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CLIQUE-SUMS, TREE-DECOMPOSITIONS AND COMPACTNESS

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We develop a technique for extending excluded minor theorems to infinite graphs, and in particular we answer a question of Neil Robertson.

1. Introduction

The aim of this paper is to extend certain results of finite graph theory to infinite graphs and to show a limitation of this.

Recall that a graph G is a *minor* of a graph H if G can be obtained from a subgraph of H by contraction of edges. There are several so-called excluded minor theorems in finite graph theory, i.e. statements describing finite graphs without minors isomorphic to members of a given list of finite graphs. The celebrated Kuratowski's theorem [4] is also of this form: Finite graphs without minors isomorphic to $K_{3,3}$ and K_5 are planar.

Note that this theorem is *exact* in the sense that it in fact characterizes the class of graphs with no $K_{3,3}$ and K_5 minor. Some other excluded minor theorems are listed in *Table 1* below; the first six of them are exact. (For the definition of k-sums see Section 2.4.)

These statements have a common feature: The describing structure involves clique-sums of graphs from a certain basic class Γ. We develop a general technical tool to treat this situation, the concept of a k-decomposition over a class Γ (for definition see 2.3). This concept covers the notion of tree-width introduced by Robertson and Seymour [6] in the sense that graphs of tree-width $\leqslant k$ are exactly those admitting a k-decomposition over

$$\Gamma_k = \{(V, E) \mid |V| \leqslant k + 1\}.$$

The first idea of such a kind was probably due to Wagner [13].

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Table 1.

Excluded minor(s)	Structure
K_3	1-sums of graphs with $\leqslant 2$ vertices (graphs of tree-width $\leqslant 1$)
K_4	2-sums of graphs with $\leqslant 3$ vertices (graphs of tree-width $\leqslant 2$)
K_5	3-sums of planar graphs and the four rung Möbius ladder [12]
$K_{3,3}$	2-sums of planar graphs and K_5 [13]
K_5 minus one edge	2-sums of wheels, $K_{3,3}$, the prism and graphs with $\leqslant 4$ vertices (see [11])
All finite 4-connected graphs	2-sums of the five-sided prism, the four rung Möbius ladder and graphs with $\leqslant 4$ vertices [1]
Arbitrary H finite planar	k-sums of graphs with $\leqslant k+1$ vertices (graphs of tree-width $\leqslant k$) for some k depending on H [6]
Arbitrary H finite	k-sums of graphs which are (p, q, h)-representable over some surface Σ in which H cannot be embedded (for some k, p, q, h depending on H) [7]

Let us briefly recall the definition of tree-width and the basic results. We start with a

Theorem 1.1. *Let G be a finite graph and let $k > 0$. Then the following conditions are equivalent*:

(i) *k is the least integer such that G can be constructed by repeated clique-sums, starting from graphs in Γ_k.*

(ii) *k is the least integer such that G admits a k-decomposition over Γ_k.*

(iii) *k is the least integer such that G is a subgraph of a chordal graph not containing K_{k+2}.*

Definition 1.2. The number k from the above theorem is called the *tree-width* of G.

It can be shown, for instance, that the graphs of tree-width $\leqslant 1$ are precisely the forests, and the graphs of tree-width $\leqslant 2$ are the series-parallel graphs. The complete graph K_n has tree-width $n-1$ and the $n \times n$ grid (adjacency graph of the $n \times n$ chessboard) has tree-width n. For tree-width the following compactness theorem holds.

Theorem 1.3 [9]. *If every finite subgraph of G has tree-width $\leqslant k$, then G has tree-width $\leqslant k$; in other words, G admits a k-decomposition over Γ_k iff each finite subgraph of G does.*

A simpler proof of 1.3 can be found in [10] or [3]. In this paper we prove a generalization of Theorems 1.1, 1.3 with Γ_k replaced by an arbitrary class Γ

satisfying certain natural assumptions (see 2.2, 3.5, 3.9). As an immediate consequence we obtain infinite versions of the theorems of Table 1.

Our results apply only to the case of k-sums for fixed k. In Section 4 we present a counterexample showing that the technique cannot be extended to the case of unbounded clique-sums, that is, Proposition 3.4 does not hold for $|V(K_1) \cap V(K_2)| \leq k$ replaced by $|V(K_1) \cap V(K_2)| < \omega$. The counterexample presented also shows that 1.1 fails for $k = \omega$. More exactly, we construct a graph G such that $k = \omega$ satisfies 1.1(iii) while the k satisfying 1.1(ii) is arbitrarily large.

2. Definitions

2.1. A *graph* may be infinite, and may have multiple edges but may not have loops. A *clique* in a graph is a complete subgraph. If M is a set, then $K(M)$ denotes the complete graph on M. A graph is chordal if every cycle with at least four vertices has a chord.

If G is a graph and $A \subseteq V(G)$, then $G \upharpoonright A$ means the graph induced by A in G. A *cut* in G is a set $C \subseteq V(G)$ such that G becomes disconnected after deleting vertices from C. A *k-cut* in G is a cut C such that $|C| \leq k$ and $G \upharpoonright C$ is complete. If $G_\alpha = (V_\alpha, E_\alpha)$ are graphs, then $\bigcap G_\alpha$, $\bigcup G_\alpha$ are the graphs $(\bigcap V_\alpha, \bigcap E_\alpha)$ and $(\bigcup V_\alpha, \bigcup E_\alpha)$, respectively.

2.2. Let Γ be a class of graphs with the following properties.

($\Gamma 1$) If G is a graph and $e_1, e_2 \in E(G)$ are two distinct edges with the same endpoints and if $G \setminus e_1 \in \Gamma$ then $G \in \Gamma$.

($\Gamma 2$) If $G \in \Gamma$ and H is a k-subsimplex of G, then $H \in \Gamma$.

($\Gamma 3$) If G is such that for every finite subgraph H of G there exists a finite subgraph $H' \in \Gamma$ of G containing H, then $G \in \Gamma$.

2.3. Let G be a graph. A *tree-decomposition* of G is a pair (T, X), where T is a tree and $X = (X^t : t \in V(T))$ is such that

($T1$) $\bigcup_{t \in V(T)} X^t = V(G)$,

($T2$) every edge e of G has both endpoints in some X^t,

($T3$) if $t, t', t'' \in V(T)$ and t' is on the path between t' and t'', then $X^t \cap X^{t''} \subseteq X^{t'}$.

We say that a tree-decomposition (T, X) is a *k-decomposition of G over Γ*, if

($T4$) $|X^t \cap X^{t'}| \leq k$ for every $\{t, t'\} \in E(T)$,

($T5$) $G^* \upharpoonright X^t \in \Gamma$ for every $t \in V(T)$,

where G^* is the graph obtained from G by joining every edge which has both its endpoints in some $X^t \cap X^{t'}$ for $\{t, t'\} \in E(T)$.

2.4. Let G_1, G_2 be two graphs such that $G_2 \upharpoonright (V(G_1) \cap V(G_2))$ is edge-less and let H_i be the graph obtained from G_i by adding new edges joining every pair of vertices in $V(G_1) \cap V(G_2)$. We say $G_1 \cup G_2$ is the *clique-sum* of H_1 and H_2 and if

$|V(G_1) \cap V(G_2)| \leqslant k$ we also say it is the *k-sum* of H_1 and H_2. We say that a graph G is *k-summable over Γ* if there exists a transfinite sequence $\{G_\alpha\}_{\alpha \leqslant \lambda}$ of graphs such that $G_0 \in \Gamma$, $G_\lambda = G$, $G_{\alpha+1}$ is a k-sum of G_α and some graph from Γ and for α a limit ordinal

(1) $V(G_\alpha) = \bigcup_{\beta<\alpha} V(G_\beta)$

(2) $E(G_\alpha) = \bigcup_{\beta<\alpha} \bigcap_{\gamma \geqslant \beta} E(G_\gamma)$.

The least λ with this property is called the *rank* of G. Relations (1) and (2) will be abbreviated $G_\alpha = \liminf_{\beta<\alpha} G_\beta$.

2.5. A graph G is called a *k-simplex* if it does not contain a k-cut. If G is a graph, then each induced subgraph H of G, which is a k-simplex, is called a *k-subsimplex*. A graph G is a *k-complex over Γ* if every finite k-subsimplex belongs to Γ.

3. The main results

Lemma 3.1. *Let G be a graph and S_1, S_2 two distinct maximal k-subsimplices. Then $S_1 \cap S_2$ is a clique of size at most k.*

Proof. Suppose not. Let K be a clique in G of size $\leqslant k$. Then $(S_1 \cap S_2) \setminus V(K) \neq \emptyset$ and both $S_1 \setminus V(K)$, $S_2 \setminus V(K)$ are connected. Hence $(S_1 \cup S_2) \setminus V(K)$ is connected and thus $S_1 \cup S_2$ is a k-simplex, contrary to the maximality of S_1 and S_2. □

Proposition 3.2. *Let G be a k-simplex and G_0 a finite subgraph. Then there exists a finite k-subsimplex of G, which contains G_0.*

Proof. Assume we have already constructed a finite induced subgraph G_n of G such that G_n contains G_0 and each of its k-cuts has size at least n. Let $C_1, \ldots, C_s$ be its k-cuts. Since G is a k-simplex, there is an induced finite subgraph of G, which contains G_n and is such that for no i, C_i is a cut of it. Let G_{n+1} denote a minimal such subgraph. If C is a k-cut of G_{n+1}, it follows by minimality that $C \cap V(G_n)$ is a k-cut of G_n and thus $C \supsetneqq C_i$ for some $i = 1, \ldots, s$. Hence every k-cut of G_{n+1} is of size at least $n+1$. Now G_{k+1} is the desired graph. □

Corollary 3.3. *If G is a k-complex over Γ, then every k-subsimplex belongs to Γ.*

Proof. Let S be a k-subsimplex of G and S' a finite subgraph of S. By Proposition 3.2 there is a finite k-subsimplex S'' of G which contains S'. Then $S'' \in \Gamma$, and hence $S \in \Gamma$ by (Γ3). □

Proposition 3.4. *Let G be a chordal graph such that $|V(K_1) \cap V(K_2)| \leq k$ for any two distinct maximal cliques K_1, K_2 of G. Then G admits a tree-decomposition (T, X) such that $G \restriction X^t$ is a maximal clique in G for any $t \in V(T)$.*

Proof. We just sketch the proof, and for the remaining details we refer to [3]. Consider the complete edge-valued graph $\Delta(G)$ on the set of all maximal cliques of G with the value of edge $\{K_1, K_2\}$ equal to $|V(K_1) \cap V(K_2)|$. Construct a "maximal" skeleton T of $\Delta(G)$ by adding at each step an edge of the maximal possible value, not to obtain a cycle. Put $X = (t \mid t \in V(T))$. Then (T, X) is the desired tree-decomposition. □

Theorem 3.5. *The following are equivalent for every integer $k \geq 2$:*
(i) *G is k-summable over Γ.*
(ii) *G is a subgraph of a k-complex over Γ.*
(iii) *G admits a k-decomposition over Γ.*

Proof. (i)→(ii). We proceed by induction on rank of G. First we show that if $G_{\alpha+1}$ is a k-sum of G_α and $G \in \Gamma$ and H_α, H are k-complexes over Γ containing G_α, G, respectively, then there exists a k-complex $H_{\alpha+1}$ over Γ such that $H_{\alpha+1}$ contains $G_{\alpha+1}$ and $H_{\alpha+1} \restriction V(G_\alpha) = H_\alpha$. Indeed, it is sufficient to put $H_{\alpha+1}$ to be the graph obtained from $H_\alpha \cup H$ by deleting those edges from H which join vertices in $V(H_\alpha) \cap V(H)$. Now if α is a limit ordinal and $G_\alpha = \liminf_{\beta<\alpha} G_\beta$, let H_β be k-complexes over Γ defined as above and put $H_\alpha = \bigcup_{\beta<\alpha} H_\beta$. It is easily seen that H_α is a k-complex over Γ which contains G_α.

(ii)→(iii). Let H be a k-complex over Γ which contains G. We put

$$H^* := \bigcup \{K(V(S)) \mid S \text{ is a } k\text{-subsimplex of } H\}.$$

Since every chordless cycle of H of length at least four is a k-subsimplex of H, we see that H^* is chordal. We claim

(∗) If K^* is a maximal clique in H^*, then $K = H \restriction V(K^*)$ is a k-subsimplex of H.

Choose a minimal system $\{S_\alpha\}$ of k-subsimplices of H with the the property that K^* is a subgraph of $K \cup \bigcup_\alpha K(V(S_\alpha))$. We shall show that $S := \bigcup_\alpha S_\alpha \cup K$ is a k-subsimplex of H. Let V be a clique in S of size $\leq k$. We shall show that $S \setminus V$ is connected. Let first $u, v \in K \setminus V$. If u, v are not joined by an edge in S, then, since they are joined in K^*, there exists an α such that $u, v \in V(S_\alpha)$. Hence they are joined by a path in $S_\alpha \setminus V$.

Now let $u \in V(S_\beta) \setminus V$. Since $\{S_\alpha\}$ is assumed minimal, it follows that $K \restriction V(S_\beta)$ is not a clique. Hence $(V(S_\beta) \cap V(K)) \setminus V \neq \emptyset$. Let $v \in (V(S_\beta) \cap V(K)) \setminus V$; now u, v are joined by a path in $S_\beta \setminus V$. This shows that $S \setminus V$ is connected, and hence S is a k-simplex. Now the maximality of K^* implies that $K = S$, proving (∗).

If K_1^*, K_2^* are distinct maximal cliques in H^*, then $K_1 = H \restriction V(K_1^*)$, $K_2 =$

$H \restriction V(K_2^*)$ are maximal k-subsimplices of H by (*), and hence $|V(K_1^*) \cap V(K_2^*)| = |V(K_1) \cap V(K_2)| \leq k$ by Lemma 3.1. Thus, the assumptions of Proposition 3.4 are satisfied.

Let (T, X) be the tree-decomposition of H^* from Proposition 3.4. If $\{t, t'\} \in E(T)$, then $H \restriction (X^t \cap X^{t'})$ is the intersection of two maximal k-subsimplices of H by (*). Hence by Lemma 3.1 $|X^t \cap X^{t'}| \leq k$ and $H \restriction (X^t \cap X^{t'})$ is a clique. From this and Corollary 3.3 it follows that if G^* is as in $(T5)$ then $G^* \restriction X^t$ is a subgraph of $H \restriction X^t \in \Gamma$. Hence (T, X) can be converted to a k-decomposition over Γ of G by adding a leaf $r(t, e)$ with $X^{r(t,e)} = e$ for every $e \in E(H \restriction X^t) \setminus E(G^*)$.

(iii)$\to$(i). Let (T, X) be a k-decomposition over Γ of G. There are subtrees $\{T_\alpha\}_{\alpha \leq \lambda}$ of T such that $T_0 = (\{t_0\}, \emptyset)$, the one-point tree, $T_\lambda = T$, $T_{\alpha+1}$ is obtained from T_α by joining a vertex $t_{\alpha+1}$ of degree one and $T_\alpha = \bigcup_{\beta<\alpha} T_\beta$ for α a limit ordinal. Let G^* be the graph obtained from G by joining all edges

$$\{e^{\{u,v\}}_{\{t,t'\}} \mid \{t, t'\} \in E(T),\ u, v \in X^t \cap X^{t'},\ u \neq v\},$$

where $e^{\{u,v\}}_{\{t,t'\}}$ has u, v as its endpoints. Let $t'_{\alpha+1}$ be that vertex of $T_{\alpha+1}$ to which $t_{\alpha+1}$ is adjacent. Put

$$G_\alpha := (G^* \restriction \bigcup \{X^t \mid t \in V(T_\alpha)\}) \setminus \{e^{\{u,v\}}_{\{t,t'\}} \mid \{t, t'\} \in E(T_\alpha),\ u, v \in X^t \cap X^{t'},\ u \neq v\}$$

$$G'_\alpha := ((G^* \restriction X^{t_{\alpha+1}}) \setminus E(G_\alpha)) \cup \{e^{\{u,v\}}_{\{t_{\alpha+1}, t'_{\alpha+1}\}} \mid u, v \in X^{t_{\alpha+1}} \cap X^{t'_{\alpha+1}},\ u \neq v\}.$$

It is easily seen that $G_0 \in \Gamma$, $G_\lambda = G$, $G_{\alpha+1}$ is the k-sum of G_α and $G'_\alpha \in \Gamma$ and that for a limit ordinal α $G_\alpha = \liminf_{\beta<\alpha} G_\beta$. Hence G is k-summable over Γ. □

Corollary 3.6. *Let Γ satisfy the following condition:*

Let G be a graph and $e \in E(G)$. If $G \setminus e \in \Gamma$ then $G \in \Gamma$. Then (i), (ii), (iii) *of the preceding theorem are equivalent to:*

(iv) *G is a subgraph of a chordal graph H such that every clique of H belongs to Γ and any two distinct maximal cliques of H have at most k vertices in common.*

Remark 3.7. The method of proof (ii)$\to$(iii) in Theorem 3.5 gives another existence theorem for prime graph decompositions of infinite graphs. To make this statement precise let us call a graph G *prime* if there is no complete subgraph of G which is a cut. A *prime graph decomposition* is a tree-decomposition (T, X) of G such that $G \restriction X^t$ is prime for any $t \in V(T)$ and $G \restriction (X^t \cap X^{t'})$ is complete for any $\{t, t'\} \in E(T)$. An infinite graph need not have a prime decomposition, but every graph without infinite cliques does (see [2]). Thus, the following corollary is another theorem of this kind.

Corollary 3.8. *Let H be a graph and k an integer such that*

$$|V(G_1) \cap V(G_2)| \leq k$$

for any two distinct maximal prime induced subgraphs G_1, G_2 of H. Then H admits a prime graph decomposition.

Proof. We use the proof of (ii)→(iii) from 3.5 with "k-simplex" replaced by "prime" and "k-subsimplex" replaced by "induced prime subgraph", we also drop the restriction on the size of V and instead of Lemma 3.1 we use our assumption. Then the tree-decomposition (T, X) of H thus produced is as desired, because for $t \in V(T)$, $H \upharpoonright X^t$ is maximal prime by (*) and for $\{t, t'\} \in E(T)$, $H \upharpoonright (X^t \cap X^{t'})$ is complete, being the intersection of two maximal prime graphs. □

Theorem 3.9. *Let G be such that every finite subgraph of G is a subgraph of a k-complex over Γ. Then G is a subgraph of a k-complex over Γ.*

Proof. We shall prove the theorem for graphs without multiple edges for simplicity. The general case is then trivial.

For each $\alpha \in \binom{V(G)}{2}$ we introduce a logical variable A_α. Consider the system of formulas

(1) A_α	for $\alpha \in E(G)$
(2) $\bigvee_{\alpha \in E} \neg A_\alpha \vee \bigvee_{\alpha \in \binom{V}{2} \setminus E} A_\alpha$	for every finite graph (V, E) which is a k-simplex not belonging to Γ such that $V \subseteq V(G)$, $E \subseteq \binom{V}{2}$.

A valuation $v : \{A_\alpha \mid \alpha \in \binom{V(G)}{2}\} \to 2$ satisfies (1), (2) iff the graph with vertex set $V(G)$ and edges $\{\alpha \mid A_\alpha[v] = 1\}$ is a k-complex over Γ which contains G. By assumption, each finite subsystem of (1), (2) can be fulfilled. Use Logical Compactness. □

3.10. Now we are able to extend the results of *Table 1* to infinite graphs. The last statement answers a question of Robertson [8] as to whether the excluded minor theorem from [7] can be extended to the infinite case. As in the finite case, the first six results are exact.

Table 2.

Excluded minor(s)	Structure
K_3	tree-width ≤1
K_4	tree-width ≤2
K_5	graphs 3-summable over the four rung Möbius ladder and graphs with every finite subgraph planar
$K_{3,3}$	2-summable over K_5 and graphs with every finite subgraph planar
K_5 minus one edge	2-summable over wheels, $K_{3,3}$, the prism and graphs with ≤4 vertices
All finite 4-connected graphs	2-summable over the five-sided prism, the four rung Möbius ladder and graphs with ≤4 vertices
Arbitrary finite planar H	bounded tree-width
Arbtirary finite H	k-summable over graphs with each finite subgraph (p, q, h)-representable over some Σ H cannot be drawn on.

4. An example

4.1. Theorem 1.1 does not hold for infinite k. In fact for any cardinal κ there exists a chordal graph G with each clique at most countable such that every tree-decomposition (T, X) contains an X^t of cardinality $\geqslant\kappa$.

Let the vertices of G be all sequences of elements of κ of length $\leqslant\omega$ including the empty sequence and let $\{u, v\} \in E(G)$ if u is a proper initial segment of v. Clearly G is a chordal graph in which all cliques are at most countable. We prove the following

Theorem 4.2. *Any tree-decomposition (T, X) of G contains an X^t of cardinality $\geqslant\kappa$.*

Proof. Suppose the contrary. Sequences of ordinals will be indicated by juxtaposition. We shall construct finite sequences $\lambda_0, \ldots, \lambda_n, \ldots \in V(G)$ and distinct vertices $t_0, \ldots, t_n, \ldots \in V(T)$ such that for all n

(i) there exist ordinals $\alpha_0, \ldots, \alpha_n, \ldots < \kappa$ such that $\lambda_n = \alpha_0, \alpha_1 \cdots \alpha_{n-1}$,

(ii) X^{t_n} contains $\lambda_0, \ldots, \lambda_n$ but does not contain any $\lambda \in V(G)$ with λ_{n+1} as an initial segment,

(iii) if t is on the path between t_n and t_{n+1} and $X^t \supseteq \{\lambda_0, \ldots, \lambda_{n+1}\}$, then $t = t_{n+1}$.

Let λ_0 be the empty sequence and choose t_0 such that $\lambda_0 \in X^{t_0}$. Now assume we have already constructed $\lambda_0, \ldots, \lambda_n, t_0, \ldots, t_n$. Since $|X^{t_n}| < \kappa$, there exists an ordinal $\alpha_n \in \kappa$ such that $\lambda \notin X^{t_n}$ whenever $\lambda_{n+1} = \lambda_n\alpha_n$ is the initial segment of λ. Now choose $t_{n+1} \in V(T)$ such that $\lambda_0, \ldots, \lambda_n, \lambda_{n+1} \in X^{t_{n+1}}$ (this is possible since $G \restriction \{\lambda_0, \ldots, \lambda_{n+1}\}$ is a complete graph) and such that the distance of t_{n+1} from t_n in T is the least possible. Obviously this choice of t_{n+1} implies (iii). This completes the construction.

We claim that for any $n \in \omega$, t_{n+1} lies on the path between t_n and t_{n+2}. Should this not be the case, we have the following possibilities:

(a) t_n is on the path between t_{n+1} and t_{n+2}. Then

$$\lambda_{n+1} \in X^{t_{n+1}} \cap X^{t_{n+2}} \subseteq X^{t_n}$$

by $(T3)$, a contradiction.

(b) t_n is not on the path between t_{n+1} and t_{n+2}. Then denote by t that vertex of T, which belongs to the path between t_{n+1} and t_{n+2} and whose distance from t_n is the least possible. By the assumptions, $t \notin \{t_n, t_{n+1}\}$ and

$$\{\lambda_0, \ldots, \lambda_{n+1}\} \subseteq X^{t_{n+1}} \cap X^{t_{n+2}} \subseteq X^t,$$

by $(T3)$, a contradiction to (iii) (realize that t lies on the path between t_n, t_{n+1}).

It follows from the claim that there is an infinite path $s_0, s_1, \ldots$ in T containing $t_0, t_1, \ldots$ in this order. Let, say $s_{i_n} = t_n$. Then $i_1 < i_2 < \cdots$. Let $\lambda \in V(G)$ be the infinite sequence with initial segments λ_n. Since $G \restriction \{\lambda_0, \ldots, \lambda_n, \lambda\}$ is a complete graph, we have for each n a vertex $u_n \in V(T)$ such that

$$\lambda_0, \ldots, \lambda_n, \lambda \in X^{u_n}.$$

Let j_n be such that the distance between u_n and s_{j_n} is the least possible. Now we distinguish two cases.

If $j_{n+1} > i_n$ for any n, take n such that $i_n > j_1$. Then we have $j_{n+1} > i_n > j_1$ and hence s_{i_n} lies on the path between u_n, u_{n+1}. By $(T3)$ we conclude

$$\lambda \in X^{u_1} \cap X^{u_{n+1}} \subseteq X^{s_{j_n}} \cap X^{s_{j_{n+1}}} \subseteq X^{s_{i_n}} = X^{t_n},$$

contradicting (ii).

If $j_{n+1} \leqslant i_n$ for some n, then by $(T3)$

$$\lambda_{n+1} \in X^{u_{n+1}} \cap X^{t_{n+1}} = X^{s_{j_{n+1}}} \cap X^{S_{i_{n+1}}} \subseteq X^{S_{i_n}} = X^{t_n},$$

again contradicting (ii). □

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