5 holds for $31(p+1)/2$-connected graphs.

The extremal functions for K_p^- minors have also been studied, where K_p^- denotes the graph obtained from K_p by removing one edge. Jakobsen [10, 11] proved that, for $p \leq 7$, every graph on $n \geq p$ vertices and at least $(p - \frac{5}{2})n - \frac{1}{2}(p-3)(p-1)$ edges has a K_p^- minor, or G is a $(K_{p-1}, p-3)$-cockade, or $p = 7$ and G is a $(K_{2,2,2,2}, K_6, 4)$-cockade. Recently, the first author [21] proved a conjecture of Jakobsen [11] that every graph on $n \geq 8$ vertices and at least $\frac{1}{2}(11n - 35)$ edges has a K_8^- minor or is a $(K_{1,2,2,2,2}, K_7, 5)$-cockade. The extremal functions for the graphs obtained from K_p by deleting two edges were determined in [9, 10] when $p = 7$ or 8. In related work Jørgensen [14] proved that every 4-connected graph on $n \geq 8$ vertices and at least $4n - 7$ edges has a $K_{4,4}$ minor.

We need to introduce more notation. If G is a graph and K is a subgraph of G, then by $N(K)$ we denote the set of vertices of $V(G) - V(K)$ that are adjacent to a vertex of K. If $V(K) = \{x\}$, then we use $N(x)$ to denote $N(K)$. By abusing notation we will also denote by $N(x)$ the graph induced by the set $N(x)$. We define $N[x] = N(x) \cup \{x\}$, and similarly will use the same symbol for the graph induced by that set. If x, y are adjacent vertices of a graph G, then we denote by G/xy the graph obtained from G by contracting the edge xy and deleting all resulting parallel edges. If u, v are distinct nonadjacent vertices of a graph G, then by $G + uv$ we denote the graph obtained from G by adding an edge with ends u and v. If u, v are adjacent or equal, then we define $G + uv$ to be G. We write $G > H$ if a graph isomorphic to H can be obtained from a subgraph of G by contracting edges. In those circumstances we say that G has an H *minor*. For a graph G we use $|G|$, $e(G)$, $\delta(G)$ to denote the number of vertices, number of edges and minimum degree of G, respectively. The degree of a vertex v in a graph is denoted by $d_G(v)$ or simply $d(v)$.

2 Outline of proof

Suppose for a contradiction that G is a counterexample to Theorem 1.3 with minimum number of vertices, say n. Since deletion or contraction of edges does not produce smaller counterexamples, it follows easily that G has minimum degree at least eight, and with some effort it can be shown that every edge of G is in at least seven triangles. It also follows by a straightforward counting argument that G is 6-connected. Also $e(G) = 7n - 27$, and hence G has a vertex x of degree at least eight and at most thirteen. Fix such a vertex, and let K be a component of $G - N[x]$. Assume for a moment that every vertex of $N(x)$ has a neighbor in K. If there exists a vertex $y \in N(x)$ such that $N(x) - y > K_7$, then by contracting the connected set $V(K) \cup \{y\}$ to a single vertex, we see that $G > K_9$. Thus $G - y \not> K_7$ for every vertex $y \in N(x)$. On the other hand, $N(x)$ has minimum degree at least seven and at most thirteen vertices. Those properties are fairly restrictive: there are only fourteen such graphs, and so they can be found explicitly. It turns out that they all have two properties in common (conditions (A) and (B) stated prior to Lemma 3.7) that allow us to find a K_9 minor in G in a different way. This is how we deal with the case when there is a component K of $G - N[x]$ satisfying $N(x) = N(K)$. In fact, the argument extends to the situation when there exists a component K of $G - N[x]$ such that $N(K') \cap M \subseteq N(K)$ for every component K' of $G - N[x]$, where M is the set of all vertices of $N(x)$ that are not adjacent to every other vertex of $N(x)$.

Thus we may assume that for no vertex x of degree at most thirteen such a component exists. In the next step we prove a lemma inspired by Claim (15) of [13], namely that if $x \in V(G)$ has degree at most 13, then there is no component K of $G - N[x]$ such that

$d_G(v) \geq 14$ for all vertices $v \in V(K)$. This follows by counting edges, for if such a component exists, then we exhibit a proper minor of G with $n' < n$ vertices and more than $7n' - 27$ edges. That minor of G has a K_9 minor by the minimality of G, and hence G has a K_9 minor, a contradiction. Finally, in the last step, we select a vertex $x \in V(G)$ of degree at most thirteen to minimize the size of a component K of $G - N[x]$. It follows easily that K does not have a vertex whose degree in G is at most 13.

3 Preliminaries

The following result of Jørgensen [13] follows from the proof of Lemma 3.7 below, but we state it separately for convenience.

Theorem 3.1 *Let G be a graph with $n \leq 11$ vertices and $\delta(G) \geq 6$ such that for every vertex x in G, $G - x$ is not contractible to K_6. Then G is one of the graphs $K_{2,2,2,2}, K_{3,3,3}$ or the complement of the Petersen graph.*

The next theorem was first proved by Jung [12]. Seymour [20] and Thomassen [25] gave a complete characterization of all (not necessarily 4-connected) graphs that satisfy the hypothesis of the theorem.

Theorem 3.2 *Let G be a 4-connected graph and let x_1, x_2, y_1, y_2 be distinct vertices in G. If G does not contain an $x_1 - y_1$ path and an $x_2 - y_2$ path that are disjoint, then G is planar and $e(G) \leq 3|G| - 7$.*

As noted in Section 2, our proof uses induction by deleting and contracting edges of G. Thus we need to investigate graphs G such that the new graph $G - xy$ or G/xy is a $(K_{1,2,2,2,2,2}, 6)$-cockade or is isomorphic to $K_{2,2,2,3,3}$. We do that next.

Lemma 3.3 *Let G be $K_{2,2,2,3,3}$ or a $(K_{1,2,2,2,2,2}, 6)$-cockade and let x and y be nonadjacent vertices in G. Then $G + xy$ is contractible to K_9.*

Proof. This is easily checked if $G = K_{2,2,2,3,3}$ or $G = K_{1,2,2,2,2,2}$. So we may assume that G is obtained from H_1 and H_2 by identifying cliques of size 6, where H_1 and H_2 are $(K_{1,2,2,2,2,2}, 6)$-cockades. If x and y are both in H_1 or H_2, then $H_1 + xy > K_9$ or $H_2 + xy > K_9$ by induction. So we may assume that $x \in V(H_1) - V(H_2)$ and $y \in V(H_2) - V(H_1)$. Note that no

$(K_{1,2,2,2,2,2}, 6)$-cockade contains K_7 as a subgraph. Therefore there exists $z \in V(H_1) \cap V(H_2)$ such that $zy \notin V(G)$. Now by contracting $V(H_1) - V(H_2)$ to the vertex z in $G + xy$, the resulting graph is $H_2 + zy$. By induction, $H_2 + zy > K_9$. ∎

Lemma 3.4 *Let G be a graph. Let $x, y \in V(G)$ be such that $xy \in E(G)$ with exactly six common neighbors. If G/xy is isomorphic to $K_{2,2,2,3,3}$, then $G > K_9$.*

Proof. Let w be the new vertex in G/xy. Since x and y have exactly six common neighbors, there exist distinct vertices $w_1, w_2, w_3, w_4 \in V(G/xy) - w$ such that $w_1w_2, w_3w_4 \notin E(G/xy)$, and w_1, w_2, w_3 are common neighbors of x and y in G. Moreover, w_4 is adjacent to x or y, say to y, in G. By contracting the edges xw_2 and yw_4 we see that G has a K_9 minor, as desired. ∎

Lemma 3.5 *Let G be a graph and let x, y be adjacent vertices of G with exactly six common neighbors. If G/xy is isomorphic to $K_{1,2,2,2,2,2}$, then G has a K_9 minor, unless G is isomorphic to $K_{2,2,2,3,3}$ and x, y have degree nine in G.*

Proof. Let w be the new vertex of G/xy, and let $z, x_1, y_1, \ldots, x_5, y_5$ be the vertices of G/xy numbered so that x_i is not adjacent to y_i. Assume first that $w \neq z$, say $w = x_1$. Since x and y have six common neighbors, we may assume that x_2, y_2, x_3 are common neighbors of x and y. Moreover, y_3 is adjacent to x or y, say to y. By contracting the edges xy_2, yy_3 and y_4y_5 we see that G has a K_9 minor, as desired.

Thus we may assume that $w = z$. Since x, y have six common neighbors, their degree is at least seven. Assume for a moment that $d_G(x) = 7$. Since x, y have six common neighbors in G, we deduce that y is adjacent to all other vertices of G and there exists an index i such that x_i, y_i are common neighbors of x, y. We may assume that $i = 1$. By contracting the edges xx_1, x_2x_3 and x_4x_5, we obtain a K_9 minor of G. Hence we may assume that $d(x), d(y) \geq 8$. We may also assume that G is not isomorphic to $K_{2,2,2,3,3}$ with x, y of degree nine, and so it follows that one of x, y is adjacent to x_i or y_i for every $i = 1, 2, 3, 4, 5$. Thus we may assume (by swapping x_i and y_i) that x is adjacent to all of X, where $X = \{x_1, \ldots, x_5\}$. Moreover, we may assume that if y is also adjacent to every vertex of X, then $d(x) \leq d(y)$. Let $Y = \{y_1, \ldots, y_5\}$. Since y has degree at least eight, there is some i such that y is adjacent to x_i and y_i. We claim that y is adjacent to at least three vertices of Y. For if not, then x is adjacent to at least three vertices of Y (the non-neighbors of y) and, since $d(y) \geq 8$, y is adjacent to all vertices of X. But then $d(x) > d(y)$, a contradiction. Thus y is adjacent to at least three vertices of Y.

Thus there exist distinct indices i, j, k such that y is adjacent to x_i, y_i, y_j, y_k. Choose such indices so that, if possible, x is not adjacent to y_i. We may assume that $i = 1$, $j = 2$ and $k = 3$. We claim that x is adjacent to at least two vertices of $Y - \{y_1\}$. For if not, then y has at least four neighbors in Y, and hence x, y have at least four common neighbors in X, and so the indices i, j, k above can be chosen so that x is not adjacent to y_i. Thus x is not adjacent to y_1, and hence x has at most one neighbor in Y, implying that $d(x) = 7$, a contradiction. Thus x has at least two neighbors in $Y - \{y_1\}$, and so we may assume that x has a neighbor in $\{y_2, y_4\}$ and a neighbor in $\{y_3, y_5\}$. By contracting the edges yy_1, y_2y_4 and y_3y_5 we see that G has a K_9 minor, as required. ■

Lemma 3.6 *Let G be a graph with $\delta(G) \geq 7$. Let $x, y \in V(G)$ be such that $xy \in E(G)$ with exactly six common neighbors. If G/xy is a $(K_{1,2,2,2,2,2,2}, 6)$-cockade, then either $G > K_9$, or G is isomorphic to $K_{2,2,2,3,3}$ and x, y have degree nine in G.*

Proof. We proceed by induction on $|G|$. By Lemma 3.5 we may assume that $G/xy = H_1 \cup H_2$, where $H_1 \cap H_2$ is a complete graph on six vertices and both H_1 and H_2 are $(K_{1,2,2,2,2,2,2}, 6)$-cockades. Let w be the new vertex of G/xy. For $i = 1, 2$ let $H_i^* = G[(V(H_i) - \{w\}) \cup \{x, y\}]$. If $w \in V(H_1) - V(H_2)$, then $H_1^* \neq K_{2,2,2,3,3}$ (because the latter graph has no K_6 subgraph) and the result follows by induction applied to H_1^*. From the symmetry we may assume that $w \in V(H_1) \cap V(H_2)$. Let $S = V(H_1) \cap V(H_2) - \{w\}$; thus $V(H_1^*) \cap V(H_2^*) = S \cup \{x, y\}$. Let Z denote the set of six common neighbors of x and y in G. If $Z \subseteq V(H_1^*)$, then by induction applied to H_1^* we may assume that H_1^* is isomorphic to $K_{2,2,2,3,3}$ and x, y have degree nine in H_1^*. Since H_1^* has no K_6 subgraph one of x, y, say x, is not adjacent to some $s \in S$ and x has at least one neighbor in $V(H_2) - V(H_1)$. By using a path with ends x and s and interior in $H_2^* - V(H_1^*)$ we deduce that $G > H_1^* + sx > K_9$ by Lemma 3.3, as desired.

Thus we may assume that $Z - V(H_1^*) \neq \emptyset \neq Z - V(H_2^*)$. Since H_2 is a $(K_{1,2,2,2,2,2,2}, 6)$-cockade, it is 6-connected. Let $k = |Z - V(H_1)|$. Since $|Z \cap V(H_2)| \leq 5$ we have $|S - Z| = 5 - |Z \cap S| \geq k$. Thus there exist k disjoint paths $P_1, P_2, \ldots, P_k$ in $H_2 - (Z \cap S) - w$ between $Z \cap V(H_2 - S)$ and $S - Z$. Consequently H_1^* has a supergraph H_1' on the same vertex set such that $H_1' < G$ and x, y have exactly six common neighbors in H_1'. By induction H_1' is isomorphic to $K_{2,2,2,3,3}$ and x, y have degree nine in H_1'. By symmetry the same holds for the analogous graph H_2'. It follows that in H_1' the vertex x has a unique non-neighbor in S, say x'. Then $x' \notin V(P_1 \cup \cdots \cup P_k)$. From the symmetry between H_1 and H_2 we may assume that $k \leq 3$. (In fact, $|Z - V(H_1)| = |Z - V(H_2)| = 3$.) It follows that the k disjoint paths $P_1, \ldots, P_k$ can each be chosen of length one, and that there exists a common neighbor of x and x' in $V(H_2^*)$, say u, that does not belong to any of the paths. Thus by contracting the edge ux' and all the edges of the paths $P_1, \ldots, P_k$ we deduce that $G > H_1' + xx' > K_9$ by Lemma 3.3, as desired. ∎

As pointed out in Section 2, we need to examine graphs G such that $|V(G)| \leq 13$, $\delta(G) \geq 7$ and $G \not> K_7 \cup K_1$. (Here $K_7 \cup K_1$ stands for a disjoint union of K_7 and K_1.) The next lemma shows that those graphs G satisfy the following properties:

(A) either G is isomorphic to $K_{1,2,2,2,2,2}$, or G has four distinct vertices a_1, b_1, a_2, b_2 such that $a_1a_2, b_1b_2 \notin E(G)$ and for $i = 1, 2$ the vertex a_i is adjacent to b_i, the vertices a_i, b_i have at most four common neighbors, and $G + a_1a_2 + b_1b_2 > K_8$,

(B) for any two sets $A, B \subseteq V(G)$ of cardinality at least five such that neither is complete and $A \cup B$ includes all vertices of G of degree at most $|G| - 2$, either

(B1) there exist $a \in A$ and $b \in B$ such that $G' > K_8$, where G' is obtained from G by adding all edges aa' and bb' for $a' \in A - \{a\}$ and $b' \in B - \{b\}$, or

(B2) there exist $a \in A - B$ and $b \in B - A$ such that $ab \in E(G)$ and the vertices a and b have at most five common neighbors in G, or

(B3) one of A and B contains the other and $G + ab > K_7 \cup K_1$ for all distinct nonadjacent vertices $a, b \in A \cap B$.

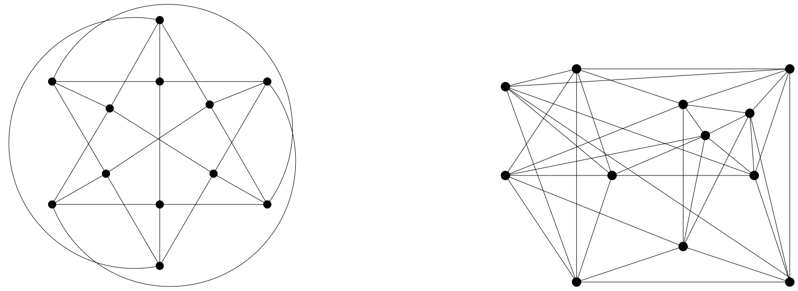

Figure 1: graph J_1 and graph J_2.

Lemma 3.7 *Let n be an integer satisfying $9 \leq n \leq 13$ and let G be a graph on n vertices with $\delta(G) \geq 7$. Then either $G > K_7 \cup K_1$, or G satisfies* (A) *and* (B).

Proof. By a computer search we have determined that the graphs G with $9 \leq n \leq 13$ vertices, $\delta(G) \geq 7$ and $G \not> K_7 \cup K_1$ are the following ones: $K_{1,2,2,2,2}$, $K_{1,3,3,3}$, $K_{3,3} + P_4$, $K_{3,3} + \overline{C}_4$, $K_{2,2,3,3}$, $K_{2,3} + C_5$, $C_5 + C_5$, $\overline{K}_3 + \overline{C}_7$, $K_{3,4,4}$, $\overline{K}_3 + \overline{V}_8$, $K_1 + \overline{P}$, $\overline{P'}$, $\overline{J_1}$ and $K_1 + J_2$. Here $H + G$ stands for the graph obtained from $G \cup H$ by adding all edges with one end in $V(H)$ and the other in $V(G)$, $\overline{G}$ denotes the complement of a graph, P_4 denotes the path of four vertices, V_8 denotes the graph obtained from C_8 by joining all four pairs of diagonally opposite vertices, P is the Petersen graph, P' denotes the graph obtained from P by subdividing one edge, and the graphs J_1 and J_2 are depicted in Figure 1. It is straightforward to check that those graphs satisfy (A) and (B). The details of this and of the computer search can be obtained from the authors' websites. ■

4 Proof of Theorem 1.3

In this section, we are going to prove Theorem 1.3 by induction on n. The only graph G with 9 vertices and $e(G) \geq 7 \times 9 - 27 = 36$ is K_9. Thus we may assume that $n \geq 10$ and that the assertion holds for smaller values of n. Throughout this section we assume that G is a graph with n vertices and $e(G) \geq 7n - 27$ but G is not contractible to K_9 and G is not $K_{2,2,2,3,3}$ or a $(K_{1,2,2,2,2,2}, 6)$-cockade. By Lemma 3.3, we may assume that $e(G) = 7n - 27$.

Suppose that G has a vertex x of degree at most 6. Then $e(G - x) \geq 7(n-1) - 26$, and hence $G > G - x > K_9$ by induction, a contradiction. Suppose now that G has two adjacent vertices x, y with at most five common neighbors. Then $e(G/xy) \geq 7(n-1) - 26$. By induction, $G > K_9$, a contradiction. Thus $\delta(G) \geq 7$ and $\delta(N(x)) \geq 6$. If G has a vertex x of degree 7, then $N(x) = K_7$ and $e(G - x) \geq 7(n-1) - 27$. Note that neither a $(K_{1,2,2,2,2,2}, 6)$-cockade nor $K_{2,2,2,3,3}$ contain K_7 as a subgraph. Thus, by induction, $G - x > K_9$, a contradiction. Hence

(1) $\delta(G) \geq 8$ and $\delta(N(x)) \geq 6$ for any $x \in V(G)$.

Let S be a separating set of vertices in G, and let G_1 and G_2 be proper subgraphs of G so that $G = G_1 \cup G_2$ and $G_1 \cap G_2 = G[S]$. Let $m_i = 7|G_i| - 27 - e(G_i)$, $i = 1, 2$. Then $7n - 27 = e(G) = e(G_1) + e(G_2) - e(G[S]) = 7n + 7|S| - 54 - m_1 - m_2 - e(G[S])$, and so

(2) $7|S| = 27 + m_1 + m_2 + e(G[S])$.

For $i = 1, 2$, let d_i be the maximum number of edges that can be added to G_{3-i} by contracting edges of G with at least one end in G_i. More precisely, let d_i be the largest integer so that G_i contains disjoint set of vertices $V_1, V_2, \ldots, V_p$ so that $G_i[V_j]$ is connected, $|S \cap V_j| = 1$ for $1 \leq j \leq p = |S|$, and so that the graph obtained from G_i by contracting $V_1, V_2, \ldots, V_p$ and deleting $V(G) - (\bigcup_j V_j)$ has $e(G[S]) + d_i$ edges. By (1), $\delta(G) \geq 8$. Thus $|G_i| \geq 9$, $i = 1, 2$. By induction, $d_1 \leq m_2$ and $d_2 \leq m_1$. By (2),

(3) $7|S| \geq 27 + d_1 + d_2 + e(G[S])$.

In particular, $|S| \geq 4$. If S is a minimal separating set, then let $v \in S$ be a vertex of minimum degree in $G[S]$. By choosing $V_1 = V(G_i) - (S - \{v\})$ and the rest of the sets V_j to be singletons, we see that $d_i \geq |S| - 1 - \delta(G[S])$ for $i = 1, 2$. Thus

(4) if S is a minimal separating set, then

$$5|S| \geq 25 + e(G[S]) - 2\delta(G[S])) \geq 25 + \frac{1}{2}(|S| - 4)\delta(G[S]).$$

Lemma 4.1 *G is 6-connected.*

Proof. Suppose G is not 6-connected. Let S be a minimal separating set of G, and let G_1, G_2, d_1, d_2 be as above. By (4) G is 5-connected and $G[S] = \overline{K_5}$. We next show that $d_1 \geq 5$. Let x and y be distinct vertices in $G_1 \backslash S$. By Menger's theorem, there exist five x-S paths $P_1, P_2, \ldots, P_5$ in G_1 which have only the vertex x in common. If all these paths have length 1, then, since there are at least four internally disjoint y-S paths in $G_1 \backslash \{x\}$, by contracting these paths we deduce that $d_1 \geq 7$. We may now assume that P_1 has length at least 2. Let $V(P_1) \cap S = \{z\}$. As $\{x, z\}$ is not a separating set in G, there is a path P from a vertex on $P_1 \backslash \{x, z\}$ to a vertex on some $P_i \backslash \{x\}$, $i \neq 1$, so that only the end vertices of P belong to $\bigcup_{j=1}^{5} P_j$. By contracting a suitable subset of the edges of $P \cup P_1 \cup \cdots \cup P_5$ we deduce that $d_1 \geq 5$, as claimed.

By symmetry, $d_2 \geq 5$ and so $d_1 + d_2 \geq 10$. However, by (3), $d_1 + d_2 \leq 8$, which is a contradiction. ∎

Lemma 4.2 *There is no separating set S with a vertex x so that $G[S - x]$ is complete.*

Proof. Suppose that $G[S - x]$ is complete and let G_1, G_2 be as above. We may assume that S is a minimal separating set. By Lemma 4.1, $|S| \geq 6$. If $|S| \geq 8$, by contracting $V(G_1) - S$

to x and $V(G_2)-S$ to a new vertex, we get a K_9 minor, a contradiction. So we may assume that $|S|=6$ or $|S|=7$.

If $|S|=6$, by (4), $5|S|\geq 25+e(G[S])-2\delta(G[S]))\geq 25+10+\delta(G[S])-2\delta(G[S])$, which implies that $G[S]=K_6$. By induction, we may assume $e(G_i)\leq 7|G_i|-27$, $i=1,2$. Since $7n-12=7n-27+15=e(G)+15=e(G_1)+e(G_2)\leq 7|G_1|-27+7|G_2|-27=7n-12$, it follows that $e(G_i)=7|G_i|-27$, $i=1,2$. Since $K_{2,2,2,3,3}$ does not contain K_6 as a subgraph, by induction, $G_i>K_9$ or G_i is a $(K_{1,2,2,2,2,2},6)$-cockade. Thus $G>K_9$ or G is a $(K_{1,2,2,2,2,2},6)$-cockade, a contradiction.

If $|S|=7$, by (4), $5|S|\geq 25+e(G[S])-2\delta(G[S]))\geq 25+15+\delta(G[S])-2\delta(G[S])$, which implies that $G[S]$ is isomorphic to K_7 or K_7 with an edge deleted. Let $e(G[S])=21-t$, where $t=0$ or 1. Suppose $e(G_1)\geq 7|G_1|-27-t$. Let G_1' be obtained from G by contracting $V(G_2)-S$ to x. Then $e(G_1')=e(G_1)+t\geq 7|G_1'|-27$. Since G_1' contains a K_7 subgraph, it is not $K_{2,2,2,3,3}$ or a $(K_{1,2,2,2,2,2},6)$-cockade, and hence by induction, $G>G_1'>K_9$. Thus $e(G_1)\leq 7|G_1|-28-t$. Similarly, we have $e(G_2)\leq 7|G_2|-28-t$. But now $e(G)=e(G_1)+e(G_2)-e(G[S])\leq 7(n+7)-28-t-28-t-21+t=7n-28-t$, which is a contradiction. ∎

Lemma 4.3 *$\delta(N(x))\geq 7$ for any $x\in V(G)$.*

Proof. Suppose $\delta(N(x))\leq 6$. By (1) there exists a vertex $y\in N(x)$ such that x and y have exactly six common neighbors. Then $e(G/xy)=7(n-1)-27$. Since $G\not> K_9$, the minimality of $|G|$ implies that G/xy is isomorphic to $K_{2,2,2,3,3}$ or is a $(K_{1,2,2,2,2,2},6)$-cockade. In either case, by Lemma 3.4 or Lemma 3.6, $G>K_9$ or $G=K_{2,2,2,3,3}$, a contradiction. ∎

Lemma 4.4 *$\delta(G)\geq 9$.*

Proof. Let $x\in V(G)$ be such that $d(x)=\delta(G)\leq 8$. By Lemma 4.3, $N(x)=K_8$ and so $G>N[x]=K_9$, a contradiction. ∎

Lemma 4.5 *If $G-N[x]$ is 2-connected or has at most two vertices, then $N(x)\neq K_{1,2,2,2,2}$.*

Proof. Suppose for a contradiction that $N(x)=K_{1,2,2,2,2}$. Let $V(N(x))=\{y,z_1,z_2,z_3,z_4,$ $w_1,w_2,w_3,w_4\}$ so that y is adjacent to all vertices in $N(x)-y$ and $z_iw_i\notin E(G)$.

We next show that z_i and w_i have no common neighbor in $G - N[x]$ for $i = 1, 2, 3, 4$. To this end suppose that there exists a vertex $v \in V(G - N[x])$ adjacent to, say z_1 and w_1. Let $K = G - N[x] - v$. Then K is not null by Lemma 4.4, because G is not isomorphic to $K_{1,2,2,2,2,2}$. Since $G - N[x]$ has no cut vertex, K is connected. If $z_i, w_i \in N(K)$ for some $i \in \{2, 3, 4\}$, then let P be a path with ends z_i and w_i and interior in K. By contracting the edge z_1v and all but one of the edges of P we see that $G > N[x] + z_1v_1 + z_iw_i > K_9$, a contradiction. Thus we may assume that $w_2, w_3, w_4 \notin N(K)$. Let $i \in \{2, 3, 4\}$. It follows from Lemma 4.3 applied to w_i that v is adjacent to w_i. By Lemma 4.3 the edge vw_i is in at least seven triangles, and hence z_2, z_3, z_4 are all adjacent to v. By Lemma 4.2 the set $N(K) - \{v\}$ is not complete, and hence $z_1, w_1 \in N(K)$. By contracting the edge vw_2 and all but one edge of a z_1-w_1 path with interior in K we deduce that $G > N[x] + z_1w_1 + z_2w_2 > K_9$, a contradiction. This proves that the vertices z_i and w_i have no common neighbor in $G - N[x]$.

Let $u \in V(G) - N[x]$ be a neighbor of z_1. By Lemma 4.3 the vertices u and z_1 have at least seven common neighbors, and so by the result of the previous paragraph z_1 has at least four neighbors in $G - N[x]$. By symmetry the same holds for all z_i and w_i.

Let $H = G - \{x, y, z_3, w_3, z_4, w_4\}$. We next show that H is 4-connected. Suppose for a contradiction that S is a minimal separating set of at most three vertices in H. Since $G - N[x]$ has no cut vertex, $|S| \geq 2$ and $|S \cap N(x)| \leq 1$. If $|S \cap N(x)| = 1$, we may assume that $w_1 \in S$. Since $z_1z_2, z_1w_2 \in E(G)$, z_1, z_2, w_2 are in the same component of $H - S$. Denote this component by K. If $w_1 \notin S$, then also $w_1 \in K$. Since z_2, w_2 have at least four neighbors in $G - N[x]$, there exist z_2' and w_2' in $G - N[x] - S$ adjacent to z_2 and w_2, respectively. Clearly, z_2' and w_2' belong to K. As $G - N[x]$ has no cut vertex, $G - N[x]$ contains two independent z_2'-w_2' paths. One of these paths is contained in $G[K \cup S]$.

Since G is not contractible to $N[x] + z_2w_2 + z_iw_i > K_9$ for $i = 3, 4$, there is no z_i-w_i path in $G[K' \cup \{z_i, w_i\}]$, where $K' \neq K$ is another component of $H - S$. But this implies that K' is separated from x by S and three pairwise adjacent vertices. We may assume that such three vertices are y, w_3, w_4. Since G is 6-connected, $|S| = 3$. Let $S = \{s_1, s_2, s_3\}$, where $s_1 = w_1$ if $w_1 \in S$, and let $S' = S \cup \{y, w_3, w_4\}$. Then S' is a minimal separating set of G. Let $H_1 = G[K' \cup S']$ and $H_2 = G - K'$. Let d_1 and d_2 be defined as in the paragraph prior to (3). Clearly, $K \cup \{x, z_3, z_4\}$ is contained in H_2. By Menger's theorem, there exist three disjoint paths between $\{x, w_1, w_2\}$ and S in $G - \{y, w_3, w_4\}$. Now by contracting those paths, we get $d_2 + e(G[S']) = e(K_6) = 15$. By Lemma 4.2, $d_1 \geq 1$. By (3), $42 = 7|S'| \geq 27 + 1 + 15 = 43$, a contradiction. Thus H is 4-connected.

Since G is not contractible to K_9, it follows from Theorem 3.2 applied to the vertices z_1, z_2, w_1, w_2 that $e(H) \leq 3|H| - 7 = 3(n-6) - 7$. Since for $i \in \{3, 4\}$ the vertices z_i and w_i have no common neighbor in $G - N[x]$, they together have at most $|G| - |N[x]| = n - 10$ neighbors in $G - N[x]$. The vertices $\{z_3, w_3, z_4, w_4\}$ are incident with 20 edges of $N(x) - y$. Thus

$$\begin{aligned} 7n - 27 = e(G) &\leq d(x) + d(y) - 1 + e(H) + 2(n-10) + 20 \\ &\leq 9 + n - 2 + 3(n-6) - 7 + 2(n-10) + 20 = 6n - 18. \end{aligned}$$

It follows that $n \leq 9$, a contradiction. ■

Lemma 4.6 *Let $x \in V(G)$ be such that $9 \leq d(x) \leq 13$. Then there is no component K of $G - N[x]$ such that $N(K') \cap M \subseteq N(K)$ for every component K' of $G - N[x]$, where M is the set of vertices of $N(x)$ not adjacent to all other vertices of $N(x)$.*

Proof. Assume such a component K exists. Among all vertices x with $9 \leq d(x) \leq 13$ for which such a component exists, choose x to be of minimal degree. We first prove that $M \subseteq N(K)$. Suppose for a contradiction that $M - N(K) \neq \emptyset$, and let $y \in M - N(K)$ be such that $d(y)$ is minimum. Clearly, $d(y) < d(x)$. Let J be the component of $G - N[y]$ containing K. Since $d(y) < d(x)$ the choice of x implies that $N(x) - N[y] \not\subseteq V(J)$, and hence some component H of $N(x) - N[y]$ is disjoint from $N(K)$. We have $d_G(z) \geq d_G(y)$ for all $z \in V(H)$ by the choice of y. Let $t = |V(H)|$. Then $t \geq 2$, for otherwise the vertex y and component H contradict the choice of x. On the other hand $t \leq d(x) - d(y) \leq 13 - 9 = 4$. From Lemma 4.3 applied to y we deduce that $N(y) \cap N(x)$ has minimum degree at least six. Let L be the subgraph of G induced by $(N[y] \cap N(x)) \cup V(H)$. Then the edge-set of L consists of edges of $N(x) \cap N(y)$, edges incident with y, and edges incident with $V(H)$. Thus

$$\begin{aligned} e(L) &\geq 3(d(y) - 1) + d(y) - 1 + t(d(y) - 1) - \frac{1}{2} t(t-1) \\ &\geq 6(d(y) + t) + (t-2)d(y) - 4 - 7t - \frac{1}{2} t(t-1) \geq 6|V(L)| - 20, \end{aligned}$$

because $d(y) \geq 9$ and $2 \leq t \leq 4$. Since $11 \leq |V(L)| \leq 13$ the graph L is not a $(K_{2,2,2,2,2}, 5)$-cockade, and hence $N(x) > L > K_8$ by Theorem 1.2. Thus $G > K_9$, a contradiction. This proves that $M \subseteq N(K)$.

If $N(x) > K_7 \cup K_1$, then $N(x)$ has a vertex y such that $N(x) - y > K_7$. If $y \notin M$, then $N(x) > K_8$. Otherwise, by contracting the connected set $V(K) \cup \{y\}$ we can contract K_8 onto $N(x)$. Thus in either case $G > K_9$, a contradiction. Thus by Lemma 3.7, we may assume that $N(x)$ satisfies properties (A) and (B).

If $G - N[x]$ is 2-connected or has at most two vertices, then by Lemma 4.5, we may assume that $N(x) \neq K_{1,2,2,2,2}$. Then by property (A) and Lemma 4.3 the set $N(x)$ has four distinct vertices a_1, b_1, a_2, b_2 such that $a_1a_2, b_1b_2 \notin E(G)$, $N(x) + a_1a_2 + b_1b_2 > K_8$ and for $i = 1, 2$ the vertex a_i is adjacent to b_i, the vertices a_i, b_i have at least two common neighbors in $G - N[x]$. Let u_1, u_2 (resp. w_1, w_2) be two distinct common neighbors of a_1 and b_1 (resp. a_2 and b_2) in $G - N[x]$. By Menger's Theorem, $G - N[x]$ contains two disjoint paths from $\{u_1, u_2\}$ to $\{w_1, w_2\}$ and so $G > N[x] + a_1a_2 + b_1b_2 > K_9$, a contradiction.

Thus we may assume that $G - N[x]$ has at least three vertices and is not 2-connected. If $G - N[x]$ is disconnected, let $H_1 = K$ and H_2 be another connected component of $G - N[x]$. If $G - N[x]$ has a cut-vertex, say w, let H_1 be a connected component of $G - N[x] - w$ and let $H_2 = G - N[x] - V(H_1)$. In either case, H_1 and H_2 are disjoint connected subgraphs of $G - N[x]$ such that $M \subseteq N(H_1) \cup N(H_2)$ (because we have shown that $M \subseteq N(K)$). For $i = 1, 2$ let $A_i = N(H_i) \cap N(x)$. By Lemma 4.2 and Lemma 4.1, A_i is not complete and $|A_i| \geq 5$ for $i = 1, 2$. By property (B), A_1 and A_2 satisfy properties (B1), (B2) or (B3).

Suppose first that A_1 and A_2 satisfy property (B1). Then there exist $a_i \in A_i$ such that $N(x) + \{a_1a : a \in A_1 - \{a_1\}\} + \{a_2a : a \in A_2 - \{a_2\}\} > K_8$. By contracting the connected sets $V(H_1) \cup \{a_1\}$ and $V(H_2) \cup \{a_2\}$ to single vertices, we see that $G > K_9$, a contradiction. Suppose next that A_1 and A_2 satisfy property (B2). Then there exist $a_1 \in A_1 - A_2$ and $a_2 \in A_2 - A_1$ such that $a_1a_2 \in E(G)$ and the vertices a_1 and a_2 have at most five common neighbors in $N(x)$. Thus $a_1, a_2 \in M$ by Lemma 4.3, and by another application of the same lemma there exists a common neighbor $u \in V(G) - N[x]$ of a_1 and a_2. But $a_1 \notin A_2$ and $a_2 \notin A_1$, and hence $u \notin V(H_1) \cup V(H_2)$. Thus $G - N[x]$ is disconnected and $H_1 = K$. But then $a_2 \in M \subseteq N(K) = N(H_1)$, a contradiction. Thus we may assume that A_1 and A_2 satisfy (B3), and hence $A_i \subseteq A_{3-i}$ for some $i \in \{1, 2\}$. As $M \subseteq A_1 \cup A_2$, we have $M \subseteq N(H_{3-i})$. Since A_i is not complete, let $a, b \in A_i$ be distinct and not adjacent. By property (B3), $N(x) + ab > K_7 \cup K_1$. Let P be an a-b path with interior in H_i. By contracting all but one of the edges of the path P and by contracting H_{3-i} similarly as above, we see that $G > K_9$, a contradiction. ■

Lemma 4.7 *$G - N[x]$ is disconnected for every vertex $x \in V(G)$ of degree at most* 13.

Proof. If $G - N[x]$ is not null, then it is disconnected by Lemma 4.6. Thus we may assume that x is adjacent to every other vertex of G. Let $H = G - x$. Then $e(H) = e(G) - n + 1 = 7n - 27 - n + 1 = 6|H| - 20$. By Theorem 1.2 applied to H the graph G has a K_9 minor or is a $(K_{1,2,2,2,2,2}, 6)$-cockade, a contradiction. ∎

Lemma 4.8 $\delta(G) \geq 10$.

Proof. Let $x \in V(G)$ be such that $d(x) = \delta(G) = 9$. By Lemma 4.3, $\delta(N(x)) \geq 7$, and hence $N(x) > K_{1,2,2,2,2}$. Let K, K' be two components of $G - N[x]$. By Lemma 4.2, $N(K)$ and $N(K')$ contain distinct pairs of nonadjacent vertices of $N(x)$, say a, b and c, d, respectively. Thus $G > N[x] + ab + cd > K_9$ by the existence of internally disjoint a-b and c-d paths with interiors in K, K' respectively, a contradiction. ∎

Lemma 4.9 *Let $x \in V(G)$ be such that $10 \leq d(x) \leq 13$. Then there is no component K of $G - N[x]$ such that $d_G(y) \geq 14$ for every vertex $y \in V(K)$.*

Proof. Assume that such a component K exists. Let $G_1 = G - K$ and $G_2 = G[K \cup N(K)]$. Let d_1 be defined as in the paragraph prior to (3). Let G_2' be a graph with $V(G_2') = V(G_2)$ and $e(G_2') = e(G_2) + d_1$ edges obtained from G by contracting edges in G_1. By Lemma 4.8, $|G_2'| \geq 11$. If $e(G_2') \geq 7|G_2'| - 26$, then by induction $G > G_2' > K_9$, a contradiction. Thus

$$e(G_2) = e(G_2') - d_1 \leq 7|G_2| - 27 - d_1 = 7|N(K)| + 7|K| - 27 - d_1.$$

By contracting the edge xz, where $z \in N(K)$ has minimum degree in $N(K)$, we see that $d_1 \geq |N(K)| - d - 1$, and hence

$$e(G_2) \leq 6|N(K)| + 7|K| - 26 + d. \tag{a}$$

Let $t = e_G(N(K), K)$ and $d = \delta(N(K))$. We have $e(G_2) = e(K) + t + e(N(K))$ and

$$2e(K) \geq 14|K| - t, \tag{b}$$

and hence

$$e(G_2) \geq 7|K| + t/2 + d|N(K)|/2. \tag{c}$$

Since $N(x)$ has minimum degree at least seven, it follows that the subgraph $N(K)$ of $N(x)$ has minimum degree at least $7-(d(x)-|N(K)|)$. Thus $d \geq 7-(d(x)-|N(K)|) \geq |N(K)|-6$. From (a) and (c) we get

$$-t/2 \geq 26 - 6|N(K)| + d(|N(K)| - 2)/2 \geq -18, \tag{d}$$

where the second inequality holds with equality only when $|N(K)| = 10$. Since G is not contractible to K_9, we deduce from (b) by induction that $|K| < 9$. The inequality $e(K) \geq 7|K| - 18$ implies $|K| \leq 3$. But every vertex of K has degree at least 14 and $N(K)$ is a proper subgraph of $N(x)$, and hence $|K| = 3$, $|N(K)| = 12$ and (d) holds with equality, contrary to our earlier observation that (d) holds with equality only when $|N(K)| = 10$. ■

By Lemma 4.8 and the fact that $e(G) = 7n - 27$ there is a vertex x of degree $10, 11, 12$ or 13 in G. Choose such a vertex x so that $G - N[x]$ has a component K of minimum order. Then choose a vertex $y \in V(K)$ of least degree in G. Thus $10 \leq d_G(y) \leq 13$ by lemmas 4.8 and 4.9. Let L be the component of $G - N[y]$ containing x. We claim that $N(L)$ contains all vertices of $N(y)$ that are not adjacent to all other vertices of $N(y)$. Indeed, let $z \in N(y)$ be not adjacent to some vertex of $N(y) - \{z\}$. We may assume that $z \notin N(x)$, for otherwise $z \in N(L)$. Thus $z \in V(K)$, and hence $d_G(z) \geq d_G(y)$ by the choice of y. Thus z has a neighbor $z' \in N[x] \cup K - N[y]$. Then $z' \in V(L)$, for otherwise the component of $G - N[y]$ containing z' would be a proper subgraph of K. Thus $z \in N(L)$. This proves our claim that $N(L)$ contains all vertices z as above, contrary to Lemma 4.6. This contradiction completes the proof of Theorem 1.3. ■

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