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A counter-example to 'Wagner's conjecture' for infinite graphs

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1. *Introduction*

Wagner made the conjecture that given an infinite sequence $G_1, G_2, \ldots$ of finite graphs there are indices $i < j$ such that G_i is a minor of G_j. (A graph is a minor of another if the first can be obtained by contraction from a subgraph of the second.) The importance of this conjecture is that it yields excluded minor theorems in graph theory, where by an excluded minor theorem we mean a result asserting that a graph possesses a specified property if and only if none of its minors belongs to a finite list of 'forbidden minors'. A widely known example of an excluded minor theorem is Kuratowski's famous theorem on planar graphs; one of its formulations says that a graph is planar if and only if it has neither K_5 nor $K_{3,3}$ as a minor. But several other excluded minor theorems have been discovered by now (see e.g. [**7**–**9**]).

The connection with Wagner's conjecture is as follows. Let P be any property of finite graphs which is closed under taking minors; that is, any minor of a graph with property P also has property P. Let L' be the set of all minor–minimal graphs not possessing property P and let $L \subseteq L'$ contain exactly one representative of each isomorphism class of L'. Then a graph has property P if and only if no graph from L occurs as a minor of it. But Wagner's conjecture implies that L is finite, since no member of L is a minor of any other, hence P can be characterized by an excluded minor theorem.

In a series of papers [**4**] Robertson and Seymour are publishing the proof of Wagner's conjecture. It is also of interest to ask about Wagner's conjecture for infinite graphs. Very little was known in this area. Nash-Williams [**3**] proved Wagner's conjecture for infinite trees, the author [**6**] proved it for any set of infinite graphs which do not have a fixed finite planar graph as a minor, and Galvin [**1**] disproved it for the class of order-theoretical trees of height $\omega+1$, but these are not graphs. In this paper we show using Galvin's idea that in general Wagner's conjecture fails for infinite graphs. It may be a surprise for those who believed that results about Wagner's conjecture for finite graphs could be transferred to infinite ones by appropriate strengthening of the methods involved.

I would like to express my thanks to David Preiss for suggesting the short proof of Theorem 2 and to Jaroslav Nešetřil for his kind help.

2. *Notation*

ω denotes the set of natural numbers, c the cardinality of the continuum.

A graph G is a pair $(V(G), E(G))$, where $V(G)$, the set of vertices, is an arbitrary set and $E(G)$, the set of edges, consists of some two-element subsets of $V(G)$. If G, H are graphs, then a surjective mapping $f: V(H) \to V(G)$ is called a contraction of H onto G provided

(i) $f^{-1}(v)$ induces a connected subgraph in H for every $v \in V(G)$,

(ii) if $\{u, v\} \in E(G)$, then there are $u' \in f^{-1}(u)$, $v' \in f^{-1}(v)$ such that $\{u', v'\} \in E(H)$, and

(iii) if $\{u, v\} \in E(H)$, then $\{f(u), f(v)\} \in E(G)$.

A graph G is a minor of a graph H if there is a subgraph H' of H and a contraction of H' onto G.

For $\alpha \in \omega+1$ we put $2^\alpha = \{x \mid x : \alpha \to 2\}$, $2^{<\alpha} = \bigcup_{\beta<\alpha} 2^\beta$.

If $x \in 2^\alpha$ and $\beta \leqslant \alpha$, then $x \restriction \beta \in 2^\beta$ denotes the restriction of x to β. For $x_0 \in 2^\omega$ and $n \in \omega$ we write

$$\mathscr{U}_n(x_0) = \{x \in 2^\omega \mid x_0 \restriction n = x \restriction n\},$$

$$\mathscr{V}_n(x_0) = \{x \in \bigcup_{n \leqslant \alpha \leqslant \omega} 2^\alpha \mid x_0 \restriction n = x \restriction n\}.$$

We consider 2^ω as a topological space endowed with the product topology, i.e. the topology whose basic open sets are the $\mathscr{U}_n(x)$. For $X \subseteq 2^\omega$ we define the graph G_X by

$$V(G_X) = \{x \restriction \alpha \mid x \in X, \alpha \in \omega+1\}$$

$$E(G_X) = \{\{x, x \restriction n\} \mid x \in X, n \in \omega\} \cup \{\{x \restriction n, x \restriction n+1\} \mid x \in X, n \in \omega\}.$$

Then for $x \in X$, $\mathscr{V}_n(x) \cap V(G_X)$ spans a connected subgraph in G_X, while $\mathscr{U}_n(x) \cap V(G_X)$ spans a discrete one. Finally, if X, $Y \subseteq 2^\omega$ we say that X almost embeds in Y if there exists a countable set S and a continuous 1–1 mapping $X \backslash S \to Y$.

3. *Main result*

The following lemma follows easily from the definition of G_{2^ω}.

LEMMA. *Let $n \in \omega$, $x \in 2^\omega$ and let G be a connected subgraph of G_{2^ω} with $x \in V(G)$. Then either $V(G) \cap 2^{<n} \neq \varnothing$, or $V(G) \subseteq \mathscr{V}_n(x)$.*

THEOREM 1. *Let G_X be a minor of G_Y. Then X almost embeds in Y.*

Proof. Assume there is a subgraph G of G_Y and a contraction $f: V(G) \to V(G_X)$. By the finiteness of $2^{<n}$ there exists for any integer n an integer a_n such that

$$f^{-1}(x \restriction a) \cap 2^{<n} = \varnothing$$

for any integer $a \geqslant a_n$. Since $2^{<\omega}$ is countable there exists an at most countable set $S \subseteq X$ such that $f^{-1}(x) \subseteq Y$ for $x \in X \backslash S$. Since the graph induced by Y in G_Y is discrete, it follows that for $x \in X \backslash S$, $f^{-1}(x)$ consists of one element only. Let us put, for $x \in X \backslash S$, $f^{-1}(x) = \{\phi(x)\}$. Then ϕ is a 1–1 mapping $X \backslash S \to Y$.

Our aim is to show that ϕ is continuous. So let $x_0 \in X \backslash S$ and let the basic open neighbourhood $\mathscr{U}_n(\phi(x_0))$ of $\phi(x_0)$ be given. Since $\mathscr{V}_{a_n}(x_0) \cap (V(G_X) \backslash S)$ spans a connected subgraph in G_X, $f^{-1}(\mathscr{V}_{a_n}(x_0) \cap (V(G_X) \backslash S))$ spans a connected subgraph in G_Y, which contains $\phi(x_0)$. By the Lemma and the choice of a_n,

$$f^{-1}(\mathscr{V}_{a_n}(x_0) \cap (V(G_X) \backslash S)) \subseteq \mathscr{V}_n(\phi(x_0)),$$

giving $\phi(\mathscr{U}_{a_n}(x_0) \cap (X \backslash S)) \subseteq \mathscr{U}_n(\phi(x_0))$, which proves the continuity of ϕ and hence the Theorem. ∎

THEOREM 2. *There exists a sequence $X_1, X_2, \ldots$ of subsets of 2^ω such that for $i < j$ X_i does not almost embed in X_j.*

Proof. Put $X_1 = 2^\omega$ and assume we have already constructed $X_1, \ldots, X_n \subseteq 2^\omega$, each of cardinality c and such that X_i does not almost embed in X_j for $1 \leqslant i < j \leqslant n$. Since there are c countable subsets of 2^ω and every subset of 2^ω admits at most c continuous images, we may (assuming the Axiom of Choice) let M_α $(\alpha \in c)$ be a well-ordering of all subsets of 2^ω which are continuous 1–1 images of some set $X_i \backslash S$ $(i = 1, \ldots, n; S$ countable). For $\alpha \in c$ we take $a_\alpha, x_\alpha \in M_\alpha \backslash \bigcup_{\beta < \alpha} \{a_\beta, x_\beta\}$, $a_\alpha \neq x_\alpha$ and let $X_{n+1} = \{x_\alpha \mid \alpha \in c\}$.

MAIN THEOREM. *There exists a sequence $G_1, G_2, \ldots$ of infinite graphs such that for $i < j$ G_i is not a minor of G_j.*

Proof. Let us take the sequence $X_1, X_2, \ldots$ of subsets of 2^ω from Theorem 2. By Theorem 1, the sequence

$$G_{X_1}, G_{X_2}, \ldots$$

is as desired. ∎

4. *Concluding remarks*

Petr Simon [5], using ideas of [2], has shown that there exists a sequence $Y_1, Y_2, \ldots$ of mutually incomparable (with respect to the almost-embedding relation) subsets of 2^ω. Then for the corresponding sequence of graphs $G_{Y_1}, G_{Y_2}, \ldots$ the conclusion of the Main Theorem holds for all $i \neq j$.

The positive results on Wagner's conjecture for infinite graphs were mentioned in the Introduction. There still remains the open question whether Wagner's conjecture holds for countable graphs.

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