EXCLUDING INFINITE CLIQUE MINORS

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ABSTRACT

For each infinite cardinal κ, we give a structural characterization of the graphs with no K_κ-minor. We also give such a characterization of the graphs with no "half-grid" minor.

1. INTRODUCTION

The first two authors proved (or claim to have proved — the proof is still being refereed) Wagner's conjecture, that for any infinite sequence G_1, G_2,... of finite graphs, there exists integers $j > i \geq 1$ such that G_i is isomorphic to a minor of G_j. (Definitions are given later.) The main step of the proof was a lemma that for any finite graph H (which we might as well assume to be complete), all finite graphs not containing H as a minor have a restricted structure; roughly, each such graph consists of pieces which "almost" have bounded genus, fitted together at small cutsets in a tree structure. This lemma, while powerful, suffers from several disadvantages:

(i) it is very difficult to state and to apply

(ii) it is even more difficult to prove

(iii) it is not necessary and sufficient for the exclusion of H, but merely necessary for the exclusion of H and sufficient for the exclusion of some other, larger, graph.

Nevertheless, given this lemma, one can attack Wagner's conjecture as follows. Let G_1, G_2,... be a sequence of finite graphs; then, since we may assume that for $i \geq 2$ G_i does not contain G_1 as a minor, it follows that G_2, G_3,... all have the structure of the lemma (taking $H = G_1$), and it therefore suffices to prove the result for sequences of graphs with this structure.

What about extending Wagner's conjecture to sequences of infinite graphs? Thomas [10] gave countably many infinite graphs none of which was a minor of another, and so Wagner's conjecture does not extend to infinite graphs in an unrestricted way. However, Thomas' graphs are uncountable, and it is an open question whether there is a countable set of countable graphs no member of which is a minor of another. Another open question is: let κ be an infinite cardinal, and let C be a set of graphs, such that $|C| \geq \kappa$ and every graph in C has $\leq \kappa$ vertices; then do there necessarily exist distinct $G, H \in C$ so that H is a minor of G?

There is not much chance of proving these conjectures because they imply that the set of all finite graphs is "better-quasi-ordered" by minor containment, which in itself seems to be a hopelessly difficult problem. Nevertheless, there is a natural first step, a generalization to infinite graphs of the finite graph lemma about the structure resulting from excluding a minor; and the main result of this paper is such a generalization. As often happens with infinite graph theory, things work better than in the finite case; the structure resulting from excluding as a minor the complete graph K_κ with κ vertices (when κ is an infinite cardinal) is relatively easy to explain and to prove, and is both necessary and sufficient.

This is one reason for interest in examining the structure resulting from excluding K_κ as a minor. Another is as follows. Let κ be a cardinal; a *haven* β of order κ is a graph G is a function assigning, to every set X of $< \kappa$ vertices, a component $\beta(X)$ of the graph obtained by deleting X, so that if $X \subseteq Y$ then $\beta(Y) \subseteq \beta(X)$. For κ and G finite, it was shown in [9] that G has a haven of order κ if and only if G cannot be expressed as a tree-structure of pieces each with $< \kappa$ vertices (that is, G has "tree-width" $\geq \kappa - 1$); and since tree-width is an interesting and useful concept for finite graphs, it seems that havens of finite order have some significance. For $\kappa = \aleph_0$, the havens in G are 1-1 correspondence with the "ends" of G, and the latter have been extensively studied. This suggests that perhaps havens of uncountable order κ, and the graphs with no such havens, might also be interesting. But it is easy to show that for κ uncountable, G has a haven of order κ if and only if G has K_κ as a minor, and so the result of this paper solves also the question of which graphs have havens of order κ. That is a second reason for interest in this topic.

A third reason for interest is that, in finite graph theory, there are a number of important theorems about the structure resulting from the exclusion of a fixed minor or set of minors — for example, the Kuratowski-Wagner theorem that a graph has no K_5 or $K_{3,3}$ minor if and only if it is planar. It is natural to ask for infinite analogues with infinite excluded minors. Along these lines, Halin [2] gave a necessary and sufficient structural description of the graphs with no 1-way infinite path; in [8] the last two authors gave a necessary and sufficient structural condition for a graph not to contain as a minor the κ-valent tree, for each cardinal κ; and in [7] we gave a structural description of the graphs not containing K_κ "topologically", for each infinite κ. (A similar result when κ is regular uncountable was independently obtained by Diestel [1].) But not much else seems to have been done, and the result of this paper is the natural next step. (For κ regular and uncountable, excluding K_κ topologically is equivalent to excluding it as a minor, as Jung [4] showed; but for κ singular or countable, the two kinds of containment are not equivalent.)

Our main result states, roughly, that for every cardinal $\kappa \geq \aleph_1$, if a graph has no K_κ-minor then it admits a kind of generalized tree-decomposition into pieces of cardinality $< \kappa$, with some additional constraints; and conversely, no graph with a K_κ-minor admits such a decomposition. The case $\kappa = \aleph_0$ is exceptional, however. When $\kappa = \aleph_0$, admitting this decomposition is sufficient for excluding K_κ-minors, but not necessary, and is in fact necessary and sufficient for excluding the ''half-grid'' as a minor. We shall deal with the case $\kappa = \aleph_0$, but the corresponding structure is quite different.

Let us be more precise.

Graphs in this paper may be infinite, and may have loops or multiple edges. We denote the complete graph with κ vertices, where κ is a cardinal, by K_κ. The *half-grid* is the graph with vertex set all pairs (x, y) of integers with $y \geq 0$, in which (x, y) and (x', y') are adjacent if $|x' - x| + |y' - y| = 1$. A graph G has a *minor* isomorphic to a graph H (or briefly an *H-minor*) if there is a function α with domain $V(H) \cup E(H)$ ($V(H)$ and $E(H)$ are the vertex- and edge-sets of H) such that

(i) for each $v \in V(H)$, $\alpha(v)$ is a non-null connected subgraph of G, and for distinct $v_1, v_2 \in V(H)$, $\alpha(v_1)$ and $\alpha(v_2)$ are vertex-disjoint

(ii) for each $e \in E(H)$, $\alpha(e) \in E(G)$, and for distinct $e_1, e_2 \in E(H)$, $\alpha(e_1) \neq \alpha(e_2)$

(iii) for each $v \in V(H)$ and $e \in E(H)$, $e \notin E(\alpha(v))$

(iv) if $e \in E(H)$ has ends v_1, v_2 then one end of $\alpha(e)$ belongs to $V(\alpha(v_1))$ and the other to $V(\alpha(v_2))$ (if $v_1 = v_2$ then $V(\alpha(v_1))$ contains both ends of $\alpha(e)$).

We say that G is a *subdivision* of H if G can be obtained from H by replacing the edges of H by internally disjoint paths joining the same ends; and that G *contains H topologically* if some subgraph of G is isomorphic to a subdivision of H. A set X of ordinals all less than κ is *cofinal* in κ if for every ordinal $\lambda < \kappa$ there exists $\mu \in X$ with $\lambda \leq \mu$; and the least order type of sets cofinal in κ is called the *cofinality* of κ and denoted by $cf(\kappa)$. We say a cardinal κ is *regular* if $cf(\kappa) = \kappa$, and *singular* otherwise.

The paper is organized as follows. Our main results are stated in section 2, and that also contains a brief discussion of the connection between the structures of this paper (''dissections'') and the more familiar ''tree-decompositions''. Sections 3 and 4 introduce ''havens'', which are a kind of escape strategy for fugitives in the graph, and which greatly simplify finding the minors we are concerned with. In section 5 we prove the easy halves of our structure theorems, that no graph containing the minor can have the stated structure. Sections 6 and 7 contain the proof that for κ uncountable graphs with no K_κ-minor have decompositions. Sections 8 and 9 contain lemmas for later use, and the structures corresponding to excluding the half-grid and $K_{\aleph_0}$ are established in sections 10 and

11 respectively. Section 12 contains a more detailed analysis of the connection between tree-decompositions and dissections, and enables us to restate some of our results in terms of tree-decompositions. Finally, in sections 13-15 we prove a strengthening of the results of sections 6 and 7.

2. DISSECTIONS

A *separation* in a graph G is a pair (A, B) of subsets of $V(G)$ such that $A \cup B = V(G)$ and no edge of G has one end in $A - B$ and the other in $B - A$. Two separations (A_1, B_1), (A_2, B_2) *cross* unless one of the following holds:

$A_1 \subseteq A_2$ and $B_2 \subseteq B_1$

$A_1 \subseteq B_2$ and $A_2 \subseteq B_1$

$B_1 \subseteq A_2$ and $B_2 \subseteq A_1$

$B_1 \subseteq B_2$ and $A_2 \subseteq A_1$.

A *dissection* of G is a set $\mathcal{D}$ of separations such that

(i) if $(A, B) \in \mathcal{D}$ then $(B, A) \in \mathcal{D}$

(ii) if $(A, B) \in \mathcal{D}$ then $A \neq V(G)$

(iii) if $(A_1, B_1), (A_2, B_2) \in \mathcal{D}$ and $A_1 \neq A_2$ then $B_1 \neq B_2$

(iv) no two members of $\mathcal{D}$ cross.

Let us observe the following useful lemma.

(2.1) *If $\mathcal{D}$ is a dissection, $(A_1, B_1), (A_2, B_2) \in \mathcal{D}$ and $A_1 \subseteq A_2$, then $B_2 \subseteq B_1$.*

Proof. Now $A_1 \not\subseteq B_1$ by (ii) above, and so $A_2 \not\subseteq B_1$. Similarly, $B_2 \not\subseteq A_2$ and so $B_2 \not\subseteq A_1$. If $A_2 \subseteq A_1$ then $A_1 = A_2$ and hence $B_1 = B_2$ by (iii) above, and so we may assume that $A_2 \not\subseteq A_1$. Since (A_1, B_1) and (A_2, B_2) do not cross the desired conclusion follows. ■

An *orientation* of a dissection $\mathcal{D}$ is a subset $\mathcal{P} \subseteq \mathcal{D}$ such that

(i) if $(A, B) \in \mathcal{D}$ then $\mathcal{P}$ contains one of (A, B), (B, A)

(ii) if $(A_1, B_1), (A_2, B_2) \in \mathcal{P}$ then $B_1 \not\subseteq A_2$.

It follows from (i) and (ii) that if $(A, B) \in \mathcal{D}$ then $\mathcal{P}$ contains exactly one of (A, B), (B, A). We call $V(G) \cap \bigcap(B : (A, B) \in \mathcal{P})$ the *centre* of $\mathcal{P}$. We say that $\mathcal{D}$ has *width* $< \kappa$, where κ is a cardinal, if for every orientation $\mathcal{P}$ of $\mathcal{D}$, the centre of $\mathcal{P}$ has cardinality $< \kappa$. The harder "only if" part of the following is a consequence of a result of an earlier paper [7].

(2.2) *For every infinite cardinal κ, a graph does not contain K_κ topologically if and only if it has a dissection of width $< \kappa$.*

Nevertheless, we shall give another proof of (2.2) (with $\kappa \geq \aleph_1$) in this paper, because it is a simple consequence of lemmas which we need to develop in any case.

Jung [4] proved –

(2.3) *If κ is a regular uncountable cardinal, then a graph has a K_κ-minor if and only if it contains K_κ topologically.*

Thus, from (2.1) and (2.3) we deduce

(2.4) *If κ is regular uncountable, then a graph G has no K_κ-minor if and only if it admits a dissection of width $< \kappa$.*

If κ is countable or singular then (2.4) fails. That is a consequence of (2.2) and the fact that (2.3) is false for κ countable or singular, but it may be helpful to see an example directly. Let G be a connected, locally finite graph with a $K_{\aleph_0}$-minor. Choose $u \in V(G)$, and for every integer $i \geq 0$, let A_i be the set of all $v \in V(G)$ such that there is a path between u and v with $\leq i$ edges. For $i \geq 1$, let $B_i = V(G) - A_{i-1}$; then (A_i, B_i) is a separation, and $A_1 \subseteq A_2 \subseteq \dots$, $B_1 \supseteq B_2 \supseteq \dots$. Let $\mathcal{D} = \{(A_i, B_i) : i \geq 1\}$; then $\mathcal{D}$ is a dissection. Let $\mathcal{P}$ be an orientation of $\mathcal{D}$, with centre W. We claim that W is finite. For let $v \in W$ and choose $i \geq 1$ such that $v \notin B_i$. It follows that $W \not\subseteq B_i$, so $(B_i, A_i) \in \mathcal{P}$, and $W \subseteq A_i$. Since A_i is finite (because G is locally finite) it follows that W is finite. Hence $\mathcal{D}$ has width $< \aleph_0$. Thus (2.4) fails for $\kappa = \aleph_0$. There are similar, more complicated counterexamples for any singular cardinal κ.

We remark that although the "only if" part of (2.4) is more difficult to prove, it is the "if" part which fails when κ is singular or countable; the "only if" part is true in general, because of (2.2).

Our problem, then, is to adapt (2.4) so that it applies when κ is singular or countable. We need to replace the condition that the dissection has width $< \kappa$ with a more restrictive condition, and the appropriate concept is that of "adhesion".

Let $\mathcal{P}$ be an orientation of a dissection $\mathcal{D}$. If $(A, B), (A', B') \in \mathcal{P}$, we say that (A', B') *cuts off* (A, B) *(from* $\mathcal{P}$*)* if $A \subseteq A'$ and $B' \subseteq B$. If κ' is a cardinal, we say that $\mathcal{D}$ has *adhesion* $\leq \kappa'$ *at* $\mathcal{P}$ if for every member of $\mathcal{P}$ there is a member of $\mathcal{P}$ cutting it off, of order $\leq \kappa'$. (The *order* of a separation (A, B) is $|A \cap B|$.) We say that $\mathcal{D}$ has *adhesion* $< \kappa$ if for every orientation $\mathcal{P}$ of $\mathcal{D}$ there exists $\kappa' < \kappa$ such that $\mathcal{D}$ has adhesion $\leq \kappa'$ at $\mathcal{P}$. Our first main result is

(2.5) *For every cardinal* $\kappa > \aleph_0$, *a graph has no* K_κ*-minor if and only if it has a dissection of width* $< \kappa$ *and adhesion* $< \kappa$.

In the case when κ is regular, (2.5) does not seem to say exactly the same thing as (2.4). The equivalence of (2.4) and (2.5) is easy if κ is a successor cardinal, but if it is not (that is, κ is "weakly inaccessible") the equivalence of (2.4) and (2.5) requires some non-trivial argument.

(2.5) fails when $\kappa = \aleph_0$, for the half-grid is a counterexample. Indeed, we shall show that a graph has no half-grid minor if and only if it has a dissection of width $< \aleph_0$ and adhesion $< \aleph_0$. This result can be stated more informatively, but to do so we need some further definitions.

A *tree* is a non-null connected graph without circuits. A *tree-decomposition* of a graph G is a pair (T, W), where $W = (W_t : t \in V(T))$ is a family of subsets of $V(G)$ satisfying

(i) $\bigcup(W_t : t \in V(T)) = V(G)$, and for every edge e of G there exists $t \in V(T)$ such that W_t contains both ends of e

(ii) if $t, t', t'' \in V(T)$ and t' lies on the path between t and t'' then $W_t \cap W_{t''} \subseteq W_{t'}$.

We say that (T, W) has *width* $< \kappa$ if $|W_t| < \kappa$ for all $t \in V(T)$, and

$$\left| \bigcup_{i \geq 1} \bigcap_{j \geq i} W_{t_j} \right| < \kappa$$

for every infinite path $t_1, t_2, \ldots$ of T. For $f \in E(T)$, the *order* of f is $|W_{t_1} \cap W_{t_2}|$, where f has ends t_1, t_2. We say that (T, W) has *adhesion* $< \kappa$ if

(i) for every $t \in V(T)$, there exists $\kappa' < \kappa$ such that every edge of T incident with t has order $\leq \kappa'$, and

(ii) for every infinite path R of T, there exists $\kappa' < \kappa$ such that infinitely many edges of R have order $\leq \kappa'$.

We shall show that

(2.6) *For a graph G, the following are equivalent:*

(i) G has no half-grid minor

(ii) G admits a dissection of width $< \aleph_0$ *and adhesion* $< \aleph_0$

(iii) G admits a tree-decomposition of width $< \aleph_0$ *and adhesion* $< \aleph_0$.

The analogous strengthening of (2.5) is false, even for κ regular; for a graph is given in [5] with κ uncountable which has no K_κ-minor and yet which has no tree-decomposition of width $< \kappa$. On the other hand, it was shown in [7] that for any $\kappa \geq \aleph_0$, every graph which does not contain K_κ topologically has a "well-founded tree-decomposition" of width $< \kappa$, where we permit order-theoretic trees rather than graph-theoretic trees in the definition of tree-decomposition. In sections 13-15 we show that this device permits a strengthening of (2.5), that is, every graph with no K_κ-minor has a "well-founded tree-decomposition" of width $< \kappa$ and adhesion $< \kappa$.

Our third and last main result concerns the structure of graphs with no $K_{\aleph_0}$-minor. This again needs some further definitions. Let $\mathcal{P}$ be an orientation of a dissection $\mathcal{D}$ of G, with centre U. We define the *torso* of G at $\mathcal{P}$ (denoted by $ts(\mathcal{P})$) to be the simple graph with vertex set U, in which distinct $u, v \in U$ are adjacent if either they are adjacent in G, or there exists $(A, B) \in \mathcal{P}$ with $u, v \in A \cap B$. Similarly, if (T, W) is a tree-decomposition of G and $t \in V(T)$, we define the *torso* of G at t (denoted $ts(t)$) to be the simple graph with vertex set W_t, in which distinct $u, v \in W_t$ are adjacent if either they are adjacent in G, or there is a neighbour t' of t in T with $u, v \in W_{t'}$. Our last main result is

(2.7) *For a graph G, the following are equivalent:*

(i) G has no $K_{\aleph_0}$*-minor*

(ii) G admits a dissection $\mathcal{D}$ *of adhesion* $< \aleph_0$, *such that for every orientation* $\mathcal{P}$ *of* $\mathcal{D}$ *there is an integer* $k \geq 0$ *such that* $ts(\mathcal{P})$ *has no* K_k*-minor.*

(iii) G admits a tree-decomposition of adhesion $< \aleph_0$, such that for every $t \in V(T)$ there is an integer $k \geq 0$ such that $ts(t)$ has no K_k-minor.

To aid the reader's intuition let us informally study the connection between tree-decompositions and dissections. Let (T, W) be a tree-decomposition of G. For each $e \in E(T)$, let T_1, T_2 be the components of $T \backslash e$, and let

$$A_i = \bigcup (W_t : t \in V(T_i)) \; (i = 1, 2).$$

Then (A_1, A_2) is a separation of G. Let $\mathcal{D}$ be the set of all such separations arising from edges of T; then $\mathcal{D}$ satisfies conditions (i) and (iv) in the definition of a dissection. It may not satisfy (ii) or (iii), but these are inessential conditions included merely for convenience, and in any case it is usually possible to modify (T, W) slightly so that they are satisfied. Thus, it is more or less true that tree-decompositions yield dissections. What about the converse? Not every dissection arises from a tree-decomposition (that is why we are using dissections, because they are more general) but the only reason why not is "trouble with infinity". More precisely, a dissection $\mathcal{D}$ arises from some tree-decomposition if and only if for all $u, v \in V(G)$ there are only finitely many $(A, B) \in \mathcal{D}$ with $u \in A - B$ and $v \in B - A$. For instance, every dissection of a finite graph (or more generally, every finite dissection of a graph) arises from a tree-decomposition. We shall show this in section 12.

Given (T, W) and $\mathcal{D}$ as before, what are the orientations of $\mathcal{D}$? Each orientation corresponds to a way of directing the edges of T so that no two edges point away from one another. (For, with e, A_1, A_2, T_1, T_2 as before, let e be directed from T_1 to T_2 if (A_1, A_2) belongs to the orientation.) Thus there is an orientation of $\mathcal{D}$ corresponding to each vertex t of T (direct all edges of T towards it), and it is easy to see that its centre is W_t. There are also orientations which correspond to no vertex, as follows. Let t_1, t_2, ... be an infinite path R of T, and for each edge e of T direct e towards the infinite part of R. The centre of this orientation is

$$\bigcup_{i \geq 1} \bigcap_{j \geq i} W_{t_j}.$$

We shall see in section 12 that every orientation of $\mathcal{D}$ is of one of these two kinds. Thus, (T, W) has width $< \kappa$ if and only if $\mathcal{D}$ has width $< \kappa$, and similarly for adhesion.

3. HAVENS AND MINORS

Let κ be a cardinal. If V is a set we denote by $[V]^{<\kappa}$ the set of all subsets of V of cardinality $< \kappa$. If G is a graph and $X \subseteq V(G)$, an *X-flap* in G is the vertex set of a component of $G \backslash X$ (the graph obtained from G by deleting X). A *haven* of order κ in G is a function β which assigns to each $X \in [V(G)]^{<\kappa}$ an X-flap $\beta(X)$, in such a way that if $X \subseteq Y \in [V(G)]^{<\kappa}$ then $\beta(Y) \subseteq \beta(X)$. An *escape* of order κ in G is a function β which assigns to each $X \in [V(G)]^{<\kappa}$ a union of X-flaps $\beta(X)$, such that $\beta(\emptyset) \neq \emptyset$, and if $X \subseteq Y \in [V(G)]^{<\kappa}$ then $\beta(X)$ includes precisely those X-flaps that have non-empty intersection with $\beta(Y)$. Escapes were discussed in [8], and a number of theorems were proved about them. Since havens are escapes, we may apply these theorems to havens. Havens are of interest to us, because as we shall prove later in this section,

(3.1) *For $\kappa > \aleph_0$, G has a haven of order κ if and only if G has a K_κ-minor.*

A *cluster* in G is a set C of non-empty subsets of $V(G)$, such that

(i) $G|X$ is connected for each $X \in C$

(ii) if $X, Y \in C$ are distinct then $X \cap Y = \emptyset$ and there is an edge of G with one end in X and the other in Y.

(If $X \subseteq V(G)$, $G|X$ denotes the restriction of G to X.) It is easy to see that if G has a K_κ-minor, and α is the corresponding function, then $\{V(\alpha(v)) : v \in V(K_\kappa)\}$ is a cluster of cardinality κ; and conversely, if G has a cluster of cardinality κ then it has a K_κ-minor.

(3.2) *Let C be a cluster in G with $|C| = \kappa$. For each $X \in [V(G)]^{<\kappa}$, let $\beta(X)$ be an X-flap which includes a member of C. Then β is a haven of order κ.*

Proof. We observe, first, that if $X \in [V(G)]^{<\kappa}$ then there is a unique X-flap which includes a member of C. For some member Y of C is disjoint from X, since $|C| > |X|$, and so Y is included in an X-flap, since $G|Y$ is connected; and no other X-flap includes a member of C, since any two members of C are joined by an edge. Now verifying that β is a haven is easy. ■

It is convenient to write β_C for the haven β arising from the cluster C as above. Havens which arise in this way from some cluster are said to be *clustered.*

Let β be a haven in G of order κ. For $\kappa' \leq \kappa$, a vertex $v \in V(G)$ is *κ'-major* if $v \in X \cup \beta(X)$ for all

$X \in [V(G)]^{<\kappa'}$. It is proved in [8, theorem (2.4)] that

(3.3) *If* β *is a haven in G of order* $\kappa > \aleph_0$, *and* κ' *is regular with* $\aleph_0 \le \kappa' \le \kappa$, *then for every* $X \in [V(G)]^{<\kappa'}$, $\beta(X)$ *contains a* κ'*-major vertex.*

(3.1) is a consequence of (3.2) and the following.

(3.4). *Every haven of order* $\ge \aleph_1$ *is clustered.*

Proof. Let β be a haven in G of order $\kappa \ge \aleph_1$. Let $\lambda = cf(\kappa)$ if κ is singular, and $\lambda = \aleph_0$ if κ is regular. We say a cluster C is *good* if

(i) $|X| \le \max(\lambda, |C|)$ for each $X \in C$

(ii) for each regular κ' with $\aleph_0 \le \kappa' \le \kappa$ and each $X \in C$, some member of X is κ'-major.

By Zorn's lemma, there is a maximal good cluster C. Suppose, for a contradiction, that $|C| < \kappa$. Let $Y = \bigcup(X : X \in C)$. Then $|Y| < \kappa$, since $|C| < \kappa$ and each $X \in C$ satisfies $|X| \le \max(\lambda, |C|) < \kappa$. If κ is singular, let I be a set of regular cardinals, cofinal in κ, with $|I| = cf(\kappa)$; and if κ is regular let $I = \{\kappa\}$. For each $\kappa' \in I$, there is by (3.3) a κ'-major vertex $v(\kappa') \in \beta(Y)$. Let

$$Z_1 = \{v(\kappa') : \kappa' \in I\}.$$

We claim that

(1) *For each* $X \in C$ *there exists* $v(X) \in \beta(Y)$ *with a neighbour in X.*

For choose κ' regular with $|Y| < \kappa' \le \kappa$. Since X contains a κ'-major vertex, it follows that $X \cap \beta(Y - X) \ne \varnothing$, and in particular $\beta(Y - X) \ne \beta(Y)$. Since $Y - X \subseteq Y$ and so $\beta(Y) \subseteq \beta(Y - X)$, we deduce that $\beta(Y)$ is not a $(Y - X)$-flap, and the claim follows.

Let $Z_2 = \{v(X) : X \in C\}$. Then $|Z_1| \le |I| \le \lambda$ and $|Z_2| \le |C|$, and so $|Z_1 \cup Z_2| \le \max(\lambda, |C|)$. Choose a minimal tree T with $V(T) \subseteq \beta(Y)$ such that $Z_1 \cup Z_2 \subseteq V(T)$. Then $|V(T)| \le \max(\lambda, |C|)$, for if $Z_1 \cup Z_2$ is infinite then $|V(T)| = |Z_1 \cup Z_2|$, and otherwise $|V(T)|$ is finite. Hence $C \cup \{V(T)\}$ is a good cluster, contrary to the maximality of C. This proves that $|C| \ge \kappa$.

Choose $C' \subseteq C$ with $|C'| = \kappa$, and let $\beta' = \beta_{C'}$. We claim that $\beta' = \beta$. For let $X \in [V(G)]^{<\kappa}$, and choose κ'

regular with $|X| < \kappa' \leq \kappa$. Choose $C \in \mathcal{C}$ with $C \cap X = \emptyset$, and choose $v \in C$, κ'-major (with respect to β). Then $C \subseteq \beta'(X)$ by definition of β', but $v \in X \cup \beta(X)$ and so $v \in \beta(X)$. Hence $\beta'(X) = \beta(X)$, as required. ■

(3.1) yields a very convenient way to produce the desired minors, because it is usually easier to construct a haven than to exhibit a minor directly. Our next objective is to similarly replace finding a half-grid minor and finding a $K_{\aleph_0}$-minor by finding certain kinds of havens. In this section we discuss the half-grid, and $K_{\aleph_0}$ in the next.

A *ray* in G is a 1-way infinite path of G. Two rays R_1, R_2 are *parallel* if for every finite $X \subseteq V(G)$, the unique X-flap F with $F \cap V(R_1)$ infinite has $F \cap V(R_2)$ infinite. This is an equivalence relation, and its equivalence classes are called the *ends* of G. These were introduced by Halin [3]. We shall see that there is a natural 1-1 correspondence between the ends of G and the havens of order $\aleph_0$ in G.

First, if R is a ray in G, for each finite $X \subseteq V(G)$ let $\beta(X)$ be the X-flap F with $F \cap V(R)$ infinite; then clearly β is a haven of order $\aleph_0$. Any other ray parallel to R yields the same haven and we call β the haven *produced* by the end Π, where Π is the end containing R. Moreover, two rays R_1, R_2 in different ends produce different havens, because there is a finite set $X \subseteq V(G)$ such that the X-flap with infinite intersection with $V(R_1)$ has only finite intersection with $V(R_2)$. It remains to show that every haven of order $\aleph_0$ is produced by some end. That is a corollary of theorem (5.4) of [8].

Suppose that G has a minor isomorphic to the half-grid H. For every edge e of H there corresponds an edge $\alpha(e)$ of G; and for every ray S of H there is a ray R of G with $\alpha(e) \in E(R)$ for every $e \in E(S)$. Moreover, all these rays of G belong to the same end. The ends which arise in this way from half-grid minors were characterized by Halin [3]. In particular, Halin showed that an end arises from a half-grid minor if and only if for every integer $k \geq 0$ there is a finite set $X \subseteq V(G)$ such that for every $Y \subseteq V(G)$ with $|Y| \leq k$, the Y-flap containing a ray of the end intersects X. Let us express this in terms of havens. We say a haven β in G is a *half-grid haven* if it has order $\aleph_0$ and the corresponding end arises from a half-grid minor. Thus, Halin's result asserts that

(3.5) *A haven β in G of order $\aleph_0$ is a half-grid haven if and only if for every integer $k \geq 0$ there exists a finite $X \subseteq V(G)$ such that $X \cap \beta(Y) \neq \emptyset$ for every $Y \subseteq V(G)$ with $|Y| \leq k$.*

4. CLUSTERED HAVENS OF ORDER $\aleph_0$

We wish to find a characterization of clustered havens of order $\aleph_0$, to facilitate producing $K_{\aleph_0}$-minors, in the same spirit as (3.4) and (3.5). If β is a haven of order κ, and $\kappa' \le \kappa$ is a cardinal, define $\beta'(X) = \beta(X)$ for all $X \in [V(G)]^{<\kappa'}$; then β' is a haven of order κ', called the *κ'-truncation* of β. Our characterization is the following.

(4.1) *Let β be a haven in G of order $\aleph_0$. Then β is clustered if and only if for every integer $k \ge 0$, the k-truncation of β is clustered.*

The proof of (4.1) will require a number of steps. We begin with the following. A *rooted path* in G is a path (finite or a ray) with a designated end, called its *root*. If it is finite, its other end is its *tip*. A *comb* is a set of mutually vertex-disjoint rooted paths. If β is a haven in G of order κ, a finite subset $X \subseteq V(G)$ is *β-free* if $|X| \le \kappa$ and $X \cap \beta(Y) \ne \emptyset$ for all $Y \subseteq V(G)$ with $|Y| < |X|$. A subset $Y \subseteq V(G)$ (or a subgraph H with $V(H) = Y$) is *major* if $Y \cap (X \cup \beta(X)) \ne \emptyset$ for all $X \in [V(G)]^{<\kappa}$. The following was proved in [6, theorem (3.18)].

(4.2) *Let β be a haven in G of order $\aleph_0$ and let $X \subseteq V(G)$ be β-free. Then there is a comb with set of roots X, every member of which is major.*

We need a refinement of (4.2). Let us say two rooted paths are *cofinal* if either they are both finite with the same tip, or they are both rays and their intersection includes a ray. If β is a haven in G, a rooted path P is *cofinally major* if every rooted path cofinal with P is major.

(4.3) *Let β be a haven in G of order $\aleph_0$, and let P be a major rooted path. There is a cofinally major rooted path Q with the same root, contained in P.*

Proof. Let P have root u. If there is a vertex $v \in V(P)$ such that $\{v\}$ is major, we may take Q to be the subpath of P between u and v. We assume then that there is no such vertex. We shall show that P itself is cofinally β-major. For let Q be a rooted path cofinal with P, and suppose that there exists $X \subseteq V(G)$ with $V(Q) \cap (X \cup \beta(X)) = \emptyset$. For each $v \in V(P)$, let $X_v \subseteq V(G)$ be finite such that $v \notin X_v \cup \beta(X_v)$. Let

$$Y = X \cup \bigcup (X_v : v \in V(P) - V(Q)).$$

Then Y is finite, because $V(P) - V(Q)$ is finite. Let $Z \subseteq Y$ be the set of vertices in Y with a neighbour in $\beta(Y)$; then $\beta(Z) = \beta(Y) \subseteq \beta(X)$. Since P is major, there exists $v \in V(P) \cap (Z \cup \beta(Z))$. Now $v \notin V(Q)$, for no vertex of Q

belongs to $\beta(X)$ or has a neighbour in $\beta(X)$, since $V(Q) \cap (X \cup \beta(X)) = \emptyset$. Hence $v \in V(P) - V(Q)$, and so $X_v \subseteq Y$ and $\beta(Y) \subseteq \beta(X_v)$. But $v \notin \beta(X_v)$ and v has no neighbour in $\beta(X_v)$, and yet either $v \in \beta(Y)$ or v has a neighbour in $\beta(Y)$, a contradiction. Thus there is no such X, and so Q is major. The result follows. ■

From (4.2) and (4.3) we immediately have

(4.4) *Let β be a haven in G of order $\aleph_0$, and let $X \subseteq V(G)$ be β-free. Then there is a comb with set of roots X, every member of which is cofinally major.*

A cluster $\mathcal{C}$ is *local* if each member of $\mathcal{C}$ is finite.

(4.5) *If $\mathcal{C}$ is a finite cluster in G, there is a local cluster $\mathcal{C}'$ with $|\mathcal{C}'| = |\mathcal{C}|$ such that every member of $\mathcal{C}$ includes a member of $\mathcal{C}'$, and hence $\beta_{\mathcal{C}'} = \beta_{\mathcal{C}}$.*

Proof. Let $\mathcal{C} = \{C_1, ..., C_k\}$ and for $1 \le i < j \le k$ let e_{ij} be an edge of G between C_i and C_j. For $1 \le i \le k$, exactly $k-1$ of these edges have an end in C_i; let T_i be a finite tree of G with $\emptyset \ne V(T_i) \subseteq C_i$, containing the ends in C_i of these $k-1$ edges. Then $\mathcal{C}' = \{V(T_i) : 1 \le i \le k\}$ is the required local cluster. ■

We shall need the following theorem of [6]. A comb $\mathcal{P}$ is said to *traverse* a cluster $\mathcal{C}$ if each member of $\mathcal{P}$ has its root, and no other vertex, in $\bigcup(C : C \in \mathcal{C})$, and no two members of $\mathcal{P}$ have their roots in the same member of $\mathcal{P}$.

(4.6) *Let $\mathcal{C}$ be a cluster in a finite graph G with $|\mathcal{C}| \ge 2k$, and let $X \subseteq V(G)$ with $|X| = k$. Suppose that there is no separation (A, B) of G of order $< k$ such that $X \subseteq A$ and $B - A$ includes a member of $\mathcal{C}$. Then there exists $\mathcal{C}' \subseteq \mathcal{C}$ with $|\mathcal{C}'| = |\mathcal{C}| - k$ and with $X \cap C = \emptyset$ for all $C \in \mathcal{C}'$, and a comb $\mathcal{P}$ traversing $\mathcal{C}'$ with $|\mathcal{P}| = k$, such that X is its set of tips.*

We apply (4.6) to deduce the following.

(4.7) *Let $\mathcal{C}$ be a finite local cluster in a graph G, with $|\mathcal{C}| \ge 2k$, and let $X \subseteq V(G)$ with $|X| = k$, where X is $\beta_{\mathcal{C}}$-free. Then there exists $\mathcal{C}' \subseteq \mathcal{C}$ with $|\mathcal{C}'| = |\mathcal{C}| - k$ and with $X \cap C = \emptyset$ for $C \in \mathcal{C}'$, and a comb $\mathcal{P}$ traversing $\mathcal{C}'$ with $|\mathcal{P}| = k$, such that X is its set of tips.*

Proof. For each $C \in \mathcal{C}$ there are k paths of G between X and C, mutually vertex-disjoint except possibly for their ends in C, by Menger's theorem. Let G' be the union of all these paths (for all $C \in \mathcal{C}$) together with $G | \bigcup(C : C \in \mathcal{C})$. Then G' is finite, and $\mathcal{C}$ is a cluster in it. Suppose that (A, B) is a separation of G' such that

$X \subseteq V(A)$ and $B - A$ includes a member of $\mathcal{C}$. Since there are k paths of G' between X and $V(C)$, mutually vertex-disjoint except for their ends in C, we deduce that $|A \cap B| \geq k$. The result follows from (4.6) applied to G'. ■

Let $\mathcal{C}$ be a cluster in G, and let β be a haven in G, of order $\geq |\mathcal{C}|$. We say that $\mathcal{C}$ is *β-free* if for every $X \subseteq V(G)$ with $|X| < |\mathcal{C}|$, $\beta(X)$ includes a member of $\mathcal{C}$; that is, $\beta_{\mathcal{C}}$ is the $|\mathcal{C}|$-truncation of β. We say that $\mathcal{C}$ is *β-combed* if there is a comb $\mathcal{P}$ traversing $\mathcal{C}$ such that every member of $\mathcal{C}$ contains a root of some member of $\mathcal{P}$, and every member of $\mathcal{P}$ is major (with respect to β). We call $\mathcal{P}$ a *β-comb for $\mathcal{C}$.*

(4.8) *With $\mathcal{C}$, β as above, if $\mathcal{C}$ is β-combed then it is β-free.*

Proof. Let $\mathcal{P}$ be a β-comb for $\mathcal{C}$, and let $\mathcal{P} = \{P(C) : C \in \mathcal{C}\}$, where $P(C)$ has its root in C. Let $X \subseteq V(G)$ with $|X| < |\mathcal{C}|$. Now the sets $C \cup V(P(C))$ $(C \in \mathcal{C})$ are mutually disjoint, and so there exists $C \in \mathcal{C}$ with $(C \cup V(P(C))) \cap X = \emptyset$. But $V(P(C)) \cap (X \cup \beta(X)) \neq \emptyset$, since $P(C)$ is major, and so $C \cup V(P(C)) \subseteq \beta(X)$, since $G|(C \cup V(P(C)))$ is connected. Hence $C \subseteq \beta(X)$, as required. ■

(4.9) *Let β be a haven in G of order $\aleph_0$, and let $\mathcal{C}$ be a finite local β-free cluster in G with $|\mathcal{C}| = 2k$. Then there exists $\mathcal{C}' \subseteq \mathcal{C}$ with $|\mathcal{C}'| = k$ which is β-combed.*

Proof. First we claim

(1) *There is a comb $\mathcal{R}$ in G with $|\mathcal{R}| = k$, every member of which is cofinally major.*

For let $X \subseteq \bigcup(C : C \in \mathcal{C})$ with $|X| = k$, such that $|X \cap C| \leq 1$ for all $C \in \mathcal{C}$. We claim that X is β-free. For let $Y \subseteq V(G)$ with $|Y| < |X|$. Then $\beta(Y)$ includes a member of $\mathcal{C}$ since $\mathcal{C}$ is β-free, and so every member of $\mathcal{C}$ intersects $Y \cup \beta(Y)$. Since at most $k - 1$ intersect Y, and $G|C$ is connected for each $C \in \mathcal{C}$, it follows that $\beta(Y)$ includes at least $k + 1$ members of $\mathcal{C}$. At least one of these intersects X, and so $X \cap \beta(Y) \neq \emptyset$. Hence X is β-free, as claimed. Then (1) follows from (4.4).

Let $\mathcal{R} = \{R_1, \ldots, R_k\}$.

(2) *For $1 \leq i \leq k$ and each $C \in \mathcal{C}$ there are k paths of G between $V(R_i)$ and C, such that any common vertex of two of these paths is an end of both.*

For suppose that (A, B) is a separation of G of order $< k$ with $V(R_i) \subseteq A - B$ and $C \subseteq B - A$. Let $X = A \cap B$. Since R_i is major, it follows that $V(R_i) \cap (X \cup \beta(X)) \neq \emptyset$, and so $\beta(X) \subseteq A - B$. Since $\mathcal{C}$ is β-free there exists

$C' \in \mathcal{C}$ with $C' \subseteq \beta(X) \subseteq A - B$. But $C \subseteq B - A$ and there is an edge between C and C', a contradiction. Thus, there is no such (A, B), and the claim follows from Menger's theorem.

Let G_1 be the union of k paths as in (2), for all $C \in \mathcal{C}$ and all i $(1 \leq i \leq k)$, together with $G | \bigcup(C : C \in \mathcal{C})$. Then G_1 is finite, and so for $1 \leq i \leq k$, there is a vertex $x_i \in V(R_i)$ such that the subpath P_i of R_i between x_i and the root of R_i contains every vertex of $V(G_1) \cap V(R_i)$. Let $G_2 = G_1 \cup P_1 \cup ... \cup P_k$, and let $X = \{x_1, ..., x_k\}$. Suppose that (A, B) is a separation of G_2 of order $< k$ with $X \subseteq A$ and $C \subseteq B - A$ for some $C \in \mathcal{C}$. Choose i with $1 \leq i \leq k$ such that $V(P_i) \cap A \cap B = \emptyset$. Since $x_i \in A$ and P_i is connected it follows that $V(P_i) \subseteq A - B$. But there are k paths of G_2 between $V(P_i)$ and C, mutually disjoint except for their ends, which is impossible. Hence there is no such (A, B).

By (4.6) applied to G_2, there is a cluster $\mathcal{C}' \subseteq \mathcal{C}$ with $|\mathcal{C}'| = k$, and a comb $\mathcal{P}'$ for $\mathcal{C}'$ in G_2 with $|\mathcal{P}'| = k$, where $\mathcal{P}' = \{P_1', ..., P_k'\}$ say, such that for $1 \leq i \leq k$ P_i' has tip x_i. Let R_i' be the union of P_i' and the path Q_i with edge set $E(R_i) - E(P_i)$ with one end x_i, and designate the root of P_i' to be the root of R_i'. Now each Q_i is major since it is cofinal with P_i, and hence each R_i' is major. It follows that $\{R_1', ..., R_k'\}$ is a β-comb for $\mathcal{C}'$, as required. ■

(4.10) *Let β be a haven in G of order $\aleph_0$, such that for every $k \geq 0$ the k-truncation of β is clustered. Let $\mathcal{C}$ be a finite local β-combed cluster. Then there is a local β-combed cluster $\mathcal{C}'$ with $|\mathcal{C}'| = |\mathcal{C}| + 1$ such that every member of $\mathcal{C}'$ except one includes a member of $\mathcal{C}$.*

Proof. Let $\mathcal{C} = \{C_1, ..., C_k\}$, and let $C_1 \cup ... \cup C_k = Z$, where $|Z| = n$. Let $\{R_1, ..., R_k\}$ be a β-comb for $\mathcal{C}$, and let $x_i \in C_i \cap V(R_i)$ $(1 \leq i \leq k)$. Thus $Z \cap V(R_i) = \{x_i\}$ $(1 \leq i \leq k)$. Let $\mathcal{C}_1$ be a β-free cluster with $|\mathcal{C}_1| = 6k + 2n + 2$. (This exists because the $(6k + 2n + 2)$-truncation of β is clustered.) By (4.5) we may assume that $\mathcal{C}_1$ is local. By (4.9), there exists $\mathcal{C}_2 \subseteq \mathcal{C}_1$ with $|\mathcal{C}_2| = 3k + n + 1$ which is β-combed. Let $\mathcal{S} = \{S(C) : C \in \mathcal{C}_2\}$ be a β-comb for $\mathcal{C}_2$, such that for each $C \in \mathcal{C}_2$, C contains the root of $S(C)$. Since $|Z| = n$, and all the sets $C \cup V(S(C))$ $(C \in \mathcal{C}_2)$ are mutually disjoint, it follows that there exists $\mathcal{C}_3 \subseteq \mathcal{C}_2$ with $|\mathcal{C}_3| = 3k + 1$ such that $Z \cap (C \cup V(S(C))) = \emptyset$ for all $C \in \mathcal{C}_3$. Let G' be $G \backslash (Z - X)$ where $X = \{x_1, ..., x_k\}$, and let β' be the haven in G' obtained from $\mathcal{C}_3$.

(1) *X is β'-free in G'.*

For let $Y' \subseteq V(G')$ with $|Y'| < |X| = k$, and let $Y = Y' \cup (Z - X)$. Now $\beta'(Y')$ is a Y-flap of G, and since it includes a member of $\mathcal{C}_3 \subseteq \mathcal{C}_1$ and $\mathcal{C}_1$ is β-free and $|Y| < |\mathcal{C}_1|$, it follows that $\beta'(Y') = \beta(Y)$. Now $R_1, ..., R_k$ are major, and so $V(R_i) \cap (Y \cup \beta(Y)) \neq \emptyset$ for $1 \leq i \leq k$. Since $V(R_i) \cap (Z - X) = \emptyset$ for $1 \leq i \leq k$, and $|Y'| < \kappa$, it follows that $V(R_i) \cap Y = \emptyset$ for some i, and hence $x_i \in V(R_i) \subseteq \beta(Y) = \beta'(Y')$. Thus $X \cap \beta'(Y') \neq \emptyset$, and so X is β'-free, as required.

From (4.7) applied to $\mathcal{C}_3$ and G', there exists $\mathcal{C}_4 \subseteq \mathcal{C}_3$ with $|\mathcal{C}_4| = 2k+1$, and a comb $\mathcal{P}$ traversing $\mathcal{C}_4$ with $|\mathcal{P}| = k$, such that X is its set of tips. Let us choose $\mathcal{C}_4$ and $\mathcal{P}$ such that the graph $H \cup \bigcup(P : P \in \mathcal{P})$ is minimal, where $H = \bigcup(S(C) : C \in \mathcal{C}_3)$. (This is possible since all the paths in $\mathcal{P}$ are finite.) Let $\mathcal{C}_5$ be the set of those members $C \in \mathcal{C}_4$ which contain a root of a member of $\mathcal{P}$.

(2) *If* $C \in \mathcal{C}_4 - \mathcal{C}_5$ *then* $(C \cup V(S(C))) \cap \bigcup(V(P) : P \in \mathcal{P}) = \emptyset$.

For suppose that this set is non-empty. Let A be a minimal path from C to $(C \cup V(S(C))) \cap \bigcup(V(P) : P \in \mathcal{P})$ with $V(A) \subseteq C \cup V(S(C))$. Let A have ends a, b where $a \in C$ and $b \in V(P)$ for some $P \in \mathcal{P}$. Let P have root $c \in C' \in \mathcal{C}_5$ and tip $x \in X$, and let B be the subpath of P between b and x. Let P' be the path $A \cup B$, rooted at a. Let $\mathcal{P}' = (\mathcal{P} - \{P\}) \cup \{P'\}$; then, by the minimality of A, $\mathcal{P}'$ is a comb traversing $\mathcal{C}_5' = (\mathcal{C}_5 - \{C'\}) \cup \{C\}$. Moreover, the first edge of P (moving away from c) which has an end not in $C' \cup V(S(C'))$ does not belong to P', and this contradicts the minimality of $H \cup \bigcup(P \in \mathcal{P})$.

Let $\mathcal{P} = \{P_1, ..., P_k\}$, $\mathcal{C}_5 = \{C_1', ..., C_k'\}$, $\mathcal{C}_4 - \mathcal{C}_5 = \{C_0'', C_1'', ..., C_k''\}$, where P_i has root in C_i' and tip in C_i $(1 \leq i \leq k)$. Define $D_0 = C_0''$,

$$D_i = C_i \cup C_i' \cup C_i'' \cup V(P_i) \ (1 \leq i \leq k).$$

Then, by (2), $\{D_0, D_1, ..., D_k\}$ is a cluster, and it is β-combed since it is traversed by the comb $\{S(C_i'') : 0 \leq i \leq k\}$, by (2), every member of which is major. The result follows. ■

Proof of (4.1).

The " only if" part is clear, for if $\beta = \beta_{\mathcal{C}}$ for some cluster $\mathcal{C}$, choose $\mathcal{C}' \subseteq \mathcal{C}$ with $|\mathcal{C}'| = k$, and then $\beta_{\mathcal{C}'}$ is the k-truncation of β. Let us prove "if". By (4.10) there is a sequence $\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2, ...$ of local β-combed clusters, such that $|\mathcal{C}_i| = i$ and each member of $\mathcal{C}_{i+1}$, except one includes a member of $\mathcal{C}_i$ $(i \geq 0)$. Let $\mathcal{C}_i = \{C_1^i, ..., C_i^i\}$, where $C_i^j \subseteq C_i^k$ for $1 \leq i \leq j \leq k$, and let $C_i = \bigcup_{j \geq i} C_i^j$. Then $\mathcal{C} = \{C_1, C_2, ...\}$ is a cluster with $|\mathcal{C}| = \aleph_0$, and we claim that $\beta = \beta_{\mathcal{C}}$. For let $X \subseteq V(G)$ be finite. Choose $i > |X|$ such that $C_i \cap X = \emptyset$. Then $C_i^i \cap X = \emptyset$, and since $\mathcal{C}_i$ is β-

combed and hence β-free by (4.8), and $|X| < |C_i|$, it follows that $C_i^i \subseteq \beta(X)$. Hence $C_i \subseteq \beta(X)$ and so $\beta = \beta_C$. ■

5. THE EASY HALVES

In this section we shall prove the easier parts of our main theorems, that if the dissection exists then the minor does not. We begin with the following.

(5.1) *Let $\mathcal{D}$ be a dissection and let $(A_0, B_0) \in \mathcal{D}$. Let*

$$\mathcal{P} = \{(A, B) \in \mathcal{D} : (A, B) \neq (B_0, A_0) \text{ and either } A \subseteq A_0 \text{ or } A \subseteq B_0\}.$$

Then $\mathcal{P}$ is an orientation of $\mathcal{D}$.

Proof. If $(A, B) \in \mathcal{D}$ then since it does not cross (A_0, B_0) it follows that $\mathcal{P}$ contains one of (A, B), (B, A). Now let $(A_1, B_1), (A_2, B_2) \in \mathcal{P}$, and suppose that $B_1 \subseteq A_2$. Since $(A_1, B_1) \in \mathcal{P}$, it follows that either $A_1 \subseteq A_0$ or $A_1 \subseteq B_0$. Since $B_2 \subseteq A_1$ by (2.1), we deduce that either $B_2 \subseteq A_0$ or $B_2 \subseteq B_0$. But since $(A_2, B_2) \in \mathcal{P}$, B_2 includes one of A_0, B_0 by (2.1). Since neither of A_0, B_0 includes the other we deduce that $B_2 = A_0$ or $B_2 = B_0$, and hence either $(A_2, B_2) = (A_0, B_0)$ or $(A_2, B_2) = (B_0, A_0)$. The second is impossible by definition of $\mathcal{P}$, and so $(A_2, B_2) = (A_0, B_0)$. Similarly $(A_1, B_1) = (A_0, B_0)$ and hence $B_0 = B_1 \subseteq A_2 \subseteq A_0$, a contradiction. Thus $\mathcal{P}$ is a orientation. ■

A dissection $\mathcal{D}$ has *order* $< \kappa$ if each member of $\mathcal{D}$ has order $< \kappa$. We deduce

(5.2) *If $\mathcal{D}$ is a dissection and either $\mathcal{D}$ has width $< \kappa$ or $\mathcal{D}$ has adhesion $< \kappa$, then $\mathcal{D}$ has order $< \kappa$.*

Proof. Let $(A_0, B_0) \in \mathcal{D}$; we shall show that $|A_0 \cap B_0| < \kappa$. Let $\mathcal{P}$ be the orientation from (5.1).

(1) *$A_0 \cap B_0$ is included in the centre of $\mathcal{P}$.*

For let $(A, B) \in \mathcal{P}$. Then either $A \subseteq A_0$ or $A \subseteq B_0$, and so either $B_0 \subseteq B$ or $A_0 \subseteq B$ by (2.1). In either case $A_0 \cap B_0 \subseteq B$, and so $A_0 \cap B_0$ is included in the centre.

From (1) we deduce that if $\mathcal{D}$ has width $< \kappa$ then $|A_0 \cap B_0| < \kappa$. We may assume then that $\mathcal{D}$ has adhesion $< \kappa$. Choose $(A, B) \in \mathcal{P}$ cutting off (A_0, B_0) from $\mathcal{P}$, of order $< \kappa$. Then $A_0 \subseteq A$; but either $A \subseteq A_0$ or $A \subseteq B_0$ since $(A, B) \in \mathcal{P}$, and so $A = A_0$ and $B = B_0$. Thus $|A_0 \cap B_0| < \kappa$, as required. ■

(5.3) *Let $\mathcal{D}$ be a dissection of G of order $< \kappa$, and let β be a haven in G of order $\kappa \geq \aleph_0$. Then*

$$\{(A, B) \in \mathcal{D} : \beta(A \cap B) \subseteq B\}$$

is an orientation of $\mathcal{D}$.

Proof. Let $\mathcal{P} = \{(A, B) \in \mathcal{D} : \beta(A \cap B) \subseteq B\}$, and let $(A, B) \in \mathcal{D}$. Since $G|\beta(A \cap B)$ is connected it follows that either $\beta(A \cap B) \subseteq A - B$ or $\beta(A \cap B) \subseteq B - A$, and so $\mathcal{P}$ contains one of (A, B), (B, A). Secondly, let (A_1, B_1), $(A_2, B_2) \in \mathcal{P}$. Since $\kappa \geq \aleph_0$ it follows that $|(A_1 \cap B_1) \cup (A_2 \cap B_2)| < \kappa$. Hence

$$\emptyset \neq \beta((A_1 \cap B_1) \cup (A_2 \cap B_2)) \subseteq \beta(A_1 \cap B_1) \cap \beta(A_2 \cap B_2) \subseteq (B_1 - A_1) \cap (B_2 - A_2)$$

and so $B_1 \not\subseteq A_2$. Thus $\mathcal{P}$ is an orientation. ■

Now we can prove the easy "if" halves of (2.2) and (2.4).

(5.4) *For any $\kappa \geq \aleph_0$, if G has a dissection of width $< \kappa$ then no subgraph of G is a subdivision of K_κ.*

Proof. Let $\mathcal{D}$ be a dissection of G of width $< \kappa$, and suppose that G has a subgraph H which is isomorphic to a subdivision of K_κ. Let Z be the set of all vertices of H with valency κ in H. For $X \in [V(G)]^{<\kappa}$, let $\beta(X)$ be the unique X-flap which intersects Z; then β is a haven of order κ. By (5.2) and (5.3)

$$\mathcal{P} = \{(A, B) \in \mathcal{D} : \beta(A \cap B) \subseteq B\}$$

is an orientation of $\mathcal{D}$. Now $Z \subseteq B$ for all $(A, B) \in \mathcal{P}$, and so the centre of $\mathcal{P}$ includes Z, which is impossible since its cardinality is $< \kappa$. ■

This completes the proof of (2.2), because the "only if" half is a consequence of [7]. But the "only if" part for $\kappa \geq \aleph_1$ is also contained in (7.7).

(5.5) *For any $\kappa > \aleph_0$, if G has a dissection of width $< \kappa$ and adhesion $< \kappa$, then G has no haven of order κ and hence no K_κ-minor.*

Proof. Let $\mathcal{D}$ be a dissection of G of width $< \kappa$ and adhesion $< \kappa$, and suppose that β is a haven in G of order κ. Let $\mathcal{P}$ be the orientation $\{(A, B) \in \mathcal{D} : \beta(A \cap B) \subseteq B\}$, and let the centre of $\mathcal{P}$ be X. Choose $\kappa' < \kappa$ such that $\mathcal{D}$ has adhesion $\leq \kappa'$ at $\mathcal{P}$, and let κ'' be regular such that $|X|$, $\kappa' < \kappa'' \leq \kappa$. By (3.3), $\beta(X)$ contains a κ''-major vertex v. Since $v \notin X$ it follows that $v \notin B$ for some $(A, B) \in \mathcal{P}$. Since $\mathcal{D}$ has adhesion $\leq \kappa'$ at $\mathcal{P}$, there exists $(A', B') \in \mathcal{P}$ cutting off (A, B) of order $\leq \kappa'$. Since $|A' \cap B'| < \kappa''$, and v is κ''-major, it follows that $v \in (A' \cap B') \cup \beta(A' \cap B')$. But $\beta(A' \cap B') \subseteq B'$ since $(A', B') \in \mathcal{P}$, and so $v \in B' \subseteq B$, a contradiction. Hence G

has no haven of order κ, and the result follows from (3.1). ■

We shall need the following lemma.

(5.6) *Let $\mathcal{D}$ be a dissection of G, let $\mathcal{P}$ be an orientation of $\mathcal{D}$ with centre W, and let T be a non-null finite connected subgraph of G with $V(T) \cap W = \emptyset$. Then there exists $(A, B) \in \mathcal{P}$ with $V(T) \cap B = \emptyset$.*

Proof. Choose $t \in V(T)$. Since $t \notin W$, there exists $(A, B) \in \mathcal{P}$ with $t \in A - B$. Choose $(A, B) \in \mathcal{P}$ with $|V(T) \cap B|$ minimum (this is possible since T is finite). Suppose that $V(T) \cap B \neq \emptyset$. Since $V(T) \not\subseteq B$ and T is connected, there exist $u \in V(T) - B$ and $v \in V(T) \cap B$, adjacent. Choose $(A', B') \in \mathcal{P}$ with $v \notin B'$. Now $B \not\subseteq B'$ since $v \in B - B'$. Also, $A \not\subseteq B'$ since $v \in A$ (because v has a neighbour in $A - B$) and $v \notin B'$. Since $\mathcal{P}$ is a orientation we deduce that $B \not\subseteq A'$. Since (A, B), (A', B') do not cross it follows that $A \subseteq A'$ and $B' \subseteq B$. This contradicts the minimality of $|V(T) \cap B|$ since $v \notin B'$. Hence $V(T) \cap B = \emptyset$. ■

We use (5.6) to prove the easy parts of (2.6) and (2.7).

(5.7) *If G has a dissection of width $< \aleph_0$ and adhesion $< \aleph_0$ then G has no half-grid minor.*

Proof. Let $\mathcal{D}$ be a dissection of width and adhesion $< \aleph_0$, and suppose that G has a half-grid minor. Let β be the corresponding haven. Now from considering the "vertical" and "horizontal" paths of the minor, we deduce that there are connected subgraphs $P_1, P_2, \ldots$ and $Q_1, Q_2, \ldots$ of G, such that $P_1, P_2, \ldots$ are mutually disjoint, $Q_1, Q_2, \ldots$ are mutually disjoint, each P_i meets each Q_j, and for every finite $X \subseteq V(G)$, $\beta(X)$ meets all of $V(P_1), V(P_2), \ldots, V(Q_1), V(Q_2), \ldots$. Let $\mathcal{P}$ be the orientation $\{(A, B) \in \mathcal{D} : \beta(A \cap B) \subseteq B\}$. Let $\mathcal{P}$ have centre W and have adhesion $\leq d$ at $\mathcal{P}$, where d is an integer. Since W is finite and $P_1, P_2, \ldots$ are mutually disjoint, we may assume that $W \cap V(P_1) = \emptyset$. Let T be a finite connected subgraph of P_1 meeting $Q_1, \ldots, Q_{d+1}$. By (5.6) there exists $(A, B) \in \mathcal{P}$ with $V(T) \cap B = \emptyset$. Since $\mathcal{D}$ has adhesion $\leq d$ at $\mathcal{P}$, there exists $(A', B') \in \mathcal{P}$ of order $\leq d$ cutting off (A, B), and hence with $V(T) \cap B' = \emptyset$. Since $|A' \cap B'| \leq d$, one of $Q_1, \ldots, Q_{d+1}$, say Q_1, is disjoint from $A' \cap B'$. Since Q_1 meets T it follows that $V(Q_1) \subseteq A' - B'$. But $\beta(A' \cap B') \subseteq B$, and $V(Q_1) \cap \beta(A' \cap B') \neq \emptyset$, a contradiction. The result follows. ■

Thus, in (2.6), statement (ii) implies (i). A similar proof shows that (iii) implies (i); we omit the almost identical details.

(5.8) *If G has a dissection $\mathcal{D}$ with adhesion $< \aleph_0$, such that for every orientation $\mathcal{P}$ of $\mathcal{D}$ there is an integer $k \geq 0$ such that $ts(\mathcal{P})$ has no K_k-minor, then G has no $K_{\aleph_0}$-minor.*

Proof. Let $\mathcal{D}$ be as above, and suppose that $\mathcal{C}$ is a cluster in G with $|\mathcal{C}| = \aleph_0$. Let $\beta = \beta_{\mathcal{C}}$; then

$$\mathcal{P} = \{(A, B) \in \mathcal{D} : \beta(A \cap B) \subseteq B\}$$

is an orientation of $\mathcal{D}$, by (5.2) and (5.3). Let W be the centre of $\mathcal{P}$, and let $\mathcal{D}$ have adhesion $\leq d$ at $\mathcal{P}$, where d is an integer.

(1) *$C \cap W \neq \emptyset$ for each $C \in \mathcal{C}$.*

For suppose that $C \cap W = \emptyset$. Choose distinct $C_1, \ldots, C_{d+1} \in \mathcal{C}$, different from C. For $1 \leq i \leq d+1$ let $v_i \in C_i$ have a neighbour $u_i \in C$. Let T be a finite connected subgraph of $G|C$ with $u_1, \ldots, u_{d+1} \in V(T)$. By (5.6) there exists $(A, B) \in \mathcal{P}$ with $V(T) \cap B = \emptyset$, and since $\mathcal{D}$ has adhesion $\leq d$ at $\mathcal{P}$ we may choose (A, B) of order $\leq d$, as in (5.7). Then one of $C_1, \ldots, C_{d+1}$, say C_1, is disjoint from $A \cap B$, and since $v_1 \in C_1$ is adjacent to $u_1 \in A - B$ and hence $v_1 \in A$, it follows that $C_1 \subseteq A - B$; and so $\beta(A \cap B) \subseteq A$, a contradiction. This proves (1).

(2) *If Q is a finite path of G between $u \in W$ and $v \in W$, and no internal vertex of Q belongs to W, then u, v are adjacent in $ts(\mathcal{P})$.*

For if $V(Q) = \{u, v\}$ then u, v are adjacent in G and hence in $ts(\mathcal{P})$. Otherwise, $Q' = Q \backslash \{u, v\}$ is non-null, and $V(Q') \cap W = \emptyset$, and so by (5.6) there exists $(A, B) \in \mathcal{P}$ with $V(Q') \cap B = \emptyset$. Since $u, v \in W \subseteq B$ and u, v have neighbours in $V(Q')$ it follows that $u, v \in A \cap B$, and so u, v are adjacent in $ts(\mathcal{P})$, as required.

(3) *For each $C \in \mathcal{C}$, $ts(\mathcal{P})|(C \cap W)$ is connected.*

For let $u, v \in C \cap W$, and let Q be a path of $G|C$ from u to v. Let the vertices of $V(Q) \cap W$ be $u = w_1, w_2, \ldots, w_k = v$ in order. By (2) applied to the subpath of Q between w_i and w_{i+1}, we deduce that w_i, w_{i+1} are adjacent in $ts(\mathcal{P})$ for $1 \leq i \leq k-1$, and hence u, v are in the same component of $ts(\mathcal{P})|(C \cap W)$, as required.

Let $\mathcal{C}' = \{C \cap W : C \in \mathcal{C}\}$.

(4) *$\mathcal{C}'$ is a cluster in $ts(\mathcal{P})$.*

For by (1) and (3), each $C \cap W$ is non-null and $ts(\mathcal{P})|(C \cap W)$ is connected. Let $C_1, C_2 \in \mathcal{C}$ be distinct. Since there is an edge of G between some vertex of C_1 and some vertex of C_2, it follows from (1) that there is a path Q of

G with $V(Q) \subseteq C_1 \cup C_2$, with ends $w_1 \in C_1 \cap W$, $w_2 \in C_2 \cap W$ and with no internal vertex in W. By (2), w_1, w_2 are adjacent in $ts(\mathcal{T})$, and the claim follows.

But (4) contradicts our hypothesis. The result follows. ■

Thus in (2.7), (ii) implies (i). A similar proof, which we omit, shows that (iii) implies (i).

6. DIVISIONS

Now we turn to the proofs of the more difficult parts of our results. In each case our approach is the same; assuming that there is no haven of the appropriate kind, as characterized in (3.4), (3.5) and (4.1) respectively, we shall construct a dissection and prove that it satisfies our requirements. In each case it is convenient to construct the dissection by constructing first a "tangential set of divisions".

If G is a graph, $X \subseteq V(G)$ and F is an X-flap, we say that F is a *full* X-flap if each $x \in X$ has a neighbour in F. A *division* in G is a triple (X, C, D) where $X \subseteq V(G)$ and C, D are distinct full X-flaps. Its *order* is $|X|$. Two divisions (X_1, C_1, D_1), (X_2, C_2, D_2) are *tangential* if $X_1 \subseteq X_2 \cup F_2$ for some X_2-flap F_2 and $X_2 \subseteq X_1 \cup F_1$ for some X_1-flap F_1; and otherwise they *cross*.

(6.1) *If (X_1, C_1, D_1), (X_2, C_2, D_2) are divisions in G which cross, then $X_1 \cap F_2 \neq \emptyset$ for every full X_2-flap F_2 and $X_2 \cap F_1 \neq \emptyset$ for every full X_1-flap F_1.*

Proof. Suppose that $X_1 \cap F_2' = \emptyset$ for some full X_2-flap F_2'. Then $F_2' \subseteq F_1$ for some X_1-flap F_1. Since every vertex of X_2 has a neighbour in F_2', it follows that $X_2 \subseteq X_1 \cup F_1$. Since there are at least two full X_1-flaps, we may choose a full X_1-flap F_1' with $F_1' \neq F_1$; then $X_2 \cap F_1' = \emptyset$. By the same argument with X_1 and X_2 exchanged, it follows that $X_1 \subseteq X_2 \cup F_2$ for some X_2-flap F_2. But then (X_1, C_1, D_1), (X_2, C_2, D_2) are tangential, contrary to the hypothesis. The result follows. ■

Throughout the remainder of the paper we shall assume that the graph G under consideration is equipped with a fixed well-ordering of its vertex set. Thus, for every non-empty $X \subseteq V(G)$, there is an element $x \in X$ which is least in the well-ordering, and we call x the *nicest* element of X. Related terminology ("x is nicer than y" etc.) is defined in the natural way.

Let (X, C, D) be a division in G, and let v be the nicest element of $C \cup D$. If $v \in C$, let $A = V(G) - D$ and

$B = X \cup D$. If $v \in D$, let $B = V(G) - C$, $A = X \cup C$. In either case (A, B) is a separation; we call it the separation *induced* by (X, C, D). We see that $A \cap B = X$, $C \subseteq A$ and $D \subseteq B$; and one of $A - X$, $B - X$ is a full X-flap and the other contains the nicest element of $C \cup D$.

(6.2) *Let* (X_1, C_1, D_1), (X_2, C_2, D_2) *be tangential divisions, inducing separations* (A_1, B_1), (A_2, B_2). *Then* (A_1, B_1), (A_2, B_2) *do not cross.*

Proof. Since (X_1, D_1, C_1) is a division inducing the separation (B_1, A_1), and (X_1, D_1, C_1), (X_2, C_2, D_2) are tangential, we may replace (X_1, C_1, D_1) by (X_1, D_1, C_1) if we wish. Thus, we may assume that the nicest element of $C_1 \cup D_1$ is $v_1 \in C_1$, and similarly that $v_2 \in C_2$ is the nicest element of $C_2 \cup D_2$. Hence, $A_i = V(G) - D_i$, $B_i = X_i \cup D_i$ $(i = 1, 2)$.

Suppose first that $X_1 \subseteq A_2$. Then $D_2 \cap X_1 = \emptyset$, and so D_2 is a subset of an X_1-flap. Hence either $D_2 \subseteq A_1 - X_1$ or $D_2 \subseteq B_1 - X_1$. Since $D_2 = B_2 - X_2$ is a full X_2-flap, it follows that either $B_2 \subseteq A_1$ or $B_2 \subseteq B_1$ respectively. In the first case, since $B_1 - A_1 \subseteq A_2 - B_2$ and since every vertex of B_1 either belongs to or has a neighbour in $B_1 - A_1$, it follows that $B_1 \subseteq A_2$, and so (A_1, B_1), (A_2, B_2) do not cross. In the second case, it follows similarly that $A_1 \subseteq A_2$ and again (A_1, B_1), (A_2, B_2) do not cross.

We may assume, therefore, that $X_1 \not\subseteq A_2$, and similarly that $X_2 \not\subseteq A_1$. Thus $X_1 \cap D_2 \neq \emptyset$; and since X_1 meets at most one X_2-flap (because (X_1, C_1, D_1) and (X_2, C_2, D_2) are tangential) it follows that $X_1 \subseteq X_2 \cup D_2 = B_2$. Similarly, $X_2 \subseteq B_1$. Now $v_i \in C_i$ is the nicest element of $C_i \cup D_i$, and $v_i \in C_i \subseteq A_i$ $(i = 1, 2)$. From the symmetry we may assume that v_1 is at least as nice as v_2. (Possibly $v_1 = v_2$.) Since v_2 is nicer than every vertex of $D_2 = B_2 - X_2$, it follows that $v_1 \notin B_2 - X_2$, and hence $v_1 \in A_2$. Since C_1 is a full X_1-flap and $X_1 \not\subseteq A_2$, there is a path P of G between v_1 and some vertex $x \in X_1 - A_2$ with $V(P) \subseteq V(C_1) \cup \{x\} \subseteq (A_1 - X_1) \cup \{x\}$. But then

$$V(P) \cap X_2 \subseteq (A_1 - X_1) \cap X_2 \subseteq (A_1 - X_1) \cap B_1 = \emptyset$$

and this contradicts that (A_2, B_2) is a separation; for one end of P is in A_2, the other is in B_2, and $V(P) \cap A_2 \cap B_2 = \emptyset$. This case (namely, $X_1 \not\subseteq A_2$ and $X_2 \not\subseteq A_1$) therefore cannot occur, and the proof is complete. ■

A *geography* in G is a set $\mathcal{G}$ of divisions, mutually tangential, such that if $(X, C, D) \in \mathcal{G}$ then $(X, D, C) \in \mathcal{G}$.

(6.3) *Let $\mathcal{G}$ be a geography in G, and let $\mathcal{D}$ be the set of all separations induced by members of $\mathcal{G}$. Then $\mathcal{D}$ is a dissection.*

Proof. Certainly if $(A, B) \in \mathcal{D}$ then $(B, A) \in \mathcal{D}$, and by (6.2) no two members of $\mathcal{D}$ cross. Moreover, if (A, B) is induced by (X, C, D) then $\emptyset \neq C \subseteq A - B$ and $\emptyset \neq D \subseteq B - A$, and so $A, B \neq V(G)$. Finally, let (A_i, B_i) be induced by (X_i, C_i, D_i) $(i = 1, 2)$ and suppose that $B_1 = B_2$. Since every vertex of $A_1 \cap B_1$ has a neighbour in $C_1 \subseteq A_1 - B_1 = A_2 - B_2$, it follows that $A_1 \cap B_1 \subseteq A_2$ and hence $A_1 \subseteq A_2$. Similarly $A_2 \subseteq A_1$ and so $A_1 = A_2$, as required. Thus, $\mathcal{D}$ is a dissection. ■

We call the dissection in (6.3) the dissection *induced* by $\mathcal{G}$.

7. LONG DIVISIONS

If κ is a cardinal, we say that $v_1, v_2 \in V(G)$ are κ-*separated* if there is a separation (A, B) of order $\leq \kappa$ with $v_1 \in A - B$ and $v_2 \in B - A$. Thus, if v_1, v_2 are adjacent they are not κ-separated for any κ. A division (X, C, D) is *long* if no two members of X are κ-separated, where $\kappa = \max(|X|, \aleph_0)$.

(7.1) *Every pair of long divisions are tangential.*

Proof. Let (X_1, C_1, D_1), (X_2, C_2, D_2) be long divisions. From the symmetry we may assume that $|X_1| \leq |X_2|$. Let (A, B) be a separation with $C_1 \subseteq A$, $D_1 \subseteq B$, $A \cap B = X_1$. Then (A, B) has order $|X_1| \leq |X_2|$, and so one of $X_2 \cap (A - B)$, $X_2 \cap (B - A)$ is empty since (X_2, C_2, D_2) is long. Hence one of $X_2 \cap C_1$, $X_2 \cap D_1$ is empty, and the result follows from (6.1). ■

(7.2) *Let $Y \subseteq V(G)$ and let F_1, F_2 be distinct Y-flaps. Then there is a division (X, C, D) with $X \subseteq Y$, $F_1 \subseteq C$, $F_2 \subseteq D$.*

Proof. Let $X' \subseteq Y$ be the set of all vertices in Y with a neighbour in F_1. Then F_1 is a full X'-flap, and F_2 is a subset of some X'-flap $D \neq F_1$. Let X be the set of all vertices in X' with a neighbour in D. Then D is a full X-flap, and F_1 is a subset of some X-flap $C \neq D$. Since F_1 is a full X'-flap and $X \subseteq X'$ it follows that C is a full X-flap, and hence (X, C, D) is the required division. ■

(7.3) *Let $X_0 \subseteq V(G)$ with $|X_0| \leq \kappa$ where κ is an infinite cardinal, and let C_0 be an X_0-flap. Let $Z \subseteq V(G)$ with $|Z - C_0| > \kappa$. Then there is a long division (X, C, D) with $|X| \leq \kappa$ such that $C_0 \subseteq C$ and $D \cap Z \neq \emptyset$.*

Proof. We shall define inductively a sequence $X_0 \subseteq X_1 \subseteq X_2 \subseteq \ldots$ of subsets of $V(G) - C_0$, as follows. Suppose that $i \geq 0$ and X_i is defined. For each pair x, y of elements of X_i which are κ-separated, let $N(x, y) = A \cap B - C_0$, where (A, B) is some separation of order $\leq \kappa$ with $x \in A - B$ and $y \in B - A$. Let

$$X_{i+1} = X_i \cup \bigcup (N(x, y) : x, y \in X_i \text{ are } \kappa\text{-separated}).$$

This completes the inductive definition. We see that since κ is infinite, $|X_0| \leq \kappa$ and each $|N(x, y)| \leq \kappa$, it follows that each $|X_i| \leq \kappa$.

Let $X^* = \bigcup (X_i : i \geq 0)$: then $|X^*| \leq \kappa$, and C_0 is a X^*-flap. Since $|Z - C_0| > \kappa$ there is an X^*-flap $F_2 \neq C_0$ with $F_2 \cap Z \neq \emptyset$. Let Y be the set of vertices in X^* with a neighbour in F_2.

(1) *If $x, y \in Y$ then x, y are not κ-separated.*

For since $x, y \in Y$, there exists $i \geq 0$ such that $x, y \in X_i$. Since x, y both have neighbours in F_2 and $F_2 \cap (X_{i+1} \cup C_0) = \emptyset$, there is no separation (A, B) of G of order $\leq \kappa$ with $x \in A - B$, $y \in B - A$ and $A \cap B \subseteq X_{i+1} \cup C_0$. By definition of X_{i+1} it follows that x, y are not κ-separated, as required.

Now F_2 is a Y-flap, and C_0 is a subset of another Y-flap F_1. The result follows from (1) and (7.2). ■

(7.4) *Let κ be an infinite cardinal, and let $X_1, X_2 \subseteq V(G)$, such that $|X_1|, |X_2| > \kappa$ and for $i = 1, 2$ no two members of X_i are κ-separated. Suppose that there is no long division (X, C, D) with $|X| \leq \kappa$ and $X_1 \subseteq X \cup C$, $X_2 \subseteq X \cup D$. Then no two members of $X_1 \cup X_2$ are κ-separated.*

Proof. Suppose, for a contradiction, that there exists $X_0 \subseteq V(G)$ with $|X_0| \leq \kappa$, such that $F_1 \cap X_1 \neq \emptyset \neq F_2 \cap X_2$ for some distinct X_0-flaps F_1, F_2. We claim that

(1) *There is a long division (X, C, D) with $|X| \leq \kappa$, $C \cap X_1 \neq \emptyset$ and $D \cap X_2 \neq \emptyset$.*

For if $|X_2 - F_1| > \kappa$ this follows from (7.3), taking $Z = X_2$, since $X_1 \cap F_1 \neq \emptyset$. Thus, we may assume that $|X_2 - F_1| \leq \kappa$ and, similarly, $|X_1 - F_2| \leq \kappa$. It follows that $|X_2 - F_2| > \kappa$, since $|X_2| > \kappa \geq \aleph_0$, and so by (7.3) there is a long division (X, C, D) with $|X| \leq \kappa$, $F_2 \subseteq C$ and $D \cap X_2 \neq \emptyset$. But $X_1 \cap F_2 \neq \emptyset$ since $|X_1 - F_2| \leq \kappa$ and $|X_1| > \kappa$, and so $X_1 \cap C \neq \emptyset$. Thus, (X, C, D) satisfies (1).

Now since no two members of X_1 are κ-separated, it follows that $X_1 \subseteq C \cup X$, and similarly $X_2 \subseteq D \cup X$, a contradiction. Thus there is no such X_0, and the result follows. ■

(7.5) *Let $\mathcal{G}$ be a geography, and let $\mathcal{D}$ be the induced dissection. Let $\mathcal{P}$ be an orientation of $\mathcal{D}$, and let W be the centre of $\mathcal{P}$. Then there is no $(X, C, D) \in \mathcal{G}$ with $C \cap W \neq \emptyset \neq D \cap W$.*

Proof. Let $(X, C, D) \in \mathcal{G}$, and let (A, B) be the separation induced by (X, C, D). Then either $(A, B) \in \mathcal{P}$ or $(B, A) \in \mathcal{P}$, and so either $W \subseteq B$ or $W \subseteq A$; hence, either $W \cap C = \emptyset$ or $W \cap D = \emptyset$, as required. ■

Let $\{\beta_i : i \in I\}$ be a set of havens in G, where β_i has order κ_i. Let $\kappa = sup(\kappa_i : i \in I)$. We say this set is *convergent* if for all $X \subseteq V(G)$ with $|X| < \kappa$ there is an X-flap $\beta(X)$ and λ with $|X| < \lambda < \kappa$ such that $\beta_i(X) = \beta(X)$ for all $i \in I$ with $\kappa_i \geq \lambda$; and β is a *limit* of $\{\beta_i : i \in I\}$.

(7.6) *If $\{\beta_i : i \in I\}$ is convergent then the limit β is unique and is a haven of order κ.*

Proof. With notation as above, suppose that β, β' are both limits and $\beta \neq \beta'$. Choose $X \in [V(G)]^{<\kappa}$ with $\beta(X) \neq \beta'(X)$. Choose λ with $|X| < \lambda < \kappa$ such that $\beta_i(X) = \beta(X)$ for all $i \in I$ with $\kappa_i \geq \lambda$; and choose λ' similarly for β'. Since $\lambda, \lambda' < \kappa = sup(\kappa_i : i \in I)$ there exists $i \in I$ such that $\lambda, \lambda' \leq \kappa_i$. Then $\beta(X) = \beta_i(X) = \beta'(X)$, a contradiction. Thus the limit is unique.

Now let $X \subseteq Y \in [V(G)]^{<\kappa}$. Choose λ with $|Y| < \lambda < \kappa$ such that $\beta_i(X) = \beta(X)$ and $\beta_i(Y) = \beta(Y)$ for all $i \in I$ with $\kappa_i \geq \lambda$. Choose $i \in I$ with $\kappa_i \geq \lambda$. Then $\beta(X) = \beta_i(X) \subseteq \beta_i(Y) = \beta(Y)$. Hence, β is a haven of order κ, as required. ■

Now we complete the proof of (2.5) (and give a second proof of (2.2) in the uncountable case) with the following.

(7.7) *Let $\kappa > \aleph_0$. If G has no dissection of width $< \kappa$ and adhesion $< \kappa$ then G has a K_κ-minor; and if G has no dissection of width $< \kappa$ then G contains K_κ topologically.*

Proof. Let $\mathcal{G}$ be the set of all long divisions of order $< \kappa$; then $\mathcal{G}$ is a geography, by (7.1). Let $\mathcal{D}$ be the induced dissection.

(1) *If $\mathcal{D}$ does not have width $< \kappa$ then G contains K_κ topologically.*

For let $\mathcal{P}$ be an orientation of $\mathcal{D}$, with centre W, and suppose that $|W| \geq \kappa$. If no pair of vertices in W are κ'-separated for any $\kappa' < \kappa$ then it follows easily that G contains K_κ topologically. We suppose, for a contradiction, that for some $\kappa' < \kappa$ some pair of vertices in W are κ'-separated. Since $\kappa > \aleph_0$ we may assume that $\kappa' \geq \aleph_0$.

Choose $X_0 \subseteq V(G)$ with $|X_0| \leq \kappa'$ such that at least two X_0-flaps intersect W, and let C_0 be an X_0-flap with $C_0 \cap W \neq \emptyset$, such that $|W - C_0| \geq \kappa > \kappa'$. By (7.3) there is a long division (X, C, D) with $|X| \leq \kappa'$, such that $C_0 \subseteq C$ and $D \cap W \neq \emptyset$. Then $C \cap W \neq \emptyset \neq D \cap W$ contrary to (7.5). This proves (1).

From (1), the second assertion of the theorem follows. To prove the first assertion we may assume, in view of (1), that $\mathcal{D}$ has width $< \kappa$ but does not have adhesion $< \kappa$. Let $\mathcal{P}$ be an orientation of $\mathcal{D}$ such that for all cardinals $\lambda < \kappa$, $\mathcal{D}$ does not have adhesion $\leq \lambda$ at $\mathcal{P}$. For each λ with $\aleph_0 \leq \lambda < \kappa$, choose $(A_\lambda, B_\lambda) \in \mathcal{P}$ such that there is no $(A, B) \in \mathcal{P}$ of order $\leq \lambda$ cutting off (A_λ, B_λ). Let $X_\lambda = A_\lambda \cap B_\lambda$.

(2) $|X_\lambda| > \lambda$, *and no two members of* X_λ *are* $|X_\lambda|$*-separated.*

For (A_λ, B_λ) cuts off itself, and so $|X_\lambda| > \lambda$; and the second assertion follows since (A_λ, B_λ) is induced by some long division.

For each λ and $X \in [V(G)]^{<\lambda}$ let $\beta_\lambda(X)$ be an X-flap which intersects X_λ; then β_λ is a haven of order λ, by (2).

(3) *The set* $\{\beta_\lambda : \aleph_0 \leq \lambda < \kappa\}$ *is convergent.*

For let $X \in [V(G)]^{<\kappa}$ and let $\lambda = \max(|X|, \aleph_0)$. Let λ_1, λ_2 satisfy $\lambda \leq \lambda_1, \lambda_2 < \kappa$, and suppose that there is a long division (Y, C, D) with $|Y| \leq \lambda$ and $X_{\lambda_1} \subseteq Y \cup C$, $X_{\lambda_2} \subseteq Y \cup D$. Let (A, B) be the separation induced by (Y, C, D); then $(A, B) \in \mathcal{D}$, and from the symmetry we may assume that $(A, B) \in \mathcal{P}$. Now (A, B) does not cut off $(A_{\lambda_1}, B_{\lambda_1})$ since $|A \cap B| = |Y| \leq \lambda_1$. But $X_{\lambda_1} \subseteq Y \cup C$ and $|X_{\lambda_1}| > |Y|$, and so $X_{\lambda_1} \not\subseteq B$; and so B includes neither A_{λ_1} nor B_{λ_1}. This contradicts that $\mathcal{P}$ is an orientation, and we deduce that there is no such (Y, C, D). By (2) and (7.4) we deduce that no two members of $X_{\lambda_1} \cup X_{\lambda_2}$ are λ-separated, and so $\beta_{\lambda_1}(X) = \beta_{\lambda_2}(X)$. This proves (3).

Then the limit of the set of (3) is a haven of order κ, by (7.6). By (3.1), G has a K_κ-minor, as required. ■

8. ROBUST DIVISIONS

Now we begin the proof of the harder parts of (2.6) and (2.7) (that is, (i) => (ii), in both cases). A division (X, C, D) is *robust* if X is finite and for every separation (A, B) of finite order,

$$\min(|A \cap X|, |B \cap X|) \leq \min(|A \cap B \cap (C \cup X)|, |A \cap B \cap (D \cup X)|).$$

A separation is *robust* if it is induced by a robust division.

(8.1) *If (A', B') is a robust separation, and (A, B) is a separation of finite order, then one of $(A \cap A', B \cup B')$, $(B \cap A', A \cup B')$ has order at most $|A \cap B|$.*

Proof. Let (A', B') be induced by (X, C, D). Since (X, C, D) is robust we may assume (by the symmetry between A and B) that

$$|A \cap A' \cap B'| = |A \cap X| \leq |A \cap B \cap (D \cup X)| \leq |A \cap B \cap B'| .$$

Thus

$$|A \cap A' \cap (B \cup B')| = |A \cap A' \cap B'| + |A \cap (B - B')| \leq |A \cap B \cap B'| + |A \cap (B - B')| = |A \cap B| .$$

The result follows. ■

If $X, Y \subseteq V(G)$ are finite, we say that X is *nicer* than Y if $X \neq Y$, $|X| \leq |Y|$, and if equality occurs then X contains the nicest element of $(X - Y) \cup (Y - X)$. A division (X, C, D) is *nicely robust* if X is finite and for every separation (A, B) of finite order, one of $A \cap X$, $B \cap X$ is at least as nice as both of $A \cap B \cap (C \cup X)$, $A \cap B \cap (D \cup X)$. Thus, nicely robust divisions are robust.

(8.2) *No two nicely robust divisions cross.*

Proof. Let (X, C, D), (X', C', D') be divisions of finite order, and suppose that they cross. By (6.1), $X' \cap C$, $X' \cap D$, $X \cap C'$, $X \cap D'$ are all non-empty, and mutually disjoint. From the symmetry we may assume that the first is nicer than the other three, and so $X' \cap (C \cup X)$ is nicer than $X \cap (C' \cup X')$ and $X \cap (D' \cup X')$. Let (A, B) be a separation with $A \cap B = X'$, $C' \subseteq A$ and $D' \subseteq B$. Then $X \cap (C' \cup X') \subseteq X \cap A$ and so $A \cap B \cap (C \cup X) = X' \cap (C \cup X)$ is nicer than $X \cap A$, and similarly nicer than $X \cap B$. Hence (X, C, D) is not nicely robust, as required. ■

(8.3) *Let β_1, β_2 be havens of order $> \frac{3}{2}k$ where $k \geq 0$ is an integer, and such that there exists $X \subseteq V(G)$ with $|X| \leq k$ such that $\beta_1(X) \neq \beta_2(X)$. Choose such a set X as nice as possible. Then $(X, \beta_1(X), \beta_2(X))$ is a nicely robust division.*

Proof. Let X' be the set of vertices in X with a neighbour in $\beta_1(X)$. Then $\beta_1(X') = \beta_1(X)$, and $\beta_2(X') \supseteq \beta_2(X)$, and so $\beta_1(X') \neq \beta_2(X')$. By the choice of X, it follows that $X = X'$. Hence $\beta_1(X)$ is a full X-flap, and similarly so is $\beta_2(X)$. Thus, $(X, \beta_1(X), \beta_2(X))$ is a division.

Let (A, B) be a separation of finite order and suppose that $A \cap B \cap (\beta_1(X) \cup X)$ is nicer than both $A \cap X$ and $B \cap X$. Let $Y = X \cup (A \cap B \cap \beta_1(X))$. Now since

$$|A \cap B \cap (\beta_1(X) \cup X)| \leq |A \cap X|, |B \cap X|$$

it follows that

$$|A \cap B \cap (\beta_1(X) \cup X)| \leq \frac{1}{2}|X| + \frac{1}{2}|A \cap B \cap X|$$

and so $|A \cap B \cap \beta_1(X)| \leq \frac{1}{2}|X|$. Thus

$$|Y| = |X| + |A \cap B \cap \beta_1(X)| \leq \frac{3}{2}|X| \leq \frac{3}{2}k,$$

and so $\beta_1(Y)$, $\beta_2(Y)$ are defined. Now

$$\beta_1(Y) \subseteq \beta_1(X) - Y = \beta_1(X) - (A \cap B) = ((A - B) \cap \beta_1(X)) \cup ((B - A) \cap \beta_1(X)).$$

Since no vertex in $A - B$ is adjacent to any in $B - A$ and $G|\beta_1(Y)$ is connected, it follows that $\beta_1(Y) \subseteq (A - B) \cap \beta_1(X)$ or $\beta_1(Y) \subseteq (B - A) \cap \beta_1(X)$, and by symmetry we may assume the former. Let $X' = (X \cap A) \cup (A \cap B \cap \beta_1(X))$. Now $X' \subseteq Y$ and no vertex in $Y - X'$ has a neighbour in $\beta_1(Y)$, for $Y - X' \subseteq B - A$ and $\beta_1(Y) \subseteq A - B$. Thus $\beta_1(X') = \beta_1(Y)$. But $\beta_2(X) = \beta_2(Y)$ since $Y \cap \beta_2(X) = \emptyset$, and $\beta_2(Y) \subseteq \beta_2(X')$ since $X' \subseteq Y$; and so $\beta_2(X) \subseteq \beta_2(X')$. But $\beta_2(X) \not\subseteq \beta_1(X')$, for $\beta_1(X') = \beta_1(Y) \subseteq \beta_1(X)$ and $\beta_2(X) \not\subseteq \beta_1(X)$. Since $\beta_2(X) \subseteq \beta_2(X')$ and $\beta_2(X) \not\subseteq \beta_1(X')$ it follows that $\beta_1(X') \neq \beta_2(X')$. From the choice of X, we deduce that X is at least as nice as X'. Since $X - B \subseteq X \cap X'$ it follows that $X - (X - B) = B \cap X$ is at least as nice as $X' - (X - B) = A \cap B \cap (\beta_1(X) \cup X)$. But this contradicts an earlier assumption. We deduce that $(X, \beta_1(X), \beta_2(X))$ is nicely robust, as required. ■

Let $X \subseteq V(G)$ be finite. An X-flap F is *complete* if there is a cluster $\mathcal{C}$ in G with $|\mathcal{C}| = |X|$, such that each $C \in \mathcal{C}$ satisfies $C \subseteq X \cup F$ and $|C \cap X| = 1$. A division (X, C, D) is *bicomplete* if X is finite and C, D are both complete X-flaps.

(8.4) *Let β_1, β_2 be clustered havens in G, both of order $> 2k$, where $k \geq 0$ is an integer. Suppose that there exists $X \subseteq V(G)$ with $|X| \leq k$ such that $\beta_1(X) \neq \beta_2(X)$, and choose such a set X, as nice as possible. Then $(X, \beta_1(X), \beta_2(X))$ is a nicely robust, bicomplete division.*

Proof. By truncating we may assume that β_1, β_2 both have order $2t+1$, where $|X| = t$. By (8.3), $(X, \beta_1(X), \beta_2(X))$ is a nicely robust division. To show it is bicomplete it suffices, therefore, to show that $\beta_1(X)$ is a complete X-flap. We claim that X is β_1-free. For let $Y \subseteq V(G)$ with $|Y| < |X|$. Since $\emptyset \neq \beta_2(X \cup Y) \subseteq \beta_2(X) \cap \beta_2(Y)$, it follows that $\beta_2(Y) \not\subseteq \beta_1(X)$. From the minimality of X, $\beta_1(Y) = \beta_2(Y)$ and so $\beta_1(Y) \not\subseteq \beta_1(X)$. Since $\beta_1(X \cup Y) \subseteq \beta_1(X)$ it follows that $\beta_1(Y) \neq \beta_1(X \cup Y)$, and so $X \cap \beta_1(Y) \neq \emptyset$. This proves that X is β_1-free.

Let $\beta_1 = \beta_C$ for some cluster C with $|C| = 2t+1$. By (4.5) we may assume that C is local. By (4.7) there exists $C' \subseteq C$ with $|C'| = t$, and with $X \cap C = \emptyset$ for all $C \in C'$, and a comb $\mathcal{P}$ traversing C' with $|\mathcal{P}| = t$, such that X is the set of its tips. Let $\mathcal{P} = \{P_1, \ldots, P_t\}$ where P_i has its root in $C_i \in C'$ $(1 \leq i \leq t)$. For $1 \leq i \leq t$, let $D_i = C_i \cup V(P_i)$, and let $\mathcal{D} = \{D_i : 1 \leq i \leq t\}$. Since $X \cap C_i = \emptyset$ for $1 \leq i \leq t$ and hence $C_i \subseteq \beta_1(X)$, it follows that $D_i \subseteq X \cup \beta_1(X)$ and that $|D_i \cap X| = 1$ $(1 \leq i \leq t)$. Hence $\beta_i(X)$ is a complete X-flap, as required. ■

(8.5) *Let (X_1, C_1, D_1), (X_2, C_2, D_2) be divisions of finite order with $(X_1 \cup C_1) \cap (X_2 \cup D_2) = X_1 \cap X_2$. Let $X \subseteq V(G)$ meet every path between X_1 and X_2, and subject to that have minimum cardinality. Then $X \cap C_1 = X \cap D_2 = \emptyset$; let C, D be the X-flaps including C_1, D_2 respectively. Then (X, C, D) is a division. Moreover*

(i) if (X_1, C_1, D_1), (X_2, C_2, D_2) are robust then so is (X, C, D)

(ii) if X is chosen as nice as possible meeting every path between X_1 and X_2 and (X_1, C_1, D_1), (X_2, C_2, D_2) are nicely robust, then so is (X, C, D)

(iii) if (X_1, C_1, D_1), (X_2, C_2, D_2) are bicomplete then so is (X, C, D).

Proof. Let $|X| = k$. By Menger's theorem there are k paths $P_1, \ldots, P_k$ of G between X_1 and X_2, mutually vertex-disjoint, each with only its first vertex in X_1 and only its last in X_2; and hence each with exactly one vertex in X. Let P_i have ends $s_i \in X_1$, $t_i \in X_2$ and let $x_i \in V(P_i) \cap X$ $(1 \leq i \leq k)$. Let the subpaths of P_i between s_i and x_i, and between x_i and t_i, be S_i and T_i respectively. Now every vertex of P_i except s_i is in the same X_1-flap, and since $t_i \notin C_1$ it follows that $V(P_i) \cap C_1 = \emptyset$, and similarly $V(P_i) \cap D_2 = \emptyset$. Thus $X \cap C_1 = X \cap D_2 = \emptyset$. This proves the first claim of the theorem. Let C, D be defined as in the theorem.

(1) $C \neq D$.

For suppose that $C = D$. Then $C \cap C_1 \neq \emptyset$, $C \cap D_2 \neq \emptyset$. Not both $C \cap X_1$, $C \cap X_2$ are non-empty since $G|C$ is connected and X meets every path between X_1 and X_2. Thus, we may assume from the symmetry that $C \cap X_1 = \emptyset$. Hence C is included in an X_1-flap and so $C = C_1$. But $C_1 \cap D_2 = \emptyset$ by hypothesis, a contradiction. This proves (1).

(2) *(X, C, D) is a division.*

It suffices to show that C is a full X-flap. Let $1 \leq i \leq k$. If $s_i = x_i$ then since s_i has a neighbour in C_1 it follows that x_i has a neighbour in C. If $s_i \neq x_i$ then x_i has a neighbour y in $V(S_i) - \{x_i\}$, and y belongs to the same X-flap as s_i; but since s_i has a neighbour in $C_1 \subseteq C$ and $s_i \notin X$ it follows that $s_i \in C$, and so $y \in C$. This proves (2).

(3) *For any separation (A, B),*

$$X' = (X - A) \cup (A \cap X_1) \cup (A \cap B \cap ((C \cup X) - (C_1 \cup X_1)))$$

meets every path between X_1 *and* X_2.

For we claim that X' meets every path from X_1 to X (and hence from X_1 to X_2). For let P be such a path. We may assume that its ends ($u \in X_1$, $v \in X$ say) are not in X'; and so $u \notin A$ and $v \in A$. We may also assume that $V(P) \cap X_1 = \{u\}$ and $V(P) \cap X = \{v\}$; and so $V(P) \cap C_1 = \emptyset$, and $V(P) - \{v\} \subseteq C$. Since $u \notin A$ and $v \in A$, there is a vertex $w \in V(P)$ with $w \in A \cap B$. Since $w \neq u$ (because $u \notin A$) and $V(P) \cap X_1 = \{u\}$ it follows that $w \notin X_1$, and so $w \in A \cap B \cap ((C \cup X) - (C_1 \cup X_1) \subseteq X'$. Hence $X' \cap V(P) \neq \emptyset$. This proves (3).

(4) *If* (X_1, C_1, D_1), (X_2, C_2, D_2) *are robust then so is (X, C, D).*

For let (A, B) be a separation of finite order. It suffices by the symmetry between X_1 and X_2 to show that

$$\min(|A \cap X|, |B \cap X|) \leq |A \cap B \cap (C \cup X)|\ .$$

Since (X_1, C_1, D_1) is robust, we may assume by the symmetry between A and B that $|A \cap X_1| \leq |A \cap B \cap (C_1 \cup X_1)|$. From (3), $|X| \leq |X'|$ and so

$$\begin{aligned} |A \cap X| &\leq |X'| - |X - A| \leq |(A \cap X_1) \cup (A \cap B \cap ((C \cup X) - (C_1 \cup X_1)))| \\ &\leq |(A \cap B \cap (C_1 \cup X_1)) \cup (A \cap B \cap ((C \cup X) - (C_1 \cup X_1)))| \\ &= |A \cap B \cap (C \cup X)| \end{aligned}$$

as required.

(5) *If* (X_1, C_1, D_1), (X_2, C_2, D_2) *are nicely robust, and* X *is chosen as nice as possible meeting every path between* X_1 *and* X_2, *then* (X, C, D) *is nicely robust.*

The argument is as for (4), comparing the "niceness" of the sets instead of their cardinalities.

(6) *If* (X_1, C_1, D_1), (X_2, C_2, D_2) *are bicomplete then so is* (X, C, D).

For it suffices to show that C is a complete X-flap. Since C_1 is a complete X_1-flap, there is a cluster $\mathcal{C}_1$ with $|\mathcal{C}_1| = |X_1|$ such that each member $F \in \mathcal{C}_1$ satisfies $F \subseteq C_1 \cup X_1$ and $|F \cap X_1| = 1$. For $1 \le i \le k$ let $s_i \in F_i \in \mathcal{C}_1$; then $\{F_i \cup V(S_i) : 1 \le i \le k\}$ is a cluster satisfying our requirements.

This completes the proof. ■

(8.6) *Let* $\mathcal{G}$ *be a geography every member of which is robust, and let* (X_1, C_1, D_1), $(X_2, C_2, D_2) \in \mathcal{G}$, *where* $(X_1 \cup C_1) \cap (X_2 \cup D_2) = X_1 \cap X_2$. *Choose* k *minimum such that there exists* $X \subseteq V(G)$ *with* $|X| \le k$ *meeting every path between* X_1 *and* X_2. *Then for some* $X \subseteq V(G) - (C_1 \cup D_2)$ *with* $|X| = k$ *there is a division* (X, C, D), *with* $X_1 \cup C_1 \subseteq X \cup C$ *and* $X_2 \cup D_2 \subseteq X \cup D$, *which crosses no member of* $\mathcal{G}$.

Proof. Let M be the set of all $X \subseteq V(G)$ with $|X| = k$ such that X meets every path between X_1 and X_2.

(1) *M is finite.*

For there are k mutually vertex-disjoint paths $P_1, \ldots, P_k$ of G between X_1 and X_2, by Menger's theorem, and each P_i is finite. Each $X \in M$ contains a vertex from each P_i and so $X \subseteq V(P_1) \cup \ldots \cup V(P_k)$. This proves (1).

From (8.5), $X \cap C_1 = \varnothing = X \cap D_2$ for each $X \in M$. Let $C(X)$, $D(X)$ be the X-flaps including C_1 and D_2 respectively; then by (8.5), $(X, C(X), D(X))$ is a division. Let $S(X)$ denote

$$\{(Y, C, D) \in \mathcal{G} : (X, C(X), D(X)) \text{ crosses } (Y, C, D)\}.$$

By (1), we may choose $X \in M$ with $S(X)$ minimal. We shall show that $S(X) = \varnothing$. For suppose that $(Y, C, D) \in S(X)$.

(2) *If* $u, v \in X \cup Y$ *and* $(Y', C', D') \in \mathcal{G}$ *is such that* $u \in C'$ *and* $v \in D'$, *then* $(Y', C', D') \in S(X)$.

For if $u, v \in X$ this is clear. We may assume then that $u \in Y - X$. Since $C' \cap Y \ne \varnothing$ and (Y, C, D), (Y', C', D') are tangential, it follows that $Y \subseteq Y' \cup C'$. Hence $v \notin Y$, since $v \in D'$, and so $v \in X - Y$. In particular, $Y \cap D' = \emptyset$; let F be the Y-flap including D'. Then $D' \cup Y' \subseteq F \cup Y$, since D' is a full Y'-flap. Choose $w \in X - (F \cup Y)$ (this is possible since $X \cap C$, $X \cap D \neq \emptyset$). Then $w \notin D' \cup Y'$, and so X meets two distinct Y'-flaps, as required.

(3) *Either* $X_1 \cap C \neq \emptyset$ *or* $X_2 \cap C \neq \emptyset$.

For suppose that $X_1 \cap C = \emptyset = X_2 \cap C$. Let (A, B) be a separation of G with $C(X) \subseteq A$, $D(X) \subseteq B$ and $A \cap B = X$. By the symmetry between X_1 and X_2 we may assume, from the robustness of (Y, C, D), that $|A \cap Y| \leq |A \cap B \cap (C \cup Y)|$. Put $X' = (A \cap Y) \cup (A \cap B - (C \cup Y))$. Then X' meets every path from X_1 to X and hence to X_2; for the first vertex of such a path in $B \cup C \cup Y$ is not in C (since $X_1 \cap C = \emptyset$) and so is either in $A \cap Y$ or in $A \cap B - (C \cup Y)$. But $|X'| \leq |X|$, and so $X' \in M$. By (2), $S(X') \subseteq S(X)$ (because $C' \cap X' \neq \emptyset \neq D' \cap X'$ for every $(Y', C', D') \in S(X')$ by (6.1)); but $(Y, C, D) \notin S(X')$ since $X' \cap C = \emptyset$. This contradicts the minimality of $S(X)$, and proves (3).

From (3) we may assume that $X_1 \cap C \neq \emptyset$. Since (X_1, C_1, D_1) and (Y, C, D) are tangential, it follows that $X_1 \subseteq Y \cup C$. By (3) it follows by the symmetry between C and D that one of $X_1 \cap D$, $X_2 \cap D$ is non-empty. Since $X_1 \subseteq Y \cup C$ we deduce that $X_2 \cap D \neq \emptyset$, and so $X_2 \subseteq Y \cup D$. Let (A, B) be a separation as before, that is with $C(X) \subseteq A$, $D(X) \subseteq B$ and $A \cap B = X$. From the symmetry between X_1 and X_2 we may assume that $|A \cap Y| \leq |B \cap Y|$. Since (Y, C, D) is robust it follows that

$$|A \cap Y| \leq |A \cap B \cap (D \cup Y)| .$$

Let $X' = (A \cap Y) \cup (A \cap B - (D \cup Y))$. Then X' meets every path from X_1 to X and hence to X_2; for the first vertex of such a path in $B \cup D \cup Y$ is not in D, since $X_1 \cap D = \emptyset$, and so is either in $A \cap Y$ or in $A \cap B - (D \cup Y)$. Since $|X'| \leq |X|$ it follows that $X' \in M$. By (2), $S(X') \subseteq S(X)$. But $(Y, C, D) \notin S(X')$ since $X' \cap D = \emptyset$. This contradicts the minimality of $S(X)$. We deduce that $S(X) = \emptyset$, as required. ■

(8.7) *Let* β *be a haven in* G *of order* $\aleph_0$, *and let* $W \subseteq V(G)$ *be such that* $W \cap \beta(X) \neq \emptyset$ *for every finite* $X \subseteq W$. *Suppose that* $k \geq 0$ *is such that for every finite* $X \subseteq W$ *there exists* $Y \subseteq W$ *with* $|Y| \leq k$ *such that* $X \cap \beta(Y) = \emptyset$; *and there exists* $X_0 \subseteq W$ *with* X_0 *finite such that* $X_0 \cap \beta(Y) \neq \emptyset$ *for all* $Y \subseteq V(G)$ *with* $|Y| \leq k - 1$. *Then there is a robust division* (X, C, D) *with* $X \subseteq W$ *and with* $C \cap W \neq \emptyset \neq D \cap W$.

Proof. Let $Z = \{v \in W : v \in Y \cup \beta(Y) \text{ for all finite } Y \subseteq W\}$.

(1) $|Z| < k$.

For suppose that there exists $X \subseteq Z$ with $|X| = k$. Since $W \cap \beta(X) \neq \varnothing$, we may choose $x \in W \cap \beta(X)$. Choose $Y \subseteq W$ with $|Y| \leq k$ such that $(X \cup \{x\}) \cap \beta(Y) = \varnothing$. Since $X \subseteq Z \subseteq Y \cup \beta(Y)$ it follows that $X \subseteq Y$, and since $|X| = k \geq |Y|$ we deduce that $X = Y$. But $x \in \beta(X)$ and $x \notin \beta(Y)$, a contradiction. This proves (1).

Choose $Y_0 \subseteq W$ with $|Y_0| \leq k$ such that $(Z \cup X_0) \cap \beta(Y_0) = \varnothing$.

(2) *$|Y_0| = k$ and Y_0 is β-free, and $Z \subseteq Y_0$, and $\beta(Y_0)$ is a full Y_0-flap.*

For certainly $|Y_0| = k$, from the choice of X_0. Since $Z \subseteq Y_0 \cup \beta(Y_0)$ and $Z \cap \beta(Y_0) = \varnothing$ we deduce that $Z \subseteq Y_0$. To see that Y_0 is β-free, let $Y \subseteq V(G)$ with $|Y| < |Y_0| = k$. Certainly $X_0 \cap \beta(Y) \neq \varnothing$, by the choice of X_0, and so $\beta(Y) \not\subseteq \beta(Y_0)$. Hence, $Y_0 \cap \beta(Y) \neq \varnothing$, and therefore Y_0 is β-free. Let Y be the set of vertices in Y_0 with a neighbour in $\beta(Y_0)$. Then $\beta(Y) = \beta(Y_0)$, and since Y_0 is free and $Y_0 \cap \beta(Y) = \varnothing$, it follows that $|Y| = k$, that is, $Y = Y_0$ and $\beta(Y_0)$ is a full Y_0-flap.

(3) *If $X \subseteq V(G)$ is finite, $G|X$ is connected, $X \cap W \neq \varnothing$ and $X \cap Z = \varnothing$, then there exists $Y \subseteq W$ with $|Y| \leq k$ such that $Z \subseteq Y$, $Y \cup \beta(Y) \subseteq \beta(X)$, and $\beta(Y)$ is a full Y-flap.*

For let $Y_v \subseteq W$ be finite, such that $v \notin Y_v \cup \beta(Y_v)$, for each $v \in X \cap W$. (This is possible since $v \notin Z$.) Let

$$X^* = Z \cup (X \cap W) \cup \bigcup (Y_v : v \in X \cap W).$$

Then $X^* \subseteq W$ and is finite. Choose $Y \subseteq W$ with $|Y| \leq k$ such that $X^* \cap \beta(Y) = \varnothing$, with Y minimal. Then $\beta(Y)$ is a full Y-flap. Since $Z \subseteq Y \cup \beta(Y)$ and $Z \cap \beta(Y) = \varnothing$ it follows that $Z \subseteq Y$. It remains to show that $Y \cup \beta(Y) \subseteq \beta(X)$. Suppose that $X \cap (Y \cup \beta(Y)) \neq \varnothing$. Since $X \not\subseteq \beta(Y)$ (because $\varnothing \neq X \cap W \subseteq X^*$) and $G|X$ is connected, we deduce that there exists $v \in X \cap Y$. Since $\beta(Y)$ is a full Y-flap, there exists $u \in \beta(Y)$ adjacent to v. Now $v \in X \cap Y \subseteq X \cap W$ and so Y_v exists. Moreover, $u \in \beta(Y) = \beta(Y \cup X^*)$ since $X^* \cap \beta(Y) = \varnothing$, and $\beta(Y \cup X^*) \subseteq \beta(Y_v)$ since $Y_v \subseteq Y \cup X^*$. Hence $u \in \beta(Y_v)$, and so $v \in Y_v \cup \beta(Y_v)$, a contradiction. We deduce that $X \cap (Y \cup \beta(Y)) = \varnothing$. Since $X \cap \beta(Y) = \varnothing$ it follows that $\beta(Y) = \beta(X \cup Y) \subseteq \beta(X)$; and since $X \cap Y = \varnothing$, $\beta(Y) \subseteq \beta(X)$ and every vertex of Y has a neighbour in $\beta(Y)$, we deduce that $Y \subseteq \beta(X)$. This proves (3).

We define a sequence $Y_0, X_1, Y_1, \ldots, X_{k+1}, Y_{k+1}$ of finite subsets of $V(G)$ as follows. Let $1 \leq i \leq k+1$ and suppose that Y_{i-1} has been defined, and $Z \subseteq Y_{i-1} \subseteq W$, and $\beta(Y_{i-1})$ is a full Y_{i-1}-flap. Choose $X_i \subseteq \beta(Y_{i-1})$, finite, with $X_i \cap W \neq \emptyset$, such that $G|X_i$ is connected and every vertex in Y_{i-1} has a neighbour in X_i. Let $X = (Y_{i-1} - Z) \cup X_i$; then X is finite and $G|X$ is connected, and $X \cap W \neq \emptyset$, $X \cap Z = \emptyset$, and so by (3) there exists $Y_i \subseteq W$ with $|Y_i| \leq k$ such that $Z \subseteq Y_i$, $Y_i \cup \beta(Y_i) \subseteq \beta(X)$, and $\beta(Y_i)$ is a full Y_i-flap. This completes the inductive definition.

We observe that each Y_i includes Z and $Y_i \cap Y_{i-1} = Z$. Moreover, since $Y_{i-1} \cap \beta(Y_i) = \emptyset$ it follows that $\beta(Y_{i-1}) \supseteq \beta(Y_i \cup Y_{i-1}) = \beta(Y_i)$ for $1 \leq i \leq k+1$. Since each member of Y_i has a neighbour in $\beta(Y_i) \subseteq \beta(Y_{i-1})$, and $Y_{i-1} \cap Y_i = Z$, it follows that $Y_i \subseteq Z \cup \beta(Y_{i-1})$ for $1 \leq i \leq k+1$. In particular, since $\beta(Y_i) \subseteq \beta(Y_{i-1}) \subseteq \ldots \subseteq \beta(Y_0)$ it follows that $X_0 \cap \beta(Y_i) = \emptyset$ and so $|Y_i| = k$.

(4) *There are major paths $R_1, \ldots, R_k$ of G, mutually vertex-disjoint, such that $R_1, \ldots, R_k$ each have an end in Y_0.*

This follows from (4.2).

Let $k - |Z| = r$; then $r \geq 1$ by (1). Let $R_1, \ldots, R_k$ be numbered so that the end of R_j in Y_0 does not belong to Z for $1 \leq j \leq r$. Now for $1 \leq j \leq k$, since R_j is major it follows that $V(R_j) \cap (Y_i \cup \beta(Y_i)) \neq \emptyset$, and since the first vertex of $V(R_j)$ is not in $\beta(Y_i)$ we deduce that R_j meets Y_i for $0 \leq i \leq k+1$. Since each $|Y_i| = k$, exactly one vertex of R_j belongs to Y_i. For $1 \leq j \leq r$ let S_j be the restriction of R_j to $V(G) - (Y_{k+1} \cup \beta(Y_{k+1}))$. For $1 \leq i \leq k$, since $Y_i \cap Y_{k+1} = Z$ and $V(R_j) \cap Z = \emptyset$ (because $Z \subseteq Y_0$) it follows that $V(S_j) \cap (Y_i - Z) \neq \emptyset$. Let

$$M = \bigcup_{1 \leq j \leq r} V(S_j) \cup \bigcup_{1 \leq i \leq k+1} X_i .$$

Then M is finite and $G|M$ is connected, and $M \cap Y_{k+1} = \emptyset$. Let C be the Y_{k+1}-flap including M. We claim that $(Y_{k+1}, C, \beta(Y_{k+1}))$ is a division satisfying the theorem. Put $D = \beta(Y_{k+1})$.

(5) *(Y_{k+1}, C, D) is a division.*

For $C \neq D$ since $X_{k+1} \subseteq M \subseteq C$ and $X_{k+1} \cap \beta(Y_{k+1}) = \emptyset$. By the construction, $D = \beta(Y_{k+1})$ is a full Y_{k+1}-flap. Let $y \in Y_{k+1}$. If $y \notin Z$, then y belongs to some R_j $(1 \leq j \leq r)$ and hence has a neighbour in $M \subseteq C$; while if $y \in Z$ then $y \in Y_0$ and has a neighbour in $X_1 \subseteq M \subseteq C$. Thus C is a full Y_{k+1}-flap. This proves (5).

(6) *For every separation (A, B) of G of finite order,*

$$\min(|A \cap Y_{k+1}|, |B \cap Y_{k+1}|) \leq |A \cap B \cap (D \cup Y_{k+1})| .$$

For let $X = (A \cap B \cap D) \cup Y_{k+1}$. Then X is finite, and $\beta(X) \subseteq \beta(Y_{k+1}) = D$. Moreover, since $\beta(X) \subseteq D$ and $(A \cap D, B \cap D)$ is a separation of $G|D$, it follows that either $\beta(X) \subseteq A \cap D$ or $\beta(X) \subseteq B \cap D$, and we assume the former without loss of generality. Since $\beta(X) \cap A \cap B = \varnothing$ it follows that $\beta(X) \subseteq (A - B) \cap D$. Let $Y = (A \cap Y_{k+1}) \cup (A \cap B \cap D)$. Then $\beta(X)$ is a Y-flap, and so $\beta(X) = \beta(Y)$. Since $Y_0 \cap D = \varnothing$ and $\beta(Y) \subseteq D$ it follows from (2) that $|Y| \geq k$. Since $|Y_{k+1}| = k$ we deduce that

$$|Y_{k+1} \cap (A-B)| + |A \cap B \cap (D \cup Y_{k+1})| = |Y| \geq |Y_{k+1}| = |Y_{k+1} \cap (A-B)| + |Y_{k+1} \cap B| .$$

This proves (6).

(7) *For every separation (A, B) of G,*

$$\min(|A \cap Y_{k+1}|, |B \cap Y_{k+1}|) \leq |A \cap B \cap (C \cup Y_{k+1})| .$$

For suppose not. Since $|Y_{k+1}| = k$ it follows that $|A \cap B \cap (C \cup Y_{k+1})| \leq k$, and hence for some i $(1 \leq i \leq k+1)$, $A \cap B \cap (C \cup Y_{k+1}) \cap X_i = \varnothing$, that is, $A \cap B \cap X_i = \varnothing$. Since $(A \cap C, B \cap C)$ is a separation of $G|C$, and $G|X_i$ is connected, we may assume without loss of generality that $X_i \subseteq (A-B) \cap C$; and hence $Y_{i-1} \subseteq A \cap (C \cup Y_k)$, since every vertex in Y_{i-1} has a neighbour in X_i. Now every path from $A \cap (C \cup Y_k)$ to Y_{k+1} intersects $Y = (A \cap Y_{k+1}) \cup (A \cap B \cap C)$, and each R_j meets both $Y_{i-1} \subseteq A \cap (C \cup Y_k)$ and Y_{k+1}; and hence $|Y| \geq k$. The claim follows as for (6).

From (5), (6) and (7) it follows that (Y_{k+1}, C, D) is a robust division; and $Y_{k+1} \subseteq W$ and $C \cap W$, $D \cap W \neq \varnothing$ by the construction. This completes the proof. ■

9. LIMITED DISSECTIONS

(9.1) *Let $\mathcal{G}$ be a geography in G and let $\mathcal{D}$ be the induced dissection. If $u, v \in V(G)$ and $k \geq 0$ is an integer, there are only finitely many members $(A, B) \in \mathcal{D}$ with $u \in A - B$ and $v \in B - A$, of order $\leq k$.*

Proof. By a *chain* we mean a set C of members of $\mathcal{D}$, such that each $(A, B) \in C$ has order $\leq k$ and $u \in A - B$, $v \in B - A$. Suppose, for a contradiction, that there is an infinite chain. Choose an infinite chain C with

$$W(C) = \bigcap (A \cap B : (A, B) \in C)$$

maximal. (This is possible, since certainly $|W(C)| \leq k$.) We may assume that

(1) *For each $(A, B) \in C$, $A - B$ is a full $A \cap B$-flap and there is a full $A \cap B$-flap $B' \subseteq B - A$, and some vertex of B' is nicer than every vertex of $A - B$.*

For since each $(A, B) \in C$ is induced by a division in $\mathcal{G}$ it follows that either (A, B) or (B, A) satisfies (1). By exchanging u and v if necessary we may therefore assume that (1) holds for all $(A, B) \in C' \subseteq C$ where C' is infinite; and then $W(C) \subseteq W(C')$, and so we may replace C by C'. The claim follows.

Let Q be the $W(C)$-flap containing u, and let u' be the nicest vertex in Q. Let P be a path between u and u' with $V(P) \subseteq Q$. For each $w \in V(P)$, $w \in A \cap B$ for only finitely many $(A, B) \in C$, by the maximality of $W(C)$; and so $A \cap B \cap V(P) \neq \emptyset$ for only finitely many $(A, B) \in C$ since P is finite. Thus, we may choose $(A, B) \in C$ with $A \cap B \cap V(P) = \emptyset$ and hence with $u' \in V(P) \subseteq A - B$. Let B' be as in (1). Some vertex $v' \in B'$ is nicer than u' by (1), and so $v' \notin Q$ by the definition of u'. Hence $B' \cap Q = \emptyset$. Since B' is a full $A \cap B$-flap, it follows that $A \cap B \cap Q = \emptyset$. But by (1), $A - B$ is a full $A \cap B$-flap, and so $A \subseteq Q \cup W(C)$. Hence $A \cap B \subseteq W(C)$. But $W(C) \subseteq A \cap B$, and so equality holds. Since $u \in A - B$ it follows from (1) that $A = Q \cup W(C)$. We have shown then that for all $(A, B) \in C$ except finitely many, $A = Q \cup W(C)$ and $B = V(G) - Q$, and so C is finite, a contradiction. Thus, there is no infinite chain, as required. ■

If $\mathcal{D}$ is a dissection and $(A_1, B_1), (A_2, B_2), (A_3, B_3) \in \mathcal{D}$, we say that (A_2, B_2) *separates* (A_1, B_1) and (A_3, B_3) if A_2 includes one of A_1, B_1 and B_2 includes one of A_3, B_3. A dissection $\mathcal{D}$ is *diffuse* if

(i) $A \cap B$ is finite for all $(A, B) \in \mathcal{D}$, and

(ii) there do not exist $(A_i, B_i) \in \mathcal{D}$ $(i = 1, 2, \ldots)$ such that for all $k \geq 1$ and all $i, j \geq k$, every member of $\mathcal{D}$ separating (A_i, B_i) and (A_j, B_j) has order $\geq k$.

We recall that a dissection $\mathcal{D}$ is *limited* if for all $u, v \in V(G)$ there are only finitely many $(A, B) \in \mathcal{D}$ with $u \in A - B$ and $v \in B - A$.

(9.2) *Let $\mathcal{D}$ be the dissection in G resulting from a geography $\mathcal{S}$. If $\mathcal{D}$ is diffuse then it is limited and has adhesion $< \aleph_0$.*

Proof. Let $u, v \in V(G)$, and let C be the set of all $(A, B) \in \mathcal{D}$ with $u \in A - B$, $v \in B - A$. By (9.1) C contains only finitely many members of order $\le k$, for any integer k. Suppose that C is infinite. By an *interval* in C we mean a subset $C' \subseteq C$ such that if $(A, B), (A', B') \in C$ and $(A'', B'') \in \mathcal{D}$ separates (A, B) and (A', B') (whence $(A'', B'') \in C$, and $A \subseteq A''$ and $B' \subseteq B''$) then $(A'', B'') \in C'$. Define $C_0 = C$. Inductively, suppose that $i \ge 0$ and C_i is an infinite interval all members of which have order $\ge i$. Since only finitely many members of C_i have order i, there is an infinite interval $C_{i+1} \subseteq C_i$ all members of which have order $\ge i+1$. By this inductive definition we have constructed intervals $C = C_0 \supseteq C_1 \supseteq C_2 \supseteq \dots$. Let $(A_i, B_i) \in C_i$ $(i = 1, 2, \dots)$. For $k \ge 1$ and $i, j \ge k$, both (A_i, B_i), and (A_j, B_j) belong to C_k, and so does every member of $\mathcal{D}$ separating (A_i, B_i) and (A_j, B_j). Hence, every such member has order $\ge k$, contradicting that $\mathcal{D}$ is diffuse. We deduce that C is finite. This proves that $\mathcal{D}$ is limited.

Now suppose that $\mathcal{P}$ is an orientation of $\mathcal{D}$, such that for each integer $i \ge 0$, $\mathcal{D}$ does not have adhesion $\le i$ at $\mathcal{P}$. For each $i \ge 1$, let $(A_i, B_i) \in \mathcal{P}$ be such that there is no $(A, B) \in \mathcal{P}$ of order $\le i$ with $A_i \subseteq A$ and $B \subseteq B_i$. For $k \ge 1$ and $i, j \ge k$, suppose that (A, B) separates (A_i, B_i) and (A_j, B_j). From the symmetry we may assume that $(A, B) \in \mathcal{P}$. Then $A_i \subseteq A$ and $B \subseteq B_i$; for it is not the case that $B_i \subseteq A$ and $B \subseteq A_i$ since $\mathcal{P}$ is an orientation. Hence (A, B) has order $> i \ge k$. This contradicts that $\mathcal{D}$ is diffuse. We deduce that there is no such $\mathcal{P}$, and so $\mathcal{D}$ has adhesion $< \aleph_0$. ■

If $\mathcal{P}$ is an orientation of $\mathcal{D}$, we say that $(A, B) \in \mathcal{P}$ is *incident* with $\mathcal{P}$ if there is no $(A', B') \in \mathcal{P}$ cutting off (A, B) with $(A', B') \ne (A, B)$.

(9.3) *If $\mathcal{P}$ is an orientation of $\mathcal{D}$ and (A, B) is incident with $\mathcal{P}$ then $A \cap B \subseteq W$, where W is the centre of $\mathcal{P}$. If in addition $\mathcal{D}$ has adhesion $\le d$ at $\mathcal{P}$ where $d \ge 0$ is an integer then $|A \cap B| \le d$.*

Proof. Let $v \in A \cap B$. If $v \notin W$, there exists $(A', B') \in \mathcal{P}$ with $v \notin B'$. Since $v \in A \cap B$, it follows that B' includes neither A nor B. Since $\mathcal{P}$ is an orientation we deduce that (A', B') cuts off (A, B), and hence $(A', B') = (A, B)$. But $v \in B - B'$, a contradiction. Thus $v \in W$, and so $A \cap B \subseteq W$. Now if $\mathcal{D}$ has adhesion $\le d$ at $\mathcal{P}$, choose $(A', B') \in \mathcal{P}$ cutting off (A, B), of order $\le d$. Then $(A', B') = (A, B)$ and so $|A \cap B| \le d$. ■

(9.4) *Let $\mathcal{D}$ be a limited dissection, and let $\mathcal{P}$ be an orientation of $\mathcal{D}$ with centre W, and let $\mathcal{D}$ have adhesion $\le d$ at $\mathcal{P}$, where $d \ge 0$ is an integer. Then for every $(A_0, B_0) \in \mathcal{P}$ there exists $(A, B) \in \mathcal{P}$ of order $\le d$, cutting off (A_0, B_0), such that either $W \subseteq A \cap B$, or (A, B) is incident with $\mathcal{P}$.*

Proof. Let $C = \{(A, B) \in \mathcal{P} : A_0 \subseteq A\}$. If $(A, B) \in \mathcal{P}$ then there exists $(A', B') \in \mathcal{P}$ cutting off (A, B) of order

$\leq d$, since $\mathcal{D}$ has adhesion $\leq d$ at $\mathcal{P}$, and so $|A \cap W| \leq d$, since $A \cap W \subseteq A' \cap W \subseteq A' \cap B'$. Moreover, if $(A_1, B_1), (A_2, B_2) \in C$ then either $A_1 \subseteq A_2$ or $A_2 \subseteq A_1$. It follows that, if

$$X = \bigcup (A \cap W : (A, B) \in C)$$

then $|X| \leq d$ and there exists $(A_1, B_1) \in C$ of order $\leq d$ with $X \subseteq A_1 \cap B_1$. If $X = W$ then $W \subseteq A_1 \cap B_1$ as required. If not, choose $v \in W - X$, and choose $u \in A_0 - B_0$. Then C is finite, since $u \in A - B$ and $v \in B - A$ for all $(A, B) \in C$ and $\mathcal{D}$ is limited, and so we may choose $(A, B) \in C$ such that $A' \subseteq A$ for all $(A', B') \in C$. Then (A, B) is incident with $\mathcal{P}$, as required. ■

10. EXCLUDING THE HALF-GRID

In this section we shall prove the remaining implications of (2.6). We shall need the following. Let us say that $Z \subseteq V(G)$ is *coherent* if Z is finite and for every separation (A, B),

$$\min(|Z \cap A|, |Z \cap B|) \leq |A \cap B| .$$

We remark that if (X, C, D) is a robust division then X is coherent.

(10.1) *Let $Z \subseteq V(G)$ be coherent, and let $k = \lceil \frac{1}{3}|Z| \rceil$. For each $X \subseteq V(G)$ with $|X| < k$ there is a unique X-flap $\beta(X)$ such that $|\beta(X) \cap Z| \geq |Z| - |X|$; and β thus defined is a haven of order k.*

Proof. Let $X \subseteq V(G)$ with $|X| < k$, and let the X-flaps be C_i $(i \in I)$. Let $|C_i \cap Z| = c_i$ $(i \in I)$.

(1) *For every $J \subseteq I$, either $\sum_{i \in J} c_i \leq |X - Z|$ or $\sum_{i \in J} c_i \geq |Z| - |X|$.*

For let $A = X \cup \bigcup_{i \in J} C_i$, $B = V(G) - \bigcup_{i \in J} C_i$. Then (A, B) is a separation. Since Z is coherent,

$$\min(|Z \cap A|, |Z \cap B|) \leq |A \cap B| = |X| .$$

If $|Z \cap A| \leq |X|$ then $\sum_{i \in J} c_i + |X \cap Z| \leq |X|$, while if $|Z \cap B| \leq |X|$ then $\sum_{i \in J} c_i = |Z - B| \geq |Z| - |X|$. This proves (1).

Now $2|X| \leq 3|X| < |Z|$, and so $2(|Z| - |X|) > |Z|$. Hence there is at most one $i \in I$ with $c_i \geq |Z| - |X|$. Suppose, for a contradiction, that there is none. By (1), taking $|J| = 1$, we deduce

(2) $c_i \le |X - Z|$ *for all* $i \in I$.

Choose $J \subseteq I$ minimal such that $\sum_{i \in J} c_i > |X - Z|$. (This is possible since $\sum_{i \in I} c_i = |Z - X| \ge |Z| - |X| > |X| \ge |X - Z|$.) By (1), $\sum_{i \in J} c_i \ge |Z| - |X|$. Now $J \ne \varnothing$; choose $j \in J$. By the minimality of J, $\sum_{i \in J - \{j\}} c_i \le |X - Z|$, and by (2), $c_j \le |X - Z|$. Thus

$$|Z| - |X| \le \sum_{i \in J} c_i = \left(\sum_{i \in J - \{i\}} c_i \right) + c_i \le |X - Z| + |X - Z| .$$

Hence $|Z| \le |X| + 2|X - Z| \le 3|X|$ and so $|X| \ge k$, a contradiction.

Thus $\beta(X)$ is defined for all $X \subseteq V(G)$ with $|X| < k$. Let $X \subseteq Y$ where $|Y| < k$. Then

$$|\beta(X) \cap Z| + |\beta(Y) \cap Z| \ge |Z| - |X| + |Z| - |Y| > |Z| ,$$

and so $\beta(X) \cap \beta(Y) \ne \varnothing$. Since $\beta(Y)$ is a subset of some X-flap it follows that $\beta(Y) \subseteq \beta(X)$. Hence β is a haven of order $\aleph_0$, as required. ■

We call β, defined as in (10.1), the *coherence haven derived from Z*.

A dissection $\mathcal{D}$ is *linked* if

(i) every member of $\mathcal{D}$ has finite order, and

(ii) for all $(A_1, B_1), (A_2, B_2) \in \mathcal{D}$ and every integer $k \ge 0$, either there are k paths of G between $A_1 \cap B_1$ and $A_2 \cap B_2$, mutually vertex-disjoint, or there exists $(A, B) \in \mathcal{D}$ of order $< k$ separating (A_1, B_1) and (A_2, B_2).

The main result of this section is the following, which immediately yields the implication (i) => (ii) of (2.6).

(10.2) *If G has no half-grid minor, then G admits a linked, limited dissection of width* $< \aleph_0$ *and adhesion* $< \aleph_0$.

Proof. By (8.2) there is a geography containing all nicely robust divisions, every member of which is robust. By Zorn's lemma, we may choose a maximal such geography $\mathcal{G}$. Thus,

(1) *$\mathcal{G}$ contains every nicely robust division, every member of $\mathcal{G}$ is robust, and every robust division not in $\mathcal{G}$ crosses some member of $\mathcal{G}$.*

Let $\mathcal{D}$ be the induced dissection. We claim that $\mathcal{D}$ satisfies the theorem.

(2) *$\mathcal{D}$ is diffuse.*

For suppose that there exist $(A_i, B_i) \in \mathcal{D}$ $(i \geq 1)$ such that for all $k \geq 1$ and all $i, j \geq k$, every member of $\mathcal{D}$ separating (A_i, B_i) and (A_j, B_j) has order $\geq k$. For each $i \geq 1$, let β_i be the coherence haven derived from $A_i \cap B_i$. (This set is coherent since (A_i, B_i) is induced by some robust separation.) Thus, β_i has order k_i say, where $k_i = \lceil \frac{1}{3}(A_i \cap B_i) \rceil \geq \frac{1}{3} i$. Let $i, j \geq 0$. Suppose that there is a nicely robust division $(X, \beta_i(X), \beta_j(X))$ with $|X| < \frac{1}{3}\min(i, j)$. Then $(X, \beta_i(X), \beta_j(X)) \in \mathcal{G}$; let it induce $(A, B) \in \mathcal{D}$. Now $\beta_i(X) \cap (A_i \cap B_i) \neq \emptyset$, and so $A_i \cap B_i \not\subseteq B$. Since (A_i, B_i) and (A, B) do not cross it follows that A includes one of A_i, B_i; and similarly B includes one of A_j, B_j. But then $(A, B) \in \mathcal{D}$ separates (A_i, B_i) and (A_j, B_j), and its order is $|X| < \frac{1}{3}\min(i, j)$. This contradicts the choice of the sequence $(A_1, B_1), (A_2, B_2), \ldots$. Hence there is no such X. By (8.3), $\beta_i(X) = \beta_j(X)$ for all $X \subseteq V(G)$ with $i, j > \frac{9}{2}|X|$; for then $k_i, k_j > \frac{3}{2}|X|$. Thus the set $\{\beta_i : i \geq 1\}$ is convergent; let β be its limit. We claim that for all $k \geq 0$ there exists a finite $X \subseteq V(G)$ such that $X \cap \beta(Y) \neq \emptyset$ for all $Y \subseteq V(G)$ with $|Y| \leq k$. For let $X = A_i \cap B_i$ for some $i > \frac{9}{2}k$. Then if $|Y| \leq k$ it follows that $\beta(Y) = \beta_i(Y)$; but $\beta_i(Y) \cap (A_i \cap B_i) \neq \emptyset$, and so $X \cap \beta(Y) \neq \emptyset$, as required. By (3.5) β is a half-grid haven, and so G has a half-grid minor, contrary to hypothesis. This proves (2).

(3) *$\mathcal{D}$ is limited and has adhesion $< \aleph_0$.*

This follows from (2) and (9.2).

(4) *$\mathcal{D}$ is linked.*

For let $(A_1, B_1), (A_2, B_2) \in \mathcal{D}$, and let $k \geq 0$ be an integer such that no k paths of G between $A_1 \cap B_1$ and $A_2 \cap B_2$ are mutually vertex-disjoint. Since $(A_1, B_1), (A_2, B_2)$ do not cross, we may assume that $A_1 \subseteq A_2$ and $B_2 \subseteq B_1$. We must show that there exists $(A, B) \in \mathcal{D}$ of order $< k$ separating (A_1, B_1) and (A_2, B_2). If $|A_1 \cap B_1| < k$ we may take $(A, B) = (A_1, B_1)$; and so we may assume that $|A_1 \cap B_1| \geq k$ and similarly $|A_2 \cap B_2| \geq k$. Let $(A_1, B_1), (A_2, B_2)$ be induced by $(X_1, C_1, D_1), (X_2, C_2, D_2) \in \mathcal{G}$. Then

$$(X_1 \cup C_1) \cap (X_2 \cup D_2) \subseteq A_1 \cap B_2 \subseteq (A_1 \cap B_1) \cap (A_2 \cap B_2) = X_1 \cap X_2 .$$

Choose $k' < k$ minimum such that there is a set of k' vertices meeting every path between X_1 and X_2 (this is possible by Menger's theorem). By (8.6) there is a division (X, C, D) with $|X| = k'$, $X_1 \cup C_1 \subseteq X \cup C$ and

$X_2 \cup D_2 \subseteq X \cup D$, which crosses no member of $\mathcal{G}$. By (8.5) (i), (X, C, D) is robust. By (1), $(X, C, D) \in \mathcal{G}$. Let it induce $(A, B) \in \mathcal{D}$. Now $X_1 \subseteq X \cup C \subseteq A$ and $X_1 \not\subseteq X = A \cap B$ (because $|X_1| > |X|$) and so $X_1 \not\subseteq B$. Hence B includes neither A_1 nor B_1. Since (A_1, B_1), $(A, B) \in \mathcal{D}$ it follows that A includes one of A_1, B_1; and similarly B includes one of A_2, B_2. Thus (A, B) separates (A_1, B_1) and (A_2, B_2), and its order is $|A \cap B| = |X| = k' < k$, as required.

To complete the proof it remains to show that $\mathcal{D}$ has width $< \aleph_0$. Let $\mathcal{P}$ be an orientation of $\mathcal{D}$ with centre W, and suppose for a contradiction that W is infinite.

(5) *For any finite $X \subseteq V(G)$, only finitely many X-flaps intersect W.*

For if possible, choose a counterexample X with $|X|$ minimum. For each $v \in X$ there are only finitely many X-flaps meeting W which contain no neighbour of v, by the minimality of W; and so there are infinitely many full X-flaps meeting W. Let C, D be two of them. By (7.5), $(X, C, D) \notin \mathcal{G}$. By (1) there exists $(X', C', D') \in \mathcal{G}$ crossing (X, C, D). By (6.1), X' meets every full X-flap; yet X' is finite, a contradiction.

(6) *For any finite $X \subseteq V(G)$, there is a unique X-flap F with $F \cap W$ infinite.*

For, if possible, choose a counterexample X, as nice as possible. By (5) $C \cap W$ is infinite for some X-flap C, and hence there is another X-flap, D say, with $D \cap W$ infinite. For each $v \in X$ both C and D contain neighbours of v, for otherwise $X - \{v\}$ would be a nicer counterexample. Hence (X, C, D) is a division. We claim that it is nicely robust. Let (A, B) be a separation of G of finite order; it suffices, by the symmetry between C and D, to show that one of $A \cap X$, $B \cap X$ is at least as nice as $A \cap B \cap (C \cup X)$. Now since $C \cap W$ is infinite it follows that one of $A \cap C \cap W$, $B \cap C \cap W$ is infinite, and we may assume the former, by the symmetry between A and B. Let $Y = (A \cap X) \cup (A \cap B \cap C)$. Then Y contains every vertex not in $(A - B) \cap C$ with a neighbour in C. Since $(A - B) \cap C$ is a union of Y-flaps and $(A - B) \cap C \cap W$ is infinite (because $A \cap B$ is finite and $A \cap C \cap W$ is infinite), it follows from (5) that some Y-flap included in $(A - B) \cap C$ has infinite intersection with W. But D is included in another Y-flap which has infinite intersection with W, and so from the choice of X, X is at least as nice as Y. Since $X - B \subseteq Y$, it follows that $B \cap X$ is at least as nice as

$$Y - (X - B) = (A \cap B \cap X) \cup (A \cap B \cap C) = A \cap B \cap (C \cup X).$$

This proves our claim that (X, C, D) is nicely robust. By (1), $(X, C, D) \in \mathcal{G}$, contrary to (7.5). This proves (6).

For each finite $X \subseteq V(G)$ let $\beta(X)$ be the unique X-flap F with $F \cap W$ infinite. By (6), β is a haven of order $\aleph_0$. Since β is not a half-grid haven, by (3.5) there exists an integer $k \geq 0$ such that for every finite $X \subseteq V(G)$ there exists $Y \subseteq V(G)$ with $|Y| \leq k$ such that $X \cap \beta(Y) = \emptyset$. In particular

(7) *For every finite $X \subseteq W$ there exists $Y \subseteq V(G)$ with $|Y| \leq k$ such that $X \cap \beta(Y) = \emptyset$.*

Let us choose $k \geq 0$ minimum such that (7) holds. It follows that

(8) *There exists $X_0 \subseteq W$, finite, such that $X_0 \cap \beta(Y) \neq \emptyset$ for all $Y \subseteq V(G)$ with $|Y| \leq k-1$.*

We shall show that

(9) *For every finite $X \subseteq W$ there exists $Y \subseteq W$ with $|Y| \leq k$ such that $X \cap \beta(Y) = \emptyset$.*

For choose Y as in (7), with $|Y - W|$ minimum. Suppose that $y \in Y - W$. By (9.4) there exists $(A', B') \in \mathcal{P}$ with $y \in A' - B'$ and $A' \cap B' \subseteq W$ (because $A' \cap B'$ is finite, and so the first alternative in (9.4) does not apply). Let $A = V(G) - \beta(Y)$, $B = Y \cup \beta(Y)$; then (A, B) is a separation with $A \cap B = Y$. Let $Y_1 = A \cap B' \cap (B \cup A')$, $Y_2 = B \cap B' \cap (A \cup A')$. Now by (5) and (6), $(A - B) \cap W$ is finite and so $\beta(Y_1) \not\subseteq A \cap B'$; and since Y_1 contains every vertex of $A \cap B'$ with a neighbour in $V(G) - A \cap B'$, it follows that $\beta(Y_1) \cap A \cap B' = \emptyset$. But $X \subseteq A \cap B'$, and so $X \cap \beta(Y_1) = \emptyset$. Moreover, since $Y_2 \subseteq B = Y \cup \beta(Y)$, it follows that $\beta(Y_2) = \beta(Y \cup Y_2) \subseteq B(Y)$, and so $X \cap \beta(Y_2) = \emptyset$. But $Y_1 - W$, $Y_2 - W \subseteq Y - (W \cup \{y\})$, since $A' \cap B' \subseteq W$ and $y \notin Y_1, Y_2$; and so from the choice of Y, $|Y_1| > k$ and $|Y_2| > k$. But (A', B') is robust, and so from (8.1), $\min(|Y_1|, |Y_2|) \leq |A \cap B| \leq k$, a contradiction. This proves (9).

From (8), (9) and (8.7), there is a robust division (X, C, D) with $X \subseteq W$ and with $C \cap W, D \cap W \neq \emptyset$. By (7.5), $(X, C, D) \notin \mathcal{G}$. By (1), there exists $(X', C', D') \in \mathcal{G}$ crossing (X, C, D). By (6.1), $X \cap C' \neq \emptyset \neq X \cap D'$. Since $X \subseteq W$ it follows that $W \cap C' \neq \emptyset \neq W \cap D'$, contrary to (7.5).

It follows that W is not infinite, and so $\mathcal{D}$ has width $< \aleph_0$, as required. ■

11. EXCLUDING $K_{\aleph_0}$

In this section we prove the implication (i) => (ii) of (2.7). We begin with

(11.1) *Let (X, C, D) be a bicomplete division. Then (X, C, D) is robust, and the coherence haven derived from X is clustered.*

Proof. First let us show that (X, C, D) is robust. Let (A, B) be a separation of G of finite order. It suffices to prove that

$$\min(|A \cap X|, |B \cap X|) \leq |A \cap B \cap (C \cup X)| .$$

Let $\mathcal{C}$ be a cluster with $|\mathcal{C}| = |X|$, such that each $F \in \mathcal{C}$ satisfies $F \subseteq C$ and $|F \cap X| = 1$. Now not both $A - B$ and $B - A$ include members of $\mathcal{C}$ since no vertex in $A - B$ has a neighbour in $B - A$. We assume then that $A - B$ includes no member of $\mathcal{C}$. Since $A \cap X$ intersects $|A \cap X|$ members of $\mathcal{C}$ it follows that $A \cap B$ intersects $\geq |A \cap X|$ members of $\mathcal{C}$; and since they are disjoint and each is included in C it follows that $|A \cap B \cap C| \geq |A \cap X|$. Hence (X, C, D) is robust.

Thus X is coherent; let β be its coherence haven, of order $k = \lceil \frac{1}{3}|X| \rceil$. We claim that β is the k-truncation of $\beta_{\mathcal{C}}$. For let $Y \subseteq V(G)$ with $|Y| < k$. Since $\beta_{\mathcal{C}}(Y)$ includes a member of $\mathcal{C}$, it follows that every member of $\mathcal{C}$ either intersects Y or is included in $\beta_{\mathcal{C}}(Y)$; and thus at least $|\mathcal{C}| - |Y|$ members of $\mathcal{C}$ are included in $\beta_{\mathcal{C}}(Y)$. Hence $|\beta_{\mathcal{C}}(Y) \cap X| \geq |X| - |Y|$ and so $\beta_{\mathcal{C}}(Y) = \beta(Y)$. Hence β is the k-truncation of $\beta_{\mathcal{C}}$ and so is clustered, as in (4.1).

■

The main result of this section is the following.

(11.2) *If G has no $K_{\aleph_0}$-minor, then G admits a linked, limited dissection $\mathcal{D}$ of adhesion $< \aleph_0$, such that for every orientation $\mathcal{P}$ of $\mathcal{D}$ there is an integer $k \geq 0$ such that $ts(\mathcal{P})$ has no K_k-minor.*

Proof. Let $\mathcal{G}$ be the set of all nicely robust bicomplete divisions. By (8.2), $\mathcal{G}$ is a geography. Let $\mathcal{D}$ be the induced dissection. We shall show that $\mathcal{D}$ satisfies the theorem.

(1) *$\mathcal{D}$ is linked.*

For let $(A_1, B_1), (A_2, B_2) \in \mathcal{D}$, and let $k \geq 0$ be such that there do not exist k disjoint paths in G between $A_1 \cap B_1$ and $A_2 \cap B_2$. We may assume that $A_1 \subseteq A_2$ and $B_2 \subseteq B_1$, and that $|A_1 \cap B_1|, |A_2 \cap B_2| \geq k$. Let $(A_1, B_1), (A_2, B_2)$ be derived from $(X_1, C_1, D_1), (X_2, C_2, D_2)$ respectively; then $(X_1 \cup C_1) \cap (X_2 \cup D_2) = X_1 \cap X_2$. By (8.5) (ii) and (iii) there is a division (X, C, D) of order $< k$ with $X_1 \cup C_1 \subseteq X \cup C$, $X_2 \cup D_2 \subseteq X \cup D$, such that (X, C, D) is nicely robust and bicomplete. Hence, $(X, C, D) \in \mathcal{G}$; let it induce $(A, B) \in \mathcal{D}$. Then as in the proof of (4) in (10.2), (A, B) separates (A_1, B_1) and (A_2, B_2) as required.

(2) *$\mathcal{D}$ is diffuse.*

For suppose that there exist $(A_i, B_i) \in \mathcal{D}$ $(i \geq 1)$ such that for all $k \geq 1$ and all $i, j \geq k$, every member of $\mathcal{D}$ separating (A_i, B_i) and (A_j, B_j) has order $\geq k$. For $i \geq 1$ let β_i be the coherence haven derived from $A_i \cap B_i$. Since $|A_i \cap B_i| \geq i$ it follows that β_i has order $\geq \lceil \frac{1}{3} i \rceil$. We claim that $\{\beta_i : i \geq 1\}$ is convergent. For let $X \subseteq V(G)$ be finite, and let $i, j > 3|X|$. Suppose for a contradiction that $\beta_i(X) \neq \beta_j(X)$. Let (A, B) be a separation with $A \cap B = X$, $\beta_i(X) \subseteq A$, $\beta_j(X) \subseteq B$.

Let $A' = A \cup (A_i \cap B_i \cap B)$, $B' = B \cup (A_j \cap B_j \cap A)$. Then (A', B') is a separation of order

$$\leq |X| + |A_i \cap B_i \cap B| + |A_j \cap B_j \cap A| \leq |X| + |A_i \cap B_i - \beta_i(X)| + |A_j \cap B_j - \beta_j(X)| \leq 3|X| .$$

But $A_i \cap B_i \subseteq A'$ and $A_j \cap B_j \subseteq B'$, and so there do not exist $3|X| + 1$ disjoint paths of G between $A_i \cap B_i$ and $A_j \cap B_j$. By (1), there exists $(A_0, B_0) \in \mathcal{D}$ of order $\leq 3|X|$ separating (A_i, B_i) and (A_j, B_j). From our choice of the sequence $(A_1, B_1), (A_2, B_2), \ldots$ it follows that $3|X| \geq \min(i, j)$, contrary to the choice of i, j. Hence, $\beta_i(X) = \beta_j(X)$, and so $\{\beta_i : i \geq 1\}$ is convergent. Let its limit be β; then β is a haven of order $\aleph_0$. Moreover, for any $k \geq 0$ the k-truncation of β is the k-truncation of β_{3k+1}, and hence is clustered, by (11.1). By (4.1), β is clustered, and so G has a $K_{\aleph_0}$-minor, a contradiction. This proves (2).

From (9.2),

(3) *$\mathcal{D}$ is limited and has adhesion $< \aleph_0$.*

Now let $\mathcal{P}$ be an orientation of $\mathcal{D}$; to complete the proof it suffices to show that for some integer $k \geq 0$, $ts(\mathcal{P})$ has no K_k-minor. Let $\mathcal{P}$ have centre W. If W is finite we may take $k = |W| + 1$, and so we may assume that W is infinite. Let $\mathcal{D}$ have adhesion $\leq d$ at $\mathcal{P}$. Let $\{(A_i, B_i) : i \in I\}$ be the set of all $(A, B) \in \mathcal{P}$ which are incident with $\mathcal{P}$. Then $A_i \subseteq B_j$ for all distinct $i, j \in I$, and for each $(A, B) \in \mathcal{P}$ there exists $i \in I$ such that (A_i, B_i) cuts off (A, B), by (9.4). For each $i \in I$ let C_i be a cluster in G such that $|C_i| = |A_i \cap B_i|$, and each $F \in C_i$ satisfies $F \subseteq A_i$ and $|F \cap A_i \cap B_i| = 1$. (This is possible since (A_i, B_i) is induced by a bicomplete division.) For each $v \in W$, let

$$N_v = \{v\} \cup \bigcup_{i \in I} \bigcup (C \in C_i : v \in C).$$

(4) *For distinct $v, v' \in W$, $N_v \cap N_{v'} = \emptyset$.*

For suppose that $u \in N_v \cap N_{v'}$. We may assume that $u \neq v$. Thus, there exists $i \in I$ and $C \in \mathcal{C}_i$ with $u, v \in C$. Since $|C \cap W| = 1$ and $v \in C \cap W$ it follows that $u \notin W$. Hence $u \neq v'$, and so similarly there exists $i' \in I$ and $C' \in \mathcal{C}_{i'}$ such that $u, v' \in C'$. Now $u \in C \cap C' \subseteq A_i \cap A_{i'}$, and $u \notin B_{i'}$, since $A_{i'} \cap B_{i'} \subseteq W$. Hence $A_i \not\subseteq B_{i'}$, and so $i = i'$. Then $C, C' \in \mathcal{C}_i$ and $C \cap C' \neq \emptyset$, and so $C = C'$. But $C \cap W = \{v\}$ and $C' \cap W = \{v'\}$, and $v \neq v'$, a contradiction. This proves (4).

Suppose, for a contradiction, that for all $k \geq 0$ there is a cluster $\mathcal{C}_k'$ in $ts(\mathcal{P})$ with $|\mathcal{C}_k'| = k$. Let

$$\mathcal{C}_k = \{ \bigcup_{v \in C} N_v : C \in \mathcal{C}_k' \} ;$$

then $\mathcal{C}_k$ is a cluster in G of cardinality k, as is easily seen. Moreover, each member of $\mathcal{C}_k$ intersects W. For each $k \geq 1$ let $\beta_k = \beta_{\mathcal{C}_k}$.

(5) *The set* $\{\beta_k : k \geq 1\}$ *is convergent.*

For let $X \subseteq V(G)$ be finite and let $k, k' > 2|X|$. If $\beta_k(X) \neq \beta_{k'}(X)$ then by (8.4) there exists $Y \subseteq V(G)$ with $|Y| \leq |X|$ such that $(Y, \beta_k(Y), \beta_{k'}(Y))$ is a nicely robust, bicomplete division. By definition of $\mathcal{G}$, $(Y, \beta_k(Y), \beta_{k'}(Y)) \in \mathcal{G}$; but $\beta_k(Y)$ and $\beta_{k'}(Y)$ include members of $\mathcal{C}_k$ and $\mathcal{C}_{k'}$ respectively, and hence both intersect W, contrary to (7.5). We deduce that $\beta_k(X) = \beta_{k'}(X)$. This proves (5).

Let β be the limit of $\{\beta_k : k \geq 1\}$. Since each β_k is clustered, so is β, by (4.1). Hence G has a $K_{\aleph_0}$-minor, contrary to hypothesis. We deduce that our assumption about $\mathcal{P}$ was incorrect, and that for some integer $k \geq 0$ no cluster $\mathcal{C}_k'$ in $ts(\mathcal{P})$ exists. The result follows. ■

12. DISSECTIONS AND TREE-DECOMPOSITIONS

Our next objective is to recast (10.2) and (11.2) in terms of tree-decompositions instead of dissections. Let (T, W) be a tree-decomposition of G. For $e \in E(T)$, let T_1, T_2 be the components of $T \backslash e$, and let

$$A_i = \bigcup (W_t : t \in V(T_i)) \ \ (i = 1, 2).$$

Then (A_1, A_2) is a separation of G, and we call (A_1, A_2) and (A_2, A_1) the separations *arising* from e.

We say that (T, W) is *proper* if

(i) for all $e \in E(T)$, if (A_1, A_2) arises from $e \in E(T)$ then $A_1 \neq V(G)$, and

(ii) for all $e, f \in E(T)$, if (A_1, A_2) and (B_1, B_2) arise from e and f respectively, and $A_1 = B_1$ then $A_2 = B_2$.

If (T, W) is proper, the set $\mathcal{D}$ of all separations arising from edges of T is clearly a dissection, which we call the *resultant dissection.*

(12.1) *If (T, W) is proper, the resultant dissection is limited.*

Proof. Let $u, v \in V(G)$ and let $u \in W_s$, $v \in W_t$ where $s, t \in V(T)$. Let P be the path of T between s and t. Suppose that $(A_1, A_2) \in \mathcal{D}$ and $u \in A_1 - A_2$, $v \in A_2 - A_1$, where (A_1, A_2) arises from some edge $e \in E(T)$ with trees T_1, T_2 as before. Now $u \notin A_2$ and so $s \notin V(T_2)$, and similarly $t \notin V(T_1)$; and so $e \in E(P)$. Since $E(P)$ is finite it follows that there are only finitely many such $(A_1, A_2) \in \mathcal{D}$, as required. ■

To find the orientations of $\mathcal{D}$ we need the following lemma. A *confluence* of a tree T is an assignment of a direction to each edge of T in such a way that no two edges are directed away from each other. Two examples of confluences are

(1) *vertex confluences*: for some $v \in V(T)$, every edge is directed towards v

(2) *end confluences*: for some end Π of T, every edge e is directed towards the members of Π not using e.

(12.2) *Every confluence in a tree T is either a vertex confluence or an end confluence.*

Proof. Suppose that for some $v \in V(T)$, every edge of T incident with T is directed towards v. For any edge $e \in E(T)$ let f be the first edge of the path of T from v to e. Since f is directed towards v and e and f are not directed away from one another, it follows that e is directed towards v. Hence, this is a vertex confluence.

Now suppose that there is no such vertex. Then there is a ray R with vertices $v_1, v_2, \ldots$ such that for all $i \geq 1$ the edge $e_i \in E(R)$ with ends v_i, v_{i+1} is directed towards v_{i+1}. For any $e \in E(T)$ choose $i \geq 1$ such that v_i and e belong to the same component of $T \backslash e_i$. Since e_i and e are not directed away from one another it follows that e is directed towards $v_i, v_{i+1}, v_{i+2}, \ldots$. Hence, this is an end confluence. ■

We deduce

(12.3) *Let (T, W) be a proper tree-decomposition and let $\mathcal{D}$ be the resultant dissection. For any cardinal κ, (T, W) has width $< \kappa$ if and only if $\mathcal{D}$ has width $< \kappa$, and (T, W) has adhesion $< \kappa$ if and only if $\mathcal{D}$ has adhesion $< \kappa$.*

Moreover, if (T, W) has adhesion $< \aleph_0$, then the following are equivalent:

(i) for every $t \in V(T)$ there is an integer $k \geq 0$ such that $ts(t)$ has no K_k-minor

(ii) for every orientation $\mathcal{P}$ of $\mathcal{D}$ there is an integer $k \geq 0$ such that $ts(\mathcal{P})$ has no K_k-minor.

Proof. There is clearly a bijection between the confluences of T and the orientations of $\mathcal{P}$, and so the first two assertions follow from (12.2). For the third, that (ii) => (i) is clear, since $ts(t) = ts(\mathcal{P})$ where $\mathcal{P}$ is the orientation corresponding to the vertex confluence defined by t. To see (i) => (ii), let $\mathcal{P}$ be an orientation of $\mathcal{D}$. If $\mathcal{P}$ corresponds to a vertex confluence defined by $t \in V(T)$ then $ts(\mathcal{P}) = ts(t)$, and so we may assume that $\mathcal{P}$ is defined by an end confluence. Since $\mathcal{D}$ has adhesion $\leq d$ at $\mathcal{P}$, for some integer d, it follows from (12.1) and (9.4) that $\mathcal{P}$ has centre of cardinality $\leq d$, and hence (ii) holds for $\mathcal{P}$ with $k = d + 1$. ■

Thus, proper tree-decompositions yield limited dissections. The converse is less obvious, and is the main topic of this section. Let $\mathcal{D}$ be a dissection, and let $(A_0, B_0) \in \mathcal{D}$. By (5.1),

$$\{(A, B) \in \mathcal{D} : (A, B) \neq (B_0, A_0) \text{ and either } A \subseteq A_0 \text{ or } A \subseteq B_0\}$$

is an orientation of $\mathcal{D}$. We call it the *head* of (A_0, B_0), and denote it by $\mathcal{P}(A_0, B_0)$.

(12.4) *If $\mathcal{D}$ is a dissection, $\mathcal{P}$ is an orientation of $\mathcal{D}$, and $(A_0, B_0) \in \mathcal{P}$, then (A_0, B_0) is incident with $\mathcal{P}$ if and only if $\mathcal{P}$ is the head of (A_0, B_0).*

Proof. Suppose that (A_0, B_0) is the head of $\mathcal{P}$, and (A_0, B_0) is not incident with $\mathcal{P}$. Choose $(A, B) \in \mathcal{P}$ cutting off (A_0, B_0) with $(A, B) \neq (A_0, B_0)$. Then $A_0 \subseteq A$ and $B \subseteq B_0$, and since $(A, B) \in \mathcal{P}$ it follows that A includes one of A_0, B_0. Since $A_0 \not\subseteq B_0$ we deduce that $A = A_0$ and hence $B = B_0$, a contradiction.

For the converse, let $(A_0, B_0) \in \mathcal{P}$ be incident with $\mathcal{P}$ and let $(A, B) \in \mathcal{P}$. We must show that $(A, B) \neq (B_0, A_0)$ and A is included in one of A_0, B_0. Certainly $(A, B) \neq (B_0, A_0)$ because $(B_0, A_0) \notin \mathcal{P}$. Since $\mathcal{P}$ is an orientation, either $A \subseteq A_0$, or $A \subseteq B_0$, or $A_0 \subseteq A$. In the third case $A_0 = A$ since (A_0, B_0) is incident with $\mathcal{P}$, and so either $A \subseteq A_0$ or $A \subseteq B_0$ as required. ■

Let $\mathcal{D}$ be a dissection. We define $S(\mathcal{D})$ to be the directed graph with vertex set the set of heads of members of $\mathcal{D}$, and edge set $\mathcal{D}$, in which $(A, B) \in \mathcal{D}$ has head as already defined, and has tail the head of (B, A). Thus, if $(A, B) \in \mathcal{D}$ then (A, B) and (B, A) are oppositely directed parallel edges of $S(\mathcal{D})$. Let $T(\mathcal{D})$ be the graph obtained from $S(\mathcal{D})$ by replacing each pair of oppositely directed parallel edges (A, B), (B, A), by one edge, which we denote by $e(A, B)$. Thus $e(A, B) = e(B, A)$. If $\mathcal{P}$ is an orientation of $\mathcal{D}$, it consists of a choice of one edge of $S(\mathcal{D})$ from each opposite pair (A, B), (B, A), and hence corresponds to a directing $o(\mathcal{P})$ of the edges of $T(\mathcal{D})$ in the natural way.

(12.5) *For each orientation $\mathcal{P}$ of $\mathcal{D}$, no two edges of $S(\mathcal{D})$ in $\mathcal{P}$ have a common tail.*

Proof. If $(A_1, B_1), (A_2, B_2) \in \mathcal{P}$, and $\mathcal{P}(B_1, A_1) = \mathcal{P}(B_2, A_2)$, then $(B_1, A_1) \in \mathcal{P}(B_2, A_2)$, and so either $B_1 \subseteq B_2$ or $B_1 \subseteq A_2$; and similarly either $B_2 \subseteq B_1$ or $B_2 \subseteq A_1$. But $B_1 \not\subseteq A_2$ and $B_2 \not\subseteq A_1$ since $(A_1, B_1), (A_2, B_2) \in \mathcal{P}$, and so $B_1 = B_2$, and hence $A_1 = A_1$, as required. ■

(12.6) *$S(\mathcal{D})$ has no circuits.*

Proof. For each $t \in V(S(\mathcal{D}))$ there corresponds a directing $o(t)$ of the edges of $S(\mathcal{D})$ as we have seen; and if $t_1, t_2 \in V(S(\mathcal{D}))$ are adjacent, that is, $t_1 = \mathcal{P}(A, B)$ and $t_2 = \mathcal{P}(B, A)$ for some $(A, B) \in \mathcal{D}$, then $o(t_1)$, $o(t_2)$ differ only on the edge $e(A, B)$. If P_1, P_2 are paths of $S(\mathcal{D})$ both between t_1, t_2, then it follows (by reversing the directions of the edges of P_1 one at a time) that $o(t_1)$ and $o(t_2)$ differ precisely on the edges of P_1, and similarly on P_2, and so $P_1 = P_2$. Thus $S(\mathcal{D})$ has no circuits. ■

(12.7) *If $\mathcal{D}$ is limited and non-empty then $T(\mathcal{D})$ is a tree.*

Proof. For $(A_1, B_1), (A_2, B_2) \in \mathcal{D}$ we denote by $d((A_1, B_1), (A_2, B_2))$ the number of $(A, B) \in \mathcal{D}$ which separate $(A_1, B_1), (A_2, B_2)$. (Since $\mathcal{D}$ is limited this number is finite.) We prove that every two edges $e(A_1, B_1)$, $e(A_2, B_2)$ of $T(\mathcal{D})$ belong to the same component of $T(\mathcal{D})$, by induction on $d((A_1, B_1), (A_2, B_2))$. We may assume that $A_1 \subseteq A_2$ and $B_2 \subseteq B_1$. If this number is 1 then $(A_1, B_1) = (A_2, B_2)$ and we are done. If it is 2, then (A_1, B_1) is incident with $\mathcal{P}(B_2, A_2)$ and so $e(A_1, B_1)$, $e(A_2, B_2)$ have a common end-vertex. Thus we may assume that $d((A_1, B_1), (A_2, B_2)) \geq 3$. Choose $(A_3, B_3) \in \mathcal{D}$ separating (A_1, B_1) and (A_2, B_2), and different from them both. Thus $A_1 \subseteq A_3 \subseteq A_2$ and $B_2 \subseteq B_3 \subseteq B_1$. If $(A, B) \in \mathcal{D}$ separates (A_1, B_1) and (A_3, B_3) then $A_1 \subseteq A \subseteq A_3 \subseteq A_2$ and $B_2 \subseteq B_3 \subseteq B \subseteq B_1$, and so (A, B) separates (A_1, B_1) and (A_2, B_2); moreover, (A_2, B_2) separates the latter pair and not $(A_1, B_1), (A_3, B_3)$, for B_2 includes neither A_3 nor B_3. Hence $d((A_1, B_1), (A_3, B_3)) < d((A_1, B_1), (A_2, B_2))$, and so from the inductive hypothesis $e(A_1, B_1)$, $e(A_3, B_3)$, belong to the same component of $T(\mathcal{D})$. Similarly so do $e(A_3, B_3)$, $e(A_2, B_2)$ and hence so do

$e(A_1, B_1)$, $e(A_2, B_2)$, as required.

Since every vertex of $T(\mathcal{D})$ is incident with some edge of $T(\mathcal{D})$, the result follows. ■

(12.8) *If $\mathcal{D}$ is limited and non-empty then a directing of $E(T(\mathcal{D}))$ is $o(\mathcal{P})$ for some orientation $\mathcal{P}$ of $\mathcal{D}$ if and only if it is a confluence of $T(\mathcal{D})$.*

Proof. By (12.5) and (12.7), $o(\mathcal{P})$ is a confluence for every orientation $\mathcal{P}$. Conversely, the members of $\mathcal{D}$ corresponding to any given confluence form an orientation of $\mathcal{D}$. ■

For each $t \in V(T(\mathcal{D}))$, let W_t be the centre of $\mathcal{D}$ at the orientation $\mathcal{P}$. Let $W = (W_t : t \in V(T(\mathcal{D})))$.

(12.9) *If $\mathcal{D}$ is limited and non-empty, then $(T(\mathcal{D}), W)$ is a tree-decomposition of G.*

Proof. Let $v \in V(G)$. We must show that $v \in W_t$ for some $t \in V(T(\mathcal{D}))$. If $v \in A \cap B$ for some $(A, B) \in \mathcal{D}$, then $v \in W_t$ where $t = \mathcal{P}(A, B)$ by (9.3). Suppose not. Since $\mathcal{D}$ is limited there exists $(A, B) \in \mathcal{D}$ with $v \in B - A$, such that there is no $(A', B') \in \mathcal{D}$ with $(A', B') \neq (A, B)$ and $A \subseteq A'$ and $v \in B'$ (for we may assume that $\mathcal{D} \neq \emptyset$). Let $t = \mathcal{P}(A, B)$. We claim that $v \in W_t$. For let $(A', B') \in \mathcal{P}(A, B)$, and suppose that $v \in A' - B'$. Then $A \not\subseteq B'$ by the choice of (A, B), and $A' \not\subseteq A$ since $v \in A'$, and yet $(A', B') \in \mathcal{P}(A, B)$, a contradiction. Thus, $v \in W_t$, as required.

Now let $v_1, v_2 \in V(G)$ be adjacent in G. If there exists $(A, B) \in \mathcal{D}$ with $v_1, v_2 \in A \cap B$ then we are done. If not, we may choose $(A, B) \in \mathcal{D}$ with A maximal such that $v_1, v_2 \in B$, as before; and then $v_1, v_2 \in W_t$ where $t = \mathcal{P}(A, B)$, as before.

Now let $t_1, t_2, t_3 \in V(T(\mathcal{D}))$ where t_2 lies on the path between t_1 and t_3. Let $v \in V(G)$ with $v \notin W_{t_2}$. We must show that $v \notin W_{t_1} \cap W_{t_3}$. Choose $(A, B) \in t_2$ with $v \notin B$. From the symmetry we may assume that $e(A, B)$ does not lie in the path of $T(\mathcal{D})$ between t_1 and t_2. Since $o(t_1)$ and $o(t_2)$ differ only on edges of this path; it follows that $(A, B) \in t_1$, and so $v \notin W_{t_1}$.

This completes the proof. ■

(12.10) *If $\mathcal{D}$ is limited and non-empty then $(T(\mathcal{D}), W)$ is a proper tree-decomposition, and $\mathcal{D}$ is the resultant dissection.*

Proof. Let $(A, B) \in \mathcal{D}$, and let T_1, T_2 be the two components of $T(\mathcal{D}) \backslash e(A, B)$, where T_2 contains the head of (A, B). Let

$$A_i = \bigcup(W_t : t \in V(T_i)) \; (i = 1, 2).$$

We claim that $(A_1, A_2) = (A, B)$. For let $t \in V(T_1)$. Then $(B, A) \in t$, and so $W_t \subseteq A$; hence $A_1 \subseteq A$, and similarly, $A_2 \subseteq B$. But $A \cap B \subseteq W_t$ where $t = \mathcal{P}(B, A)$, by (9.3), and so $A \cap B \subseteq A_1$; and similarly $A \cap B \subseteq A_2$. Since $A_1 \subseteq A$, $A_2 \subseteq B_1$, $A \cap B \subseteq A_1 \cap A_2$ and $A \cup B = A_1 \cup A_2$, it follows that $(A, B) = (A_1, A_2)$. Both claims of the theorem follow. ■

In view of these results we can prove strengthened forms of (2.6) and (2.7). Let us say a tree-decomposition (T, W) is *linked* if

(i) for each $e \in E(T)$, the separations arising from e have finite order, and

(ii) for all $e_1, e_2 \in E(T)$ and all integers $k \geq 0$, if (A_1, B_1) and (A_2, B_2) arise from e_1, e_2 respectively, then either there are k paths of G between $A_1 \cap B_1$ and $A_2 \cap B_2$, mutually disjoint, or for some edge $e_3 \in E(T)$ on the path between e_1, e_2, the separation arising from e_3 has order $< k$.

Then from (5.7), (10.2), (12.3), (12.9) and (12.10), we have

(12.11) *For a graph G, the following are equivalent:*

(i) G has no half-grid minor

(ii) G admits a dissection of width $< \aleph_0$ and adhesion $< \aleph_0$

(iii) G admits a proper linked tree-decomposition of width $< \aleph_0$ and adhesion $< \aleph_0$.

From (5.8), (11.2), (12.3), (12.9) and (12.10) we have

(12.12) *For a graph G, the following are equivalent:*

(i) G has no $K_{\aleph_0}$-minor

(ii) G admits a dissection $\mathcal{D}$ of adhesion $< \aleph_0$, such that for every orientation $\mathcal{P}$ there is an integer $k \geq 0$ such that $ts(\mathcal{P})$ has no K_k-minor

(iii) G admits a proper linked tree-decomposition (T, W) of adhesion $< \aleph_0$ such that for every $t \in V(T)$ there is an integer $k \geq 0$ such that $ts(t)$ has no K_k-minor.

13. TOPOLOGICAL TREES

Our final objective is to prove a strengthening of (2.5) in terms of "well-founded tree-decompositions". Our method is indirect with an intermediate step. In this section we introduce "topological trees" and prove that if a graph has a dissection of width $< \kappa$ and adhesion $< \kappa$, then it has a 'topological tree-decomposition" of width $< \kappa$ and adhesion $< \kappa$. In the next section we prove a lemma, that every "topological tree" has a kind of well-founded decomposition, called a "tree-labelling". In the final section, we use this to show that if a graph has a "topological tree-decomposition" of width $< \kappa$ and adhesion $< \kappa$, then it has a "well-founded tree-decomposition" of width $< \kappa$ and adhesion $< \kappa$; and we also show that if a graph has a "well-founded tree-decomposition" of width $< \kappa$ and adhesion $< \kappa$, then it has no K_κ minor, thereby completing the cycle of implications.

A *topological tree* is a pair $X = (V(X), X[\cdot, \cdot])$, where $V(X)$ is a non-empty set and $X[\cdot, \cdot]$ is a mapping from $V(X) \times V(X)$ to subsets of $V(X)$ such that for any three elements $x_1, x_2, x_3 \in V(X)$

(i) $\{x_1, x_2\} \subseteq X[x_1, x_2] = X[x_2, x_1]$,

(ii) if $x_2 \in X[x_1, x_3]$ then $X[x_1, x_3] = X[x_1, x_2] \cup X[x_2, x_3]$, and

(iii) the set $X[x_1, x_2] \cap X[x_2, x_3] \cap X[x_3, x_1]$ has exactly one element, called the *centroid* of x_1, x_2, x_3.

(13.1) *Let X be a topological tree and let $x, x_1, x_2 \in V(X)$. If $x_1 \in [x_2, x]$ and $x_2 \in [x_1, x]$ then $x_1 = x_2$.*

Proof. Since $x_1, x_2 \in X[x_1, x_2] \cap X[x_2, x] \cap X[x_1, x]$, we deduce from (iii) that $x_1 = x_2$. ■

We now prove a proposition which gives some insight into the definition of a topological tree and also explains the relation to well-founded trees to be introduced in the next section.

(13.2) *Let X be a topological tree and let $x_0 \in X$. For $x, x' \in X$ we define $x \le x'$ if $x \in X[x_0, x']$. Then $(V(X), \le)$ is a partially ordered set such that*

(i) *the set $\{x' \in V(X) : x' \le x\}$ is linearly ordered for every $x \in X$, and*

(ii) *every two elements $x, x' \in V(X)$ have an infimum.*

Proof. The relation $\le$ is clearly reflexive. To prove the transitivity let $x_1 \le x_2$ and $x_2 \le x_3$. Then $x_1 \in X[x_0, x_2] \subseteq X[x_0, x_3]$, and hence $x_1 \le x_3$. If $x \le x'$ and $x' \le x$ then $x = x'$ by (13.1). Thus $\le$ is a partial ordering.

To prove (i) let $x \in V(X)$, and let $x_1, x_2 \in [x_0, x]$. Suppose for a contradiction that $x_1 \notin X[x_0, x_2]$ and $x_2 \notin X[x_0, x_1]$. Since $X[x_0, x] = X[x_0, x_i] \cup X[x_i, x]$ $(i = 1, 2)$ we deduce that $x_2 \in X[x_1, x]$ and $x_1 \in X[x, x_2]$. Hence $x_1 = x_2$ by (13.1), a contradiction.

To prove (ii) let $x_1, x_2 \in V(X)$ and let x be the centroid of x_0, x_1, x_2. We claim that x is the infimum of x_1 and x_2. Indeed, $x \le x_1$ and $x \le x_2$ from the definition of a centroid. Let $y \in V(X)$ be such that $y \le x_1$ and $y \le x_2$. Then

$$y \in X[x_0, x_1] \cap X[x_0, x_2] = (X[x_0, x] \cup X[x, x_1]) \cap (X[x_0, x] \cup X[x, x_2])$$
$$= X[x_0, x] \cup (X[x, x_1] \cap X[x, x_2]) \subseteq X[x_0, x] \cup (X[x_0, x_1] \cap X[x_0, x_2] \cap X[x_1, x_2]) = X[x_0, x],$$

and hence $y \le x$. Thus x is the infimum of x_1 and x_2. ■

The partial ordering from (13.2) will be called the x_0-*domination relation.*

(13.3) *Let X be a topological tree, let $x_0, x, y, z \in V(X)$, let $\le$ be the x_0-domination relation, and assume that $x \le z$. Then $x \le y \le z$ if and only if $y \in X[x, z]$.*

Proof. Assume first that $x \in X[x_0, z]$ and $y \in X[x, z]$. Then $y \in X[x, z] \subseteq X[x_0, z]$, and $x \in X[x_0, z] = X[x_0, y] \cup X[y, z]$. Now $x \notin X[y, z] - \{y\}$ by (13.1), and so $x \in X[x_0, y]$, and hence $x \le y \le z$, as desired.

Now assume that $x \le y \le z$, that is $x, y \in X[x_0, z]$ and $x \in X[x_0, y]$. Then $y \in X[x_0, z] = X[x_0, x] \cup X[x, z]$, and if $y \in X[x_0, x]$ then $x = y$ by (13.1). Thus $y \in X[x, z]$, as desired. ■

Let X be a topological tree. A transfinite sequence $\{x_\alpha\}_{\alpha<\lambda}$ of elements of $V(X)$ is said to be *monotone* if it is (non-strictly) increasing in the x_0-domination relation. If it has a supremum (in the same partial ordering), we denote this supremum by $\lim_{\alpha<\lambda} x_\alpha$, and say that $\{x_\alpha\}_{\alpha<\lambda}$ *has a limit.* We say that X is *complete* if every monotone transfinite sequence has a limit. It follows from (13.1) that if a sequence $\{x_\alpha\}_{\alpha<\lambda}$ is monotone if and only if $x_\beta \in X[x_\alpha, x_\gamma]$ for all $\alpha \le \beta \le \gamma < \lambda$.

(13.4) *Let $\mathcal{D}$ be a non-empty dissection of a graph G. Let $V(X)$ be the set of all orientations of $\mathcal{D}$, and for $\mathcal{P}_1, \mathcal{P}_2 \in V(X)$ let $X[\mathcal{P}_1, \mathcal{P}_2]$ be the set of all orientations $\mathcal{P}$ of $\mathcal{D}$ with $\mathcal{P}_1 \cap \mathcal{P}_2 \subseteq \mathcal{P}$. Then X is a complete topological tree.*

Proof. We need to verify the axioms. Certainly $V(X) \neq \emptyset$, and clearly $\{\mathcal{P}_1, \mathcal{P}_2\} \subseteq X[\mathcal{P}_1, \mathcal{P}_2] = X[\mathcal{P}_2, \mathcal{P}_1]$, for all $\mathcal{P}_1, \mathcal{P}_2 \in V(X)$. Let $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3 \in V(X)$, and let $\mathcal{P}_2 \in X[\mathcal{P}_1, \mathcal{P}_3]$. Clearly $X[\mathcal{P}_1, \mathcal{P}_2] \cup X[\mathcal{P}_2, \mathcal{P}_3] \subseteq X[\mathcal{P}_1, \mathcal{P}_3]$. To prove the converse inclusion let $\mathcal{P} \in X[\mathcal{P}_1, \mathcal{P}_3] - X[\mathcal{P}_1, \mathcal{P}_2]$. Then $\mathcal{P}_1 \cap \mathcal{P}_3 \subseteq \mathcal{P}_2 \cap \mathcal{P}$, and $\mathcal{P}_1 \cap \mathcal{P}_2 \not\subseteq \mathcal{P}$. We must show that $\mathcal{P}_2 \cap \mathcal{P}_3 \subseteq \mathcal{P}$. Let $(A, B) \in \mathcal{P}_1 \cap \mathcal{P}_2 - \mathcal{P}$; then $(B, A) \in \mathcal{P}_3$, because otherwise $(A, B) \in \mathcal{P}_1 \cap \mathcal{P}_3 \subseteq \mathcal{P}$. Let $(C, D) \in \mathcal{P}_2 \cap \mathcal{P}_3$. We shall show that $(C, D) \in \mathcal{P}$. If $(C, D) \in \mathcal{P}_1$ then $(C, D) \in \mathcal{P}_1 \cap \mathcal{P}_3 \subseteq \mathcal{P}$, as desired, and so we may assume that $(D, C) \in \mathcal{P}_1$. Now $A \not\subseteq C$ because $(B, A), (C, D) \in \mathcal{P}_3$, $B \not\subseteq C$ because $(A, B), (C, D) \in \mathcal{P}_2$, and $B \not\subseteq D$ because $(A, B), (D, C) \in \mathcal{P}_1$. Since $(A, B), (C, D)$ do not cross it follows that $A \subseteq D$, and hence $(C, D) \in \mathcal{P}$, because $(B, A) \in \mathcal{P}$. Thus $\mathcal{P}_2 \cap \mathcal{P}_3 \subseteq \mathcal{P}$, as desired. This verifies the second axiom.

For the centroid axiom let $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3 \in V(X)$, and let $\mathcal{P}$ be the set of all $(A, B) \in \mathcal{D}$ such that (A, B) belongs to at least two of $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$. Then $\mathcal{P}$ is an orientation of $\mathcal{D}$, and it is the only element of $X[\mathcal{P}_1, \mathcal{P}_2] \cap X[\mathcal{P}_1, \mathcal{P}_3] \cap X[\mathcal{P}_2, \mathcal{P}_3]$, as desired.

Thus X is a topological tree. To prove completeness let $\{\mathcal{P}_\alpha\}_{\alpha<\lambda}$ be a monotone transfinite sequence of orientation, and let $\leq$ be the $\mathcal{P}_0$-domination relation. We define $\mathcal{P}$ to be the set of all $(A, B) \in \mathcal{D}$ such that there exists $\lambda' < \lambda$ with $(A, B) \in \mathcal{P}_\alpha$ for every α with $\lambda' \leq \alpha < \lambda$. Since $\mathcal{P}_\alpha \cap \mathcal{P}_\gamma \subseteq \mathcal{P}_\beta$ for all $\alpha \leq \beta \leq \gamma < \lambda$ by (13.3), we deduce that $\mathcal{P}$ is an orientation. Let $(A, B) \in \mathcal{P} \cap \mathcal{P}_0$, and let $\alpha < \lambda$ be such that $(A, B) \in \mathcal{P}_\alpha$. Then $(A, B) \in \mathcal{P}_\alpha \cap \mathcal{P}_0 \subseteq \mathcal{P}_\beta$ for every $\beta < \alpha$. Thus $\mathcal{P} \cap \mathcal{P}_0 \subseteq \mathcal{P}_\alpha$ for every $\alpha < \lambda$, and so $\mathcal{P}_\alpha \leq \mathcal{P}$ for every $\alpha < \lambda$. Let $\mathcal{P}' \in V(X)$ be such that $\mathcal{P}_\alpha \leq \mathcal{P}'$ for every $\alpha < \lambda$. Then $\mathcal{P}' \cap \mathcal{P}_0 \subseteq \mathcal{P}_\alpha$ for every $\alpha < \lambda$, and so $\mathcal{P}' \cap \mathcal{P}_0 \subseteq \mathcal{P}$. Thus $\mathcal{P}$ is the supremum of $\{\mathcal{P}_\alpha\}_{\alpha<\lambda}$. ■

Let G be a graph. A *topological tree-decomposition* of G is a pair (X, Φ), where X is a complete topological tree and $\Phi = (\Phi_x : x \in V(X))$ is such that

(i) $\bigcup_{x \in V(X)} \Phi_x = V(G)$, and every edge of G has both endpoints in some Φ_x,

(ii) if $x, x', x'' \in V(X)$ and $x' \in X[x, x'']$, then $\Phi_x \cap \Phi_{x''} \subseteq \Phi_{x'}$, and

(iii) if $\{x_\alpha\}_{\alpha<\lambda}$ is a monotone transfinite sequence in X with limit $x \in V(X)$, then

$$\bigcup_{\alpha<\lambda} \bigcap_{\alpha \leq \beta < \lambda} \Phi_{x_\beta} \subseteq \Phi_x .$$

Let κ be a cardinal. We say that a topological tree-decomposition (X, Φ) has *width* $< \kappa$ if $|\Phi_x| < \kappa$ for every $x \in V(X)$. We say that (X, Φ) has *adhesion* $\leq \kappa$ *at* $x \in V(X)$ if for every $x' \in V(X) - \{x\}$ there exist distinct $x_1, x_2 \in X[x, x']$ with $\{x_1, x_2\} = X[x_1, x_2]$ and such that $|\Phi_{x_1} \cap \Phi_{x_2}| \leq \kappa$. We say that (X, Φ) has adhesion $< \kappa$ if for every $x \in V(X)$ there exists a cardinal $\kappa' < \kappa$ such that (X, Φ) has adhesion $\leq \kappa'$ at x.

(13.5) *Let $\mathcal{D}$ be a dissection of a graph G of width $< \kappa$ and adhesion $< \kappa$, and let X be as in* (13.4). *For $x \in V(X)$ let Φ_x be the centre of x, and let $\Phi = (\Phi_x : x \in V(X))$. Then (X, Φ) is a topological tree-decomposition of G of width $< \kappa$ and adhesion $< \kappa$.*

Proof. We know from (13.4) that X is a complete topological tree. It remains to show that Φ satisfies the above three axioms. For the first let $V \subseteq V(G)$ such that either $|V| = 1$, or $V = \{u, v\}$, where u, v are adjacent in G. If $V \subseteq A \cap B$ for some $(A, B) \in \mathcal{D}$, then V is contained in the centre of $\mathcal{P}(A, B)$ by (9.3) and (12.4). If $V \not\subseteq A \cap B$ for every $(A, B) \in \mathcal{D}$, let $\mathcal{P}$ be the set of all $(A, B) \in \mathcal{D}$ such that $V \subseteq B$. Then $\mathcal{P}$ is an orientation and V is contained in its centre. This proves the first axiom.

For the second axiom let $x, x', x'' \in V(X)$ with $x' \in X[x, x'']$, and let $v \in V(G) - \Phi_{x'}$. Then $v \in A - B$ for some $(A, B) \in x'$. It follows that either $(A, B) \in x$ or $(A, B) \in x''$. Hence $v \notin \Phi_x \cap \Phi_{x''}$, as desired.

For the third axiom, let $\{x_\alpha\}_{\alpha<\lambda}$ be a monotone transfinite sequence with limit x, and let $v \in \bigcap_{\alpha \leq \beta < \lambda} \Phi_{x_\beta}$ for some $\alpha < \lambda$. Let $(A, B) \in x$; then $(A, B) \in x_\beta$ for some β with $\alpha \leq \beta < \lambda$, and so $v \in B$. Thus $v \in \Phi_x$, as desired. Hence (X, Φ) is a topological tree-decomposition.

The statement about width follows immediately, and it remains to prove the one about adhesion. Let $x \in V(X)$; then $\mathcal{D}$ has adhesion $\leq \kappa'$ at x, for some cardinal $\kappa' < \kappa$. We claim that (X, Φ) has adhesion $\leq \kappa'$ at x as well. Indeed, let $x' \in V(X) - \{x\}$ and choose $(A, B) \in x - x'$. Let $(A', B') \in x$ cut off (A, B) from x with $|A' \cap B'| \leq \kappa'$; then $(B', A') \in x'$, because $(B, A) \in x'$ and $A \subseteq A'$. Let x_1 be $\mathcal{P}(A', B')$ and let x_2 be $\mathcal{P}(B', A')$. Then $|\Phi_{x_1} \cap \Phi_{x_2}| \leq |A' \cap B'| \leq \kappa'$, and $\{x_1, x_2\} = X[x_1, x_2]$, because $|(x_1 - x_2) \cup (x_2 - x_1)| = 1$. We claim that $x_1, x_2 \in X[x, x']$. Indeed, let $(C, D) \in x \cap x'$. Then $D \not\subseteq A'$ because $(A', B'), (C, D) \in x$, and $D \not\subseteq B'$ because $(B', A'), (C, D) \in x'$. Thus $(C, D) \in x_1 \cap x_2$, and hence $x \cap x' \subseteq x_1 \cap x_2$. Thus $x_1, x_2 \in X[x, x']$, as claimed. This proves that (X, Φ) has adhesion $< \kappa$. ■

(13.6) *Let X be a topological tree and let $S \subseteq V(X)$. We define a relation $\sim$ on $V(X) - S$ by saying that $x \sim y$ if $X[x, y] \cap S = \emptyset$. Then $\sim$ is an equivalence relation.*

Proof. Reflexivity and symmetry are clear. For the transitivity let $x_1 \sim x_2$ and $x_2 \sim x_3$, and let c be the centroid of x_1, x_2, x_3. Then $X[x_1, x_3] = X[x_1, c] \cup [c, x_3] \subseteq X[x_1, x_2] \cup X[x_2, x_3] \subseteq V(X) - S$, and so $x_1 \sim x_3$, as required. ■

The equivalence classes of the equivalence relation from (13.6) will be called *S-flaps*.

Let G be a graph, let (X, Φ) be a topological tree-decomposition of G, and let κ be a cardinal. We say that (X, Φ) has *strong adhesion* $\leq \kappa$ at $x \in V(X)$ if for every $x' \in V(X) - \{x\}$ there exists $x'' \in X[x, x'] - \{x\}$ with $|\Phi_{x''}| \leq \kappa$. We say that (X, Φ) has *strong adhesion* $< \kappa$ if for every $x \in V(X)$ there exists a cardinal $\kappa' < \kappa$ such that (X, Φ) has strong adhesion $\leq \kappa'$ at x. We say that (X, Φ) is *conservative* if for every monotone transfinite sequence $\{x_\alpha\}_{\alpha<\lambda}$ in X with limit x,

$$|\Phi_x| \leq (\liminf_{\alpha<\lambda} |\Phi_{x_\alpha}|)^+ .$$

[If μ is a cardinal, then μ^+ is the least cardinal $> \mu$]. The following is the main result of this section.

(13.7) *Let G be a graph and let κ be an infinite cardinal. If G has a dissection of width $< \kappa$ and adhesion $< \kappa$, then G has a conservative topological tree-decomposition of width $< \kappa$ and strong adhesion $< \kappa$.*

Proof. By (13.5) G has a topological tree-decomposition (X, Φ) of width $< \kappa$ and adhesion $< \kappa$. We modify (X, Φ) so that it will be conservative and will have strong adhesion $< \kappa$.

Let $S(X)$ be the set of all pairs (s, S), where $s \in V(X)$ and S is an $\{s\}$-flap. For $x, y \in V(X)$ we put $\sigma(x, y) = \{(s, S) \in S(X) : s \in X[x, y], S \cap \{x, y\} \neq \emptyset\}$, and for $x, y \in V(X)$ and $S \subseteq V(X)$ we define $\{x\}\mathbf{1}_{y \in S}$ to be $\{x\}$ if $y \in S$ and $\emptyset$ otherwise. We define $V(Y) = V(X) \cup S(X)$ and

$$Y[x, y] = X[x, y] \cup \sigma(x, y) \qquad (x, y \in V(X)),$$

$$Y[(s, S), x] = Y[x, (s, S)] = (X[s, x] - \{s\}\,\mathbf{1}_{x \in S}) \cup \sigma(s, x) \qquad (x \in V(S), (s, S) \in S(X)),$$

$$Y[(s_1, S_2), (s_2, S_2)] = (X[s_1, s_2] - \{s_1\}\,\mathbf{1}_{s_2 \in S_1} - \{s_2\}\,\mathbf{1}_{s_1 \in S_2}) \cup \sigma(s_1, s_2) \qquad ((s_1, S_1), (s_1, s_2) \in S(X)).$$

It is straightforward but tedious to verify that Y is a topological tree. We omit the details.

Now we prove that Y is complete. For $y \in V(Y)$ we define $\bar{y} \in V(X)$ as follows. If $y \in V(X)$ we put $\bar{y} = y$, and if $y = (s, S) \in S(X)$ we put $\bar{y} = s$. Let $\{y_\alpha\}_{\alpha<\lambda}$ be a monotone transfinite sequence in Y. We may assume that it is not eventually constant. The sequence $\{\bar{y}_\alpha\}_{\alpha<\lambda}$ is a monotone sequence in X; let s denote its limit, and let S be the $\{s\}$-flap in X containing $\bar{y}_\alpha$ $(\alpha < \lambda)$. It follows that (s, S) is the limit of $\{y_\alpha\}_{\alpha<\lambda}$ in Y, as desired. Thus Y is complete.

For $y \in V(X)$ we put $\psi_y = \Phi_y$, and for $y = (s, S) \in S(X)$ we define

$$\psi_{(s,S)} = \Phi_s \cap \bigcup_{x \in S} \bigcap \{\Phi_{x'} : x' \in X[s, x] - \{s\}\},$$

and put $\psi = (\psi_y : y \in V(Y))$. We shall show that (Y, ψ) is the desired topological tree-decomposition.

We begin by verifying the axioms of a topological tree-decomposition. The first axiom is clear. For the second let $y_1, y_2, y_3 \in V(Y)$, and let $y_2 \in Y[y_1, y_3] - \{y_1, y_3\}$. Since $\psi_{y_1} \cap \psi_{y_3} \subseteq \Phi_{\bar{y}_1} \cap \Phi_{\bar{y}_3} \subseteq \Phi_{\bar{y}_2}$, and $\Phi_{\bar{y}_2} = \psi_{y_2}$ if $y_2 \in V(X)$, we may assume that $y_2 \in S(X)$, say $y_2 = (s, S)$. From the symmetry we may assume that $\bar{y}_1 \in S$. Then $\Phi_{\bar{y}_1} \cap \Phi_{\bar{y}_3} \subseteq \Phi_s \cap \bigcap \{\Phi_y : y \in X[\bar{y}_1, s] - \{s\}\} \subseteq \psi_{(s,S)}$ and so $\psi_{y_1} \cap \psi_{y_3} \subseteq \psi_{y_2}$, as desired.

For the third axiom let $\{y_\alpha\}_{\alpha<\lambda}$ be a monotone transfinite sequence in Y. We may assume that it is not eventually constant; let $(s, S) \in V(Y)$ be the limit. We have

$$\bigcup_{\alpha<\lambda} \bigcap_{\beta\le\alpha<\lambda} \psi_{y_\beta} \subseteq \bigcup_{\alpha<\lambda} \bigcap_{\beta\le\alpha<\lambda} \Phi_{\bar{y}_\beta} \subseteq \Phi_s \cap \bigcup_{\alpha<\lambda} \bigcap \{\Phi_y : y \in X[y_\alpha, s] - \{s\}\} \subseteq \psi_{(s,S)},$$

as desired. Thus (Y, ψ) is a topological tree-decomposition.

Since $\psi_y \subseteq \Phi_{\bar{y}}$ for every $y \in V(Y)$, it follows that (Y, ψ) has width $< \kappa$. We now prove that it has strong adhesion $< \kappa$. Let $y_0 \in V(Y)$, let $\kappa' < \kappa$ be such that $|\Phi_{\bar{y}_0}| \le \kappa'$ and such that (X, Φ) has adhesion $\le \kappa'$ at $\bar{y}_0$, and let $y_1 \in V(Y) - \{y_0\}$. If $\bar{y}_0 = \bar{y}_1$ then $|\psi_{y_1}| \le |\Phi_{\bar{y}_1}| = |\Phi_{\bar{y}_0}| \le \kappa'$, and so we may assume that $\bar{y}_0 \ne \bar{y}_1$. Let $x \in Y[y_0, y_1] \cap V(X)$ if such an x exists, and otherwise let $x = \bar{y}_1$. These exist distinct $x_1, x_2 \in X[\bar{y}_0, x]$ such that $\{x_1, x_2\} = X[x_1, x_2]$ and such that $|\Phi_{x_1} \cap \Phi_{x_2}| \le \kappa'$. We may assume that $x_1 \in X[\bar{y}_0, x_2]$. Let S be the $\{x_2\}$-flap containing x_1 (and hence containing $\bar{y}_0$); then $|\psi_{(x_2,S)}| = |\Phi_{x_1} \cap \Phi_{x_2}| \le \kappa'$, as desired. This proves that (Y, ψ) has strong adhesion $\le \kappa'$ at y_0, and since y_0 was arbitrary it proves that (Y, ψ) has strong adhesion $< \kappa$.

It remains to show that (Y, ψ) is conservative. Let $\{y_\alpha\}_{\alpha<\lambda}$ be a monotone transfinite sequence in Y. We may assume that it is not eventually constant. Let $(s, S) \in V(Y)$ be the limit, and let $v \in \psi_{(s,S)}$. Then there is an $x \in S$ such that $v \in \Phi_{x'}$ for every $x' \in X[s, x] - \{s\}$. Let c be the centroid of s, x, y_0; then $c \in S$. There exists $\alpha < \lambda$ such that $y_\beta \in Y[c, (s, S)]$ for every $\beta \geq \alpha$, and so $v \in \bigcap_{\alpha \leq \beta < \lambda} \psi_{y_\beta}$. Thus $\psi_{(s,S)} \subseteq \bigcup_{\alpha<\lambda} \bigcap_{\alpha \leq \beta < \lambda} \psi_{y_\beta}$, and hence $|\psi_{(s,S)}| \leq (\liminf_{\beta<\lambda} |\psi_{y_\beta}|)^+$, as desired. ■

14. WELL-FOUNDED TREES

In this section we prove a lemma which will be used in the next section to convert dissections to ''well-founded tree-decompositions''. It states, roughly, that every topological tree has a kind of ''well-founded tree-decomposition'' into pieces of cardinality ≤ 3.

We begin with several lemmas about topological trees. Let X be a topological tree. If $F \subseteq V(X)$, an element $s \in V(X) - F$ is an *attachment* of F if $X[s, u] \subseteq F \cup \{s\}$ for some $u \in F$. Let us say that $F \subseteq V(X)$ is *convex* if $X[x, y] \subseteq F$ for all $x, y \in F$.

(14.1) *Let X be a topological tree, let $F \subseteq V(X)$ be convex, and let s be an attachment of F. Then $X[s, x] \subseteq F \cup \{s\}$ for every $x \in F$.*

Proof. Let $u \in F$ be such that $X[s, u] \subseteq F \cup \{s\}$, and let $x \in F$. Let c be the centroid of x, u, s. Then $X[x, c] \subseteq X[x, u] \subseteq F$, since F is convex and $x, u \in F$; and $X[c, s] \subseteq X[u, s] \subseteq F \cup \{s\}$; and $X[x, s] = X[x, c] \cup X[c, s]$ since $c \in X[x, s]$. Consequently, $X[x, s] \subseteq F \cup \{s\}$, as required. ■

(14.2) *Let X be a topological tree, let $x_1, x_2 \in V(X)$ be distinct, and let $K \subseteq V(X)$ be convex, such that both x_1, x_2 are attachments of K. Then $\emptyset \neq X[x_1, x_2] - \{x_1, x_2\} \subseteq K$.*

Proof. Take $x \in K$ and let c be the centroid of x, x_1, x_2. Since $X[x, x_1] - \{x_1\} \subseteq K$ and $X[x, x_2] - \{x_2\} \subseteq K$ by (14.1), and c belongs to at least one of these sets (since $x_1 \neq x_2$), we deduce that $c \notin \{x_1, x_2\}$. Thus $c \in X[x_1, x_2] - \{x_1, x_2\}$ and $c \in X[x_1, x] - \{x_1\} \subseteq K$. Let $y \in X[x_1, x_2] - \{x_1, x_2\}$. Then $y \in (X[x_1, c] - \{x_1\}) \cup (X[c, x_2] - \{x_2\}) \subseteq K$, as desired. ■

(14.3) *Let X be a topological tree, and let $K, S, S' \subseteq V(X)$ be such that K is an S-flap and $S \subseteq S'$. Then $K - S'$ is a union of S'-flaps.*

Proof. Since every $x \in K - S'$ belongs to an S'-flap, it is enough to show that if K' is an S'-flap with $K \cap K' \neq \varnothing$ then $K' \subseteq K$. Let $x \in K \cap K'$ and let $y \in K'$. Then $\varnothing = X[x, y] \cap S' \supseteq X[x, y] \cap S$, and so $y \in K$, as desired. ■

(14.4) *Let X be a topological tree, let $S \subseteq V(X)$, let K be an S-flap, and let $s \in S$. If s is not an attachment of K then K is an $(S - \{s\})$-flap.*

Proof. Suppose that K is not a $(S - \{s\})$-flap. Then there exist $x \in K$ and $y \in V(X) - K$ such that $X[x, y] \cap S = \{s\}$. For any $z \in X[x, s] - \{s\}$, $X[x, z] \cap S = \varnothing$ by (13.1), and so $X[x, z] \subseteq K$, since $x \in K$ and K is an S-flap. In particular, $z \in K$, and so $X[x, s] \subseteq K \cup \{s\}$. Hence s is an attachment of K, as required. ■

Let X be a topological tree, and let $\{\tau_\alpha\}_{\alpha<\lambda}$ be a transfinite sequence of subsets of $V(X)$. We say that $x \in X$ is a *limit point* of $\{\tau_\alpha\}_{\alpha<\lambda}$ if there exists $\Lambda \subseteq \lambda$, cofinal in λ, and there exists $x_\alpha \in \tau_\alpha$ for all $\alpha \in \Lambda$, such that the sequence $\{x_\alpha\}_{\alpha\in\Lambda}$ is monotone with limit x; and we denote the set of all limit points of $\{\tau_\alpha\}_{\alpha<\lambda}$ by $\lim\{\tau_\alpha\}_{\alpha<\lambda}$. The sequence $\{\tau_\alpha\}_{\alpha<\lambda}$ is said to be *monotone* if $\tau_{\alpha'} \cap X[x, x''] \neq \varnothing$ for all $\alpha \leq \alpha' \leq \alpha'' < \lambda$ and all $x \in \tau_\alpha$ and $x'' \in \tau_{\alpha''}$. We shall need some lemmas about limit points of monotone sequences, as follows.

(14.5) *Let $\{\tau_\alpha\}_{\alpha<\lambda}$ be a monotone sequence of finite subsets of a topological tree X, and let $x \in \lim\{\tau_\alpha\}_{\alpha<\lambda}$. Then there exists $\alpha_0 < \lambda$ and $x_\alpha \in \tau_\alpha$ for all α with $\alpha_0 \leq \alpha < \lambda$, such that $\{x_\alpha\}_{\alpha_0\leq\alpha<\lambda}$ is monotone with limit x.*

Proof. Choose $\Lambda \subseteq \lambda$, cofinal in λ and $y_\alpha \in \tau_\alpha$ for each $\alpha \in \Lambda$, such that $\{y_\alpha\}_{\alpha\in\Lambda}$ is monotone with limit x. Then $\Lambda \neq \varnothing$; let $\alpha_0 \in \Lambda$ be minimum. Since $\{y_\alpha\}_{\alpha\in\Lambda}$ is monotone with limit x, it follows that $y_\alpha \in X[y_{\alpha_0}, x]$ for all $\alpha \in \Lambda$. Let $L = X[y_{\alpha_0}, x]$.

(1) *For all α_1, α_2 with $\alpha_0 \leq \alpha_1 \leq \alpha_2 < \lambda$ and all $z_1 \in \tau_{\alpha_1} \cap L$, there exists $z_2 \in \tau_{\alpha_2} \cap L$ with $z_2 \in X[z_1, x]$.*

For since $\{y_\alpha\}_{\alpha\in\Lambda}$ has limit x, there exists $\alpha_3 \in \Lambda$ such that $\alpha_3 \geq \alpha_2$ and such that $y_{\alpha_3} \in X[z_1, x]$. Since $\{\tau_\alpha\}_{\alpha<\lambda}$ is monotone, and $\alpha_1 \leq \alpha_2 \leq \alpha_3$ and $z_1 \in \tau_{\alpha_1}$ and $y_{\alpha_3} \in \tau_{\alpha_3}$, there exists $z_2 \in X[z_1, y_{\alpha_3}] \cap \tau_{\alpha_2}$. Since $X[z_1, y_{\alpha_3}] \subseteq X[z_1, x] \subseteq L$, the claim follows.

For each $\alpha < \lambda$ with $\alpha \geq \alpha_0$, it follows (by (1)) that $\tau_\alpha \cap L \neq \varnothing$. Since τ_α is finite, there exists $x_\alpha \in \tau_\alpha$ such that $X[x_\alpha, x] \cap \tau_\alpha = \{x_\alpha\}$. For $\alpha_0 \leq \alpha_1 \leq \alpha_2 < \lambda$ it follows from (1) that $x_{\alpha_2} \in X[x_{\alpha_1}, x]$, and so the sequence $\{x_\alpha\}_{\alpha_0\leq\alpha<\lambda}$ is monotone; and it has limit x, since $x_\alpha \in X[y_\alpha, x]$ for all $\alpha \in \Lambda$, as required. ■

From (14.5) we have immediately (we omit the proof):

(14.6) *Let* $\{\tau_\alpha\}_{\alpha<\lambda}$ *be a monotone sequence of finite subsets of* $V(X)$, *and let* $\Lambda \subseteq \lambda$ *be cofinal. Then* $\lim\{\tau_\alpha\}_{\alpha<\lambda} = \lim\{\tau_\alpha\}_{\alpha\in\Lambda}$.

(14.7) *If* $\{\tau_\alpha\}$ *is a monotone sequence of finite subsets of V(X), and* $\liminf_{\alpha<\lambda} |\tau_\alpha|$ *is finite, then*

$$|\lim\{\tau_\alpha\}_{\alpha<\lambda}| \le \liminf_{\alpha<\lambda} |\tau_\alpha| \;.$$

Proof. Let $k = \liminf_{\alpha<\lambda} |\tau_\alpha|$. By (14.6) we may assume that $|\tau_\alpha| \le k$ for all $\alpha < \lambda$. Suppose that $v^1, \ldots, v^{k+1}$ are distinct limit points of $\{\tau_\alpha\}_{\alpha<\lambda}$. By (14.5), for $1 \le i \le k+1$ there exists $\alpha^i < \lambda$ and $x^i_\alpha \in \tau_\alpha$ ($\alpha^i \le \alpha < \lambda$) such that $\{x^i_\alpha\}_{\alpha^i\le\alpha<\lambda}$ is monotone with limit v^i. For $1 \le i < j \le k+1$, the set

$$\{\alpha : \alpha^i, \alpha^j \le \alpha < \lambda \text{ and } x^i_\alpha = x^j_\alpha\}$$

is not cofinal in λ, for otherwise the sequences $\{x^i_\alpha\}_{\alpha^i\le\alpha<\lambda}$ and $\{x^j_\alpha\}_{\alpha^j\le\alpha<\lambda}$ have a common cofinal subsequence, and hence have the same limit, contradicting $v^i \ne v^j$. Thus we may choose $\beta < \lambda$ such that $\alpha_i \le \beta$ for $1 \le i \le k$ and all i, j with $1 \le i < j \le k+1$, $x^i_\beta \ne x^j_\beta$. Yet $x^1_\beta, \ldots, x^{k+1}_\beta \in \tau_\beta$, and $|\tau_\beta| \le k$, a contradiction. The result follows. ■

(14.8) *Let X be a topological tree, and for all* $\alpha < \lambda$ *let* $\tau_\alpha \subseteq V(X)$ *be finite, and let* $K_\alpha \subseteq V(X)$ *be such that* $K_\alpha - \tau_\alpha$ *is a union of* τ_α*-flaps. Suppose that for* $\alpha < \beta < \lambda$, $K_\beta \subseteq K_\alpha - \tau_\alpha$ *and* $\tau_\beta \subseteq K_\alpha \cup \tau_\alpha$. *Then:*

(i) $\{\tau_\alpha\}_{\alpha<\lambda}$ *is monotone*

(ii) $\lim\{\tau_\alpha\}_{\alpha<\lambda} \subseteq \bigcap_{\alpha<\lambda} (K_\alpha \cup \tau_\alpha)$, *and*

(iii) *if for each* $\alpha < \lambda$, τ_α *contains every attachment of* K_α, *then* $\lim\{\tau_\alpha\}_{\alpha<\lambda}$ *contains every attachment of* $\bigcap_{\alpha<\lambda} K_\alpha$.

Proof. Let $\alpha \le \alpha' \le \alpha'' < \lambda$, let $x \in \tau_\alpha$ and $x'' \in \tau_{\alpha''}$, and suppose that $X[x, x''] \cap \tau_{\alpha'} = \emptyset$. In particular, $\alpha < \alpha' < \alpha''$ and $x'' \notin \tau_{\alpha'}$, and so $x'' \in \tau_{\alpha''} - \tau_{\alpha'} \subseteq K_{\alpha'} - \tau_{\alpha'}$. Since $X[x, x''] \cap \tau_{\alpha'} = \emptyset$ and $K_{\alpha'} - \tau_{\alpha'}$ is a union of $\tau_{\alpha'}$-flaps, it follows that $x \in K_{\alpha'} - \tau_{\alpha'} \subseteq K_{\alpha'} \subseteq K_\alpha - \tau_\alpha$, a contradiction. Thus $\{\tau_\alpha\}_{\alpha<\lambda}$ is monotone, and (i) holds.

To prove (ii), let $v \in \lim\{\tau_\alpha\}_{\alpha<\lambda}$, and suppose that $v \notin K_{\alpha_0} \cup \tau_{\alpha_0}$ for some $\alpha_0 < \lambda$. Choose $\alpha_1 < \lambda$ and $x_\alpha \in \tau_\alpha$ for $\alpha_1 \le \alpha < \lambda$, such that $\{x_\alpha\}_{\alpha_1\le\alpha<\lambda}$ is monotone with limit v. We may assume that $\alpha_1 \ge \alpha_0$. Since

$v \notin K_{\alpha_0} - \tau_{\alpha_0}$, and $x_{\alpha_1} \in \tau_{\alpha_1} \subseteq K_{\alpha_0} - \tau_{\alpha_0}$, and $K_{\alpha_0} - \tau_{\alpha_0}$ is a union of τ_{α_0}-flaps, it follows that $X[x_{\alpha_1}, v] \cap \tau_{\alpha_0} \neq \emptyset$. Choose $y \in X[x_{\alpha_1}, v] \cap \tau_{\alpha_0}$ with $X[y, v]$ minimal. (This is possible since τ_{α_0} is finite.) Since v is the limit of $\{x_\alpha\}_{\alpha_1 \leq \alpha < \lambda}$ and $y \neq v$ (since $v \notin \tau_{\alpha_0}$) and $x_\alpha \in X[x_{\alpha_1}, v]$ for $\alpha_1 \leq \alpha < \lambda$, it follows that there exists α_2 with $\alpha_1 \leq \alpha_2 < \lambda$ such that $x_{\alpha_2} \in X[y, v] - \{y\}$. Since $x_{\alpha_2} \in \tau_{\alpha_2} \subseteq \tau_{\alpha_0} \cup K_{\alpha_0}$ and $v \notin \tau_{\alpha_0} \cup K_{\alpha_0}$, and $K_{\alpha_0} - \tau_{\alpha_0}$ is a union of τ_{α_0}-flaps, there exists $y' \in X[x_{\alpha_2}, v] \cap \tau_{\alpha_0}$. But then

$$X[y', v] \subseteq X[x_{\alpha_2}, v] \subset X[y, v]$$

contrary to the choice of y. Hence (ii) holds.

To prove (iii), let y be an attachment of $\bigcap_{\alpha<\lambda} K_\alpha$. Choose $x \in \bigcap_{\alpha<\lambda} K_\alpha$ so that $X[x, y] \subseteq (\bigcap_{\alpha<\lambda} K_\alpha) \cup \{y\}$. There exists $\alpha_0 < \lambda$ so that $y \notin K_{\alpha_0}$, since $y \notin \bigcap_{\alpha<\lambda} K_\alpha$. Let α satisfy $\alpha_0 \leq \alpha < \lambda$. Then $y \notin K_\alpha$ since $K_\alpha \subseteq K_{\alpha_0}$. But $X[x, y] - \{y\} \subseteq K_\alpha$, and so y is an attachment of K_α. Consequently $y \in \tau_\alpha$, for $\alpha_0 \leq \alpha < \lambda$, and hence $y \in \lim\{\tau_\alpha\}_{\alpha<\lambda}$. Hence (iii) holds. ■

(14.9) *Let X be a complete topological tree, and for all $\alpha < \lambda$ let $\tau_\alpha \subseteq V(X)$ be finite, and let $K_\alpha \subseteq V(X)$ be such that $K_\alpha - \tau_\alpha$ is a union of τ_α-flaps. Suppose that for $\alpha < \beta < \lambda$, $K_\beta \subseteq K_\alpha - \tau_\alpha$ and $\tau_\beta \subseteq K_\alpha \cup \tau_\alpha$. Let $\tau = \lim\{\tau_\alpha\}_{\alpha<\lambda}$ and $K = \bigcap_{\alpha<\lambda} K_\alpha$. Then $K - \tau$ is a union of τ-flaps.*

Proof. Let $x \in K - \tau$ and $y \in V(X) - (K \cup \tau)$; we must show that $X[x, y] \cap \tau \neq \emptyset$. Since $y \notin K$, there exists $\alpha_0 < \lambda$ such that $y \notin K_{\alpha_0}$. For $\alpha_0 \leq \alpha < \lambda$, since $y \notin K_{\alpha_0} \supseteq K_\alpha$ and $x \in K_\alpha$ and $K_\alpha - \tau_\alpha$ is a union of τ_α-flaps, it follows that $X[x, y] \cap \tau_\alpha \neq \emptyset$. Choose $x_\alpha \in X[x, y] \cap \tau_\alpha$ with $X[x_\alpha, x]$ minimal (this is possible since τ_α is finite). For $\alpha < \beta < \lambda$, since $x_\alpha \notin K_\alpha - \tau_\alpha \supseteq K_\beta$ and $x \in K_\beta$, it follows that $X[x_\alpha, x] \cap \tau_\beta \neq \emptyset$, and so $x_\beta \in X[x_\alpha, x]$, by the choice of x_β. Consequently, $\{x_\alpha\}_{\alpha<\lambda}$ is monotone. Since X is complete, this sequence has a limit x^* say. Then $x^* \in \tau$ by definition of τ, and yet $x^* \in X[x, y]$ since $x_\alpha \in X[x, y]$ for $\alpha_0 \leq \alpha < \lambda$; and so $\tau \cap X[x, y] \neq \emptyset$, as required. ■

Let X be a topological tree. A *triad* (in X) is a set $\tau \subseteq V(X)$ with $|\tau| \leq 3$ such that if $x, y, z \in \tau$ then either $x \in X[y, z]$, or $y \in X[x, z]$, or $z \in X[x, y]$.

(14.10) *Let X be a topological tree, let $\tau \subseteq V(X)$ be a triad, and let $F \subseteq V(X) - \tau$ be convex. Then F has at most* *two attachments in* τ.

Proof. We may assume that $|\tau| = 3$; let $\tau = \{x, y, z\}$, where $y \in X[x, z]$. Then not both x, z are attachments of F by (14.2), as desired. ■

A *well-founded tree* is a non-empty partially ordered set $T = (V(T), \leq)$, such that for every two elements $t_1, t_2 \in V(T)$ their infimum $inf(t_1, t_2)$ exists, and such that $\{t' \in V(T) : t' \leq t\}$ is well-ordered by $\leq$ for every $t \in V(T)$. A *trunk* of T is a non-empty subset $K \subseteq V(T)$ such that K is linearly ordered by $\leq$, and K contains every $t' \in V(T)$ such that $t' \leq t$ for some $t \in V(T)$. There are three kinds of trunks: *unbounded* trunks, those for which $sup\, K$ does not exist; *bounded open* trunks, those for which $sup\, K$ exists and $sup\, K \notin K$; and *closed* trunks, those with $sup\, K \in K$. If $t_1, t_2 \in V(T)$, we define

$$T(t_1, t_2) = \{t \in V(T) : t \leq t_1 \text{ or } t \leq t_2, \text{ and } t \geq inf(t_1, t_2)\}.$$

Let X be a topological tree. A *tree-labeling* of X is a pair (T, τ), where T is a well-founded tree and $\tau = (\tau_t : t \in V(T))$ is such that

(i) each τ_t is a triad in X and $\bigcup_{t \in V(T)} \tau_t = V(X)$

(ii) for all $t, t', t'' \in V(T)$ and all $x \in \tau_t$, $x'' \in \tau_{t''}$, if $t' \in T[t, t'']$ then $X[x, x''] \cap \tau_{t'} \neq \emptyset$, and

(iii) for every bounded open trunk K of T, $\lim\{\tau_t\}_{t \in K} = \tau_{sup\, K}$.

We say that $\nu = (\nu(x) : x \in V(X))$ is a *weighting* of a topological tree X if each $\nu(x)$ is a cardinal, and $\nu(x) \leq (\liminf_{\alpha < \lambda} \nu(x_\alpha))^+$ for every monotone sequence $\{x_\alpha\}_{\alpha < \lambda}$ with limit x. If κ is an infinite cardinal, we say that a weighting ν has *adhesion* $< \kappa$ if for every $x \in V(X)$ there is a cardinal $\kappa' < \kappa$ such that

(i) $\nu(x) \leq \kappa'$ and

(ii) for every $x' \in V(X) - \{x\}$ there exists $x'' \in X[x, x'] - \{x\}$ with $\nu(x'') \leq \kappa'$.

We say that a tree-labeling (T, τ) of X has ν-*adhesion* $< \kappa$ if for every trunk K of T there exists a cardinal $\kappa' < \kappa$ such that for every $t \in K$ there exists $t' \in K$ with $t' \geq t$ such that $\nu(x') \leq \kappa'$ for every $x' \in \tau_{t'}$.

Now we prove the main result of this section, the following.

(14.11) *Let X be a complete topological tree, and let ν be a weighting of X of adhesion $< \kappa$, where κ is an infinite cardinal. Then there exists a tree-labelling of X of ν-adhesion $< \kappa$.*

Proof. For each ordinal α, we shall construct, inductively, a set Z_α, a set $\mathcal{K}_\alpha$, and for each $K \in \mathcal{K}_\alpha$, a triad $\tau(K, \alpha)$ of X, and they will satisfy (1)-(4) below.

(1) *$Z_\alpha = \bigcup(\tau(K, \beta) : \beta < \alpha, K \in \mathcal{K}_\beta)$, and $\mathcal{K}_\alpha$ is the set of all Z_α-flaps in X.*

(2) *If $K \in \mathcal{K}_\alpha$ then $\tau(K, \alpha) \subseteq K \cup Z_\alpha$, $K - \tau(K, \alpha)$ is a union of $\tau(K, \alpha)$-flaps, and $\tau(K, \alpha)$ contains every attachment of K.*

(3) *For $\alpha < \beta$, if $K' \in \mathcal{K}_\alpha$, $K \in \mathcal{K}_\beta$ and $K \subseteq K'$, then $\tau(K, \beta) \subseteq K' \cup \tau(K', \alpha)$.*

(4) *For every α, if $K' \in \mathcal{K}_\alpha$, $K \in \mathcal{K}_{\alpha+1}$ and $K \subseteq K'$, then $\tau(K, \alpha+1)$, $\tau(K', \alpha)$ are different and one includes the other.*

The inductive definition is as follows. $V(X)$ is non-empty, and so we may choose $x_0 \in V(X)$. We define $\mathcal{K}_0 = \{V(X)\}$, $Z_0 = \emptyset$ and $\tau(V(X), 0) = \{x_0\}$. Inductively, we assume that for some ordinal $\gamma > 0$, we have defined Z_α, $\mathcal{K}_\alpha$ and the $\tau(K, \alpha)$ for all $\alpha < \gamma$, and we have verified that (1)-(4) hold whenever they are meaningful. We define Z_γ and $\mathcal{K}_\gamma$ to satisfy (1); and in order to complete the inductive definition we must define $\tau(K, \gamma)$ for each $K \in \mathcal{K}_\gamma$, and verify that (1)-(4) are still satisfied whenever they are meaningful. First, let $K \in \mathcal{K}_\gamma$, and let us define $\tau(K, \gamma)$. There are two cases, depending whether γ is a successor ordinal or a limit ordinal.

Suppose first that $\gamma = \alpha + 1$. Since $Z_\alpha \subseteq Z_\gamma$ and K is a Z_γ-flap, there is a Z_α-flap K' with $K \subseteq K'$. Now every attachment of K' is in $\tau(K', \alpha)$ by (2). Moreover, for each $K'' \in \mathcal{K}_\alpha$ with $K'' \neq K'$, $\tau(K'', \alpha) \subseteq K'' \cup Z_\alpha$ by (2), and since $K'' \cap K' = \emptyset$, it follows that $\tau(K'', \alpha) \cap K' = \emptyset$. Consequently, $Z_\gamma \cap K' \subseteq \tau(K', \alpha)$. Let A be the set of attachments of K; then $A \subseteq Z_\gamma$. Every element of A is either an attachment of K' or belongs to $Z_\gamma \cap K'$, and so $A \subseteq \tau(K', \alpha)$. Since $\tau(K', \alpha)$ is a triad it follows that so is A. But K is convex since it is a Z_γ-flap, and so by (14.10), $|A| \leq 2$. If $A \neq \tau(K', \alpha)$ we define $\tau(K, \gamma) = A$. If $A = \tau(K', \alpha)$, we choose $x \in K$ as follows:

(i) If $|A| = 2$, $A = \{a_1, a_2\}$ say, then x is chosen with $\nu(x)$ minimum subject to $x \in X[a_1, a_2] - \{a_1, a_2\}$. (This is possible by (14.2).)

(ii) If $|A| \leq 1$ (and hence $|A| = 1$), then x is chosen with $\nu(x)$ minimum subject to $x \in K$.

We define $\tau(K, \gamma) = A \cup \{x\}$. This completes the inductive definition when γ is a successor. Let us verify that (1)-(4) remain satisfied. (1) and (4) are clear. For (2), let $K \in \mathcal{K}_\gamma$, and let α, K', A be as above. Then $\tau(K, \gamma) \subseteq A \cup K$, and $A \subseteq \tau(K', \alpha) \subseteq Z_\gamma$, and so $\tau(K, \gamma) \subseteq K \cup Z_\gamma$. Now we show that $K - \tau(K, \gamma)$ is a union of $\tau(K, \gamma)$-flaps. For $K' - \tau(K', \alpha)$ is a union of $\tau(K', \alpha)$-flaps, by (2) applied to K', α. But for $x \in K$ and $y \in K' - (\tau(K', \alpha) \cup K)$, $X[x, y] \cap Z_\gamma \neq \emptyset$ since K is a Z_γ-flap, and yet $X[x, y] \subseteq K'$, and $Z_\gamma \cap K' \subseteq \tau(K', \alpha)$; and hence $X[x, y] \cap \tau(K', \alpha) \neq \emptyset$. It follows that K is a union of $\tau(K', \alpha)$-flaps, since $K \subseteq K' - \tau(K', \alpha)$. By (14.4), K is a union of A-flaps. By (14.3), $K - \tau(K, \gamma)$ is a union of $\tau(K, \gamma)$-flaps, as required. This proves (2), for the third assertion of (2) is obvious. For (3), let α, K', A be as above. Let $\beta < \gamma$, and let $K'' \in \mathcal{K}_\beta$ such that $K \subseteq K''$. We must show that $\tau(K, \gamma) \subseteq K'' \cup \tau(K'', \beta)$. But $\tau(K, \gamma) \subseteq K' \cup \tau(K', \alpha)$, and $K' \subseteq K''$, and $\tau(K', \alpha) \subseteq K'' \cup \tau(K'', \beta)$ (by (3) applied to β, α if $\beta < \alpha$, since $K' \subseteq K''$, and trivially if $\beta = \alpha$). Thus $\tau(K, \gamma) \subseteq K'' \cup \tau(K'', \beta)$. This proves (3). Hence (1)-(4) still hold.

Now suppose that γ is a limit ordinal, and again let us define $\tau(K, \gamma)$ where $K \in \mathcal{K}_\gamma$. For each $\alpha < \gamma$, let $K_\alpha \in \mathcal{K}_\alpha$ such that $K \subseteq K_\alpha$; and define $\tau(K, \gamma) = \lim_{\alpha < \gamma} \tau(K_\alpha, \alpha)$. This completes the inductive definition, but we must show that $\tau(K, \gamma)$ is a triad, and that (1)-(4) remain satisfied. For brevity, let us write σ_α for $\tau(K_\alpha, \alpha)$. Now for $\alpha < \beta < \gamma$, $K_\beta \subseteq K_\alpha - \sigma_\alpha$, and $\sigma_\beta \subseteq K_\alpha \cup \sigma_\alpha$ by (2) applied to α, β. Consequently, $\{\sigma_\alpha\}_{\alpha<\gamma}$ is monotone by (14.8)(i). But $\liminf_{\alpha<\gamma} |\sigma_\alpha| \leq 2$ by (4) applied to α, and so $|\tau(K, \gamma)| \leq 2$ by (14.7). Hence $\tau(K, \gamma)$ is a triad.

It remains to verify (1)-(4). Now (1) and (4) are obviously still satisfied. For (2), let K, $K_\alpha(\alpha < \gamma)$, σ_2 be as before. Then

$$\tau(K, \gamma) = \lim \{\sigma_\alpha\}_{\alpha<\gamma} \subseteq \bigcap_{\alpha<\gamma} (K_\alpha \cup \sigma_\alpha) \subseteq \bigcap_{\alpha<\gamma} K_\alpha \cup Z_\gamma$$

by (14.8)(ii). But $\bigcap_{\alpha<\gamma} K_\alpha$ is convex since each K_α is convex, and $\bigcap_{\alpha<\gamma} K_\alpha \subseteq V(X) - Z_\gamma$, and so $\bigcap_{\alpha<\gamma} K_\alpha$ is a subset of a Z_γ-flap. Since K is a Z_γ-flap and $K \subseteq \bigcap_{\alpha<\gamma} K_\alpha$ it follows that $K = \bigcap_{\alpha<\gamma} K_\alpha$; and so $\tau(K, \gamma) \subseteq K \cup Z_\gamma$. This verifies the first assertion of (2). The second assertion follows from (14.9), and the third from (14.8)(iii). Hence, (2) still holds. For (3), let K, K_α $(\alpha < \gamma)$ $\sigma_\alpha(\alpha < \gamma)$ be as before, and let $\alpha' < \gamma$. We must show that $\tau(K, \gamma) \subseteq K_{\alpha'} \cup \sigma_{\alpha'}$. But this follows from (14.8)(ii), and so (3) still holds. Thus, (1)-(4) all still hold.

We observe from the construction that $Z_\alpha \subseteq Z_\beta$ for $\alpha \le \beta$, and if $Z_\alpha \neq V(X)$ then either $Z_\alpha \neq Z_{\alpha+1}$, or $Z_{\alpha+1} \neq Z_{\alpha+2}$. Consequently, if $Z_\alpha \neq V(X)$ then $\alpha < |V(X)|^+$. Hence there is a (least) ordinal λ such that $Z_\lambda = V(X)$; and then $\mathcal{K}_\alpha = \emptyset$ for all $\alpha \ge \lambda$. We define $V(T) = \bigcup_{\alpha<\lambda} \{(K, \alpha) : K \in \mathcal{K}_\alpha\}$, and for $(K, \alpha), (K', \alpha') \in V(T)$, we define $(K, \alpha) \le (K', \alpha')$ if $\alpha \le \alpha'$ and $K' \subseteq K$. We put $T = (V(T), \le)$; then it is easy to see that T is a well-founded tree (we omit the details). For $t \in V(T)$, let $t = (K, \alpha)$ say; we denote $\tau(K, \alpha)$ by τ_t, and let $\tau = (\tau_t : t \in V(T))$.

(5) *(T, τ) is a tree-labeling of X.*

We must verify the axioms for a tree-labeling. The first holds since $Z_\lambda = V(X)$, and the third is immediate from the construction. For the second, let $t = (K, \alpha)$, $t' = (K', \alpha')$, $t'' = (K', \alpha'') \in V(T)$, let $t' \in T[t, t'']$, and let $x \in \tau_t$, $x'' \in \tau_{t''}$. We must show that $X[x, x''] \cap \tau_{t'} \neq \emptyset$. We show this first in the special case when $t \le t' \le t''$. We may assume that $x, x'' \notin \tau_{t'}$, for otherwise we are done. Consequently $t' \neq t, t''$. Thus, $\alpha < \alpha' < \alpha''$ and $K'' \subseteq K' \subseteq K$. By (3), $\tau_{t''} \subseteq K' \cup \tau_{t'}$, and so $x'' \in K'$. But K' is a $Z_{\alpha'}$-flap, and $\tau_t \subseteq Z_{\alpha'}$, and so $K' \cap \tau_t = \emptyset$, and consequently $x \notin K'$. Since $x'' \in K'$ and $x \notin K'$, and $K' - \tau_{t'}$ is a union of $\tau_{t'}$-flaps by (2), it follows from that $X[x, x''] \cap \tau_{t'} \neq \emptyset$, as required.

Now we turn to the general case. Let t_0 be the infimum of t and t''; we may assume that $t_0 \notin \{t, t''\}$, for otherwise we are in the previous case. Let $t_0 = (L_0, \beta)$ and let $L, L'' \in \mathcal{K}_{\beta+1}$ include K and K'', respectively. Then $L \neq L''$. We claim that $X[x, x''] \cap \tau_{t_0} \neq \emptyset$. If $x \in \tau_{t_0}$ or $x'' \in \tau_{t_0}$ then our claim is true, and so we may assume that $x, x'' \notin \tau_{t_0}$. Then $x \in L$ and $x'' \in L''$ by (3), and thus $X[x, x''] \cap \tau_{t_0} \neq \emptyset$, which proves our claim. Let $x_0 \in X[x, x''] \cap \tau_{t_0}$. Now either $t' \in T[t_0, t]$, or $t' \in T[t_0, t'']$, and from the symmetry we may assume the former. Then $X[x, x_0] \cap \tau_{t'} \neq \emptyset$ by the first part of this proof, and so

$$\emptyset \neq X[x, x_0] \cap \tau_{t'} \subseteq X[x, x''] \cap \tau_{t'}.$$

This proves (5).

It remains to prove that (T, τ) has ν-adhesion $< \kappa$. For that we shall use the following lemma.

(6) *Let I be a non-empty ordinal interval (so if $\alpha \le \beta \le \gamma$ and $\alpha, \gamma \in I$ then $\beta \in I$). For $\alpha \in I$, let $K_\alpha \in \mathcal{K}_\alpha$, so that $K_\alpha \subseteq K_\beta$ for $\beta \le \alpha$. Write $\sigma_\alpha = \tau(K_\alpha, \alpha)$ for $\alpha \in I$, and suppose that $|\sigma_\alpha| \ge 2$ for all $\alpha \in I$ with $\alpha \neq sup(I)$. Then*

there exists a, b ∈ $V(X)$ so that $\sigma_\alpha \subseteq X[a, b]$ for all $\alpha \in I$. Moreover, for every cardinal μ, either

(i) *there exists α with $\alpha, \alpha+1 \in I$ so that $|\sigma_\alpha| = 2$, $\sigma_\alpha = \{x, y\}$ say, and $\nu(z) > \mu$ for all $z \in X[x, y] - \{x, y\}$, and $\sigma_\beta \subseteq X[x, y]$ for all $\beta \geq \alpha$ with $\beta \in I$, or*

(ii) *there exists $x \in \bigcap_{\alpha \in I} \sigma_\alpha$ with $\nu(x) > \mu^+$, or*

(iii) *there exists $\alpha \in I$ such that $\nu(x) \leq \mu^+$ for all $x \in \sigma_\alpha$.*

For choose $\alpha_0 \in I$, minimum, and choose $a, b \in \alpha_0$ so that $\sigma_{\alpha_0} \subseteq X[a, b]$. Then it follows easily by transfinite induction (we omit the details) that $\sigma_\alpha \subseteq X[a, b]$ for all $\alpha \in I$, because $|\sigma_\alpha| \geq 2$ for all $\alpha \in I$.

Let Z be the set of all $z \in X[a, b]$ such that there exists $\alpha \in I$ with $\alpha + 1 \in I$ so that $z \in \sigma_{\alpha+1} - \sigma_\alpha$. It follows from (4) that $\sigma_\alpha \subseteq \sigma_{\alpha+1}$ and $|\sigma_\alpha| \leq 2$; and consequently, $|\sigma_\alpha| = 2$, $\sigma_\alpha = \{x, y\}$ say. By the same transfinite induction argument, $\sigma_\beta \subseteq X[x, y]$ for all $\beta \geq \alpha$ with $\beta \in I$. Moreover, it follows from the construction that $z \in X[x, y] - \{x, y\}$, and $\nu(z) \leq \nu(z')$ for all $z' \in X[x, y] - \{x, y\}$. If $\nu(z) > \mu$ then $\nu(z') > \mu$ for all $z' \in X[x, y] - \{x, y\}$, and so (i) holds. We may therefore assume that $\nu(z) \leq \mu$, for all $z \in Z$. Let

$$Z' = \bigcup_{\alpha \in I} \sigma_\alpha - (Z \cup \sigma_{\alpha_0}).$$

From the construction, it follows easily that every element of Z' is the limit of a monotone sequence of members of Z, since $\sigma_\alpha \subseteq X[a, b]$ for all $\alpha \in I$. But ν is a weighting; and so $\nu(z) \leq \mu^+$ for every $z \in Z'$, and hence $\nu(z) \leq \mu^+$ for every $z \in (\bigcup_{\alpha \in I} \sigma_\alpha) - \sigma_{\alpha_0}$. Choose $\alpha \in I$ with $\sigma_\alpha \cap \sigma_{\alpha_0}$ minimal. We may assume that there exists $x \in \sigma_\alpha$ with $\nu(x) > \mu^+$, for otherwise (iii) holds. Consequently, $x \in \sigma_{\alpha_0}$, and from the minimality of $\sigma_\alpha \cap \sigma_{\alpha_0}$ and the monotonicity of $\{\sigma_{\alpha'}\}_{\alpha' \in I}$, it follows that $x \in \bigcap_{\alpha' \in I} \sigma_{\alpha'}$, and (ii) holds. This proves (6).

Now let us show that (T, τ) has ν-adhesion $< \kappa$. Let $\{(K_\alpha, \alpha) : \alpha < \gamma\}$ be a trunk of T, and let $\sigma_\alpha = \tau(K_\alpha, \alpha)$ for $\alpha < \gamma$. We must show that there is a cardinal $\kappa' < \kappa$ such that for every $\alpha < \gamma$ there exists β with $\alpha \leq \beta < \gamma$ such that $\nu(x) \leq \kappa'$ for every $x \in \sigma_\beta$. If κ is a successor cardinal, say $\kappa = \mu^+$, then $\nu_x \leq \mu$ for all $x \in V(X)$ since ν has adhesion $< \kappa$, and we may therefore set $\kappa' = \mu$. We assume then that κ is not a successor cardinal. If γ is a successor ordinal, say $\gamma = \beta + 1$, there exists a cardinal $\mu < \kappa$ such that $\nu(x) \leq \mu$ for all $x \in \sigma_\beta$, since ν has adhesion $< \kappa$ and σ_β is finite; and then we may set $\kappa' = \mu$. Thus we may assume that γ is a limit ordinal.

Since ν has adhesion $< \kappa$, there is an infinite cardinal $\mu < \kappa$ such that

(a) $\nu(x) \leq \mu$ for all $x \in \lim\{\sigma_\alpha\}_{\alpha<\lambda}$, and

(b) for all $x \in \lim\{\sigma_\alpha\}_{\alpha<\lambda}$ and $y \in V(X) - \{x\}$ there exists $x' \in X[x, y] - \{x\}$ such that $\nu(x') \leq \mu$.

Set $\kappa' = \mu^+$. We shall show that κ' satisfies our requirements. Now certainly $\kappa' < \kappa$, since $\mu < \kappa$ and κ is not a successor cardinal. It remains to show the following.

(7) *For every* $\beta_0 < \gamma$ *there exists* β *with* $\beta_0 \leq \beta < \gamma$ *such that* $\nu(x) \leq \mu^+$ *for every* $x \in \sigma_\beta$.

Suppose first that for some $\beta < \gamma$, $|\sigma_\alpha| \geq 2$ for all α with $\beta \leq \alpha < \gamma$. Let $I = \{\alpha : \beta \leq \alpha < \gamma\}$. By (6), one of three possible cases must occur:

(i) There exists α with $\alpha, \alpha+1 \in I$ so that $|\sigma_\alpha| = 2$, $\sigma_\alpha = \{x, y\}$ say, and $\nu(z) > \mu$ for all $z \in X[x, y] - \{x, y\}$. Let $z \in \lim\{\sigma_\alpha\}_{\alpha<\gamma}$ (such an element exists since X is complete). Then $z \in X[x, y]$, since by (6), $\sigma_{\alpha'} \subseteq X[x, y]$ for all α' with $\alpha \leq \alpha' < \gamma$. By (a) above, $\nu(z) \leq \mu$, and so $z \notin X[x, y] - \{x, y\}$, and so we may assume that $z = x$, that is, $\nu(x) \leq \mu$ and $x \in \lim\{\sigma_{\alpha'}\}_{\alpha'<\gamma}$. Since $X[x, y] \neq \{x, y\}$ by (14.2), it follows from (b) that there exists $x' \in X[x, y] - \{x, y\}$ with $\nu(x') \leq \mu$, a contradiction.

(ii) There exists $x \in \bigcap_{\alpha \in I} \sigma_\alpha$ with $\nu(x) > \mu^+$. But then $x \in \lim\{\sigma_\alpha\}_{\alpha<\gamma}$, contrary to (a).

(iii) There exists $\alpha \in I$ such that $\nu(x) \leq \mu^+$ for all $x \in \sigma_\alpha$; but then (7) holds, as required.

We may assume, therefore, that $\Lambda \subseteq \gamma$ is cofinal in γ, where $\Lambda = \{\alpha : \alpha < \gamma, |\sigma_\alpha| = 1\}$. For $\alpha \in \Lambda$, let $\sigma_\alpha = \{x_\alpha\}$. By (14.8)(i), $\{\sigma_\alpha\}_{\alpha<\gamma}$ is monotone, and hence so is $\{x_\alpha\}_{\alpha<\gamma}$. Since X is complete, $\{x_\alpha\}_{\alpha<\gamma}$ has a limit x say, and $x \in \lim\{\sigma_\alpha\}_{\alpha<\gamma}$. Moreover, $x_\beta \in X[x_\alpha, x]$ for $\alpha \leq \beta$ and $\alpha, \beta \in \Lambda$.

Now let $\beta_0 < \gamma$; we must show that it satisfies (7). Since Λ is cofinal in γ, there exists $\beta_1 \in \Lambda$ with $\beta_0 \leq \beta_1$. If $\nu(x_{\beta_1}) \leq \mu^+$ then (7) is satisfied by choosing $\beta = \beta_1$; and so we assume that $\nu(x_{\beta_1}) > \mu^+$. Then $x_{\beta_1} \neq x$ since $\nu(x) \leq \mu$ by (a). By (b), there exists $z \in X[x_{\beta_1}, x] - \{x\}$ with $\nu(z) \leq \mu$. Let $\beta_4 \in \Lambda$ be minimum such that $x_{\beta_4} \in X[z, x]$. If $\nu(x_{\beta_4}) \leq \mu^+$ we are done as before, and so we may assume that $\nu(x_{\beta_4}) > \mu^+$, and in particular, $x_{\beta_4} \neq z$. Let $\beta_2 = sup\{\beta \in \Lambda : \beta < \beta_4\}$. Since $\beta_1 \in \Lambda$ and $\beta_1 < \beta_4$, it follows that $\beta_1 \leq \beta_2$. We claim that $\beta_2 \in \Lambda$ and $\beta_2 < \beta_4$. For if $\beta_2 \neq sup(\Lambda \cap \beta_2)$ then certainly $\beta_2 \in \Lambda$ and $\beta_2 < \beta_4$. If $\beta_2 = sup(\Lambda \cap \beta_2)$, then β_2 is a limit ordinal, and so $\sigma_{\beta_2} = \lim\{\sigma_\beta\}_{\beta<\beta_2}$; but $\liminf_{\beta<\beta_2} |\sigma_\beta| = 1$ since $\beta_2 = sup(\Lambda \cap \beta_2)$, and so by (14.7), $|\sigma_{\beta_2}| \leq 1$, that is, $\beta_2 \in \Lambda$.

Moreover, $x_\beta \in X[x_{\beta_1}, z]$ for all $\beta \in \Lambda$ with $\beta < \beta_4$, by definition of β_4, and so the limit of $\{x_\beta\}_{\beta \in \Lambda \cap \beta_4}$ belongs to $X[x_{\beta_1}, z]$, that is, $x_{\beta_2} \in X[x_{\beta_1}, z]$, and so $\beta_2 \neq \beta_4$. This proves our claim that $\beta_2 \in \Lambda$ and $\beta_2 < \beta_4$. Thus $\beta_0 \leq \beta_1 \leq \beta_2 < \beta_4$, and $z \in X[x_{\beta_2}, x_{\beta_4}]$. Moreover, there is no $\beta \in \Lambda$ with $\beta_2 < \beta < \beta_4$ from the definition of β_2. As before, we may assume that $\nu(x_{\beta_2}) > \mu^+$, and in particular $x_{\beta_2} \neq z$. Since $x_{\beta_2} \notin \sigma_{\beta_4}$, there exists a minimum β_3 with $\beta_2 \leq \beta_3 \leq \beta_4$ such that $x_{\beta_2} \notin \sigma_{\beta_3}$. Then $\beta_2 < \beta_3$; let $I = \{\alpha : \beta_2 + 1 \leq \alpha \leq \beta_3\}$. Then $|\sigma_\alpha| \geq 2$ for all $\alpha \in I$ with $\alpha \neq sup(I) = \beta_3$ (and also for $\alpha = \beta_3$ unless $\beta_3 = \beta_4$). Since $I \neq \emptyset$, we deduce from (6) that there are three possibilities:

(i) There exists α with $\alpha, \alpha + 1 \in I$ so that $|\sigma_\alpha| = 2, \sigma_\alpha = \{x, y\}$ say, and $\nu(z') > \mu$ for all $z' \in X[x, y] - \{x, y\}$. Since $\alpha + 1 \in I$ it follows that $\alpha < \beta_3$ and so $x_{\beta_2} \in \sigma_\alpha$. But since $|\sigma_\beta| \geq 2$ for $\beta_2 < \beta < \beta_4$ it follows from (6) that $x_{\beta_4} \in X[x, y]$. But $x_{\beta_2} \in \sigma_\alpha = \{x, y\}$, and so $X[x_{\beta_2}, x_{\beta_4}] \subseteq X[x, y]$, and hence $z \in X[x, y] - \{x, y\}$, since $z \neq x_{\beta_2}, x_{\beta_4}$. This is impossible since $\nu(z) \leq \mu$.

(ii) There exists $y \in \bigcap_{\alpha \in I} \sigma_\alpha$ with $\nu(y) > \mu^+$. In particular, $y \in \sigma_{\beta_2+1} \cap \sigma_{\beta_3}$. Since $|\sigma_{\beta_2}| = 1$, it follows from the construction that $\sigma_{\beta_2+1} = \{x_{\beta_2}, y\}$ and $y \in K_{\beta_2+1}$ and $\nu(y) \leq \nu(y')$ for all $y' \in K_{\beta_2+1}$. Since $x_{\beta_4} \in K_{\beta_2+1}$, and hence

$$z \in X[x_{\beta_2}, x_{\beta_4}] \subseteq K_{\beta_2+1} \cup \{x_{\beta_2}\},$$

it follows that $z \in K_{\beta_2+1}$. From one of the properties of y, we deduce that $\nu(y) \leq \nu(z)$. But $\nu(z) \leq \mu$, and so $\nu(y) \leq \mu$, a contradiction.

(iii) There exists $\alpha \in I$ such that $\nu(y) \leq \mu^+$ for all $y \in \sigma_\alpha$. This is the desired conclusion, and (7) holds.

This completes the proof of (7).

From (7), we deduce that (T, τ) has ν-adhesion $< \kappa$, and the proof is complete. ■

15. WELL-FOUNDED TREE-DECOMPOSITION

If $T = (V(T), \leq)$ is a well-founded tree, and $t, t' \in V(T)$, we say that t' is a *successor* of t and t is a *predecessor* of t' if $t < t'$ and there is no $t'' \in V(T) - \{t, t'\}$ with $t \leq t'' \leq t'$. We define $E(T)$, the *set of edges* of T, to be the set of all pairs (t, t'), where $t, t' \in V(T)$ and t' is a successor of t. A *well-founded tree-decomposition* of a graph G is a pair (T, W), where T is a well-founded tree and $W = (W_t : t \in V(T))$ satisfies

(i) $\bigcup_{t \in V(T)} W_t = V(T)$, and every edge of G has both endpoints in some W_t,

(ii) if $t' \in T[t, t'']$ then $W_t \cap W_{t''} \subseteq W_{t'}$, and

(iii) if $K \subseteq V(T)$ is a bounded open trunk, then $\lim_K W_t \subseteq W_{sup(K)}$.

Here $\lim_K W_t = \bigcup_{t \in K} \bigcap \{W_{t'} : t \in K, t' \geq t\}$. We say that (T, W) has *width* $< \kappa$ if $|\lim_K W_t| < \kappa$ for every trunk $K \subseteq V(T)$. It follows if (T, W) has width $< \kappa$ then $|W_t| < \kappa$ for every $t \in V(T)$ (apply the width definition to closed trunks). We say it has *adhesion* $< \kappa$ if for every trunk $K \subseteq V(T)$ there exists a cardinal $\kappa' < \kappa$ such that for every $t_0 \in K$ with $t_0 \neq sup\, K$, there exists $t, t' \in K$ such that $t, t' \geq t_0$, t' is a successor of t, and $|W_t \cap W_{t'}| \leq \kappa'$. We say it has *strong adhesion* $< \kappa$ if for every trunk $K \subseteq V(T)$ there exists a cardinal $\kappa' < \kappa$ such that for every $t_0 \in K$ with $t_0 \neq sup\, K$, there exists $t \in K$ such that $t \geq t_0$, $t \neq t_0$ and $|W_t| \leq \kappa'$. It is easy to see that if (T, W) has strong adhesion $< \kappa$ then it has adhesion $< \kappa$. The following is the desired generalization of (2.5).

(15.1) *For a graph G and an uncountable cardinal* κ, *the following are equivalent:*

(i) *G has no minor isomorphic to* K_κ,

(ii) *G has a dissection of width* $< \kappa$ *and adhesion* $< \kappa$,

(iii) *G has a well-founded tree-decomposition of width* $< \kappa$ *and adhesion* $< \kappa$.

(iv) *G has a well-founded tree-decomposition of width* $< \kappa$ *and strong adhesion* $< \kappa$.

We have already seen in (7.7) that (i) and (ii) are equivalent and that (iv) implies (iii). We first show, in (15.3), that (iii) implies (i) and then, in (15.5), that (ii) implies (iv). Let T be a well-founded tree and let $e_1 = (t_1, t_1')$, $e_2 = (t_2, t_2')$ be two distinct edges of T. Then either $t_1 \leq t_1' \leq t_2 \leq t_2'$ or $t_2 \leq t_2' \leq t_1 \leq t_1'$ or $t_1' \not\leq t_2' \not\leq t_1'$. Now assume that both e_1, e_2 are assigned a direction in such a way that if $t_1 \leq t_1' \leq t_2 \leq t_2'$ then e_1 is directed toward t_1 and e_2 is directed toward t_2', if $t_2 \leq t_2' \leq t_1 \leq t_1'$ then e_1 is directed toward t_1' and e_2 is directed toward t_2, and if $t_1' \not\leq t_2' \not\leq t_1'$ then e_i is directed toward $t_i'(i = 1, 2)$. We say that e_1, e_2 are *directed away from each other.*

A *confluence* of a well-founded tree T is an assignment of a direction to each edge of T in such a way that no two edges are directed away from each other. Let $K \subseteq V(T)$ be a trunk, such that if K is closed then either $sup(K)$

has a predecessor or $|K| = 1$. For $(t, t') \in E(T)$, if $t' \in K$ we direct (t, t') towards t', and otherwise towards t. This defines a confluence, called the confluence *derived from* K. The following generalizes (12.2).

(15.2) *Every confluence of a well-founded tree T arises from some trunk.*

Proof. Let K contain $\inf(V(T))$, and all $t \in V(T)$ such that $t \in \{t_1, t_2\}$ for some edge $(t_1, t_2) \in E(T)$ directed towards t_2. Then it is easy to check that K is a trunk, and if K is closed then either $sup\, K$ has a predecessor or $|K| = 1$; and the given confluence arises from K. ■

If $e = (t, t') \in V(T)$ we define T^e to be the set of all $t'' \in V(T)$ such that $t' \leq t''$, and $T_e = V(T) - T^e$. Let (T, W) be a well-founded tree-decomposition, let $e \in E(T)$ and let $A^e = \bigcup_{t \in T^e} W_t$ and $A_e = \bigcup_{t \in T_e} W_t$. Then (A^e, A_e) is a separation of G and $A^e \cap A_e = W_t \cap W_{t'}$.

(15.3) *Let G be a graph and* κ *an uncountable cardinal. If G contains a minor isomorphic to* K_κ, *then G has no well-founded tree-decomposition of width* $< \kappa$ *and adhesion* $< \kappa$.

Proof. If G contains a minor isomorphic to K_κ, then G has a haven β, say of order κ, by (3.1). Suppose for a contradiction that G has a well-founded tree-decomposition (T, W) of width $< \kappa$ and adhesion $< \kappa$. We direct each edge $e = (t, t') \in E(T)$ as follows. If $\beta(W_t \cap W_{t'}) \subseteq A^e$ we direct e toward t', and if $\beta(W_t \cap W_{t'}) \subseteq A_e$ we direct e toward t. This assignment is a confluence, as is easily seen, and hence by (15.2) it arises from some trunk K, such that if K is closed then either $sup\, K$ has a predecessor or $|K| = 1$. Let X be $\lim_K W_t$ if K does not have a supremum and W_k if $k = sup\, K$. Choose $\kappa' < \kappa$ as in the definition of adhesion, with the trunk K. Let κ'' be regular such that $\max(|X|, \kappa') < \kappa'' \leq \kappa$. By (3.3), $\beta(X)$ contains a κ''-major vertex v. Let $t_0 \in V(T)$ be such that $v \in W_{t_0}$. We claim that $t \not\leq t_0$ for some $t \in K$. For otherwise K has a supremum, say k, and $k \leq t_0$. Since $v \notin W_k$ it follows that $k < t_0$. Let k' be the successor of k with $k \leq k' \leq t_0$. Then $v \in W_{t_0} \subseteq A^{(k, k')}$, but (since v is κ''-major and $|W_k \cap W_{k'}| \leq |W_k| < \kappa''$)

$$v \in \beta(W_k \cap W_{k'}) \subseteq A_{(k, k')} - W_k \cap W_{k'} ,$$

a contradiction which proves our claim that $t \not\leq t_0$ for some $t \in K$.

Choose $t_1 \in K$ with $t_1 \not\leq t_0$, and let $t_2 = inf(t_0, t_1)$. Then $t_2 \in K$, and $t_2 \leq t_0$, and so $t_2 < t_1$. Let E be the set of all $(t, t') \in E(T)$ with $t, t' \in K$ and $t_2 \leq t < t'$, and let $(t, t') \in E$. Since $t \neq sup\, K$, there exists $e = (s, s') \in E$ with

$t \leq s$ such that $|W_s \cap W_{s'}| \leq \kappa''$. Then $t_0 \in T_e$, and hence $v \in W_{t_0} \subseteq A_e$. But e is directed towards t', and so $v \in A_e \cap A^e = W_s \cap W_{s'}$. In particular $v \in W_{s'}$. Since $t, t' \in T[t_0, s']$, and $v \in W_{t_0}$ and $v \in W_{s'}$, it follows that $v \in W_t$ and $v \in W_{t'}$. We have shown, then, that $v \in W_z$ for all $z \in Z$, where

$$Z = \bigcup \{\{t, t'\} : (t, t') \in E\}.$$

But $|K| \geq 2$, and so if *sup* (K) exists and belongs to K then it has a predecessor in Z, and every other element of $\{t \in K : t \geq t_0\}$ has a successor in Z. Consequently, $Z = \{t \in K : t \geq t_0\}$, and so $v \in \lim_K W_t$, a contradiction. ■

We shall need the following.

(15.4) *Let X be a topological tree, let $\{\tau_\alpha\}_{\alpha<\lambda}$ be a monotone sequence of finite subsets of V(X), and let $Z \subseteq V(X)$, such that Z is convex and $\{\alpha < \lambda : Z \cap \tau_\alpha \neq \varnothing\}$ is cofinal in λ. Then there exists $\lambda_0 < \lambda$ and $x_\alpha \in Z \cap \tau_\alpha$ for all α with $\lambda_0 \leq \alpha < \lambda$, such that $\{x_\alpha\}_{\lambda_0 \leq \alpha < \lambda}$ is monotone.*

Proof. We may assume that $Z \cap \tau_0 \neq \varnothing$; choose $z \in Z \cap \tau_0$. For each $v \in V(X)$ with $v \neq z$, we define $X^v = \{x \in V(X) : v \in X[z, x]\}$. We say that $v \in V(X) - \{z\}$ is *big* if

$$\{\alpha < \lambda : X^v \cap Z \cap \tau_\alpha \neq \varnothing\}$$

is cofinal in λ. Let $\mathcal{P}$ be the set of all pairs (v, α) such that $v \in V(X) - \{z\}$ is big, $\alpha < \lambda$ and $v \in Z \cap \tau_\alpha$.

(1) *For each $(v, \alpha) \in \mathcal{P}$ and each α' with $\alpha \leq \alpha' < \lambda$, there exists $(v', \alpha') \in \mathcal{P}$ such that $v' \in X^v$.*

For let $\Lambda = \{\beta < \lambda : X^v \cap Z \cap \tau_\beta \neq \varnothing\}$; then Λ is cofinal in λ. Let $X^v \cap Z \cap \tau_{\alpha'} = \{y_1, ..., y_k\}$ where k is finite. For $1 \leq i \leq k$ let

$$\Lambda_i = \{\beta < \lambda : X^{y_i} \cap Z \cap \tau_\beta \neq \varnothing\}.$$

We claim that $\Lambda \subseteq \alpha' \cup \Lambda_1 \cup ... \cup \Lambda_k$. For let $\beta \in \Lambda$ with $\beta \geq \alpha'$, and choose $u \in X^v \cap Z \cap \tau_\beta$; then $X[u, v] \cap \tau_{\alpha'} \neq \varnothing$, by monotonicity, and $X[u, v] \subseteq Z$ by (i), and so we may assume that $y_i \in X[u, v]$. Since $v \in X[z, u]$ and $y_i \in X[u, v]$ it follows that $y_i \in X[z, u]$, that is, $u \in X^{y_i}$. Consequently, $\beta \in \Lambda_i$. This proves our claim that $\Lambda \subseteq \alpha' \cup \Lambda_1 \cup ... \cup \Lambda_k$. Hence some Λ_i, say Λ_1, is cofinal in λ, since Λ is cofinal in λ and $\alpha' < \lambda$. Consequently y_1 is big, and so $(y_1, \alpha') \in \mathcal{P}$, and $y_1 \in X^v$. This proves (1).

But from (1), the result follows easily by transfinite induction. ■

Now we complete the proof of (15.1) by proving the following.

(15.5) *Let G be a graph and let κ be an infinite cardinal. If G has a dissection of width $< \kappa$ and adhesion $< \kappa$, then G has a well-founded tree-decomposition of width $< \kappa$ and strong adhesion $< \kappa$.*

Proof. By (13.7) G has a conservative topological tree-decomposition (X, Φ) of width $< \kappa$ and strong adhesion $< \kappa$. For $x \in V(X)$ let $\nu_x = |\Phi_x|$, and let $\nu = (\nu_x : x \in V(X))$. Then ν is a weighting of adhesion $< \kappa$. Now X is complete, since (X, Φ) is a topological tree-decomposition. By (14.11) there exists a tree-labeling (T, τ) of X of ν-adhesion $< \kappa$. For $t \in V(T)$ let $W_t = \bigcup_{x \in \tau_t} \Phi_x$, and let $W = (W_t : t \in V(T))$. We shall show that (T, W) is a well-founded tree-decomposition of G of width $< \kappa$ and strong adhesion $< \kappa$.

The first axiom is clearly satisfied. For the second let $t, t', t'' \in V(T)$ with $t' \in \overline{T[t, t'']}$, and let $v \in W_t \cap W_{t''}$. Let x, x'' be such that $x \in \tau_t, x'' \in \tau_{t''}$ and $v \in \Phi_x \cap \Phi_{x''}$. From the second tree-labeling axiom there exists $x' \in X[x, x''] \cap \tau_{t'}$. Then $v \in \Phi_x \cap \Phi_{x''} \subseteq \Phi_{x'} \subseteq W_{t'}$ as desired.

For the third axiom let $K \subseteq V(T)$ be a bounded open trunk of T, with $\sup K = k$. Let $v \in \lim_K W_t$, and let $Z = \{z \in V(X) : v \in \Phi_z\}$. Then there exists $t_0 \in K$ such that $Z \cap \tau_t \neq \varnothing$ for all $t \in K$ with $t \geq t_0$, since $v \in \lim_K W_t$. But Z is convex, from the second axiom for topological decompositions. From (15.4) applied to $\{\tau_t\}_{t \in K}$, there exists $t_1 \in K$ and $x_t \in Z \cap \tau_t$ for all $t \in K$ with $t \geq t_1$, such that $\{x_t\}_{t \in K, t \geq t_1}$ is monotone. Let x be its limit; then $x \in \tau_k$. Now $v \in \Phi_x$, from the third axiom for topological tree-decompositions, but $\Phi_x \subseteq W_k$ since $x \in \tau_k$, and so $v \in W_k$. Thus the third axiom holds.

This completes the proof that (T, W) is a well-founded tree-decomposition. To show that it has width $< \kappa$, let $K \subseteq V(T)$ be a trunk, and let $Z = \lim\{\tau_t\}_{t \in K}$. As above we deduce that $\lim_K W_t \subseteq \bigcup_{x \in Z} \Phi_x$, and hence

$$|\lim_K W_t| \leq |\bigcup_{x \in Z} \Phi_x| < \kappa,$$

because $|Z| \leq 3$ by (14.7), and $|\Phi_x| < \kappa$ for every $x \in V(X)$. Finally, the strong adhesion condition follows immediately since (T, τ) has ν-adhesion $< \kappa$. ■

REFERENCES

1. R. Diestel, "The structure of TK_a-free graphs", submitted.

2. R. Halin, "Charakterisierung der Graphen ohne unendliche Wege", *Arch. Math.* 16 (1965), 227-231.

3. R. Halin, "Über unendliche Wege in Graphen", *Math. Ann.* 157 (1964), 125-137.

4. H. A. Jung, "Zusammenzüge und Unterteilungen von Graphen", *Math. Nachr.* 35 (1967), 241-268.

5. I. Kříž and R. Thomas, "Clique-sums, tree-decompositions, and compactness", *Discrete Math.*, 81 (1990), 177-185.

6. N. Robertson and P. D. Seymour, "Graph minors. XIII. The disjoint paths problem", submitted.

7. N. Robertson, P. D. Seymour and R. Thomas, "Excluding subdivisions of infinite cliques", *Trans. Amer. Math. Soc.*, 332 (1992), 211-223.

8. P. D. Seymour and R. Thomas, "Excluding infinite trees", *Trans. Amer. Math. Soc.*, to appear.

9. P. D. Seymour and R. Thomas, "Graph searching, and a min-max theorem for tree-width", *J. Combinatorial Theory, Ser. B*, to appear.

10. R. Thomas, "A counterexample to "Wagner's conjecture" for infinite graphs", *Math. Proc. Camb. Phil. Soc.* 103 (1988), 55-57.