CLIQUE MINORS IN GRAPHS AND THEIR COMPLEMENTS

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ABSTRACT

A graph H is a *minor* of a graph G if H can be obtained from a subgraph of G by contracting edges. Let $t \geq 1$ be an integer, and let G be a graph on n vertices with no minor isomorphic to K_{t+1}. Kostochka conjectures that there exists a constant $c = c(k)$ independent of G such that the complement of G has a minor isomorphic to K_s, where $s = \lceil \frac{1}{2}(1 + 1/t)n - c \rceil$. We prove that Kostochka's conjecture is equivalent to the conjecture of Duchet and Meyniel that every graph with no minor isomorphic to K_{t+1} has an independent set of size at least n/t. We deduce that Kostochka's conjecture holds for all integers $t \leq 5$, and that a weaker form with s replaced by $s' = \lceil \frac{1}{2}(1 + 1/(2t))n - c \rceil$ holds for all integers $t \geq 1$.

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1. INTRODUCTION

The following is a well-known conjecture of Hadwiger [2].

(1.1) Conjecture. *For every integer $t \geq 1$, if a graph G has no minor isomorphic to K_{t+1}, then it is t-colorable.*

All graphs in this note are simple (that is, they have no loops or parallel edges) and finite. By considering the largest color class Hadwiger's conjecture implies a conjecture of Duchet and Meyniel [1], the following.

(1.2) Conjecture. *For every integer $t \geq 1$, if a graph G has no minor isomorphic to K_{t+1}, then it has an independent set of size at least $|V(G)|/t$.*

Hadwiger's conjecture is known to be true for all $t \leq 5$ [6], and is open for all $t \geq 6$. The same holds for conjecture (1.2). However, Duchet and Meyniel [1] proved the following.

(1.3) Theorem. *For every integer $t \geq 1$, if a graph G has no minor isomorphic to K_{t+1}, then it has an independent set of size at least $|V(G)|/(2t)$.*

Theorem (1.3) was generalized by Reed and Seymour [5], who showed that every graph with no minor isomorphic to K_{t+1} has fractional chromatic number at most $2t$. Kostochka [4] made the following conjecture, and proved that it implies (1.2).

(1.4) Conjecture. *For every integer $t \geq 1$ there exists a constant $c = c(t)$ such that if a graph G has no minor isomorphic to K_{t+1}, then the complement of G has a minor isomorphic to K_s, where $s = \lceil \frac{1}{2}(1 + 1/t)|V(G)| - c \rceil$.*

Let us see now that (1.4) implies (1.2). Let an integer $t \geq 1$ be given, let c and s be as in (1.4) and let H be a graph with no K_{t+1} minor. Let k be an integer such that $\frac{1}{t}|V(H)| - \frac{2c}{k} > \lceil \frac{1}{t}|V(H)| \rceil - 1$ and let G be the graph consisting of k disjoint copies of H. By (1.4) applied to G there are disjoint subsets $X_1, X_2, \ldots, X_s \subseteq V(G)$ such that for all indices i, j with $1 \leq i < j \leq s$ there are vertices $u \in X_i$ and $v \in X_j$ that are not adjacent in G. We may assume that $|X_1| = |X_2| = \cdots = |X_{s'}| = 1$ and $|X_i| \geq 2$ for all $i = s'+1, s'+2, \ldots, s$. But $s' + 2(s - s') \leq |V(G)|$, and hence $s' \geq \frac{1}{t}|V(G)| - 2c$. Since $X_1 \cup X_2 \cup \cdots \cup X_{s'}$ is an independent set in G, we have thus shown that G has an independent set I of size at least $\frac{1}{t}|V(G)| - 2c$. Now the intersection of I with some component H_0 of G has size at least $\frac{1}{k}\left(\frac{1}{t}|V(G)| - 2c\right) > \lceil \frac{1}{t}|V(H)| \rceil - 1$. Thus H_0, and hence H, has an independent set of size at least $\frac{1}{t}|V(H)|$, as desired. This completes the proof that (1.4) implies (1.2).

Stiebitz [7] showed that (1.4) holds for all $t \leq 3$. Our objective is to show that (1.2) holds for an integer t if and only if (1.4) holds for t. Thus, since (1.2) holds for all $t \leq 5$, it follows that (1.4) holds for all $t \leq 5$. Moreover, our proof implies the following weaker version of (1.4).

(1.5) Theorem. *For every integer $t \geq 1$ there exists a constant $c = c(t)$ such that if a graph G has no minor isomorphic to K_{t+1}, then the complement of G has a minor isomorphic to K_s, where $s = \lceil \frac{1}{2}(1 + 1/(2t))|V(G)| - c \rceil$.*

Kostochka [4] showed that for all $\epsilon > 0$ Theorem (1.5) holds with s replaced by $\lceil \frac{1}{2}(1 - \epsilon + 1/(2t))|V(G)| - c \rceil$.

2. PROOF

We will use the following result of Thomason [8], an improvement of a result of Kostochka [3]. In fact, all we need is that the conclusion holds for some function of p, which is easy to prove, but (2.1) gives a better constant.

(2.1) Theorem. *Let p be a sufficiently large integer, and let G be a graph of average degree at least $5.36p\sqrt{\log_2 p}$. Then G has a K_p minor.*

Our results follow immediately from (2.1) and the following lemma. If G is a graph, we denote by G^2 the graph with vertex-set $V(G)$ in which two vertices are adjacent either if they are adjacent in G, or if they have a common neighbor in G. Our interest in G^2

is motivated by the fact that if I is an independent set in G and M is a matching in the complement of $G^2\backslash I$, then contracting all the edges in M yields a $K_{|I|+|M|}$ minor in the complement of G.

(2.2) Lemma. *Let d be a sufficiently large integer, let G be a graph on n vertices with maximum degree at most $n/(5d^3)$ and no minor isomorphic to K_d, and let H be an induced subgraph of the complement of G^2 on at least $n/2$ vertices. Then H has a matching of size at least $\lfloor \frac{1}{2}(|V(H)| - d)\rfloor$.*

Proof. Let d, n, G and H be as stated, and suppose for a contradiction that the matching does not exist. Then by Tutte's theorem [9] there exists a set $X \subseteq V(H)$ such that $H\backslash X$ has at least $|X| + d + 1$ odd components. Let $C_1, C_2, \dots, C_p$ be the vertex-sets of all components of $H\backslash X$ listed so that $|C_1| \le |C_2| \le \cdots \le |C_p|$. Then $p \ge d + 1$. Let $Y \subseteq C_1 \cup C_2 \cup \ldots \cup C_d$ contain precisely one element of each C_i $(i = 1, 2, \dots, d)$, and let $Z = \bigcup_{i=d+1}^{p} C_i$. Since $p \ge |X| + d + 1$ we see that $|X| \le |V(H)|/2$. Clearly $|C_1|, |C_2|, \dots, |C_d| \le |C_{d+1}| \le |Z|$; hence

$$|Z|(d+1) \ge \sum_{i=1}^{d} |C_i| + |Z| \ge \left|\bigcup_{i=1}^{p} C_i\right| = |V(H)| - |X| \ge |V(H)|/2,$$

and so

$$|Z| \ge \frac{1}{2(d+1)}\,|V(H)| \ge \frac{n}{4(d+1)}\,.$$

Since every vertex of G has degree at most $n/(5d^3)$, the set Z has a subset Z' of size at least $|Z| - n/(5d^2)$ such that G has no edge with one end in Y and the other in Z'. On the other hand, no vertex of Y is adjacent in H to a vertex of Z', and hence every $y \in Y$ and every $z \in Z'$ have a common neighbor in G. Let p_{yz} denote one such common neighbor. We have

$$|Z'| \ge |Z| - \frac{n}{5d^2} \ge \frac{n}{4(d+1)} - \frac{n}{5d^2} \ge \frac{n}{5d}\,,$$

because d is sufficiently large. We claim that Z' has a subset Z'' of size d such that $p_{yz} \ne p_{y'z'}$ for all $y, y' \in Y$ and all distinct vertices $z, z' \in Z''$. Indeed, let $Z_1 = Z'$, let $z_1 \in Z_1$ be arbitrary, and assume that for some $k = 1, 2, \dots, d-1$ we have already constructed distinct vertices $z_1, z_2, \dots, z_k$ and sets $Z_1, Z_2, \dots, Z_k$ such that $|Z_k| \ge n(d-k+1)/(5d^2)$

and for all $y \in Y$ and all $i = 1, 2, \ldots, k-1$, no neighbor of p_{yz_i} in G belongs to Z_k. We let Z_{k+1} be the set obtained from Z_k by deleting, for all $y \in Y$, all the neighbors of p_{yz_k} in G (at most $n/(5d^2)$ vertices altogether), and we pick $z_{k+1} \in Z_{k+1}$ arbitrarily. It follows that $Z'' = \{z_1, z_2, \ldots, z_d\}$ is as desired.

By contracting the edges with ends z and p_{yz} for all $y \in Y$ and all $z \in Z''$ we deduce that G has a minor isomorphic to $K_{d,d}$, and hence it has a minor isomorphic to K_d, contrary to hypothesis. □

(2.3) Lemma. *Let p be a sufficiently large integer, let $\delta = 5.36p\sqrt{\log_2 p}$, let $\alpha, n \geq 2$ be integers with $n \geq 36\delta^4$ and $n \geq 2\alpha + 6\delta^4$, and let G be a graph on n vertices with an independent set of size α and no minor isomorphic to K_p. Then the complement of G has a minor isomorphic to K_s, where $s = \lceil \frac{1}{2}(n+\alpha) - 4953p^4 \log_2^2 p \rceil$.*

Proof. By (2.1) the graph G has average degree less than δ. Let D be the set of all vertices of G of degree at least $n/(6\delta^3)$; then $|D| \leq 6\delta^4$. Let I be an independent set in G of size α, and let H be the complement of $(G\backslash D)^2$. By (2.2) applied to $\lceil \delta \rceil$, $G\backslash D$ and $H\backslash I$ we deduce that $H\backslash I$ has a matching M of size at least $\lfloor \frac{1}{2}(n - |D \cup I| - \lceil \delta \rceil) \rfloor$. Let K be the subgraph of H induced by $I - D$ and vertices incident with edges in M; by contracting every edge of K that belongs to M we get a minor of H (and hence a minor of the complement of G) isomorphic to K_s, as desired. □

From (2.3) we deduce that (1.2) implies (1.4), and from (1.3) and (2.3) we deduce that (1.5) holds.

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