On Possible Counterexamples to Negami's Planar Cover Conjecture

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Abstract

A simple graph $\boldsymbol{H}$ is a cover of a graph $\boldsymbol{G}$ if there exists a mapping φ from $\boldsymbol{H}$ onto $\boldsymbol{G}$ such that φ maps the neighbors of every vertex v in $\boldsymbol{H}$ bijectively to the neighbors of $\varphi(v)$ in $\boldsymbol{G}$. Negami conjectured in 1986 that a connected graph has a finite planar cover if and only if it embeds in the projective plane. The conjecture is still open. It follows from the results of Archdeacon, Fellows, Negami, and the first author that the conjecture holds as long as the graph $\boldsymbol{K}_{1,2,2,2}$ has no finite planar cover. However, those results seem to say little about counterexamples if the conjecture was not true. We show that there are, up to obvious constructions, at most 16 possible counterexamples to Negami's conjecture. Moreover, we exhibit a finite list of sets of graphs such that the set of excluded minors for the property of having finite planar cover is one of the sets in our list.

1 Introduction

All *graphs* in this paper are finite, and may have loops or multiple edges. The vertex set of a graph $\boldsymbol{G}$ is denoted by $V(\boldsymbol{G})$, the edge set by $E(\boldsymbol{G})$. A

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graph $\boldsymbol{H}$ is a *cover* of a graph $\boldsymbol{G}$ if there exist a pair of onto mappings (φ, ψ), called a (cover) *projection*, $\varphi : V(\boldsymbol{H}) \to V(\boldsymbol{G})$, $\psi : E(\boldsymbol{H}) \to E(\boldsymbol{G})$, such that ψ maps the edges incident with any vertex v in $\boldsymbol{H}$ bijectively onto the edges incident with $\varphi(v)$ in $\boldsymbol{G}$. In particular, this definition implies that for any edge e in $\boldsymbol{H}$ with endvertices u, v, the edge $\psi(e)$ in $\boldsymbol{G}$ has endvertices $\varphi(u), \varphi(v)$. Therefore if we deal with simple graphs, it is enough to specify the vertex cover projection φ that maps the neighbors of each vertex v in $\boldsymbol{H}$ bijectively onto the neighbors of $\varphi(v)$ in $\boldsymbol{G}$ (a traditional approach). A cover is called *planar* if it is a finite planar graph. (Notice that every graph can be covered by an infinite tree, but that is not what we are looking for.)

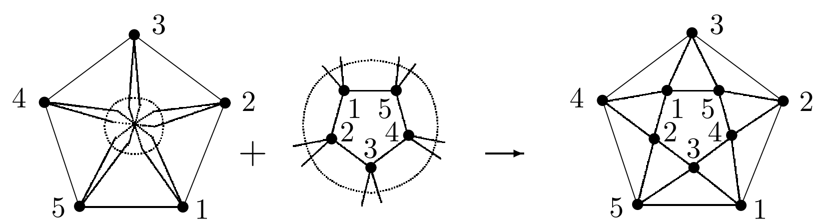


Figure 1: A double planar cover of $\boldsymbol{K}_5$, constructed by lifting its projective embedding into the sphere. The projection is determined by labels of vertices.

Every planar graph has a planar cover by the identity projection, but there are also nonplanar graphs having planar covers. If a graph $\boldsymbol{G}$ has an embedding in the projective plane, then the lifting of the embedding of $\boldsymbol{G}$ into the universal covering surface of the projective plane (the sphere) is a planar cover of $\boldsymbol{G}$, see an example in Fig. 1. Thus every projective-planar graph has a planar cover. The converse is false, because for instance the graph consisting of two disjoint copies of $\boldsymbol{K}_5$ has a planar cover, and yet has no embedding in the projective plane. On the other hand, in 1986 Negami made the following interesting conjecture.

Conjecture 1.1 (S. Negami, [9]) *A connected graph has a finite planar cover if and only if it has an embedding in the projective plane.*

Curiously, in order to prove the conjecture it suffices to prove that a certain graph has no planar cover. Let us explain that now. A graph $\boldsymbol{F}$ is a *minor* of a graph $\boldsymbol{G}$ if $\boldsymbol{F}$ can be obtained from a subgraph of $\boldsymbol{G}$ by contracting edges. We say that $\boldsymbol{G}$ has an $\boldsymbol{F}$ minor if some graph isomorphic to $\boldsymbol{F}$ is a minor of $\boldsymbol{G}$. The following is easy to see.

Lemma 1.2 *If a graph $\boldsymbol{G}$ has a planar cover, then so does every minor of $\boldsymbol{G}$.*

Glover, Huneke and Wang [3] found a family Λ' of 35 graphs such that each member of Λ' has no embedding in the projective plane, and is minor-minimal with that property. (See Appendix A for a complete list of Λ'.)

Archdeacon [1] then proved that those are the only such graphs. Three members of Λ' are disconnected; let Λ denote the remaining 32 connected members of Λ'. The next statement follows easily from Archdeacon's result.

Theorem 1.3 (D. Archdeacon, [1]) *A connected graph has no embedding in the projective plane if and only if it has a minor isomorphic to a member of* Λ.

Thus in order to prove Conjecture 1.1 it suffices to show that no member of Λ has a planar cover. The number of graphs to check can be further reduced using $Y\Delta$-transformations, defined as follows. A vertex of degree 3 with three distinct neighbors is called *cubic*. If w is a cubic vertex in a graph with neighbors v_1, v_2, v_3, then the operation of deleting the vertex w and adding three new edges forming a triangle on the vertices v_1, v_2, v_3 is called a $Y\Delta$-*transformation* (of w). The following two facts are reasonably easy to see. They were first noticed by Archdeacon [2] in about 1987, although he did not publish proofs until 2002.

Proposition 1.4 *Let $\boldsymbol{G}$ be a graph, and let e be an edge of $\boldsymbol{G}$ such that some cubic vertex of $\boldsymbol{G}$ is adjacent to both endvertices of e. If $\boldsymbol{G} - e$ has a planar cover, then so does $\boldsymbol{G}$.*

Corollary 1.5 *Let a graph $\boldsymbol{G}$ be obtained from a graph $\boldsymbol{H}$ by a sequence of $Y\Delta$-transformations. If $\boldsymbol{H}$ has a planar cover, then so does $\boldsymbol{G}$.*

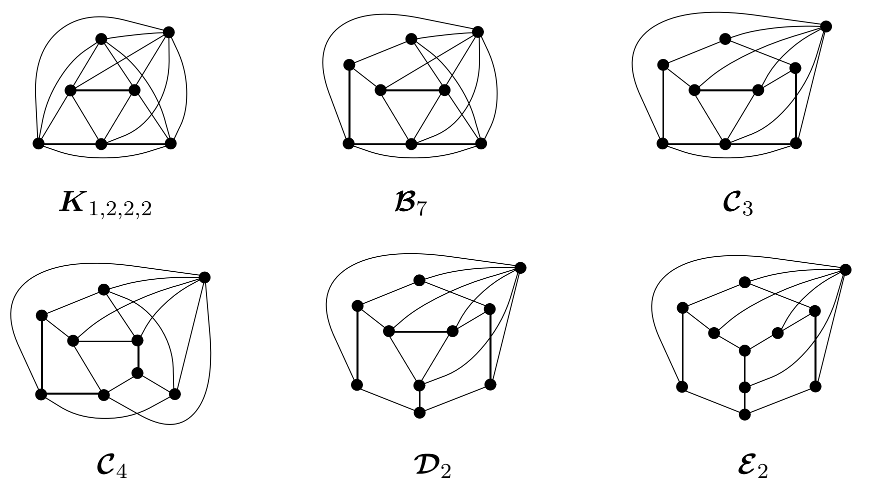


Figure 2:

Let $\boldsymbol{K}_{1,2,2,2}, \boldsymbol{\mathcal{B}}_7, \boldsymbol{\mathcal{C}}_3, \boldsymbol{\mathcal{C}}_4, \boldsymbol{\mathcal{D}}_2, \boldsymbol{\mathcal{E}}_2 \in \Lambda$ be the graphs depicted in Fig. 2. (Our notation of these graphs mostly follows [3].) Archdeacon [2], Fellows, Negami [10], and the first author [4] have shown the following.

Theorem 1.6 (D. Archdeacon, M. Fellows, S. Negami, P. Hliněný) *No member of the family* $\Lambda - \{\boldsymbol{K}_{1,2,2,2}, \boldsymbol{\mathcal{B}}_7, \boldsymbol{\mathcal{C}}_3, \boldsymbol{\mathcal{C}}_4, \boldsymbol{\mathcal{D}}_2, \boldsymbol{\mathcal{E}}_2\}$ *has a planar cover.*

Since each of the graphs $\boldsymbol{\mathcal{B}}_7, \boldsymbol{\mathcal{C}}_3, \boldsymbol{\mathcal{C}}_4, \boldsymbol{\mathcal{D}}_2, \boldsymbol{\mathcal{E}}_2$ can be $Y\Delta$-transformed to $\boldsymbol{K}_{1,2,2,2}$, Theorem 1.6 and Corollary 1.5 imply that Conjecture 1.1 is equivalent to the statement that the complete four-partite graph $\boldsymbol{K}_{1,2,2,2}$ has no finite planar cover. Thus it is tempting to say that Negami's conjecture is almost proven, but that is not quite accurate. Testing whether $\boldsymbol{K}_{1,2,2,2}$ has a planar cover does not seem to be a finite problem, because no apriori bound on the size of a planar cover is known. Moreover, the arguments outlined above seem to imply little about possible counterexamples.

Our main result, Theorem 2.2 on page 6, states that there are, up to obvious constructions, at most 16 possible counterexamples to Conjecture 1.1. The result (with a slightly different proof) is included in the Ph.D. thesis of the first author [6]. Some possible extensions of Conjecture 1.1 are discussed in [5]. Moreover, the arguments developed throughout this paper are used to exhibit a list of sets of graphs such that one of them is the set of excluded minors for the property of having finite planar cover (Theorem 6.2 on page 22).

The following statement [7, 6] is used in the proof.

Theorem 1.7 (P. Hliněný) *The graphs* $\boldsymbol{\mathcal{C}}_4$ *and* $\boldsymbol{\mathcal{E}}_2$ *have no planar covers.*

The reader will certainly notice that if someone found an *ad hoc* argument showing that $\boldsymbol{K}_{1,2,2,2}$ has no planar cover, then the present paper would become irrelevant. Why, then, did we bother with proving Theorem 2.2 rather than proving directly that $\boldsymbol{K}_{1,2,2,2}$ has no planar cover? In fact, we tried moderately hard to do that, and failed. While we hope that someone will be more successful, we decided to step back at look at possible strategies. Showing that $\boldsymbol{K}_{1,2,2,2}$ has no planar cover appears beyond reach at the moment (at least to us), but how about an easier problem?

Proposition 6.1 orders the 16 possible counterexamples in order of their relative difficulty with respect to showing that they have no planar cover. Thus if one was to look for the next easiest step toward proving Negami's conjecture, it would be showing that $\boldsymbol{\mathcal{D}}_2^{\circ}$ or $\boldsymbol{\mathcal{D}}_2^{\bullet}$ has no planar cover. A more ambitious plan would be to show that $\boldsymbol{\mathcal{D}}_2'''$ has no planar cover, and so on. (See Appendix B for pictures of these graphs.) In fact, it was work on the present paper that lead the first author to prove Theorem 1.7. By finding an easier (and hopefully solvable) problem one might be able to develop techniques that could settle Negami's conjecture itself.

In fact, there is an even weaker question which is related to the proof of Theorem 1.7. The proofs for the two graphs in [7] are completely different, while a proof of Conjecture 1.1 would necessarily have to include a unified argument for those two graphs. Thus a problem even weaker than those in the previous paragraph would be to find a unified proof that the graphs $\boldsymbol{\mathcal{C}}_4$ and $\boldsymbol{\mathcal{E}}_2$ have no planar cover.

2 Separations and Expansions

If $\boldsymbol{G}$ is a graph and X is a subset of its vertices, then $\boldsymbol{G} \restriction X$ denotes the subgraph of $\boldsymbol{G}$ induced by the vertex set X, and $\boldsymbol{G} - X$ denotes the subgraph of $\boldsymbol{G}$ induced on the vertex set $V(\boldsymbol{G}) - X$. For a graph $\boldsymbol{G}$, and an edge $e \in E(\boldsymbol{G})$, $\boldsymbol{G} - e$ denotes the graph obtained by deleting e from $\boldsymbol{G}$. If u, v are two vertices of $\boldsymbol{G}$, then $\boldsymbol{G} + \{u, v\}$ or $\boldsymbol{G} + uv$ denotes the graph obtained from $\boldsymbol{G}$ by adding a new edge with ends u and v (possibly parallel to an existing edge). If there is no danger of misunderstanding between parallel edges, then $\{u, v\}$, or shortly uv, is used for an edge with endvertices u and v.

A *separation* in a graph $\boldsymbol{G}$ is a pair of sets (A, B) such that $A \cup B = V(\boldsymbol{G})$ and there is no edge in $\boldsymbol{G}$ between the sets $A - B$ and $B - A$. A separation (A, B) is *nontrivial* if both $A - B$ and $B - A$ are nonempty. The *order of a separation* (A, B) equals $|A \cap B|$. A separation (A, B) in $\boldsymbol{G}$ is called *flat* if the graph $\boldsymbol{G} \restriction B$ has a planar embedding with all the vertices of $A \cap B$ incident with the outer face.

Let $\boldsymbol{G}$ be a graph. Let $\boldsymbol{F}$ be a connected planar graph on the vertex set $V(\boldsymbol{F})$ disjoint from $V(\boldsymbol{G})$, and let $x_1 \in V(\boldsymbol{F})$. If y_1 is a vertex of $\boldsymbol{G}$, and the graph $\boldsymbol{H}_1$ is obtained from $\boldsymbol{G} \cup \boldsymbol{F}$ by identifying the vertices x_1 and y_1, then $\boldsymbol{H}_1$ is called a 1-*expansion* of $\boldsymbol{G}$. Let $x_1, x_2 \in V(\boldsymbol{F})$ be two distinct vertices that are incident with the same face in a planar embedding of $\boldsymbol{F}$. If $e = y_1 y_2$ is an edge of $\boldsymbol{G}$, and the graph $\boldsymbol{H}_2$ is obtained from $(\boldsymbol{G} - e) \cup \boldsymbol{F}$ by identifying the vertex pairs (x_1, y_1) and (x_2, y_2), then $\boldsymbol{H}_2$ is called a 2-*expansion* of $\boldsymbol{G}$. Let $x_1, x_2, x_3 \in V(\boldsymbol{F})$ be three distinct vertices such that $\boldsymbol{F} - \{x_1, x_2, x_3\}$ is connected. Moreover, let each of the vertices x_1, x_2, x_3 be adjacent to some vertex of $V(\boldsymbol{F} - \{x_1, x_2, x_3\})$, and let all three vertices x_1, x_2, x_3 be incident with the same face in a planar embedding of $\boldsymbol{F}$. If w is a cubic vertex of $\boldsymbol{G}$ with the neighbors y_1, y_2, y_3, and the graph $\boldsymbol{H}_3$ is obtained from $(\boldsymbol{G} - w) \cup \boldsymbol{F}$ by identifying the vertex pairs (x_1, y_1), (x_2, y_2) and (x_3, y_3), then $\boldsymbol{H}_3$ is called a 3-*expansion* of $\boldsymbol{G}$.

A graph $\boldsymbol{H}$ is an *expansion* of a graph $\boldsymbol{G}$ if there is a sequence of graphs $\boldsymbol{G}_0 = \boldsymbol{G}, \boldsymbol{G}_1, \ldots, \boldsymbol{G}_l = \boldsymbol{H}$ such that $\boldsymbol{G}_i$ is a 1-,2-, or 3-expansion of $\boldsymbol{G}_{i-1}$ for all $i = 1, \ldots, l$. The following is easy to see.

Lemma 2.1 *Let $\boldsymbol{H}$ be an expansion of a graph $\boldsymbol{G}$.*
(a) $\boldsymbol{G}$ has an embedding in the projective plane if and only if so does $\boldsymbol{H}$.
(b) $\boldsymbol{G}$ has a planar cover if and only if so does $\boldsymbol{H}$.
(c) $\boldsymbol{G}$ is a minor of $\boldsymbol{H}$.

∎

A graph $\boldsymbol{G}$ would be a counterexample to Conjecture 1.1 if $\boldsymbol{G}$ had a planar cover but no projective embedding. Thus if $\boldsymbol{K}_{1,2,2,2}$ had a planar cover, then Lemma 2.1 would enable us to generate infinitely many counterexamples to Conjecture 1.1. However, our main result is:

Theorem 2.2 *Let* Π *be the family of 16 graphs listed in Appendix B. If a connected graph* $\boldsymbol{G}$ *has a planar cover but no embedding in the projective plane, then* $\boldsymbol{G}$ *is an expansion of some graph from* Π.

Before proving Theorem 2.2 in Section 5, a lot of preparatory work needs to be done. Let $\Lambda_0 = \Lambda - \{\boldsymbol{K}_{1,2,2,2},\ \boldsymbol{\mathcal{B}}_7, \boldsymbol{\mathcal{C}}_3, \boldsymbol{\mathcal{D}}_2\}$ denote the family of all connected minor-minimal nonprojective graphs which are known to have no planar cover. An immediate corollary of Theorems 1.3,1.6,1.7 is:

Corollary 2.3 *If* $\boldsymbol{G}$ *is a connected graph that has a planar cover but no embedding in the projective plane, then* $\boldsymbol{G}$ *has a minor isomorphic to one of* $\boldsymbol{K}_{1,2,2,2},\ \boldsymbol{\mathcal{B}}_7, \boldsymbol{\mathcal{C}}_3, \boldsymbol{\mathcal{D}}_2$*, but* $\boldsymbol{G}$ *has no minor isomorphic to a member of* Λ_0.

A graph $\boldsymbol{G}$ is *internally* 4*-connected* if it is simple and 3-connected, has at least five vertices, and for every separation (A, B) of order 3, either $\boldsymbol{G} \restriction A$ or $\boldsymbol{G} \restriction B$ has at most three edges. The following lemma and its corollary show that the search for a possible counterexample to Negami's conjecture may be restricted to internally 4-connected graphs. However, proving this lemma is quite a technical task, which needs several additional results, and so the proof of the lemma is postponed until Section 3.

Lemma 2.4 *Suppose* $\boldsymbol{G}$ *is a connected graph that has no embedding in the projective plane, and that has no minor isomorphic to a member of* Λ_0*. If* $k \in \{1, 2, 3\}$ *is the least integer such that there is a nontrivial separation* (A, B) *of order* k *in* $\boldsymbol{G}$*, then either* (A, B) *or* (B, A) *is flat.*

Corollary 2.5 *If* $\boldsymbol{G}$ *is a connected graph that has no embedding in the projective plane, and that has no minor isomorphic to a member of* Λ_0*, then there exists an internally* 4*-connected graph* $\boldsymbol{F}$ *with the same properties such that* $\boldsymbol{G}$ *is an expansion of* $\boldsymbol{F}$.

Proof. The proof proceeds by induction on the number of edges in $\boldsymbol{G}$. Since the graph $\boldsymbol{K}_6$ has a projective embedding, $\boldsymbol{G}$ has at least 7 vertices. If $\boldsymbol{G}$ is an internally 4-connected graph, then there is nothing to prove. If $\boldsymbol{G}$ is not simple, then it is an expansion of its underlying simple graph. So suppose, for some $k \in \{1, 2, 3\}$, that $\boldsymbol{G}$ is a simple k-connected graph, and that there exists a nontrivial separation (A, B) in $\boldsymbol{G}$ of order k. If $k = 3$, then also suppose that both $\boldsymbol{G} \restriction A$ and $\boldsymbol{G} \restriction B$ have more than three edges. By Lemma 2.4, it can be assumed that (A, B) is flat, and hence the graph

$\boldsymbol{G} \restriction B$ has a planar embedding in which all vertices of $A \cap B$ are incident with the same face.

Observe that $\boldsymbol{G} \restriction B$ is connected. If $k = 1$, then $\boldsymbol{G}$ is a 1-expansion of $\boldsymbol{G}' = \boldsymbol{G} \restriction A$. If $k = 2$, and say $A \cap B = \{u_1, u_2\}$, then $\boldsymbol{G}$ is a 2-expansion of $\boldsymbol{G}' = (\boldsymbol{G} \restriction A) + u_1u_2$. So suppose that $k = 3$, and $A \cap B = \{u_1, u_2, u_3\}$. Since $\boldsymbol{G}$ is 3-connected, every vertex $v \in B - A$ is connected by three disjoint (except for v) paths to each of the vertices u_1, u_2, u_3. In particular, each of u_1, u_2, u_3 is adjacent to some vertex of $B - A$. If the graph $\boldsymbol{G} \restriction (B - A)$ is not connected, let us say that v, w belong to distinct components of $\boldsymbol{G} \restriction (B - A)$, then the six paths connecting each of v, w with u_1, u_2, u_3 are pairwise disjoint except for their ends, and hence they form a subdivision of $\boldsymbol{K}_{2,3}$. But then there is no planar embedding of $\boldsymbol{G} \restriction B$ in which u_1, u_2, u_3 are incident with the same face, a contradiction. Thus the planar graph $\boldsymbol{F} = \boldsymbol{G} \restriction B$ fulfills all conditions in the definition of the 3-expansion, and hence $\boldsymbol{G}$ is a 3-expansion of the graph $\boldsymbol{G}'$, obtained from $\boldsymbol{G}$ by contracting the set $B - A$ into one vertex and deleting all edges with both ends in $A \cap B$.

In all three cases outlined above, the graph $\boldsymbol{G}'$ has no embedding in the projective plane by Lemma 2.1, and it is a proper minor of $\boldsymbol{G}$. So the statement follows by induction. ∎

3 Assorted Lemmas

Recall the definition of a $Y\Delta$-transformation from page 3. If a graph $\boldsymbol{G}$ is obtained from $\boldsymbol{G}'$ by $Y\Delta$-transforming a vertex $w \in V(\boldsymbol{G}')$, then we write $\boldsymbol{G} = \boldsymbol{G}' \,{\scriptstyle Y\Delta}\, \{w\}$.

Understanding the following convention is important for the next definition: Formally, a graph is a triple consisting of a vertex set, an edge set, and an incidence relation between vertices and edges. Contracting an edge e in a graph $\boldsymbol{G}$ means deleting the edge and identifying its ends. Thus if $\boldsymbol{F}$ denotes the resulting graph, then $E(\boldsymbol{F}) = E(\boldsymbol{G}) - \{e\}$. Now suppose that a simple graph $\boldsymbol{G}$ is obtained from a simple graph $\boldsymbol{G}_s$ by contracting an edge $e = uv \in E(\boldsymbol{G}_s)$ to a vertex $w \in V(\boldsymbol{G})$. If the degrees of u, v in $\boldsymbol{G}_s$ are at least 3, then $\boldsymbol{G}_s$ is said to be obtained from $\boldsymbol{G}$ by *splitting the vertex* w. The graph $\boldsymbol{G}_s$ is formally denoted by $\boldsymbol{G} \cdot \{w \angle \frac{N_u}{N_v}\}$ where N_u, N_v are the neighborhoods of u, v, respectively, excluding u, v themselves.

Let $\boldsymbol{K}_{3,5}, \boldsymbol{K}_7 - \boldsymbol{C}_4, \boldsymbol{\mathcal{D}}_3, \ \boldsymbol{K}_{4,4} - e, \boldsymbol{K}_{4,5} - \boldsymbol{M}_4, \boldsymbol{\mathcal{D}}_{17} \ \in \Lambda_0$ denote the graphs depicted in Fig. 3. Recall also the graphs $\boldsymbol{\mathcal{C}}_4, \boldsymbol{\mathcal{E}}_2 \ \in \Lambda_0$ from Fig. 2. Let Φ' be the family of all simple graphs $\boldsymbol{G}$ such that one of the graphs $\boldsymbol{K}_7 - \boldsymbol{C}_4$, $\boldsymbol{\mathcal{D}}_3$, or $\boldsymbol{\mathcal{D}}_{17}$ can be obtained from $\boldsymbol{G}$ by a sequence of $Y\Delta$-transformations; and let $\Phi = \Phi' \cup \{\boldsymbol{K}_{4,4} - e, \boldsymbol{K}_{3,5}, \boldsymbol{K}_{4,5} - \boldsymbol{M}_4\}$. Note that Φ' includes only finitely many nonisomorphic graphs since a $Y\Delta$-transformation preserves the number of edges.

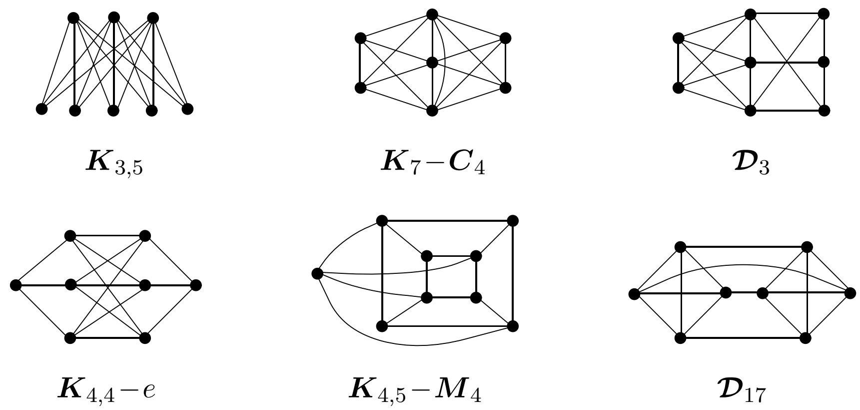


Figure 3:

Lemma 3.1 *Suppose that $\boldsymbol{G}'$ is a simple graph obtained from a graph $\boldsymbol{G}$ by a $Y\Delta$-transformation, and that a simple graph $\boldsymbol{F}'$ is a minor of $\boldsymbol{G}'$. Then $\boldsymbol{G}$ has an $\boldsymbol{F}$ minor, where $\boldsymbol{F} = \boldsymbol{F}'$ or $\boldsymbol{F}$ can be $Y\Delta$-transformed to $\boldsymbol{F}'$.*

Proof. Let $\boldsymbol{G}' = \boldsymbol{G}\,{\scriptstyle Y\Delta}\,\{w\}$ be obtained by a $Y\Delta$-transformation carrying a cubic vertex w of $\boldsymbol{G}$ to a triangle τ of $\boldsymbol{G}'$. If all three edges of τ are in $E(\boldsymbol{F}')$, then they form a triangle in $\boldsymbol{F}'$ since $\boldsymbol{F}'$ is simple. Then $\boldsymbol{F}$ is constructed from $\boldsymbol{F}'$ by adding a new vertex adjacent to the vertices of τ and deleting the edges of τ. Clearly, $\boldsymbol{F}$ is a minor of $\boldsymbol{G}$.

If an edge e of τ is not in $E(\boldsymbol{F}')$, let v be the vertex of τ not incident with e. Then $\boldsymbol{G}' - e$ has an $\boldsymbol{F}'$ minor, and $\boldsymbol{G}' - e$ is obtained from $\boldsymbol{G}$ by contracting the edge $\{v, w\}$. Thus $\boldsymbol{G}$ has an $\boldsymbol{F} = \boldsymbol{F}'$ minor. ∎

Lemma 3.2 *If $\boldsymbol{G} \in \Phi'$, then $\boldsymbol{G}$ has a minor isomorphic to some member of Λ_0.*

Proof. This statement follows from the arguments in [3], even though it is not explicitly stated there. Let $\boldsymbol{J}_0 = \boldsymbol{G}, \boldsymbol{J}_1, \ldots, \boldsymbol{J}_t$ be a sequence of graphs such that $\boldsymbol{J}_t \in \{\boldsymbol{K}_7 - \boldsymbol{C}_4, \boldsymbol{\mathcal{D}}_3, \boldsymbol{\mathcal{D}}_{17}\}$, and for $i = 1, \ldots, t$, $\boldsymbol{J}_i$ is obtained from $\boldsymbol{J}_{i-1}$ by a $Y\Delta$-transformation. It is easy to check, using Lemma 3.1, that if $\boldsymbol{J}_t = \boldsymbol{K}_7 - \boldsymbol{C}_4$ or $\boldsymbol{J}_t = \boldsymbol{\mathcal{D}}_3$, then $\boldsymbol{J}_i$, $i = 0, 1, \ldots, t-1$ has a minor isomorphic to one of $\boldsymbol{\mathcal{D}}_3, \boldsymbol{K}_{3,5}, \boldsymbol{\mathcal{E}}_5, \boldsymbol{\mathcal{F}}_1$; and if $\boldsymbol{J}_t = \boldsymbol{\mathcal{D}}_{17}$, then $\boldsymbol{J}_i$, $i = 0, 1, \ldots, t-1$ has a minor isomorphic to one of $\boldsymbol{\mathcal{E}}_{20}, \boldsymbol{\mathcal{G}}_1, \boldsymbol{\mathcal{F}}_4$. (See Appendix A for pictures of these graphs.) ∎

Lemma 3.3 *Let $\boldsymbol{G}$ be a graph, and let a simple graph $\boldsymbol{G}'$ be obtained from $\boldsymbol{G}$ by a sequence of $Y\Delta$-transformations. If $\boldsymbol{G}'$ has a minor isomorphic to some member of Φ, then so does $\boldsymbol{G}$. Consequently, $\boldsymbol{G}$ has a minor isomorphic to some member of Λ_0.*

Proof. Notice that each of the graphs $\boldsymbol{K}_{4,4}-e, \boldsymbol{K}_{3,5}, \boldsymbol{K}_{4,5}-\boldsymbol{M}_4$ is triangle-free. If $\boldsymbol{G}'$ has a minor isomorphic to any one of them, then so does $\boldsymbol{G}$ by Lemma 3.1. Otherwise, $\boldsymbol{G}'$ has an $\boldsymbol{F}'$ minor for some $\boldsymbol{F}' \in \Phi'$. By Lemma 3.1, the graph $\boldsymbol{G}$ has a minor isomorphic to a member $\boldsymbol{F}''$ of Φ'. By Lemma 3.2, the graph $\boldsymbol{F}''$ has a minor isomorphic to a member of Λ_0, and hence so does $\boldsymbol{G}$. ∎

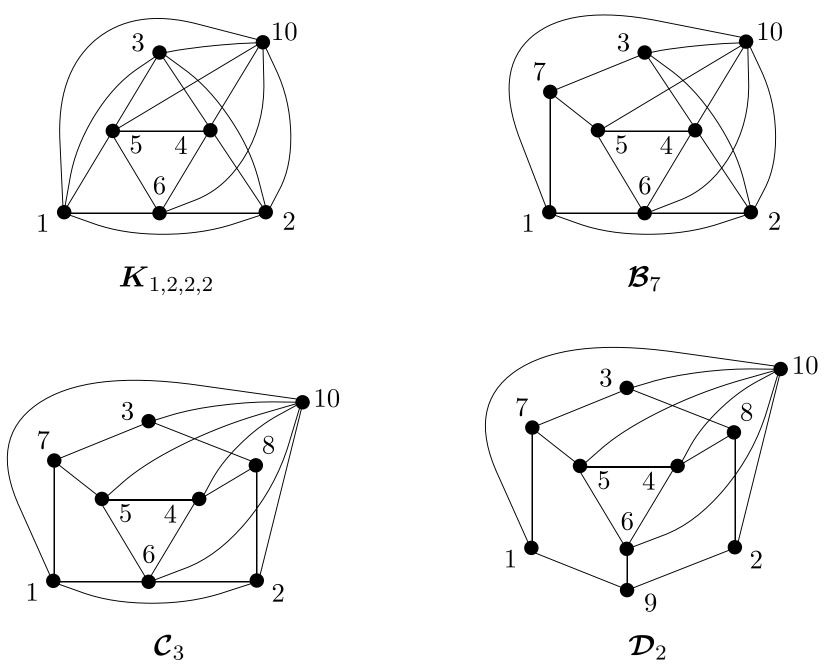


Figure 4:

Let e be an edge of a graph $\boldsymbol{G}$, and v a vertex of degree 3 in $\boldsymbol{G}$ adjacent to both ends of e. Then v is called a *violating vertex*, e is a *violating edge*, and (v, e) is a *violating pair* in $\boldsymbol{G}$. The reason for this terminology is that such a vertex and edge violate the definition of being internally 4-connected. Let the vertices of the graphs $\boldsymbol{K}_{1,2,2,2}, \boldsymbol{\mathcal{B}}_7, \boldsymbol{\mathcal{C}}_3, \boldsymbol{\mathcal{D}}_2$ be numbered as in Fig. 4.

Lemma 3.4 *Let* $\{\boldsymbol{F}_i\}_{i=1}^4$ *be a sequence of graphs defined by* $\boldsymbol{F}_1 = \boldsymbol{K}_{1,2,2,2}$, $\boldsymbol{F}_2 = \boldsymbol{\mathcal{B}}_7$, $\boldsymbol{F}_3 = \boldsymbol{\mathcal{C}}_3$, $\boldsymbol{F}_4 = \boldsymbol{\mathcal{D}}_2$. *For* $i = 1, 2, 3, 4$, *the following statements hold.*
a) Suppose that $\boldsymbol{F}' = \boldsymbol{F}_i + xy$ *is obtained from* $\boldsymbol{F}_i$ *by adding an edge joining two distinct nonadjacent vertices* x, y *of* $\boldsymbol{F}_i$. *If* xy *is not violating in* $\boldsymbol{F}'$, *then* $\boldsymbol{F}'$ *has a minor isomorphic to a member of* Λ_0, *unless* $i = 2$ *(*$\boldsymbol{F}_i = \boldsymbol{\mathcal{B}}_7$*) and* $\{x, y\} = \{7, 8\}$.
b) Suppose that $\boldsymbol{F}'$ *is obtained by splitting a vertex* w *in* $\boldsymbol{F}_i$. *Then either* $\boldsymbol{F}'$ *has a minor isomorphic to a member of* Λ_0, *or* $i \leq 3$ *and* $w \neq 7$ *and* $\boldsymbol{F}'$ *has a* $\boldsymbol{F}_{i+1}$ *subgraph.*

Proof. The proof proceeds along the sequence $\boldsymbol{F}_1, \boldsymbol{F}_2, \boldsymbol{F}_3, \boldsymbol{F}_4$, considering parts (a) and (b) together. Up to symmetry, there is only one possibility to add an edge to $\boldsymbol{F}_1 = \boldsymbol{K}_{1,2,2,2}$.

– $\boldsymbol{F}' = \boldsymbol{K}_{1,2,2,2} + \{1,4\}$ has a $\boldsymbol{K}_7 - \boldsymbol{C}_4$ subgraph.

The four possibilities to split vertex 7, up to symmetry, are discussed as follows:

– $\boldsymbol{F}' = \boldsymbol{K}_{1,2,2,2} \cdot \{7\angle\,^{1,2,3}_{4,5,6}\}$ has a $\boldsymbol{\mathcal{D}}_{17}$ subgraph.

– $\boldsymbol{F}' = \boldsymbol{K}_{1,2,2,2} \cdot \{7\angle\,^{1,2,4}_{3,5,6}\}$ has a $\boldsymbol{K}_{4,4} - e$ subgraph.

– $\boldsymbol{F}' = \boldsymbol{K}_{1,2,2,2} \cdot \{7\angle\,^{1,2}_{3,4,5,6}\}$ has a $\boldsymbol{\mathcal{D}}_3$ subgraph.

– $\boldsymbol{F}' = \boldsymbol{K}_{1,2,2,2} \cdot \{7\angle\,^{1,4}_{2,3,5,6}\}$ has a $\boldsymbol{K}_{3,5}$ subgraph.

All vertices other than 7 are symmetric in $\boldsymbol{K}_{1,2,2,2}$, so it suffices to consider the three possible nonsymmetric splittings of vertex 1.

– $\boldsymbol{F}' = \boldsymbol{K}_{1,2,2,2} \cdot \{1\angle\,^{3,6}_{2,5,7}\}$ and $\boldsymbol{F}' = \boldsymbol{K}_{1,2,2,2} \cdot \{1\angle\,^{2,7}_{3,5,6}\}$ have $\boldsymbol{\mathcal{D}}_3$ subgraphs.

– $\boldsymbol{F}' = \boldsymbol{K}_{1,2,2,2} \cdot \{1\angle\,^{3,5}_{2,6,7}\}$ has a $\boldsymbol{F}_2 = \boldsymbol{\mathcal{B}}_7$ subgraph.

Since the graphs $\boldsymbol{K}_7 - \boldsymbol{C}_4, \boldsymbol{\mathcal{D}}_{17}, \boldsymbol{K}_{4,4} - e, \boldsymbol{\mathcal{D}}_3, \boldsymbol{K}_{3,5}$ are members of Λ_0, the statement is proved for $\boldsymbol{F}_1$.

If $\boldsymbol{F}' = \boldsymbol{\mathcal{B}}_7 + xy$ where $\{x,y\}$ is one of $\{1,4\},\{2,5\},\{3,6\}$, then the graph $\boldsymbol{F}'\,\mathrm{Y}\!\Delta\,\{8\}$ equals to the graph $\boldsymbol{K}_{1,2,2,2} + xy$. So, using Lemma 3.3, the arguments in the previous paragraph imply that $\boldsymbol{F}'$ has a minor isomorphic to a graph from Λ_0. The remaining possible edge addition, up to symmetry, is covered next.

– $\boldsymbol{F}' = \boldsymbol{\mathcal{B}}_7 + \{2,8\}$ has a $\boldsymbol{K}_7 - \boldsymbol{C}_4$ minor via contracting $\{5,8\}$.

Let a *step splitting* be the splitting operation $\boldsymbol{F}_i \cdot \{w\angle\,^{u_1,u_2}_{u_3,\dots,u_k}\}$, $i \in \{1,2,3,4\}$ such that $w, u_1, u_2 \in \{1,2,3,4,5,6\}$, and $\{w, u_1, u_2\}$ does not contain any one of the pairs $\{1,4\},\{2,5\},\{3,6\}$. Notice that all non-step splittings in $\boldsymbol{F}_1$ produce members of Φ. The vertex 8 cannot be split in $\boldsymbol{\mathcal{B}}_7$, so if $\boldsymbol{F}'$ results by a non-step splitting in $\boldsymbol{\mathcal{B}}_7$, then the graph $\boldsymbol{F}'\,\mathrm{Y}\!\Delta\,\{8\}$ results by a non-step splitting in $\boldsymbol{K}_{1,2,2,2}$; hence $\boldsymbol{F}'$ has a Λ_0 minor by Lemma 3.3. All possible step splittings in $\boldsymbol{\mathcal{B}}_7$ are discussed as follows.

– $\boldsymbol{F}' = \boldsymbol{\mathcal{B}}_7 \cdot \{3\angle\,^{2,4}_{7,8}\}$ and $\boldsymbol{F}' = \boldsymbol{\mathcal{B}}_7 \cdot \{2\angle\,^{3,4}_{1,6,7}\}$ have $\boldsymbol{F}_3 = \boldsymbol{\mathcal{C}}_3$ subgraphs.

– $\boldsymbol{F}' = \boldsymbol{\mathcal{B}}_7 \cdot \{2\angle\,^{1,3}_{4,6,7}\}$ has a $\boldsymbol{K}_{4,5} - \boldsymbol{M}_4$ subgraph.

– $\boldsymbol{F}' = \boldsymbol{\mathcal{B}}_7 \cdot \{2\angle\,^{4,6}_{1,3,7}\}$ has a $\boldsymbol{\mathcal{C}}_4$ subgraph.

The discussion is continued in the same manner for $\boldsymbol{F}_3 = \boldsymbol{\mathcal{C}}_3$. If $\boldsymbol{F}' = \boldsymbol{\mathcal{C}}_3 + xy$ and xy is not violating in $\boldsymbol{F}'$, then xy is incident with at most one of the vertices $8, 9$, say 8, and it is not $\{7, 8\}$. Thus $\boldsymbol{F}' \,{\scriptstyle Y}\!\Delta\, \{9\} = \boldsymbol{\mathcal{B}}_7 + xy$ has a Λ_0 minor by the previous analysis, and so does $\boldsymbol{F}'$ by Lemma 3.3. Similarly, if $\boldsymbol{F}'$ results by a non-step splitting in $\boldsymbol{\mathcal{C}}_3$, then $\boldsymbol{F}' \,{\scriptstyle Y}\!\Delta\, \{9\}$ results by a non-step splitting in $\boldsymbol{\mathcal{B}}_7$, so $\boldsymbol{F}'$ has a Λ_0 minor again. The possible step splittings in $\boldsymbol{\mathcal{C}}_3$ are covered next.

– $\boldsymbol{F}' = \boldsymbol{\mathcal{C}}_3 \cdot \{1 \angle\, ^{2,6}_{7,8}\}$ and $\boldsymbol{F}' = \boldsymbol{\mathcal{C}}_3 \cdot \{6 \angle\, ^{1,2}_{4,5,7}\}$ have $\boldsymbol{F}_4 = \boldsymbol{\mathcal{D}}_2$ subgraphs.

– $\boldsymbol{F}' = \boldsymbol{\mathcal{C}}_3 \cdot \{6 \angle\, ^{1,5}_{2,4,7}\}$ has a $\boldsymbol{K}_{4,5} - \boldsymbol{M}_4$ minor via contracting $\{3, 9\}$.

Finally, the same arguments as above apply for the cases of an edge addition or a non-step splitting in $\boldsymbol{F}_4 = \boldsymbol{\mathcal{D}}_2$. So there is only one step splitting remaining to be checked.

– $\boldsymbol{J}' = \boldsymbol{\mathcal{D}}_2 \cdot \{5 \angle\, ^{4,6}_{7,8}\}$ has an $\boldsymbol{\mathcal{E}}_2$ subgraph. ∎

The following statement about planar embeddings of graphs appears more or less explicitly in [11, 13, 14, 15].

Theorem 3.5 (N. Robertson, P.D. Seymour, Y. Shiloach, C. Thomassen) *Let $\boldsymbol{G}$ be a 3-connected graph, and let u, v, w be three distinct vertices of $\boldsymbol{G}$. If $\boldsymbol{G}$ has no planar embedding in which u, v and w are all incident with the outer face, then $\boldsymbol{G}$ has an $\boldsymbol{F}$ minor such that $\boldsymbol{F}$ is isomorphic to $\boldsymbol{K}_{2,3}$, and the vertices u, v, w are contracted into three distinct vertices u', v', w' which form the part of size three in $\boldsymbol{F}$.*

Suppose that $\boldsymbol{G}$ is a graph, and that $v_1, v_2, v_3 \in V(\boldsymbol{G})$ are three distinct vertices of $\boldsymbol{G}$. Let $\boldsymbol{G} \leftarrow \{v_1, v_2, v_3\}$ denote the graph $\boldsymbol{H}$ defined as follows: If there exists a cubic vertex $w \in V(\boldsymbol{G})$ with the neighbors v_1, v_2, v_3, then $\boldsymbol{H}$ results from $\boldsymbol{G}$ by adding one new vertex t adjacent to all three vertices v_1, v_2, v_3. Otherwise, $\boldsymbol{H}$ results from $\boldsymbol{G}$ by adding two new vertices s, t both adjacent to all three vertices v_1, v_2, v_3, and by deleting all edges with both ends in $\{v_1, v_2, v_3\}$.

Lemma 3.6 *Let $\boldsymbol{G}$ be a 3-connected graph, and let (A, B) be a non-flat separation of order three in $\boldsymbol{G}$. Let $\boldsymbol{F}_0$ be a simple 3-connected graph. Suppose that $\boldsymbol{F} \subseteq \boldsymbol{G}$ is a subgraph of $\boldsymbol{G}$ isomorphic to a subdivision of $\boldsymbol{F}_0$, and that $W \subseteq V(\boldsymbol{F})$ is the subset of vertices that have degree more than 2 in $\boldsymbol{F}$. If $|W \cap (B - A)| \leq 1$, then $\boldsymbol{G}$ contains a minor isomorphic to the graph $\boldsymbol{F}_0 \leftarrow \{w_1, w_2, w_3\}$ for some three vertices $w_1, w_2, w_3 \in V(\boldsymbol{F}_0)$.*

Proof. Let $A \cap B = \{b_1, b_2, b_3\}$. By Theorem 3.5, there is a minor $\boldsymbol{G}'$ of $\boldsymbol{G}$, and a 3-separation (A', B') in $\boldsymbol{G}'$, such that $A = A'$, $\boldsymbol{G}' \restriction A' = \boldsymbol{G} \restriction A$, and

$B' - A' = \{s, t\}$ where each of s, t is adjacent to all three vertices b_1, b_2, b_3. (Hence $\boldsymbol{F} \restriction A$ is a subgraph of $\boldsymbol{G}'$.)

Suppose that $|W \cap B| = 1$. Let $W \cap (B - A) = \{w\}$, and let w' be the vertex of $\boldsymbol{F}_0$ corresponding to w. Let Q_e denote the path in $\boldsymbol{F}$ that corresponds to an edge $e \in E(\boldsymbol{F}_0)$. Since $\boldsymbol{F}_0$ is 3-connected, there are at least three edges incident with w' in $\boldsymbol{F}_0$. On the other hand, the vertex $w \in B$ can be connected by at most three disjoint paths with vertices in $W - \{w\} \subset A$. Hence w' is a cubic vertex in $\boldsymbol{F}_0$, and the edges incident with w' can be denoted by $e_1, e_2, e_3 \in E(\boldsymbol{F}_0)$ so that $b_i \in V(Q_{e_i})$ for $i = 1, 2, 3$. Let $\boldsymbol{G}''$ be the graph obtained from $\boldsymbol{G}'$ by contracting each of the paths $Q_{e_i} \restriction A$, $i = 1, 2, 3$, into one vertex. Then $\boldsymbol{G}''$ contains a subgraph isomorphic to a subdivision of $\boldsymbol{F}_0 \between \{v_1, v_2, v_3\}$ where v_1, v_2, v_3 are the neighbors of w in $\boldsymbol{F}_0$.

So suppose that $W \cap (B - A) = \emptyset$. Since $\boldsymbol{G}$ is 3-connected, there exist, by Menger's theorem, three vertices $d_1, d_2, d_3 \in W$, and three vertex-disjoint paths P_1, P_2, P_3 such that P_i has ends b_i and d_i for $i = 1, 2, 3$. For a path P, let $P[u, v]$ denote the subpath of P connecting the vertices $u, v \in V(P)$. Let c_i be the vertex of $V(P_i) \cap V(\boldsymbol{F})$ closest to b_i in P_i, and let $P_i' = P_i[b_i, c_i]$, for $i = 1, 2, 3$. (It may happen that $c_i = b_i$.)

First suppose the case that not all c_1, c_2, c_3 belong to the same path Q_e, $e \in E(\boldsymbol{F}_0)$. Then, for $i = 1, 2, 3$, there exists an edge $e_i \in E(\boldsymbol{F}_0)$ such that $c_i \in V(Q_{e_i})$; and one of the ends of the path Q_{e_i} can be denoted by x_i, so that the path $Q_i' = Q_{e_i}[x_i, c_i]$ does not intersect the set $\{c_1, c_2, c_3\} - \{c_i\}$. Also, x_1, x_2, x_3 can be chosen all distinct since $\boldsymbol{F}_0$ has no multiple edges. Let $\boldsymbol{G}''$ denote the graph obtained from $\boldsymbol{G}'$ by contracting each of the paths $Q_1' \cup P_1'$, $Q_2' \cup P_2'$, and $Q_3' \cup P_3'$ into one vertex. One can easily check that $\boldsymbol{G}''$ has a subgraph isomorphic to a subdivision of $\boldsymbol{F}_0 \between \{x_1', x_2', x_3'\}$, where x_1', x_2', x_3' are the vertices of $\boldsymbol{F}_0$ corresponding to x_1, x_2, x_3. (If for some $e \in E(\boldsymbol{F}_0)$ there exists a path Q_e in $\boldsymbol{G}$ intersecting $B - A$, then both ends of Q_e are in $\{x_1, x_2, x_3\}$. Thus, if there is no cubic vertex adjacent to x_1', x_2', x_3' in $\boldsymbol{F}_0$, the edge e is not present in $\boldsymbol{F}_0 \between \{x_1', x_2', x_3'\}$; and otherwise, the path Q_e can be replaced by a path Q_e' that uses one of the vertices s, t in $\boldsymbol{G}'$.)

Next, suppose that there is an edge $e \in E(\boldsymbol{F}_0)$ such that $c_1, c_2, c_3 \in V(Q_e)$. Let x, y be the ends of the path Q_e, and let $U = V(Q_e) - \{x, y\}$. For $i = 1, 2, 3$, let a_i be the vertex of $V(P_i) \cap (V(\boldsymbol{F}) - U)$ closest to b_i in P_i, and let $P_i'' = P_i[b_i, a_i]$. Assume, without loss of generality, that, for some vertex $v \in V(P_1'') \cap V(Q_e)$, one of the paths $Q_e[v, x], Q_e[v, y]$ is disjoint from $P_2'' \cup P_3''$. Let $v_1 \in V(P_1'') \cap V(Q_e)$ be such a vertex which is the one closest to b_1 in P_1''; and assume that the path $Q_e[v_1, x]$ is disjoint from $P_2'' \cup P_3''$. Further, let $v_2 \in V(Q_e) \cap (V(P_2'') \cup V(P_3''))$ be the vertex closest to y in Q_e, and assume that $v_2 \in V(P_2'')$. Then the path $P_1^o = P_1''[b_1, v_1] \cup Q_e[v_1, x]$ is disjoint from P_2'' and P_3'', and the path $P_2^o = P_2''[b_2, v_2] \cup Q_e[v_2, y]$ is disjoint from P_1^o and P_3''. In particular, $a_3 \notin \{x, y\}$. Let $e' \in E(\boldsymbol{F}_0)$ be the edge such that $a_3 \in V(Q_{e'})$; and let z be an end of $Q_{e'}$ such that $z \notin \{x, y\}$, and that $z = a_3$ if a_3 is an end of $Q_{e'}$. Let $P_3^o = P_3'' \cup Q_{e'}[a_3, z]$. Let $\boldsymbol{G}''$ denote the graph obtained from $\boldsymbol{G}'$ by contracting each of the paths P_1^o, P_2^o, and P_3^o into one vertex. Then $\boldsymbol{G}''$ has a subgraph isomorphic to a subdivision of $(\boldsymbol{F}_0 - e) \looparrowleft \{x', y', z'\} = \boldsymbol{F}_0 \looparrowleft \{x', y', z'\}$, where x', y', z' are the vertices of $\boldsymbol{F}_0$ corresponding to x, y, z. ∎

Finally, the postponed proof of Lemma 2.4 about flat separations from Section 2 is presented.

Proof of Lemma 2.4. Assume first that $k = 1$, and let $\boldsymbol{G}_A = \boldsymbol{G} \restriction A$, $\boldsymbol{G}_B = \boldsymbol{G} \restriction B$. If neither (A, B) nor (B, A) are flat, then both graphs $\boldsymbol{G}_A$ and $\boldsymbol{G}_B$ are nonplanar, and thus by Kuratowski's theorem they contain subgraphs $\boldsymbol{F}_A \subseteq \boldsymbol{G}_A$ and $\boldsymbol{F}_B \subseteq \boldsymbol{G}_B$ isomorphic to subdivisions of $\boldsymbol{K}_5$ or $\boldsymbol{K}_{3,3}$. Since $\boldsymbol{G}$ is connected, it is easy to contract $\boldsymbol{F}_A$ and $\boldsymbol{F}_B$ into a minor isomorphic to one of $\boldsymbol{K}_5 \cdot \boldsymbol{K}_5, \boldsymbol{K}_5 \cdot \boldsymbol{K}_{3,3}, \boldsymbol{K}_{3,3} \cdot \boldsymbol{K}_{3,3} \in \Lambda_0$, a contradiction.

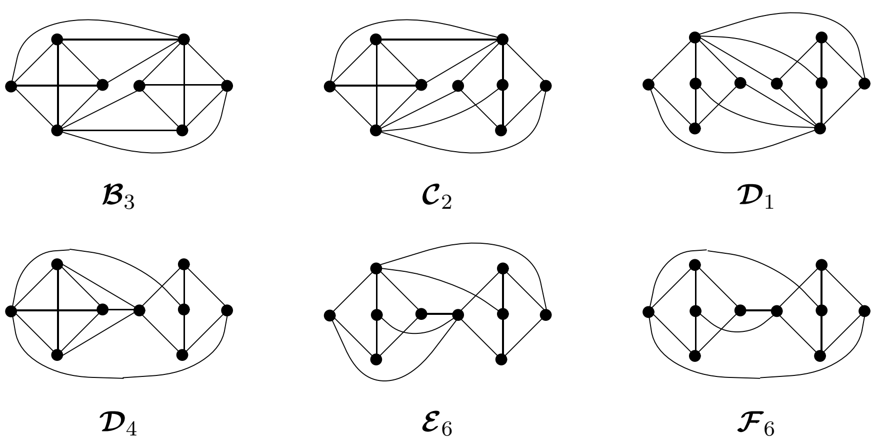


Figure 5:

The case of $k = 2$ is settled similarly. Let $A \cap B = \{u, v\}$, and let $\boldsymbol{G}_A = (\boldsymbol{G} \restriction A) + uv$, $\boldsymbol{G}_B = (\boldsymbol{G} \restriction B) + uv$. If neither (A, B) nor (B, A) are flat, then both graphs $\boldsymbol{G}_A$ and $\boldsymbol{G}_B$ are nonplanar, and thus by Kuratowski's theorem they contain subgraphs $\boldsymbol{F}_A \subseteq \boldsymbol{G}_A$ and $\boldsymbol{F}_B \subseteq \boldsymbol{G}_B$ isomorphic to subdivisions of $\boldsymbol{K}_5$ or $\boldsymbol{K}_{3,3}$. Without loss of generality, let us focus on the graph $\boldsymbol{G}_A$. Since $\boldsymbol{G}$ is 2-connected in this case, there exist two disjoint paths P_u, P_v between $\{u, v\}$ and $V(\boldsymbol{F}_A)$. Clearly, the graph $\boldsymbol{F}_A \cup P_u \cup P_v + uv$ has a minor $\boldsymbol{F}'_A$ isomorphic to $\boldsymbol{K}_5$ or $\boldsymbol{K}_{3,3}$ such that u, v are two distinct vertices of $\boldsymbol{F}'_A$. The graph $\boldsymbol{F}'_B$ is found in the same way as a minor of $\boldsymbol{G}_B$. So the graph $\boldsymbol{G}' = \boldsymbol{F}'_A \cup \boldsymbol{F}'_B - uv$ is a minor of $\boldsymbol{G}$. It is easy to show that $\boldsymbol{G}'$ is isomorphic to some member of Λ_0: If $\boldsymbol{F}'_A \simeq \boldsymbol{K}_5$, there is only one nonsymmetric choice of $u, v \in V(\boldsymbol{F}'_A)$. If $\boldsymbol{F}'_A \simeq \boldsymbol{K}_{3,3}$, there are two nonsymmetric choices of $u, v \in V(\boldsymbol{F}'_A)$ (either from the same part, or from different ones). Therefore $\boldsymbol{G}'$ is isomorphic to $\mathcal{B}_3 \in \Lambda_0$ for $\boldsymbol{F}'_A \simeq \boldsymbol{F}'_B \simeq \boldsymbol{K}_5$,

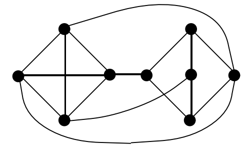
$\boldsymbol{G}'$ is isomorphic to one of $\boldsymbol{\mathcal{D}}_1, \boldsymbol{\mathcal{E}}_6, \boldsymbol{\mathcal{F}}_6 \in \Lambda_0$ for $\boldsymbol{F}'_A \simeq \boldsymbol{F}'_B \simeq \boldsymbol{K}_{3,3}$, and $\boldsymbol{G}'$ is isomorphic to one of $\boldsymbol{\mathcal{C}}_2, \boldsymbol{\mathcal{D}}_4 \in \Lambda_0$ for $\boldsymbol{F}'_A \simeq \boldsymbol{K}_5$, $\boldsymbol{F}'_B \simeq \boldsymbol{K}_{3,3}$, which is a contradiction to our assumption. (See Fig. 5 for these graphs.)

Next, let $k = 3$, and suppose for a contradiction that neither (A, B) nor (B, A) are flat. Notice that the assumptions guarantee that $\boldsymbol{G}$ is 3-connected in this case. By Theorem 1.3, $\boldsymbol{G}$ has a minor isomorphic to $\boldsymbol{M} = \boldsymbol{F}_i$ for some $i \in \{1, 2, 3, 4\}$, where $\boldsymbol{F}_1 = \boldsymbol{K}_{1,2,2,2}, \boldsymbol{F}_2 = \boldsymbol{\mathcal{B}}_7, \boldsymbol{F}_3 = \boldsymbol{\mathcal{C}}_3, \boldsymbol{F}_4 = \boldsymbol{\mathcal{D}}_2$ are as in Lemma 3.4. Let i be the maximum such integer. Since $\boldsymbol{G}$ has no minor isomorphic to a member of Λ_0, we deduce from Lemma 3.4(b) that $\boldsymbol{G}$ has a subgraph $\boldsymbol{M}_1$ isomorphic to a subdivision of $\boldsymbol{M}$. Let $W \subseteq V(\boldsymbol{M}_1)$ be the set of the vertices that have degree more than two in $\boldsymbol{M}_1$. Since $\boldsymbol{M}$ is internally 4-connected, one of the sets $A - B, B - A$, say $B - A$, contains at most one vertex of W. Thus, by Lemma 3.6, $\boldsymbol{G}$ has a minor isomorphic to $\boldsymbol{N} = \boldsymbol{M} \,\langle\!\!-\, \{b_1, b_2, b_3\}$ for some three distinct vertices $b_1, b_2, b_3 \in V(\boldsymbol{M})$.

We are going to show that the graph $\boldsymbol{N}$ (and hence also $\boldsymbol{G}$) has a minor isomorphic to some member of Λ_0 for any choice of $\{b_1, b_2, b_3\}$ from $V(\boldsymbol{M})$. That will be the desired contradiction. Let the vertices of $\boldsymbol{M}$ be numbered as in Fig. 4. Suppose first that no cubic vertex in $\boldsymbol{M}$ has the neighbors b_1, b_2, b_3.

- If $\{b_1, b_2, b_3\}$ includes any one of the pairs $\{1, 4\}$ or $\{2, 5\}$ or $\{3, 6\}$, say $\{b_1, b_2\}$, then $\boldsymbol{N}$ contains an $\boldsymbol{M} + b_1b_2$ minor. Since b_1b_2 is not an edge of $\boldsymbol{M}$, and it is not violating in $\boldsymbol{M} + b_1b_2$, it follows from Lemma 3.4(a) that $\boldsymbol{M} + b_1b_2$ has a minor isomorphic to a member of Λ_0.

- If $\boldsymbol{M} - \{b_1, b_2, b_3\}$ contains a subgraph $\boldsymbol{G}_0$ isomorphic to $\boldsymbol{K}_4$, then, by the 3-connectivity of $\boldsymbol{M}$, for some $v \in V(\boldsymbol{G}_0)$ there exist three disjoint paths between the sets $V(\boldsymbol{G}_0) - \{v\}$ and $\{b_1, b_2, b_3\}$ in $\boldsymbol{M} - v$. Moreover, since $\boldsymbol{M}$ is internally 4-connected, there exists a path between v and some vertex of $\{b_1, b_2, b_3\}$ avoiding $V(\boldsymbol{G}_0) - \{v\}$. Thus $\boldsymbol{N}$ contains an $\boldsymbol{\mathcal{E}}_{19} \in \Lambda_0$ minor, see Fig. 6.



$\boldsymbol{\mathcal{E}}_{19}$

Figure 6:

Let $\boldsymbol{N}'$ ($\boldsymbol{M}'$) be the graph obtained from $\boldsymbol{N}$ ($\boldsymbol{M}$) by $Y\Delta$-transformations of all vertices from the set $(\{8, 9, 10\} \cap V(\boldsymbol{M})) - \{b_1, b_2, b_3\}$. If $\boldsymbol{M}' \simeq \boldsymbol{K}_{1,2,2,2}$, and none of the two above general cases apply, then there is just one possibility, up to symmetry.

– For $\{b_1, b_2, b_3\} = \{1, 2, 7\}$, the resulting graph $\boldsymbol{N}'$ has a $\boldsymbol{\mathcal{D}}_3$ minor via contracting the edge $\{2, 3\}$.

If $\boldsymbol{M}' \simeq \boldsymbol{\mathcal{B}}_7$, then $|\{8, 9, 10\} \cap \{b_1, b_2, b_3\}| = 1$, and so $8 \in \{b_1, b_2, b_3\}$ by symmetry. Then all three remaining nonsymmetric possibilities are as follows.

– For $\{b_1, b_2, b_3\} = \{4, 6, 8\}$ or $\{b_1, b_2, b_3\} = \{6, 7, 8\}$, the graph $\boldsymbol{N}'$ has a $\boldsymbol{K}_{3,5}$ minor via contracting the edges $\{2, 6\}$ and $\{4, 7\}$.

– For $\{b_1, b_2, b_3\} = \{5, 7, 8\}$, the graph $\boldsymbol{N}'$ has a $\boldsymbol{\mathcal{D}}_3$ minor via contracting the edges $\{1, 8\}$ and $\{3, 8\}$.

If $\boldsymbol{M}' \simeq \boldsymbol{\mathcal{C}}_3$, then $8, 9 \in \{b_1, b_2, b_3\}$ by symmetry. There are two nonsymmetric possibilities remaining to be checked.

– For $\{b_1, b_2, b_3\} = \{6, 8, 9\}$ or $\{b_1, b_2, b_3\} = \{7, 8, 9\}$, the graph $\boldsymbol{N}'$ has a $\boldsymbol{K}_{3,5}$ minor via contracting the edges $\{1, 2\}$, $\{4, 5\}$, $\{6, 7\}$.

And if $\boldsymbol{M}' \simeq \boldsymbol{\mathcal{D}}_2$, then $\{b_1, b_2, b_3\} = \{8, 9, 10\}$, but the graph $\boldsymbol{M}' - \{8, 9, 10\}$ has a $\boldsymbol{K}_4$ subgraph, so this case was already covered above. Since $\boldsymbol{N}'$ has a Λ_0 minor, so does $\boldsymbol{N}$ by Lemma 3.3, a contradiction.

Finally, consider the case that b_1, b_2, b_3 are the neighbors of a cubic vertex w in $\boldsymbol{M}$. Similarly as above, let $\boldsymbol{N}'$ be the graph obtained from $\boldsymbol{N}$ by $Y\Delta$-transformations of the vertices $(\{8, 9, 10\} \cap V(\boldsymbol{M})) - \{w, b_1, b_2, b_3\}$. It is easy to see that there are only two nonsymmetric possibilities to consider.

– The graph $\boldsymbol{N}'$ constructed from $\boldsymbol{\mathcal{B}}_7$ by adding a new vertex t adjacent to the neighbors of $s = 8$ has a $\boldsymbol{\mathcal{D}}_3$ minor via contracting the edge $\{2, 4\}$.

– The graph $\boldsymbol{N}'$ constructed from $\boldsymbol{\mathcal{C}}_3$ by adding a new vertex t adjacent to the neighbors of $s = 3$ has a $\boldsymbol{K}_{3,5}$ minor via contracting the edges $\{1, 8\}$, $\{4, 9\}$.

So it follows from Lemma 3.3 that $\boldsymbol{N}$ (and hence also $\boldsymbol{G}$) has a minor isomorphic to some member of Λ_0, a contradiction. ∎

4 A Splitter Theorem

This section presents a key tool that is used, together with the above lemmas, in the search for possible counterexamples to Negami's conjecture. Recall the definition of a violating edge and pair from page 9. First, few more specific terms are introduced. Let $\boldsymbol{G}$ be an internally 4-connected graph, and let $\boldsymbol{G}_0 = \boldsymbol{G}, \boldsymbol{G}_1, \ldots, \boldsymbol{G}_t = \boldsymbol{H}$ be a sequence of graphs such that, for $i = 1, \ldots, t$,

– $\boldsymbol{G}_i$ is a simple graph obtained by adding an edge to $\boldsymbol{G}_{i-1}$,

– no edge is violating in both $\boldsymbol{G}_{i-1}$ and $\boldsymbol{G}_i$,

– $\boldsymbol{G}_i$, $i < t$ has at most one violating pair, and $\boldsymbol{H}$ is internally 4-connected.

Then $\boldsymbol{H}$ is called a *(t-step) addition extension* of $\boldsymbol{G}$. An example of an addition extension with intermediate violating edges is presented in Fig. 7.

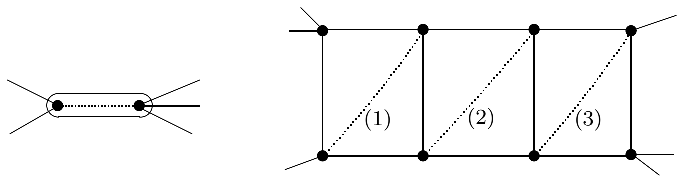


Figure 7: An illustration of a vertex splitting, and of a 3-step addition extension.

Let $\boldsymbol{G}$ be an internally 4-connected graph. Suppose that u_1, u_2, u_3, u_4 are four cubic vertices forming a cycle of length four in $\boldsymbol{G}$. The graph $\boldsymbol{H}$ obtained from $\boldsymbol{G}$ by adding a new vertex x connected to the vertices u_1, u_2, u_3, u_4 is called a *quadrangular extension* of $\boldsymbol{G}$, and it is denoted by $\boldsymbol{H} = \boldsymbol{G} \boxtimes \{u_1, u_2, u_3, u_4\}$.
Suppose that v_1, v_2, v_3, v_4, v_5 in this order are five vertices of a 5-cycle in $\boldsymbol{G}$, and that v_2, v_5 have degree three. Then the graph $\boldsymbol{H}$ obtained from $\boldsymbol{G}$ by subdividing the edge v_3v_4 with a new vertex y and adding the edge v_1y is called a *pentagonal extension* of $\boldsymbol{G}$, and it is formally denoted by $\boldsymbol{H} = \boldsymbol{G} ⬠ \{v_1;\, v_3, v_4\}$.
Suppose that $w_1, w_2, w_3, w_4, w_5, w_6$ in this order are six vertices of a 6-cycle in $\boldsymbol{G}$. Moreover, assume that w_1, w_3, w_5 have degree three, and that no cubic vertex of $\boldsymbol{G}$ has neighbors w_2, w_4, w_6. Then the graph $\boldsymbol{H}$ obtained from $\boldsymbol{G}$ by adding a new cubic vertex z connected to the vertices w_2, w_4, w_6 is called a *hexagonal extension* of $\boldsymbol{G}$, and it is formally denoted by $\boldsymbol{H} = \boldsymbol{G} ⬡ \{w_2, w_4, w_6\}$.

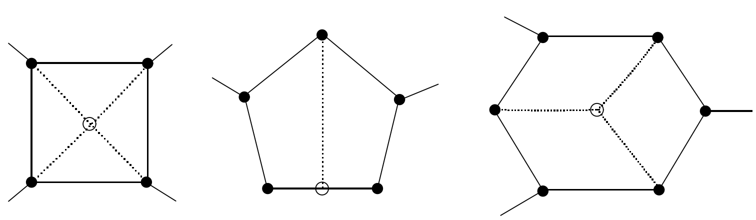


Figure 8: An illustration of a quadrangular, pentagonal, and hexagonal extensions.

Notice that each of the above operations of addition, quadrangular, pentagonal and hexagonal extension preserves internal 4-connectivity; but this is not always true for the vertex splitting operation. The following theorem is a simplified version of a result proved in [8].

Theorem 4.1 (T. Johnson, R. Thomas, 1999) *Suppose that $\boldsymbol{G}$ and $\boldsymbol{H}$ are internally 4-connected graphs, that $\boldsymbol{G}$ is a proper minor of $\boldsymbol{H}$, and that $\boldsymbol{G}$ has no embedding in the projective plane. Assume further that each component of the subgraph of $\boldsymbol{G}$ induced by cubic vertices is a tree or a cycle. Then either $\boldsymbol{H}$ is an addition extension of $\boldsymbol{G}$, or there exists a minor $\boldsymbol{H}'$ of $\boldsymbol{H}$ satisfying one of the following: $\boldsymbol{H}'$ is a 1-step addition extension of $\boldsymbol{G}$, or $\boldsymbol{H}'$ is a quadrangular, pentagonal or hexagonal extension of $\boldsymbol{G}$, or $\boldsymbol{H}'$ is obtained by splitting a vertex of $\boldsymbol{G}$.*

Please note that if the graph $\boldsymbol{H}'$ is obtained by splitting a vertex of $\boldsymbol{G}$, then $\boldsymbol{H}'$ need not be internally 4-connected. There is a different version of Theorem 4.1 (see [8]) in which all the outcome graphs $\boldsymbol{H}'$ are internally 4-connected. However, the above stated theorem is better suited for our application.

5 The Generation Process

Our objective is to prove that if an internally 4-connected graph $\boldsymbol{H}$ has a minor isomorphic to one of $\boldsymbol{K}_{1,2,2,2}, \boldsymbol{\mathcal{B}}_7, \boldsymbol{\mathcal{C}}_3, \boldsymbol{\mathcal{D}}_2$, then either $\boldsymbol{H}$ itself is isomorphic to one of the 16 specific graphs defined later in this section, or $\boldsymbol{H}$ has a minor isomorphic to a graph from Λ_0. For the readers' convenience, the proof is divided into four steps in Lemmas 5.1, 5.2, 5.3, and 5.4.

Lemma 5.1 *Let $\boldsymbol{H}$ be an internally 4-connected graph having no minor isomorphic to a graph from Λ_0. If $\boldsymbol{H}$ contains a $\boldsymbol{K}_{1,2,2,2}$ minor, then either $\boldsymbol{H}$ is isomorphic to $\boldsymbol{K}_{1,2,2,2}$ or it contains a $\boldsymbol{\mathcal{B}}_7$ minor.*

Proof. Assume that $\boldsymbol{K}_{1,2,2,2}$ is a proper minor of $\boldsymbol{H}$. Note that $\boldsymbol{K}_{1,2,2,2}$ has no cubic vertices; hence, by Theorem 4.1, $\boldsymbol{H}$ contains a minor $\boldsymbol{H}'$ obtained by adding an edge or by splitting a vertex in $\boldsymbol{K}_{1,2,2,2}$. But $\boldsymbol{H}'$ cannot have a minor isomorphic to a member of Λ_0, thus $\boldsymbol{H}'$ has a $\boldsymbol{\mathcal{B}}_7$ subgraph by Lemma 3.4. ∎

Suppose that the vertices of $\boldsymbol{\mathcal{B}}_7$ are numbered as in Fig. 4 (p. 9). Let $\boldsymbol{\mathcal{B}}'_7$ denote the graph obtained from $\boldsymbol{\mathcal{B}}_7$ by adding the edge $\{7,8\}$, and let $\boldsymbol{\mathcal{B}}''_7$ denote the graph obtained from $\boldsymbol{\mathcal{B}}'_7$ by adding the edge $\{1,5\}$. (See Appendix B.)

Lemma 5.2 *Let $\boldsymbol{H}$ be an internally 4-connected graph having no minor isomorphic to a graph from Λ_0. If $\boldsymbol{H}$ contains a $\boldsymbol{\mathcal{B}}_7$ minor, then either $\boldsymbol{H}$ is isomorphic to one of $\boldsymbol{\mathcal{B}}_7, \boldsymbol{\mathcal{B}}'_7, \boldsymbol{\mathcal{B}}''_7$, or it contains a $\boldsymbol{\mathcal{C}}_3$ minor.*

Proof. If the graph $\boldsymbol{H}$ is isomorphic to $\boldsymbol{\mathcal{B}}_7$, or $\boldsymbol{H}$ has a $\boldsymbol{\mathcal{C}}_3$ minor, then the proof is finished. So assume that $\boldsymbol{\mathcal{B}}_7$ is a proper minor of $\boldsymbol{H}$, and that $\boldsymbol{\mathcal{C}}_3$ is not a minor of $\boldsymbol{H}$. Note that $\boldsymbol{\mathcal{B}}_7$ has only one cubic vertex, so Theorem 4.1 can be applied. Consider first the case that $\boldsymbol{H}$ is an addition extension of $\boldsymbol{\mathcal{B}}_7$: If the set $E(\boldsymbol{H}) - E(\boldsymbol{\mathcal{B}}_7)$ contained any other edge than $\{1,3\},\{1,5\},\{3,5\}$ or $\{7,8\}$, then $\boldsymbol{H}$ would have a Λ_0 minor by Lemma 3.4(a). Moreover, since the graph $\boldsymbol{\mathcal{B}}_7 + \{7,8\} + \{1,3\} + \{1,5\}$ has a $\boldsymbol{K}_7{-}\boldsymbol{C}_4 \in \Lambda_0$ minor via contracting the edge $\{3,4\}$, there are only two possibilities $\boldsymbol{H} = \boldsymbol{\mathcal{B}}_7'$ or $\boldsymbol{H} = \boldsymbol{\mathcal{B}}_7''$, up to symmetry.

Otherwise, the graph $\boldsymbol{H}$ has a minor $\boldsymbol{H}'$ obtained from $\boldsymbol{\mathcal{B}}_7$ by a 1-step addition extension or by splitting a vertex. (There are not enough cubic vertices in $\boldsymbol{\mathcal{B}}_7$ to apply the other extension operations.) ¿From Lemma 3.4 and the fact that $\boldsymbol{H}$ has no Λ_0 minor, it follows that $\boldsymbol{H}' = \boldsymbol{\mathcal{B}}_7'$. If $\boldsymbol{H} \neq \boldsymbol{H}'$, then, by Theorem 4.1 again, $\boldsymbol{H}$ has a minor $\boldsymbol{H}''$ obtained as a 1-step addition extension or by splitting a vertex of $\boldsymbol{H}' = \boldsymbol{\mathcal{B}}_7'$. In the latter case when $\boldsymbol{H}'' = \boldsymbol{\mathcal{B}}_7' \cdot \{w \angle {}^{N_1}_{N_2}\}$ for $w \neq 7,8$, or for $w = 7$ and $|N_1 - \{8\}| \geq 2$ and $|N_2 - \{8\}| \geq 2$, the same splitting operation can be applied to $\boldsymbol{\mathcal{B}}_7$, and hence the statement follows from Lemma 3.4(b). So it remains to check, up to symmetry, the following possibilities.

- $\boldsymbol{H}'' = \boldsymbol{\mathcal{B}}_7' \cdot \{8 \angle {}^{1,7}_{3,5}\}$ and $\boldsymbol{H}'' = \boldsymbol{\mathcal{B}}_7' \cdot \{7 \angle {}^{1,8}_{2,3,4,5,6}\}$ have $\boldsymbol{\mathcal{C}}_3$ subgraphs.
- $\boldsymbol{H}'' = \boldsymbol{\mathcal{B}}_7' \cdot \{7 \angle {}^{2,8}_{1,3,4,5,6}\}$ has a $\boldsymbol{K}_7{-}\boldsymbol{C}_4$ minor via contracting $\{2,7\}$, $\{5,8\}$.

Therefore, consider that $\boldsymbol{H}''$ is a 1-step addition extension of $\boldsymbol{\mathcal{B}}_7'$. Using the same arguments as above, it follows that actually $\boldsymbol{H}'' \simeq \boldsymbol{\mathcal{B}}_7''$. Without loss of generality, assume that $\boldsymbol{\mathcal{B}}_7''$ is a proper minor of $\boldsymbol{H}$, and apply Theorem 4.1 once again. Since $\boldsymbol{H}$ cannot be an addition extension of $\boldsymbol{\mathcal{B}}_7''$ by the above arguments, there exists a minor $\boldsymbol{H}^o$ of $\boldsymbol{H}$ obtained by splitting a vertex in $\boldsymbol{\mathcal{B}}_7''$. All nonsymmetric possibilities not covered by Lemma 3.4(b) are listed next.

- $\boldsymbol{H}^o = \boldsymbol{\mathcal{B}}_7'' \cdot \{5 \angle {}^{1,8}_{4,6,7}\}$ has a $\boldsymbol{\mathcal{C}}_3$ subgraph.
- $\boldsymbol{H}^o = \boldsymbol{\mathcal{B}}_7'' \cdot \{5 \angle {}^{1,6}_{4,7,8}\}$ has a $\boldsymbol{K}_{4,5}{-}\boldsymbol{M}_4 \in \Lambda_0$ subgraph.
- $\boldsymbol{H}^o = \boldsymbol{\mathcal{B}}_7'' \cdot \{5 \angle {}^{1,7}_{4,6,8}\}$ has a $\boldsymbol{\mathcal{D}}_3 \in \Lambda_0$ minor via contracting $\{5,8\}$.
- $\boldsymbol{H}^o = \boldsymbol{\mathcal{B}}_7'' \cdot \{5 \angle {}^{1,4}_{6,7,8}\}$ has a $\boldsymbol{\mathcal{D}}_3 \in \Lambda_0$ minor via contracting $\{3,8\}$. ∎

Suppose that the vertices of $\boldsymbol{\mathcal{C}}_3$ are numbered as in Fig. 4. Let $\boldsymbol{\mathcal{C}}_3'$ be the graph obtained from $\boldsymbol{\mathcal{C}}_3$ by adding the edges $\{7,8\}$, $\{3,5\}$, let $\boldsymbol{\mathcal{C}}_3''$ be the graph obtained from $\boldsymbol{\mathcal{C}}_3'$ by adding the edge $\{7,9\}$, and let $\boldsymbol{\mathcal{C}}_3^{\circ}$, $\boldsymbol{\mathcal{C}}_3^{\bullet}$ be the graphs obtained from $\mathcal{C}_3''$ by adding the edges $\{2,4\}$, $\{2,3\}$, respectively. (See Appendix B.)

Lemma 5.3 *Let $\boldsymbol{H}$ be an internally 4-connected graph having no minor isomorphic to a graph from Λ_0. If $\boldsymbol{H}$ contains a $\mathcal{C}_3$ minor, then either $\boldsymbol{H}$ is isomorphic to one of $\mathcal{C}_3, \mathcal{C}_3', \mathcal{C}_3'', \mathcal{C}_3^\bullet, \mathcal{C}_3^\circ$, or it contains a $\mathcal{D}_2$ minor.*

Proof. Assume that $\mathcal{C}_3$ is a proper minor of $\boldsymbol{H}$, and that $\boldsymbol{H}$ has no $\mathcal{D}_2$ minor. Since $\mathcal{C}_3$ is internally 4-connected, and the cubic vertices in $\mathcal{C}_3$ induce a path of length two, the assumptions of Theorem 4.1 are fulfilled. Notice that a quadrangular or hexagonal extension cannot be applied to $\mathcal{C}_3$. Thus, if $\boldsymbol{H}$ is not an addition extension of $\mathcal{C}_3$, then there exists a minor $\boldsymbol{H}'$ of $\boldsymbol{H}$ obtained from $\mathcal{C}_3$ by splitting a vertex, or as a pentagonal extension, or as a 1-step addition extension. However, by Lemma 3.4 and the fact that $\boldsymbol{H}$ has no Λ_0 minor, the graph $\boldsymbol{H}'$ neither can be obtained by splitting a vertex, nor can be a 1-step addition extension of $\mathcal{C}_3$.

In order to apply a pentagonal extension to a graph, one must find, in particular, two cubic vertices with a common neighbor. It is easy to see that there is just one such pair in $\mathcal{C}_3$ – the vertices $8,9$ with a common neighbor 3. So, up to symmetry, there is only one possibility to check.

– $\boldsymbol{H}' = \mathcal{C}_3 ⌂ \{3;\, 4,5\}$ has a $\boldsymbol{K}_{4,5} - \boldsymbol{M}_4 \in \Lambda_0$ minor via contracting $\{1,8\}$.

Thus the graph $\boldsymbol{H}$ is an addition extension of $\mathcal{C}_3$. Lemma 3.4(a) implies that only a subset of the edges $\{1,3\},\{1,5\},\{3,5\},\{2,3\},\{2,4\},\{3,4\}$, $\{7,8\},\{7,9\},\{8,9\}$ may be added in $\boldsymbol{H}$. If the edge $\{8,9\}$ was added in $\boldsymbol{H}$, then also some edge incident with 3 would have to be added to keep $\boldsymbol{H}$ internally 4-connected. Up to symmetry, there is only the following possibility:

– $\boldsymbol{H} = \mathcal{C}_3 + \{8,9\} + \{1,3\}$ has a $\boldsymbol{K}_7 - \boldsymbol{C}_4$ minor via contracting $\{2,9\}$, $\{8,5\}$.

Now suppose, for a contradiction, that at least two edges $\{x,y\}$ and $\{x',y'\}$ from $\{1,3\},\{1,5\},\{3,5\}$ were added in $\boldsymbol{H}$. Since $\boldsymbol{H}$ is internally 4-connected, another edge incident with vertex 8 has to be added in $\boldsymbol{H}$, and so it is the edge $\{8,7\}$. However, the graph $(\mathcal{C}_3 + xy + x'y' + \{8,7\})$ Y∆ $\{9\}$ is isomorphic to the graph $\mathcal{B}_7 + \{7,8\} + \{1,3\} + \{1,5\}$ which has a $\boldsymbol{K}_7 - \boldsymbol{C}_4$ minor by the proof of Lemma 5.2, and hence $\boldsymbol{H}$ would have a Λ_0 minor by Lemma 3.3. The situation is symmetric for the edges $\{2,3\},\{2,4\},\{3,4\}$.

Therefore the graph $\boldsymbol{H}$ is obtained from $\mathcal{C}_3$ by adding at most one of the edges $\{1,3\},\{1,5\},\{3,5\}$, at most one of the edges $\{2,3\},\{2,4\},\{3,4\}$, and a subset of the edges $\{7,8\},\{7,9\}$. Moreover, if one of $\{1,3\},\{1,5\},\{3,5\}$ is added, then $\{7,8\}$ must be added in order to keep $\boldsymbol{H}$ internally 4-connected. Similarly, if $\{7,8\}$ is added, then one of $\{1,3\},\{5,3\},\{2,3\},\{4,3\}$ must be added, too. So the possibilities for $\boldsymbol{H}$ are as follows, up to symmetry.

- $\boldsymbol{\mathcal{C}}_3 + \{3,5\} + \{7,8\}$ is the graph $\boldsymbol{\mathcal{C}}_3'$.
- $\boldsymbol{\mathcal{C}}_3 + \{3,5\} + \{7,8\} + \{7,9\}$ is the graph $\boldsymbol{\mathcal{C}}_3''$.
- $\boldsymbol{\mathcal{C}}_3 + \{3,5\} + \{3,2\} + \{7,8\} + \{7,9\}$ is the graph $\boldsymbol{\mathcal{C}}_3^{\bullet}$.
- $\boldsymbol{\mathcal{C}}_3 + \{3,5\} + \{2,4\} + \{7,8\} + \{7,9\}$ is the graph $\boldsymbol{\mathcal{C}}_3^{\circ}$.
- $\boldsymbol{\mathcal{C}}_3 + \{3,5\} + \{3,4\} + \{7,8\} + \{7,9\}$ is isomorphic to the graph $\boldsymbol{\mathcal{C}}_3^{\circ}$. ∎

Suppose that the vertices of $\boldsymbol{\mathcal{D}}_2$ are numbered as in Fig. 4. Let $\boldsymbol{\mathcal{D}}_2'$ be the graph obtained from $\boldsymbol{\mathcal{D}}_2$ by adding the edges $\{7,8\}$, $\{1,3\}$, let $\boldsymbol{\mathcal{D}}_2''$ be obtained from $\boldsymbol{\mathcal{D}}_2'$ by adding the edges $\{7,9\}$, $\{2,3\}$, and let $\boldsymbol{\mathcal{D}}_2'''$ be obtained from $\boldsymbol{\mathcal{D}}_2''$ by adding the edge $\{7,10\}$. Let $\boldsymbol{\mathcal{D}}_2^{\bullet}$, $\boldsymbol{\mathcal{D}}_2^{\circ}$ be the graphs obtained from $\boldsymbol{\mathcal{D}}_2'''$ by adding the edges $\{1,2\}$, $\{1,6\}$, respectively. Let $\boldsymbol{\mathcal{D}}_2^{\star}$ be the graph obtained from $\boldsymbol{\mathcal{D}}_2$ by adding the edges $\{1,5\},\{3,4\},\{2,6\},\{7,8\},\{7,9\},\{7,10\}$. (See Appendix B.)

Lemma 5.4 *Let $\boldsymbol{H}$ be an internally 4-connected graph having no minor isomorphic to a graph from Λ_0. If $\boldsymbol{H}$ contains a $\boldsymbol{\mathcal{D}}_2$ minor, then $\boldsymbol{H}$ is isomorphic to one of $\boldsymbol{\mathcal{D}}_2, \boldsymbol{\mathcal{D}}_2', \boldsymbol{\mathcal{D}}_2'', \boldsymbol{\mathcal{D}}_2'''$, $\boldsymbol{\mathcal{D}}_2^{\bullet}, \boldsymbol{\mathcal{D}}_2^{\circ}, \boldsymbol{\mathcal{D}}_2^{\star}$.*

Proof. Assume that $\boldsymbol{\mathcal{D}}_2$ is a proper minor of $\boldsymbol{H}$. Since $\boldsymbol{\mathcal{D}}_2$ is internally 4-connected, and the cubic vertices in $\boldsymbol{\mathcal{D}}_2$ induce a cycle of length six, the assumptions of Theorem 4.1 are fulfilled. Note that no quadrangular extension can be applied to $\boldsymbol{\mathcal{D}}_2$. Moreover, by Lemma 3.4, neither a vertex splitting nor a 1-step addition extension is allowed in this particular case. Thus, if $\boldsymbol{H}$ is not an addition extension of $\boldsymbol{\mathcal{D}}_2$, then there exists a minor $\boldsymbol{H}'$ of $\boldsymbol{H}$ obtained from $\boldsymbol{\mathcal{D}}_2$ as a pentagonal or hexagonal extension.

The proof follows the same steps as the previous proof. In order to apply a pentagonal or hexagonal extension to a graph, one must find, in particular, two cubic vertices with a common neighbor. Up to symmetry, there are two such pairs in the graph $\boldsymbol{\mathcal{D}}_2$ – the vertices $1,3$ with common neighbors 8 and 7, or the vertices $8,9$ with a common neighbor 3. Since only the second pair allows a pentagonal extension, and since a hexagonal extension is not possible for the triple $\{7,9,10\}$, the following cases have to be checked:

- $\boldsymbol{H}' = \boldsymbol{\mathcal{D}}_2$ ⬠$\{3;4,5\}$ has a $\boldsymbol{K}_{4,5}-\boldsymbol{M}_4$ minor via contracting $\{1,8\}$, $\{2,10\}$.
- $\boldsymbol{H}' = \boldsymbol{\mathcal{D}}_2$ ⬡$\{1,2,3\}$ and $\boldsymbol{H}' = \boldsymbol{\mathcal{D}}_2$ ⬡$\{8,9,10\}$ have $\boldsymbol{K}_{4,5}-\boldsymbol{M}_4$ minors via contracting $\{4,5\},\{4,6\}$.

Thus the graph $\boldsymbol{H}$ is an addition extension of $\boldsymbol{\mathcal{D}}_2$. Similarly as in the previous proof, Lemma 3.4(a) implies that only a subset of the edges $\{1,3\},\{1,5\},\{3,5\}$, $\{2,3\},\{2,4\},\{3,4\}$, $\{1,2\},\{1,6\},\{2,6\}$,

$\{7,8\},\{7,9\},\{7,10\}$, $\{8,9\},\{8,10\},\{9,10\}$ may be added in $\boldsymbol{H}$. If the edge $\{8,9\}$ was present in $\boldsymbol{H}$, then also some edge xy incident with 3 would have to be added to keep $\boldsymbol{H}$ internally 4-connected. However, it was shown in the proof of Lemma 5.3 that the graph $(\boldsymbol{\mathcal{D}}_2+\{8,9\}+xy)\ \mathrm{Y}\!\Delta\ \{10\} \simeq \boldsymbol{\mathcal{C}}_3+\{8,9\}+\{1,3\}$ has a $\boldsymbol{K}_7{-}\boldsymbol{C}_4$ minor, which contradicts our assumptions by Lemma 3.3. By symmetry the same applies to the edges $\{9,10\}$ and $\{8{,}10\}$. Similarly, if two distinct edges $\{x,y\}$ and $\{x',y'\}$ from $\{1,3\},\{1,5\},\{3,5\}$ belonged to $\boldsymbol{H}$, then also the edge $\{7,8\}$ would have to be in $\boldsymbol{H}$; but the graph $(\boldsymbol{\mathcal{D}}_2+xy+x'y'+\{7,8\})\ \mathrm{Y}\!\Delta\ \{9\}\ \mathrm{Y}\!\Delta\ \{10\}$ has a $\boldsymbol{K}_7{-}\boldsymbol{C}_4$ minor by the proof of Lemma 5.2, again a contradiction.

Therefore the graph $\boldsymbol{H}$ is obtained from $\boldsymbol{\mathcal{D}}_2$ by adding at most one of the edges $\{1,3\},\{1,5\},\{3,5\}$, at most one of the edges $\{2,3\},\{2,4\},\{3,4\}$, at most one of the edges $\{1,2\},\{1,6\},\{2,6\}$, and a subset of the edges $\{7,8\},\{7,9\},\{7,10\}$. Moreover, if one of $\{1,3\},\{1,5\},\{3,5\}$ is added, then $\{7,8\}$ must be added in order to keep $\boldsymbol{H}$ internally 4-connected. Similarly, if $\{7,8\}$ is added, then either the edge $\{1,3\}$, or two other edges – one incident with 1 and one incident with 3, must be added in $\boldsymbol{H}$, too. Therefore, using symmetry, the possibilities for the graph $\boldsymbol{H}$ are as follows, ordered by the number of edges.

- $\boldsymbol{\mathcal{D}}_2+\{1,3\}+\{7,8\}$ is the graph $\boldsymbol{\mathcal{D}}_2'$.
- $\boldsymbol{\mathcal{D}}_2'+\{2,3\}+\{7,9\}$ is the graph $\boldsymbol{\mathcal{D}}_2''$.
- $\boldsymbol{\mathcal{D}}_2'+\{2,4\}+\{7,9\}$ is isomorphic to the graph $\boldsymbol{\mathcal{D}}_2''$.
- $\boldsymbol{\mathcal{D}}_2''+\{7,10\}$ is the graph $\boldsymbol{\mathcal{D}}_2'''$.
- $\boldsymbol{\mathcal{D}}_2'''+\{1,2\}$ is the graph $\boldsymbol{\mathcal{D}}_2^\bullet$.
- $\boldsymbol{\mathcal{D}}_2'''+\{1,6\}$ is the graph $\boldsymbol{\mathcal{D}}_2^\circ$.
- $\boldsymbol{\mathcal{D}}_2'+\{3,4\}+\{7,9\}+\{2,6\}+\{7,10\}$ is isomorphic to the graph $\boldsymbol{\mathcal{D}}_2^\circ$.
- $\boldsymbol{\mathcal{D}}_2+\{1,5\}+\{7,8\}+\{3,4\}+\{7,9\}+\{2,6\}+\{7,10\}$ is the graph $\boldsymbol{\mathcal{D}}_2^\star$.

∎

Proof of Theorem 2.2. Setting $\Pi=\{\boldsymbol{K}_{1,2,2,2}, \boldsymbol{\mathcal{B}}_7, \boldsymbol{\mathcal{B}}_7', \boldsymbol{\mathcal{B}}_7'', \boldsymbol{\mathcal{C}}_3, \boldsymbol{\mathcal{C}}_3', \boldsymbol{\mathcal{C}}_3'', \boldsymbol{\mathcal{C}}_3^\circ, \boldsymbol{\mathcal{C}}_3^\bullet, \boldsymbol{\mathcal{D}}_2, \boldsymbol{\mathcal{D}}_2', \boldsymbol{\mathcal{D}}_2'', \boldsymbol{\mathcal{D}}_2''', \boldsymbol{\mathcal{D}}_2^\circ, \boldsymbol{\mathcal{D}}_2^\bullet, \boldsymbol{\mathcal{D}}_2^\star\}$, the statement of the theorem immediately follows from Corollaries 2.3, 2.5, and Lemmas 5.1, 5.2, 5.3, 5.4.

∎

6 Excluded minors

Theorem 2.2 describes all possible counterexamples to Conjecture 1.1. Furthermore, Lemma 1.2 and Proposition 1.4 allow us to deduce certain implications among members of Π. Let us write $\boldsymbol{G}\xrightarrow[\text{NC}]{}\boldsymbol{H}$ to mean that "if $\boldsymbol{G}$ has no planar cover, then neither has $\boldsymbol{H}$".

Proposition 6.1 *The following relations hold among the members of* Π.

$$\boldsymbol{K}_{1,2,2,2} \xrightarrow[NC]{} \boldsymbol{\mathcal{B}}_7 \xrightarrow[NC]{} \boldsymbol{\mathcal{B}}_7' \xrightarrow[NC]{} \boldsymbol{\mathcal{B}}_7'' \xrightarrow[NC]{} \boldsymbol{\mathcal{C}}_3 \xrightarrow[NC]{} \boldsymbol{\mathcal{C}}_3' \xrightarrow[NC]{} \boldsymbol{\mathcal{C}}_3'' \ldots$$

$$\begin{array}{ccccccccccccccc}
 & \nearrow_{NC} & \to\ \boldsymbol{\mathcal{C}}_3^{\bullet}\ \to & & & \searrow^{NC} & & & & & & \nearrow_{NC} & \boldsymbol{\mathcal{D}}_2^{\bullet} \\
\ldots \boldsymbol{\mathcal{C}}_3'' & \xrightarrow[NC]{} & \boldsymbol{\mathcal{C}}_3^{\circ} & \xrightarrow[NC]{} & \boldsymbol{\mathcal{D}}_2 & \xrightarrow[NC]{} & \boldsymbol{\mathcal{D}}_2' & \xrightarrow[NC]{} & \boldsymbol{\mathcal{D}}_2'' & \xrightarrow[NC]{} & \boldsymbol{\mathcal{D}}_2''' & \xrightarrow[NC]{} & \boldsymbol{\mathcal{D}}_2^{\circ} \\
 & & & & \updownarrow_{NC} & & & & & & & & \\
 & & & & \boldsymbol{\mathcal{D}}_2^{\star} & & & & & & & &
\end{array}$$

Proof. The relations in the first row of the diagram follow immediately from Lemma 1.2, or from the next argument:
$\boldsymbol{\mathcal{B}}_7''\xrightarrow[\text{NC}]{}\boldsymbol{\mathcal{C}}_3$: Notice that $\boldsymbol{\mathcal{B}}_7'' = \boldsymbol{\mathcal{C}}_3 + \{1,5\} + \{7,8\}$ $Y\Delta$ $\{9\}$, and that the edges $\{1,5\}$ and $\{7,8\}$ are added between neighbors of cubic vertices. Thus if $\boldsymbol{\mathcal{C}}_3$ had a planar cover, then so would have $\boldsymbol{\mathcal{B}}_7''$ by Proposition 1.4.

The relations in the second row of the diagram follow similarly, except for:
$\boldsymbol{\mathcal{D}}_2^{\star}\xrightarrow[\text{NC}]{}\boldsymbol{\mathcal{D}}_2$: As in the previous case, $\boldsymbol{\mathcal{D}}_2^{\star}$ can be obtained from $\boldsymbol{\mathcal{D}}_2$ by subdividing the edges $\{1,10\}$, $\{2,9\}$, $\{3,8\}$ with new vertices $11, 12, 13$ respectively, then adding the edges $\{11,7\}, \{12,7\}, \{13,7\}$, and finally $Y\Delta$-transforming the vertices $8, 9, 10$. Thus if $\boldsymbol{\mathcal{D}}_2$ had a planar cover, then so would have $\boldsymbol{\mathcal{D}}_2^{\star}$ by Proposition 1.4. ∎

It is possible that more relations between the graphs from Π can be derived in a similar fashion, but we did not pursue this possibility.

By Lemma 1.2 and the Graph Minor Theorem of Robertson and Seymour [12], there exists a finite set Σ of graphs such that a connected graph has no planar cover if and only if it has a minor isomorphic to a member of Σ. Since it is not known if Conjecture 1.1 holds, it is not even known whether $\boldsymbol{K}_{1,2,2,2} \in \Sigma$. Recall that Λ_0 denotes the family of all connected minor-minimal nonprojective graphs which are known to have no planar cover. The lemmas developed in this paper imply the following:

Theorem 6.2 *The set* Σ *is the union of* Λ_0 *and a subset* $\Pi_0 \subseteq \Pi - \{\boldsymbol{\mathcal{D}}_2^{\star}\}$ *such that no graph in* Π_0 *is a subgraph of another graph in* Π_0.

Proof. Clearly, no graph in Σ is a subgraph of another member of Σ, and all graphs in Σ are connected and have no embedding in the projective plane. Moreover, all graphs in Λ_0 are minor-minimal with respect to the property of having no planar cover, since all of their proper minors are projective-planar, and hence have double planar covers.

Let $\boldsymbol{G} \in \Sigma$. If $\boldsymbol{G}$ happens to be one of the graphs in Λ_0, the proof is done; otherwise $\boldsymbol{G}$ contains one of $\boldsymbol{K}_{1,2,2,2}, \boldsymbol{\mathcal{B}}_7, \boldsymbol{\mathcal{C}}_3, \boldsymbol{\mathcal{D}}_2$ as a minor by Theorem 1.3. In such a case it follows from Corollary 2.5 that $\boldsymbol{G}$ is internally 4-connected. The statement then follows from Lemmas 5.1, 5.2, 5.3, 5.4. Moreover, by Proposition 6.1, $\boldsymbol{\mathcal{D}}_2^\star$ is not a minor-minimal graph having no planar cover, since it has a planar cover if and only if so does $\boldsymbol{\mathcal{D}}_2 \subset \boldsymbol{\mathcal{D}}_2^\star$. ∎

(Notice that Theorem 6.2 does *not* seem to be a consequence of Theorem 2.2.)

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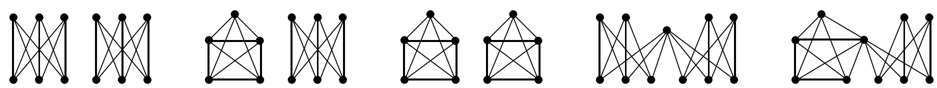
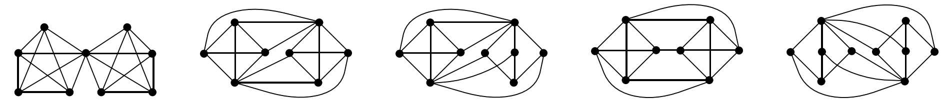
Appendix A: 35 minor minimal nonprojective graphs.

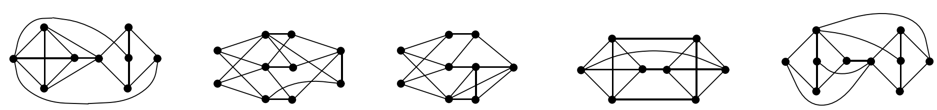
This section presents a list of all 35 minor minimal nonprojective graphs, as found in [3, 1]. (Our notation of these graphs mostly follows [3].)



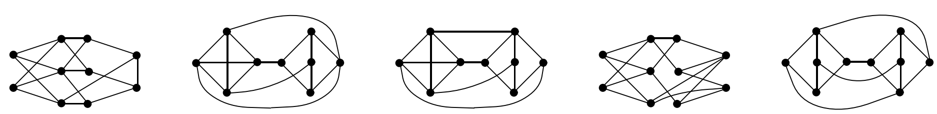
$\boldsymbol{K}_{3,3}+K_{3,3}$ $\quad$ $\boldsymbol{K}_5+\boldsymbol{K}_{3,3}$ $\quad$ $\boldsymbol{K}_5+\boldsymbol{K}_5$ $\quad$ $\boldsymbol{K}_{3,3}\cdot\boldsymbol{K}_{3,3}$ $\quad$ $\boldsymbol{K}_5\cdot\boldsymbol{K}_{3,3}$



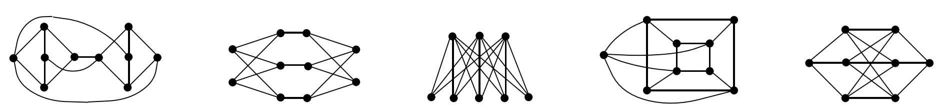
$\boldsymbol{K}_5\cdot\boldsymbol{K}_5$ $\quad$ $\boldsymbol{\mathcal{B}}_3$ $\quad$ $\boldsymbol{\mathcal{C}}_2$ $\quad$ $\boldsymbol{\mathcal{C}}_7$ $\quad$ $\boldsymbol{\mathcal{D}}_1$



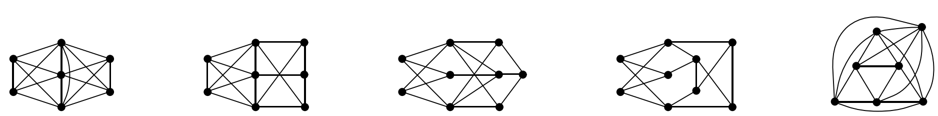
$\boldsymbol{\mathcal{D}}_4$ $\quad$ $\boldsymbol{\mathcal{D}}_9$ $\quad$ $\boldsymbol{\mathcal{D}}_{12}$ $\quad$ $\boldsymbol{\mathcal{D}}_{17}$ $\quad$ $\boldsymbol{\mathcal{E}}_6$



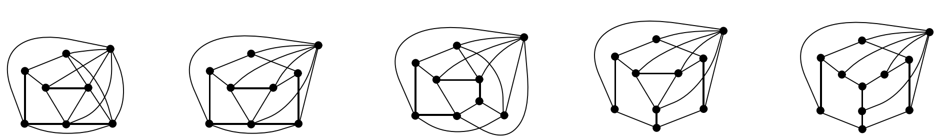
$\boldsymbol{\mathcal{E}}_{11}$ $\quad$ $\boldsymbol{\mathcal{E}}_{19}$ $\quad$ $\boldsymbol{\mathcal{E}}_{20}$ $\quad$ $\boldsymbol{\mathcal{E}}_{27}$ $\quad$ $\boldsymbol{\mathcal{F}}_4$



$\boldsymbol{\mathcal{F}}_6$ $\quad$ $\boldsymbol{\mathcal{G}}_1$ $\quad$ $\boldsymbol{K}_{3,5}$ $\quad$ $\boldsymbol{K}_{4,5}-\boldsymbol{M}_4$ $\quad$ $\boldsymbol{K}_{4,4}-e$



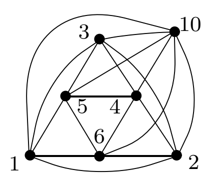
$\boldsymbol{K}_7-\boldsymbol{C}_4$ $\quad$ $\boldsymbol{\mathcal{D}}_3$ $\quad$ $\boldsymbol{\mathcal{E}}_5$ $\quad$ $\boldsymbol{\mathcal{F}}_1$ $\quad$ $\boldsymbol{K}_{1,2,2,2}$



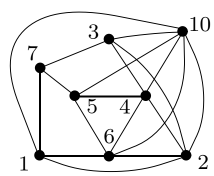
$\boldsymbol{\mathcal{B}}_7$ $\quad$ $\boldsymbol{\mathcal{C}}_3$ $\quad$ $\boldsymbol{\mathcal{C}}_4$ $\quad$ $\boldsymbol{\mathcal{D}}_2$ $\quad$ $\boldsymbol{\mathcal{E}}_2$

Appendix B: Possible counterexamples to Negami's conjecture.

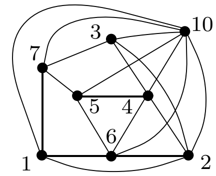
This section lists 16 internally 4-connected graphs that have no embedding in the projective plane, but possibly might have a planar cover (cf. Theorem 2.2).



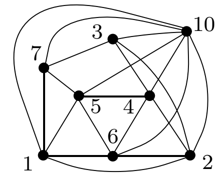
$K_{1,2,2,2}$



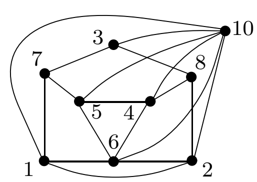
$\mathcal{B}_7$



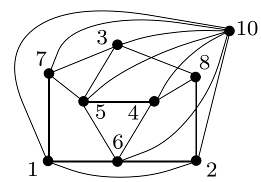
$\mathcal{B}_7'$



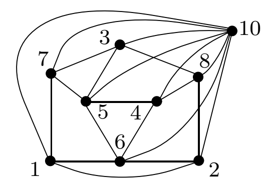
$\mathcal{B}_7''$



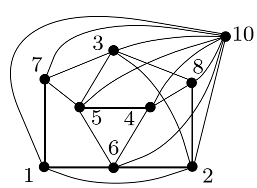
$\mathcal{C}_3$



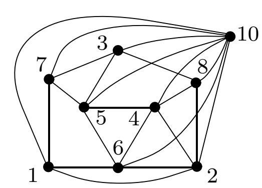
$\mathcal{C}_3'$



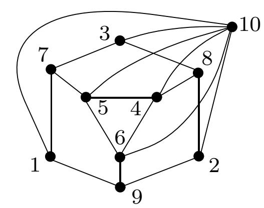
$\mathcal{C}_3''$



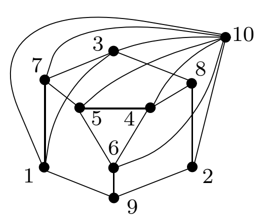
$\mathcal{C}_3^{\bullet}$



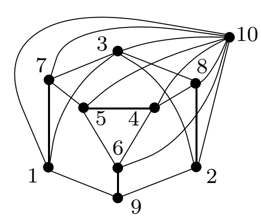
$\mathcal{C}_3^{\circ}$



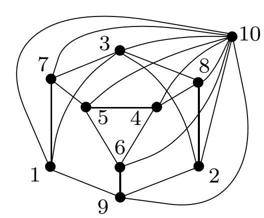
$\mathcal{D}_2$



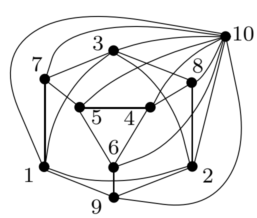
$\mathcal{D}_2'$



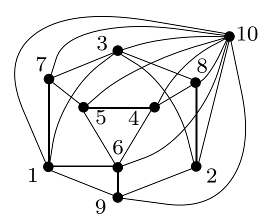
$\mathcal{D}_2''$



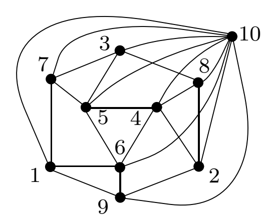
$\mathcal{D}_2'''$



$\mathcal{D}_2^{\bullet}$



$\mathcal{D}_2^{\circ}$



$\mathcal{D}_2^{\star}$