Nested Cycles in Large Triangulations and Crossing-Critical Graphs

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Abstract

We show that every sufficiently large plane triangulation has a large collection of nested cycles that either are pairwise disjoint, or pairwise intersect in exactly one vertex, or pairwise intersect in exactly two vertices. We apply this result to show that for each fixed positive integer k, there are only finitely many k-crossing-critical simple graphs of average degree at least six. Combined with the recent constructions of crossing-critical graphs given by Bokal, this settles the question of for which numbers $q > 0$ there is an infinite family of k-crossing-critical simple graphs of average degree q.

1 Introduction

All *graphs* in this paper are finite, and may have loops and parallel edges. The *crossing number* of a graph G, denoted by $\mathrm{cr}(G)$, is the minimum, over all drawings γ of G in the plane, of the number of crossings in γ. (We will formalize the notion of a drawing later.)

A graph G is k-*crossing-critical* if the crossing number of G is at least k and $\mathrm{cr}(G-e) < k$ for every edge e of G. The study of crossing-critical graphs is a central part of the emerging structural theory of crossing numbers. Good examples of this aspect of crossing numbers are Hliněný's proof that k-crossing-critical graphs have bounded path-width [5]; Fox and Tóth's work on the decay of crossing numbers [3]; and Dvořák and Mohar's ingenious construction, for each integer $k \geq 171$, of k-crossing-critical graphs of arbitrarily large maximum degree [9].

The earliest interesting, nontrivial construction of k-crossing-critical graphs is due to Širáň [17], who gave examples of infinite families of k-crossing-critical graphs for fixed values

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of k. These constructions involve graphs with parallel edges. Shortly afterwards, Kochol [7] gave an infinite family of 2-crossing-critical, simple 3-connected graphs.

In their influential paper on crossing-critical graphs, Richter and Thomassen [11] proved that k-crossing-critical graphs have bounded crossing number. Richter and Thomassen also investigated regular simple crossing-critical graphs. They used their aforementioned result to prove that for each fixed k, there are only finitely many k-crossing-critical 6-regular simple graphs, and also constructed an infinite family of 3-crossing-critical, simple 4-connected 4-regular graphs.

We note that degree two vertices affect neither the crossing number nor the crossing criticality of a graph. Also, the crossing number of a disconnected graph is clearly the sum of the crossing numbers of its connected components. Thus the interest in crossing-critical graphs is focused on connected graphs with minimum degree at least 3.

The construction of Richter and Thomassen was generalized in [16], where it was shown that for every rational number $q \in [4, 6)$, there exists an integer k_q such that there is an infinite family of k_q-crossing-critical simple graphs with average degree q. Pinontoan and Richter [10] extended the range to every rational $q \in [3.5, 6)$, and recently Bokal [2] used his novel technique of zip products to describe a construction that yields an infinite family for every rational $q \in (3, 6)$.

What about $q = 3$ or $q \geq 6$? Let G and H be simple 3-regular graphs. Since G has a subgraph isomorphic to a subdivision of H if and only if H is isomorphic to a minor of G, the Graph Minor Theorem [15] implies that for every integer $k \geq 1$ there are only finitely many k-crossing-critical 3-regular simple graphs. In fact, this does not need the full strength of the Graph Minor Theorem; by Hliněný's result [5] that k-crossing-critical graphs have bounded path-width all that is needed is the fact that graphs of bounded path-width are well-quasi-ordered, which is a lot easier that the general Graph Minor Theorem. On the other hand, it follows easily from the techniques in [11] that for each fixed positive integer k and rational $q > 6$ there are only finitely many k-crossing-critical simple graphs with average degree q. Thus the only remaining open question is whether for some k there exists an infinite family of k-crossing-critical simple graphs of average degree six. In this paper we answer this question in the negative, as follows.

Theorem 1.1. *For each fixed positive integer k, the collection of k-crossing-critical simple graphs with average degree at least six is finite.*

In fact, we prove in Theorem 3.5 below that the conclusion holds for graphs of average degree at least $6 - c/n$, where c is an absolute constant, and n is the number of vertices of the graph. The assumption that G be simple cannot be omitted: as shown in [11], for each integer $p \geq 1$ there is an infinite family of $4p$-regular $3p^2$-crossing-critical (nonsimple) graphs. Moreover, by adding edges (some of them parallel) to the 4-regular 3-crossing-critical graphs H_m in [11], it is possible to obtain an infinite family of 6-regular 12-crossing-critical (nonsimple) graphs.

The crucial new result behind the proof of Theorem 1.1 is the following theorem, which may be of independent interest. Let γ be a planar drawing of a graph G, and let H be a subgraph of G. We say that H is *crossing-free in* γ if no edge of H is crossed in γ by another edge of G. A sequence $C_1, C_2, \ldots, C_t$ of cycles in G is a *nest in* γ if the cycles are pairwise edge-disjoint, each of them is crossing-free in γ, and for each $i = 1, 2, \ldots, t-1$ the cycle $\gamma(C_{i+1})$ is contained in the closed disk bounded by $\gamma(C_i)$. We say that t is the *size* of the nest. If $X \subseteq V(G)$, $s := |X|$ and $V(C_i) \cap V(C_j) = X$ for every two distinct indices $i, j = 1, 2, \ldots, t$, then we say that $C_1, C_2, \ldots, C_t$ is an *s-nest*.

Theorem 1.2. *For every integer k there exists an integer n such that every planar triangulation on at least n vertices has an s-nest of size at least k for some $s \in \{0, 1, 2\}$.*

To deduce Theorem 1.1 from Theorem 1.2 we prove that a k-crossing-critical graph cannot have a large s-nest for any $s \in \{0, 1, 2\}$. For $s = 2$ this was shown in [6], but we give a shorter proof with a slightly better bound.

We formalize the notion of a planar drawing as follows. By a *polygonal arc* we mean a set $A \subseteq \mathbb{R}^2$ which is the union of finitely many straight line segments and is homeomorphic to the interval $[0, 1]$. The images of 0 and 1 under the homeomorphism are called the *ends* of A. A *polygon* is a set $B \subseteq \mathbb{R}^2$ which is the union of finitely many straight line segments and is homeomorphic to the unit circle $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. Let G be a graph. A *drawing* of G is a mapping γ with domain $V(G) \cup E(G)$ such that

(i) $\gamma(v) \in \mathbb{R}^2$ for every $v \in V(G)$,

(ii) $\gamma(v) \neq \gamma(v')$ for distinct $v, v' \in V(G)$,

(iii) for every non-loop edge $e \in E(G)$ with ends u and v there exists a polygonal arc $A \subseteq \mathbb{R}^2$ with ends $\gamma(u)$ and $\gamma(v)$ such that $\gamma(e) = A - \{u, v\} \subseteq \mathbb{R}^2 - \gamma(V(G))$,

(iv) for every loop $e \in E(G)$ incident with $u \in V(G)$ there exists a polygon $P \subseteq \mathbb{R}^2$ containing $\gamma(u)$ such that $\gamma(e) = P - \{u\} \subseteq \mathbb{R}^2 - \gamma(V(G))$, and

(v) if $e, e' \in E(G)$ are distinct, then $\gamma(e) \cap \gamma(e')$ is finite.

If $e, e' \in E(G)$ are distinct and $\gamma(e) \cap \gamma(e') \neq \emptyset$, then we say that e and e' *cross in* γ and that every point of $\gamma(e) \cap \gamma(e')$ is a *crossing*. (Thus a point where $\gamma(e)$ and $\gamma(e')$ "touch" also counts as a crossing.) If H is a subgraph of G, then by $\gamma(H)$ we denote the image of H under γ; that is, the set of points in $\mathbb{R}^2$ that either are equal to $\gamma(v)$ for some $v \in V(H)$ or belong to $\gamma(e)$ for some $e \in E(H)$. A *plane graph* is a graph G such that $V(G) \subseteq \mathbb{R}^2$, every edge of G is a subset of $\mathbb{R}^2$, and the identity mapping $V(G) \cup E(G) \to V(G) \cup E(G)$ is a drawing of G with no crossings.

We are restricting ourselves to piecewise linear drawings merely for convenience. This restriction does not change the class of graphs that admit drawings with a specified number of crossings, while piecewise linear drawings are much easier to handle.

We prove Theorem 1.2 in Section 2 and Theorem 1.1 in Section 3.

2 Finding a nest

A *tree decomposition* of a graph G is a triple (T, W, r) where T is a tree, $r \in V(T)$ and $W = (W_t : t \in V(T))$ is a collection of subsets of $V(G)$ such that

(T1) $\bigcup_{t \in V(T)} W_t = V(G)$ and every edge of G has both ends in some W_t, and

(T2) if $t, t', t'' \in V(T)$ and t' belongs to the unique path in T connecting t and t'', then $W_t \cap W_{t''} \subseteq W_{t'}$.

The *width* of the tree-decomposition (T, W, r) is the maximum of $|W_t| - 1$ over all $t \in V(T)$. Now let G be a plane graph. We say that the tree-decomposition (T, W, r) of G is *standard* if

(T3) for every edge $e = tt' \in E(T)$ the set $W_t \cap W_{t'}$ is the vertex-set of a cycle C_e in G, and

(T4) if $e, e' \in E(T)$ are distinct, and e lies on the unique path from r to e', then $C_{e'} \neq C_e$ and $C_{e'}$ belongs to the closed disk bounded by C_e.

The cycles C_e will be called the *rings* of (T, W, r).

We will need the following lemma.

Lemma 2.1. *Let $k \geq 1$ be an integer, and let G be a triangulation of the plane. Then G has either a 0-nest of size k, or a standard tree-decomposition of width at most $12k - 1$.*

Proof. We may assume that G has no 0-nest of size k. Let (T, W, r) be a standard tree-decomposition of G such that

(a) T has at least one edge and maximum degree at most three,

(b) $|W_t| \leq 12k$ if $t = r$ or if t is not a leaf of T,

(c) each ring of (W, T, r) has length at most $8k$,

(d) if $t \in V(T) - \{r\}$ and t' is the unique neighbor of t in the path in T from t to r, then W_t consists precisely of the vertices of G drawn in the closed disk bounded by $C_{tt'}$, and subject to (a)–(d)

(e) T is maximal.

Such a choice is possible, because of the following construction. Let T be a tree with vertex-set $\{t_1, t_2\}$, let C be the triangle bounding the outer face of G, let $W_{t_1} = V(C)$, and let $W_{t_2} = V(G)$. Then (T, W, t_1) satisfies (a)–(d).

So let (T, W, r) satisfy (a)–(e). We claim that (T, W, r) has width at most $12k - 1$. To prove that suppose to the contrary that $|W_{t_0}| > 12k$ for some $t_0 \in V(T)$. Then by (b) $t_0 \neq r$ and t_0 is a leaf of T. Let t_1 be the unique neighbor of t_0 in T, and let C denote the ring $C_{t_0 t_1}$. Then $|V(C)| \leq 8k$ by (c). Let Δ denote the closed disk bounded by C, and let H be the near-triangulation consisting of all vertices and edges of G drawn in Δ. By (d) we have $V(H) = W_{t_0}$. For $u, v \in V(C)$, let $c(u, v)$ (respectively, $d(u, v)$) be the number of edges in the shortest path of C (respectively, H) between u and v.

(1) *$c(u, v) = d(u, v)$ for all $u, v \in V(C)$.*

To prove (1) we certainly have $d(u, v) \leq c(u, v)$ since C is a subgraph of H. If possible, choose a pair $u, v \in V(C)$ with $d(u, v)$ minimum such that $d(u, v) < c(u, v)$. Let P be a path of H between u and v, with $d(u, v)$ edges. Suppose that some internal vertex w of P belongs to $V(C)$. Then

$$d(u, w) + d(w, v) = d(u, v) < c(u, v) \leq c(u, w) + c(w, v)$$

and so either $d(u, w) < c(u, w)$ or $d(w, v) < c(w, v)$, in either case contrary to the choice of u, v. Thus there is no such w. Let C, C_1, C_2 be the three cycles of $C \cup P$, let $\Delta, \Delta_1, \Delta_2$ be the closed disks they bound, and for $i = 1, 2$ let H_i be the subgraph of H consisting of all vertices and edges drawn in Δ_i. Then C_1 and C_2 have length at most $8k$. Let T' be the tree obtained from T by adding two vertices r_1, r_2, both joined to t_0. For $t \in V(T) - \{t_0\}$ let $W'_t = W_t$, let $W'_{t_0} = V(C \cup P)$, let $W'_{r_i} = V(H_i)$, and let $W' = (W'_t : t \in V(T'))$. Then (T', W', r) satisfies (a)-(d), contrary to (e). This proves (1).

(2) *C has length exactly $8k$.*

To prove (2) suppose for a contradiction that C has length at most $8k - 1$. Let uv be an edge of C, and let w be the third vertex of the face incident with uv and contained in the disk bounded by C. Then $w \notin V(C)$ by (1) and the fact that $|W_{t_0}| > 12k$. Let T' be obtained from T by adding a new vertex r_0 joined to t_0, for $t \in V(T) - \{t_0\}$ let $W'_t = W_t$, let $W'_{t_0} = V(C) \cup \{w\}$, let $W'_{r_0} = W_{t_0}$, and let $W' = (W'_t : t \in V(T'))$. Then (T', W', r) is a standard tree-decomposition satisfying (a)–(d), contrary to (e). This proves (2).

Now let $v_1, v_2, \ldots, v_{8k}$ be the vertices of C in order. By (1) and [14, Theorem (3.6)] there exist $2k$ disjoint paths from $\{v_1, v_2, \ldots, v_{2k}\}$ to $\{v_{4k+1}, v_{4k+2}, \ldots, v_{6k}\}$, and $2k$ disjoint paths from $\{v_{2k+1}, v_{2k+2}, \ldots, v_{4k}\}$ to $\{v_{6k+1}, v_{6k+2}, \ldots, v_{8k}\}$. Using those sets of paths it is easy to construct a 0-nest in G of size k. In fact, using the argument of [12, Theorem (4.1)] it can be shown that G has a $2k \times 2k$ grid minor, and hence a 0-nest of size k. $\square$

Proof of Theorem 1.2. Let $k \geq 1$ be a given integer, let h be an integer such that for every coloring of the edges of the complete graph on h vertices using at most $12k$ colors, there is a monochromatic clique of size $24k^2$, and let $n = 36k \cdot 2^{h+1}$. The integer h exists by Ramsey's theorem. We claim that n satisfies the conclusion of the theorem. To prove the claim let G be a triangulation of the plane on at least n vertices. By Lemma 2.1 we may assume that G has a standard tree-decomposition (T, W, r) of width at most $12k$. It follows that T has at least $n/(12k)$ vertices. Thus $|V(T)| > 3 \cdot 2^{h+1} - 2$, and hence T has a path on $h + 1$ vertices starting in r. Let $t_0 = r, t_1, \ldots, t_h$ be the vertices of one such path, and for $i = 1, 2, \ldots, h$ let C_i denote the ring $C_{t_{i-1}t_i}$. Then by (T3) and (T4) $C_1, C_2, \ldots, C_h$ is a sequence of distinct cycles such that for indices i, j with $1 \leq i \leq j \leq h$ the cycle C_j belongs to the closed disk bounded by C_i. We shall refer to the latter condition as the nesting property. Let K be a complete graph with vertex-set $\{1, 2, \ldots, h\}$. We color the edges of K by saying that the edge ij is colored using $|V(C_i) \cap V(C_j)|$. By the choice of n there exist a subsequence $D_1, D_2, \ldots, D_{24k^2}$ of $C_1, C_2, \ldots, C_h$ and an integer $t \in \{0, 1, \ldots, 12k - 1\}$ such that $|V(D_i) \cap V(D_j)| = t$ for every pair of distinct integers $i, j \in \{1, 2, \ldots, 24k^2\}$. Since the sequence $D_1, D_2, \ldots, D_{24k^2}$ satisfies the nesting property, we deduce that there exists a set X such that $V(D_i) \cap V(D_j) = X$ for every pair of distinct integers $i, j \in \{1, 2, \ldots, 24k^2\}$. If $|X| \leq 1$, then the sequence $D_1, D_2, \ldots, D_k$ satisfies the conclusion of the theorem. We may therefore assume that $|X| \geq 2$. Let the elements of X be numbered $x_1, x_2, \ldots, x_t = x_0$ in such a way that they appear on D_1 in the order listed. It follows that they appear on each cycle D_j in the order listed. Now for $i = 1, 2, \ldots, t$ and $j = 1, 2, \ldots, 24k^2$ let P_{ij} be the subpath of D_j with ends x_{i-1} and x_i that is disjoint from $X - \{x_{i-1}, x_i\}$ (if $|X| = 2$ we number the two subpaths of D_j arbitrarily). Since the cycles D_j are pairwise distinct and

$t \leq 12k-1$, we deduce that there exists an index $i \in \{1, 2, \ldots, t\}$ such that the path P_{ij} has at least one internal vertex for at least $2k$ distinct integers $j \in \{1, 2, \ldots, 24k^2\}$. Let us fix this index i, and let $Q_1, Q_2, \ldots, Q_{2k}$ be a subsequence of $P_{i1}, P_{i2}, \ldots$ such that each Q_j has at least one internal vertex. It follows that the paths $Q_1, Q_2, \ldots, Q_{2k}$ are internally disjoint and pairwise distinct. Thus $Q_1 \cup Q_{2k}, Q_2 \cup Q_{2k-1}, \ldots, Q_k \cup Q_{k+1}$ is a 2-nest in G of size k, as desired. □

3 Using a nest

To prove Theorem 1.1 we need several lemmas, but first we need a couple of definitions. We say that an s-nest $C_1, C_2, \ldots, C_t$ in a drawing γ of a graph G is *clean* if every crossing in γ belongs either to the open disk bounded by $\gamma(C_t)$, or to the complement of the closed disk bounded by $\gamma(C_1)$. We say that a drawing γ of a graph G is *generic* if it satisfies (i)-(v) and

(vi) every point $x \in \mathbb{R}^2$ belongs to $\gamma(e)$ for at most two edges $e \in E(G)$, and

(vii) if $\gamma(e) \cap \gamma(e') \neq \emptyset$ for distinct edges $e, e' \in E(G)$, then e and e' are not adjacent.

Lemma 3.1. *For every three integers ℓ, $r, t \geq 0$ there exists an integer n_0 such that for every simple graph G on $n \geq n_0$ vertices of average degree at least $6-r/n$ and every generic drawing γ of G with at most ℓ crossings there exists an s-nest in γ of size t for some $s \in \{0, 1, 2\}$.*

Proof. Let ℓ, t, r be given, and let n_0 be an integer such that Theorem 1.2 holds when k is replaced by $t' := t + 2\ell + r - 6$ and n is replaced by n_0. We will prove that n_0 satisfies the conclusion of the theorem. To that end let G be a simple graph on $n \geq n_0$ vertices of average degree at least $6 - r/n$ and let γ be a generic drawing of G with at most ℓ crossings. We will prove that γ has a desired s-nest. Let G' denote the plane graph obtained from γ by converting each crossing into a vertex. Let V_4 be the set of these new vertices. Then $|V_4| \leq \ell$. By (vi) each vertex in V_4 has degree four in G', and since G is simple it follows from (vii) that G' is simple. Let $\deg(v)$ denote the degree of v in G', let $\mathcal{F}$ denote the set of faces of G', and for $f \in \mathcal{F}$ let $|f|$ denote the length of the boundary of f; that is, the sum of the lengths of the walks forming the boundary of f. By Euler's formula we have

$$\sum_{v \in V(G')} (6 - \deg(v)) + \sum_{f \in \mathcal{F}} 2(3 - |f|) = 12.$$

But $\sum_{v \in V(G)-V_4} (6 - \deg(v)) \leq r$ by hypothesis, and so

$$\sum_{f \in \mathcal{F}} (|f| - 3) \leq \frac{1}{2} \sum_{v \in V_4} (6 - \deg(v)) - 6 + r = |V_4| - 6 + r \leq \ell + r - 6,$$

because every vertex in V_4 has degree four in G'. Thus G' has at most $\ell + r - 6$ non-triangular faces, each of size at most $\ell + r - 3$.

Let G'' be the triangulation obtained from G' by adding a vertex into each non-triangular face and joining it to each vertex on the boundary of that face. Thus every added vertex has degree in G'' at most $\ell + r - 3$. By Theorem 1.2 the triangulation G'' has an s-nest $C_1, C_2, \ldots, C_{t'}$ of size t' for some $s \in \{0, 1, 2\}$. Let X be the set of vertices every two distinct cycle C_i and C_j have in common. Then every vertex of X has degree at least $2t'$, and hence belongs to $V(G)$, because every vertex of $V(G'') - V(G)$ has degree four or at most $\ell + r - 3$. Thus at most $\ell + (\ell + r - 6) = 2\ell + r - 6$ cycles C_i contain a vertex not in G, and by removing all those cycles we obtain a desired s-nest in γ. □

Lemma 3.2. *Let $k \geq 0$ and $t \geq 1$ be integers, let $s \in \{0, 1, 2\}$, let G be a graph, and let γ be a drawing of G with at most k crossings and an s-nest of size $(k + 1)\ (t - 1) + 1$. Then γ has a clean s-nest of size t.*

Proof. Let $C_1, C_2, \ldots, C_{(k+1)(t-1)+1}$ be an s-nest in G. For $i = 1, 2, \ldots, (k + 1)(t - 1)$ let Ω_i denote the subset of $\mathbb{R}^2$ obtained from the closed disk bounded by $\gamma(C_i)$ by removing the open disk bounded by $\gamma(C_{i+1})$. Since there are at most k crossings in γ, it follows that Ω_i includes no crossing of γ for $t - 1$ consecutive integers in $\{1, 2, \ldots, (k + 1)(t - 1)\}$, say $i, i + 1, \ldots, i + t - 2$. Then $C_i, C_{i+1}, \ldots, C_{i+t-1}$ is a clean s-nest of size t, as desired. □

Lemma 3.3. *Let $k \geq 1$ be an integer, let $s \in \{0, 1\}$ and let γ be a drawing of a graph G with a clean s-nest of size $4k + 1$. If $\mathrm{cr}(G - e) < k$ for all $e \in E(G)$, then $\mathrm{cr}(G) < k$.*

Proof. If C is a cycle of G that is crossing-free in γ, then we denote by $\Delta(C)$ the disk bounded by $\gamma(C)$. Let $D_1, D_2, \ldots, D_{4k+1}$ be a clean s-nest in γ of size $4k + 1$. We may assume that the s-nest is chosen so that

(1) for $i = 2, 3, \ldots, 2k$, if D is a cycle in G such that D_{i-1}, D, D_{i+1} is an s-nest in γ and $\Delta(D_i) \subseteq \Delta(D)$, then $D_i = D$, and

(2) for $i = 2k + 2, 2k + 3, \ldots, 4k$, if D is a cycle in G such that D_{i-1}, D, D_{i+1} is an s-nest in γ and $\Delta(D) \subseteq \Delta(D_i)$, then $D_i = D$.

Let $e \in E(D_{2k+1})$. By hypothesis there exists a drawing γ' of $G - e$ with at most $k - 1$ crossings. Thus at most $2k - 2$ cycles among $D_1, D_2, \ldots, D_{2k}$ include an edge that is crossed by another edge in γ', and similarly for $D_{2k+2}, D_{2k+3}, \ldots, D_{4k+1}$. Hence there exist indices $i_2 \in \{2, 3, \ldots, 2k\}$ and $i_4 \in \{2k + 2, 2k + 3, \ldots, 4k\}$ such that D_{i_2} and D_{i_4} are crossing-free in γ'. Let $C_1 := D_{i_2-1}$, $C_2 := D_{i_2}$, $C_3 := D_{2k+1}$, $C_4 := D_{i_4}$, and $C_5 := D_{i_4+1}$. Then C_1, C_2, C_3, C_4, C_5 is a clean s-nest in γ. Let $H := C_2 \cup C_4$, let B_1 be the H-bridge containing C_1, and let B_5 be the H-bridge containing C_5. Since H is crossing-free in γ we see that $B_1 \neq B_5$. Let Ω be the face of $\gamma(H)$ that is incident with edges of both C_2 and C_4. Thus if $s = 0$, then Ω is an annulus, and if $s = 1$ it is a "pinched annulus". An H-bridge B of G will be called *interior* if $\gamma(B)$ is a subset of the closure of Ω and it will be called *exterior* otherwise. We need the following claim.

(3) *If B is an exterior H-bridge of G with at least two attachments, then either $B = B_1$, or $B = B_5$.*

To prove (3) let B be an exterior H-bridge of G with at least two attachments. Since C_2 and C_4 are crossing-free in γ it follows that either all attachments of B belong to C_2, or they all belong to C_4. From the symmetry we may assume the former. We may also assume that

$B \neq B_1$, for otherwise the claim holds. Thus B is disjoint from C_1. Since B has at least two attachments it includes a path P with both ends in C_2 and otherwise disjoint from it. Since C_1, C_2, C_3, C_4, C_5 is a clean s-nest, no edge of P is crossed by another edge in γ. Thus $C_2 \cup P$ includes a cycle D disjoint from $C_1 = D_{i_2-1}$ with $\Delta(C_2) \subseteq \Delta(D)$, contrary to (1). This proves (3).

We may assume, by composing γ' with a homeomorphism of the plane, that $\gamma(H) = \gamma'(H)$. We now define a new drawing δ of G as follows. For every vertex and edge x that belongs to H or to an interior H-bridge of G we define $\delta(x) = \gamma(x)$. If $\gamma'(B_5) \subseteq \Delta(C_4)$, then for every $x \in V(B_5) \cup E(B_5)$ we define $\delta(x) = \gamma'(x)$; otherwise we use circular inversion to redraw $\gamma'(B_5)$ in $\Delta(C_4)$ and use the inversion of $\gamma'(B_5)$ to define $\delta(B_5)$. We define $\delta(B_1)$ analogously. Finally, for an H-bridge B with at most one attachment we define $\delta(B)$ by scaling $\gamma'(B)$ suitably so that it does not intersect any other H-bridge of G. Thus every crossing of δ is also a crossing of γ', and hence δ has at most $k-1$ crossings, as desired. $\square$

We also need a version of Lemma 3.3 for 2-nests. Such a lemma follows from [6, Theorem 1.3], but we give a proof from first principles, because we have already done a lot of the needed work in the previous lemma. Furthermore, Lemmas 3.2 and 3.4 imply a small numerical improvement to [6, Theorem 1.3].

Lemma 3.4. *Let $k \geq 1$ be an integer, and let γ be a drawing of a graph G with a clean 2-nest of size $4k+1$. If $\mathrm{cr}(G-e) < k$ for all $e \in E(G)$, then $\mathrm{cr}(G) < k$.*

Proof. We proceed similarly as in Lemma 3.3. Let $\Delta(C)$ be as in Lemma 3.3. We select our 2-nest so that it satisfies (1) and (2) from Lemma 3.3. We pick $e \in E(D_{2k+1})$, but we now require that e be incident with a vertex in X. Let γ' be a drawing of $G-e$ with at most $k-1$ crossings. We choose a clean nest C_1, C_2, C_3, C_4, C_5 in the same way as before, with one caveat: the index i_4 can be chosen so that there exists an index i_5 such that $i_4 < i_5 \leq 4k+1$ and D_{i_5} is crossing-free in γ'. We put $C_6 := D_{i_5}$. Now $\gamma(H)$ has four faces, and we define Ω to be the one containing $\gamma(e)$. For $i = 1, 2, \ldots, 5$ let P_i be a subpath of C_i with ends in X chosen as follows: P_2 and P_4 are defined by saying that $P_2 \cup P_4$ is the boundary of Ω, P_3 is defined by $\gamma(P_3) \subseteq \Omega$, and P_1 and P_5 are defined by saying that $\Delta(P_1 \cup P_2)$ and $\Delta(P_4 \cup P_5)$ are disjoint from Ω. We define B_1 as the H-bridge of G containing P_1 and B_5 as the H-bridge containing P_5. It is now possible that $B_1 = B_5$.

Let us say that an H-bridge is *singular* if its set of attachments is X. The following is an analogue of claim (3) from Lemma 3.3. The proof follows the same lines, and so we omit it.

(3) *If B is an exterior H-bridge of G with at least two attachments, then either $B = B_1$, or $B = B_5$, or B is singular.*

Again, we may assume that $\gamma(H) = \gamma'(H)$. If $B_1 \neq B_5$, then the argument proceeds in the same way as in Lemma 3.3. All singular H-bridges can be drawn outside Ω so that they will be disjoint from each other and from all other H-bridges.

Thus we may assume that $B_1 = B_5$. Now we may assume, by applying a circular inversion with respect to H if necessary, that $\gamma'(B_1)$ lies in the complement of Ω. Finally, we may assume that, subject to the conditions already imposed on γ',

(4) *the number of H-bridges B with $\gamma'(B)$ contained in the closure of Ω is minimum.*

Let d_1 be the maximum number of edge-disjoint paths in $G - X$ from P_2 to P_4 that are contained in interior H-bridges, and let d_2 be the maximum number of edge-disjoint such paths contained in exterior H-bridges. Let d'_1, d'_2 be defined analogously, but with respect to the drawing γ'.

(5) *$d_1 = d'_1$ and $d_2 = d'_2$*

To prove (5) we notice that a path in $G - X$ from P_2 to P_4 belongs to a H-bridge that has an attachement in both $P_2 - X$ and $P_4 - X$. Let us call such an H-bridge *global*. Then B_1 is the only global exterior bridge by (3), and $\gamma'(B_1) \cap \Omega = \emptyset$, because we chose γ' that way. Conversely, if B is a global interior bridge, then $\gamma'(B - e)$ lies in the closure of Ω, for otherwise $\gamma'(B - e)$ would have to intersect $\gamma'(C_6)$ (because C_6 is crossing-free in γ'), and hence $B = B_1$, contrary to the fact that B is interior. Since e is incident with X, claim (5) follows.

(6) *If there exists an exterior singular H-bridge, then $d_1 \geq d_2$.*

To prove (6) let B be an exterior singular bridge. Let d be the maximum number of edge-disjoint paths in B that join the vertices of X. Then $d > 0$, because B is singular. In γ the bridges B and B_1 cross at least dd_2 times by definition of d and d_2. By Menger's theorem there is a set of edges $F \subseteq E(B)$ of size d such that $B - F$ includes no path joining the two vertices in X. Let J be the union of H and all interior H-bridges. Similarly, there is a set of edges $F_1 \subseteq E(J)$ of size d_1 such that $J - F_1$ has no edge from P_2 to P_4. Now γ can be changed by redrawing B inside Ω. This can be done in such a way that every edge of F crosses every edge of F_1, and these are the only crossings of an edge of B with an edge not in B. Thus the redrawing of B removes the at least dd_2 crossings of B and B_1 and introduces exactly dd_1 new crossings. Since G has crossing number exactly k, the new drawing has at least k crossings, and hence $d_1 \geq d_2$, as desired. This proves (6).

(7) *Every exterior H-bridge B satisfies $\gamma'(B) \cap \Omega = \emptyset$.*

To prove (7) let B be an exterior H-bridge. If B has only one attachment, then it clearly satisfies the conclusion of (7), for otherwise it can be redrawn outside of Ω, contrary to (4). Thus B has at least two attachments, and hence $B = B_1$, or B is singular by (3). If $B = B_1$, then the conclusion of (7) holds, and so we may assume that B is singular. By (6) $d_1 \geq d_2$, and hence $d'_1 \geq d'_2$ by (5). If $\gamma'(B) \cap \Omega \neq \emptyset$, then using the argument of (6) we can change the drawing γ' by drawing B in the complement of Ω, contrary to (4). This proves (7).

Let δ be a drawing of G defined to coincide with γ on H and all interior H-bridges, and to coincide with γ' for all other H-bridges. By (7) this is a well-defined drawing of G, and every crossing of δ is a crossing of γ', because there are no crossings in Ω. Thus δ has at most $k - 1$ crossings, as desired. $\square$

We are now ready to prove Theorem 1.1, which we restate in a slightly stronger form.

Theorem 3.5. *For all integers $k \geq 1$, $r \geq 0$ there is an integer $n_0 := n_0(k, r)$ such that if G is a k-crossing-critical simple graph on n vertices with average degree at least $6 - r/n$, then $n < n_0$.*

Proof. Let $k \geq 1$, $r \geq 0$ be integers, let $\ell := 2.5k + 16$, let $t := 4k(\ell + 1) + 1$, let n_0 be an integer such that Lemma 3.1 holds, and let G be a k-crossing-critical simple graph on n vertices with average degree at least $6 - r/n$. We claim that $n < n_0$. To prove the claim suppose to the contrary that $n \geq n_0$, and let γ be a drawing of G with at most ℓ crossings; such a drawing exists by [11, Theorem 3]. By a standard and well-known argument we may assume that γ is generic. By Lemma 3.1 there is an integer $s \in \{0, 1, 2\}$ and an s-nest in γ of size t, and by Lemma 3.2 there is a clean s-nest in γ of size $4k + 1$. That contradicts Lemma 3.3 if $s \in \{0, 1\}$, or Lemma 3.4 if $s = 2$. Thus $n < n_0$, as desired. ☐

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