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AN END-FAITHFUL SPANNING TREE COUNTEREXAMPLE

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ABSTRACT. We find an infinitely-connected graph in which every spanning tree has a 2-way infinite path. This disproves Halin's well-known "end-faithful spanning tree" conjecture and also disproves a recent conjecture of Širáň.

1. INTRODUCTION

A *ray* in a graph G is a 1-way infinite path. (In this paper, graphs may be infinite and may have loops or multiple edges.) Two rays R_1, R_2 in G are *parallel* if for every finite $X \subseteq V(G)$, the unique component of $G\backslash X$ that has infinite intersection with R_1 also has infinite intersection with R_2. ($G\backslash X$ is the graph obtained from G by deleting X.) Parallelness is an equivalence relation, and its equivalence classes are called the *ends* of G. These were first investigated by Halin [4], who proposed the following "end-faithful spanning tree conjecture," which we shall disprove.

(1.1) **Conjecture.** *In every connected graph G there is a spanning tree T such that each end of G includes a unique end of T.*

Halin [3, 4] proved that (1.1) holds if G is countable and that it holds if G does not contain $K_{\aleph_0}$. (We denote by K_κ the complete graph with κ vertices, when κ is a cardinal. A graph G *contains* a graph H if some subgraph of G is isomorphic to a subdivision of H—that is, a graph obtained from H by replacing its edges by internally disjoint paths.) However, we shall see that (1.1) is false in general. A counterexample has independently been obtained by C. Thomassen [9].

Let us say that G is *infinitely-connected* if $V(G)$ is infinite and $G\backslash X$ is connected for every finite $X \subseteq V(G)$. Since an infinitely-connected graph has a unique end, a consequence of (1.1) would be the following:

(1.2) **Conjecture.** *In every infinitely-connected graph there is a spanning tree with a unique end.*

We shall give a counterexample to (1.2) and hence to (1.1).

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The following conjecture was proposed at a recent (1989) conference in Cambridge by Širáň. (A tree is *rayless* if it has no ray.)

(1.3) **Conjecture.** *Let G be a connected graph, and suppose that for every ray R there is a vertex v such that, for every finite $X \subseteq V(G)-\{v\}$, the component of $G\backslash X$ containing v has infinite intersection with R. Then G has a rayless spanning tree.*

Širáň [8] proved this for countable graphs G, but we shall see that it is false in general. Since every infinitely-connected graph satisfies the hypothesis of (1.3), a consequence of (1.3) would be the following:

(1.4) **Conjecture.** *In every infinitely-connected graph there is a rayless spanning tree.*

Our counterexample to (1.1) and (1.2) is also a counterexample to (1.4) and hence to (1.3). (Indeed, we shall show that a graph satisfies (1.2) if and only if it satisfies (1.4).)

Let $T_{\aleph_1}$ be the tree in which every vertex has valency $\aleph_1$. We shall show the following:

(1.5). *There is an infinitely-connected graph G with $|V(G)| = 2^{\aleph_0}$ such that every spanning tree contains $T_{\aleph_1}$.*

In particular, every spanning tree has $\geq \aleph_1$ ends, contrary to (1.2) and (1.4). We shall also show that

(1.6). *There is an infinitely-connected graph G with $|V(G)| = 2^{\aleph_0}$, which does not contain $K_{\aleph_1}$, such that every spanning tree contains $T_{\aleph_1}$.*

We do not know whether (1.5) remains true with $2^{\aleph_0}$ replaced by $\aleph_1$. However, we shall see that the existence of G as in (1.6) with $|V(G)| = \aleph_1$ rather than $2^{\aleph_0}$ is independent of ZFC (Zermelo–Fraenkel set theory together with the axiom of choice). Finally, we shall show that

(1.7). *Every infinitely-connected graph which does not contain $T_{\aleph_1}$ has a rayless spanning tree.*

Before we begin the main proofs, let us see the equivalence of (1.2) and (1.4).

(1.8). *Let G be an infinitely-connected graph. Then G has a rayless spanning tree if and only if it has a spanning tree with exactly one end.*

Proof. Let T be a rayless spanning tree, and let R be a ray. Extend R to a spanning tree T' of $R \cup T$; then T' has exactly one end (for any ray not parallel to R includes a ray disjoint from R, and so contains a ray in T). For the converse, let T be a spanning tree with only one end, and let R be a ray of T, with $V(R) \neq V(G)$. Choose $v \in V(G)-V(R)$. Since G is infinitely-connected, there are infinitely many finite paths of G from v to $V(R)$, mutually disjoint except for v. Let the union of these paths be F, and extend F to a spanning tree T' of $F \cup T$. Suppose that R' is a ray of T'. Since F is connected and includes no ray, and $F \cup R'$ includes no circuit, it follows that R' includes a ray disjoint from F, so we may assume that R' is disjoint from F. Hence $R' \subseteq T$, and so R, R' are parallel in T. Since T is a tree, it follows that $R'' = R \cap R'$ is a ray. But $R' \cap F$ is null, and $(V(R) - V(R'')) \cap V(F)$ is finite, and yet $V(R) \cap V(F)$ is infinite, a contradiction. Thus T' is rayless, as required. □

We state a stronger form of (1.4) for countable graphs that we shall apply later. We omit the proof, which is easy. Let G be a graph, let $v \in V(G)$ and let $\{P_n\}_{n=1}^{\infty}$ be a collection of finite paths in G, each with at least one edge, with one endpoint v and otherwise disjoint. The tree $R = P_1 \cup P_2 \cup \cdots$ is called an *ω-star* in G with *center* v.

(1.9). *Every countable, infinitely-connected graph has a spanning ω-star with center any specified vertex.*

2. The counterexample

A *hypergraph* is a pair (V, M), where V is a set and M is a set of subsets of V. Two hypergraphs (V, M), (V', M') are *isomorphic* if there is a bijection $\alpha : V \to V'$ mapping M to M' (and α is an *isomorphism*).

Let (V, M) be a hypergraph such that

(i) every member of M has cardinality $\aleph_0$,
(ii) for every partition $(X_1, X_2, \dots)$ of V into countably many sets, some X_i includes some member of M, and
(iii) $|V|, |M| \geq \aleph_1$.

(In fact, condition (iii) is implied by (i) and (ii), as the reader may verify.) For example, we could take $V = \aleph_1$ and let M be the set of all countable subsets of V. We shall specify (V, M) later.

Let Σ be the set of all finite sequences of members of M. If $\sigma = (\mu_1, \dots, \mu_k) \in \Sigma$ and $\mu \in M$, we denote the sequence $(\mu_1, \dots, \mu_k, \mu) \in \Sigma$ by $\sigma + (\mu)$. For each $\sigma \in \Sigma$, let (V_σ, M_σ) be a hypergraph isomorphic to (V, M) such that $V_\sigma \cap V_{\sigma'} = \emptyset$ for all distinct $\sigma, \sigma' \in \Sigma$, and let $\alpha_\sigma : V \to V_\sigma$ be an isomorphism. For each $\mu \in M$, we denote $\{\alpha_\sigma(v) : v \in \mu\}$ by μ_σ; thus, $\mu_\sigma \in M_\sigma$.

Let G be the graph with $V(G) = \bigcup(V_\sigma : \sigma \in \Sigma)$, in which $u, v \in V(G)$ are adjacent if $u \in \mu_\sigma$ and $v \in V_{\sigma'}$, for some $\sigma \in \Sigma$ and $\mu \in M$, where $\sigma' = \sigma + (\mu)$. We shall show that every spanning tree of G contains $T_{\aleph_1}$.

We shall need the following lemma, which is very similar to a result of Laver [5] and which can be proved similarly (we omit the proof).

(2.1). *Let T be a tree which does not contain $T_{\aleph_1}$, and let $v_0 \in V(T)$. There is a function ϕ assigning an ordinal $\phi(v)$ to each $v \in V(T)$, such that*

(i) *if $u, v \in V(T)$ are adjacent and u lies on the path of T between v_0 and v, then $\phi(v) \leq \phi(u)$, and*

(ii) *for all* $u \in V(T)$ *the set of all* $v \in V(T)$ *as in* (i) *with* $\phi(v) = \phi(u)$ *has cardinality* $\leq \aleph_0$.

(2.2). *With* G *as defined earlier, every spanning tree of* G *contains* $T_{\aleph_1}$.

Proof. Suppose that T is a spanning tree of G not containing $T_{\aleph_1}$. Choose $v_0 \in V_{\sigma_0}$, where σ_0 is the null sequence, and for $u, v \in V(G)$, let us say that u is *before* v if u lies on the path of T between v_0 and v.

For each $\sigma \in \Sigma$, let B_σ denote $\bigcup V_{\sigma'}$, the union being taken over all $\sigma' \in \Sigma$ of which σ is an initial subsequence. Let us say that $u \in V(G)$ *dominates* $\sigma \in \Sigma$ if u is before every $v \in B_\sigma$, and let us say that $u \in V(G)$ is *big* if it dominates some $\sigma \in \Sigma$. Thus, v_0 is big. Let ϕ be as in (2.1), and choose a big vertex $u \in V(G)$ with $\phi(u)$ minimum.

(1) *There are only countably many* $v \in V(G)$ *such that* u *is before* v *and* $\phi(u) \leq \phi(v)$.

For $\phi(u) = \phi(v)$ for every such v. Let X be the set of all such v, and let R be the minimal subtree of T with $X \subseteq V(R)$. Then $u \in V(R)$ and u lies before every other vertex of R. Since every vertex of R lies on a path between u and some vertex v of X, and the ϕ-values on such a path do not increase (by (2.1)(i)), and $\phi(v) = \phi(u)$, it follows that every vertex of this path belongs to X, and in particular $V(R) = X$. Thus every vertex of R has valency $\leq \aleph_0$ (by (2.1)(ii)), and so $|V(R)| \leq \aleph_0$. This proves (1).

Since u dominates some $\sigma \in \Sigma$ and hence dominates all extensions of σ in Σ, we may choose $\sigma \in \Sigma$ such that u dominates σ and $u \notin B_\sigma$. Since M is uncountable, there are uncountably many 1-term extensions σ' of σ, and the corresponding sets $B_{\sigma'}$ are mutually disjoint. Thus by (1) we may choose $\sigma \in \Sigma$ such that, in addition, there is no $v \in B_\sigma$ with $\phi(u) \leq \phi(v)$. Choose $\mu \in M$, and let $\sigma' = \sigma + (\mu)$.

Let S be the minimal subtree of T with $\mu_\sigma \subseteq V(S)$. Since μ_σ is countable, it follows that so is $V(S)$. For each $s \in V(S)$, let X_s be the set of all $v \in V_{\sigma'}$ such that there is a path of T between s and v with no vertex in $V(S)$ except s. Thus $(X_s : s \in V(S))$ is a partition of $V_{\sigma'}$ into countably many sets, and so there exists $s \in V(S)$ and $\mu' \in M$ such that $\mu'_{\sigma'} \subseteq X_s$. Let $\sigma'' = \sigma' + (\mu')$. We claim that

(2) *s dominates* σ''.

For let $v \in B_{\sigma''}$, and let P be the path of T between v_0 and v. Since $v_0 \in V_{\sigma_0}$, it follows that $V(P) \cap \mu'_{\sigma'} \neq \emptyset$. Let $x \in V(P) \cap \mu'_{\sigma'}$. Since $V(P) \cap \mu_\sigma \neq \emptyset$, and hence $V(P) \cap V(S) \neq \emptyset$, it follows that P includes the unique minimal path of T between x and $V(S)$. Since $x \in \mu'_{\sigma'} \subseteq X_s$, it follows that $s \in V(P)$. Thus s is before v, as required.

(3) $s \in B_\sigma$.

For since X_s, is infinite and $X_s \cap V(S) \subseteq \{s\}$, there exists $v \in X_s - V(S)$. Thus $v \in V_{\sigma'}$. Let P be the path of T between v and s. Since no vertex of P is in $V(S)$ except s (because $v \in X_s$) and $\mu_\sigma \subseteq V(S)$, it follows that $V(P) \cap \mu_\sigma \subseteq \{s\}$, and so either $s \in B_{\sigma'}$ or $s \in \mu_\sigma$. In either case, $s \in B_\sigma$ as required.

Now we chose σ such that there is no $v \in B_\sigma$ with $\phi(u) \leq \phi(v)$, and so from (3) we deduce that $\phi(s) < \phi(u)$. But s is big by (2), contrary to the choice of u. This completes the proof. □

So far we have not specified the collection M of sets used in the construction of the graph G above. As we saw before, we can take $V = \aleph_1$ and M to be the collection of all countable subsets of V, and the graph G we construct satisfies (1.5). This proves (1.5). However, that graph contains $K_{\aleph_1}$, and in view of Halin's theorem that every counterexample to (1.1) and (1.2) contains $K_{\aleph_0}$, it is natural to ask if every counterexample also contains $K_{\aleph_1}$. The answer is no, as we shall see by a more complicated choice of M.

A *well-founded tree* is a poset $T = (V(T), \leq)$, such that for every pair $t, t' \in V(T)$ their infimum $\inf(t, t')$ exists, and for every $t \in V(T)$ the set $\{t' \in V(T) : t' \leq t\}$ is well ordered by $\leq$. It follows that every well-founded tree T has a minimum element, called the *root* of T and denoted by $\mathrm{root}(T)$. Let V be the set of all transfinite sequences of distinct positive integers, and let us say for such sequences s_1, s_2, that $s_1 \leq s_2$ if s_1 is an initial segment of s_2. It was shown in the Ph.D. thesis of D. Kurepa (and later, independently, by R. Laver; see [1]) that the well-founded tree $(V, \leq)$ cannot be partitioned into countably many antichains, and $|V| = 2^{\aleph_0}$. Let M be the collection of all infinite chains of $(V, \leq)$; we claim that (V, M) satisfies the requirements at the start of this section. Let $(X_1, X_2, \ldots)$ be a partition of V; we must show that some X_i includes an infinite chain of $(V, \leq)$. Suppose not; then each X_i can be partitioned into countably many antichains, and hence so can V, a contradiction. Thus (V, M) satisfies our requirements, and so the corresponding graph G satisfies (2.2). Moreover, it follows from the results of [6] that G does not contain $K_{\aleph_1}$. This proves (1.6). □

3. An independence result

We have seen that (1.2) and (1.4) are

(i) true for all countable graphs and for all graphs that do not contain $K_{\aleph_0}$,

(ii) false for a graph with $2^{\aleph_0}$ vertices which does not contain $K_{\aleph_1}$.

What about graphs with $\aleph_1$ vertices? We do not know whether such a graph can satisfy (1.5) (and hence falsify (1.2) and (1.4)), but for (1.6) we have an independence result. Let us consider the truth of the following statement:

(3.1). *In every infinitely-connected graph G with $|V(G)| = \aleph_1$ that does not contain $K_{\aleph_1}$, there is a rayless spanning tree.*

We shall see that (3.1) is independent of ZFC. For we observe from (1.6) that

(3.2). *If the continuum hypothesis holds, then* (3.1) *is false.*

On the other hand, Baumgartner, Malitz, and Reinhardt [2] proved that the following statement is consistent with ZFC (although not with the continuum hypothesis):

(3.3). *If* $(V, \leq)$ *is a well-founded tree with* $|V| \leq \aleph_1$, *and every chain of* $(V, \leq)$ *has order type* $< \omega_1$, *then* V *may be partitioned into countably many antichains.*

In the rest of this section we shall prove that (3.3) implies (3.1). If $T = (V(T), \leq)$ is a well-founded tree and $t_1, t_2 \in V(T)$, we say that $t \in V(T)$ *is between* t_1 and t_2 if $\inf(t_1, t_2) \leq t$, and either $t \leq t_1$, or $t \leq t_2$. We say that t_1 is a *predecessor* of t_2 if $t_1 \leq t_2$ and there is no $t \in V(T) - \{t_1, t_2\}$ such that $t_1 \leq t \leq t_2$. A *well-founded tree-decomposition* of a graph G is a pair (T, W), where T is a well-founded tree and $W = (W_t : t \in V(T))$ is a collection of sets such that

(W1) $\bigcup(W_t : t \in V(T)) = V(G)$, *and every edge of* G *has both its ends in some* W_t;

(W2) *if* t' *is between* t *and* t'' *in* T, *then* $W_t \cap W_{t''} \subseteq W_{t'}$; *and*

(W3) *if* t *has no predecessor, then* $W_t \supseteq \bigcup_{t'<t} \bigcap_{t' \leq t'' < t} W_{t''}$.

We need the following structure theorem [6, (2.7)].

(3.4). *Let* G *be an infinitely-connected graph with* $|V(G)| \leq \aleph_1$ *that does not contain* $K_{\aleph_1}$. *Then there exists a well-founded tree-decomposition* (T, W) *of* G *such that*

(i) $|V(T)| \leq \aleph_1$,
(ii) *every chain of* T *has order type* $< \omega_1$, *and*
(iii) W_t *induces a countable-infinitely connected graph in* G *for every* $t \in V(T)$.

(3.5). *If* (3.3) *holds, then* (3.1) *holds.*

Proof. Let G be an infinitely-connected graph with $|V(G)| = \aleph_1$ which does not contain $K_{\aleph_1}$, and let (T, W) be a well-founded tree-decomposition of G as in (3.4).

(1) *For every* $v \in V(G)$, *there exists a unique minimal element* $t \in V(T)$ *with* $v \in W_t$.

For there exists at least one such t by (W1), and the minimal one is unique by the existence of infima and (W2).

For $v \in V(G)$, the element $t \in V(T)$ as in (1) will be denoted by $t(v)$. From (i) and (ii) of (3.4) and (3.3), $V(T)$ can be partitioned into countably many antichains, say $A_1, A_2, \ldots$.

Let t_0 be the root of T. An *ideal* of T is a subset $S \subseteq V(T)$ such that $t_0 \in S$ and such that $s \in S$ for every $s \in V(T)$ with $s \leq t$ for some $t \in S$. Choose $v_0 \in W_{t_0}$. A *sprout* is a triple (S, R, ρ), where S is an ideal of T, R a tree of G with $\bigcup_{s \in S} W_s = V(R)$, and ρ is a function from $V(R)$ into $\{1, 2, \dots\}$ such that the following hold:

(2) *If $v \neq v_0$ is between v_0 and v' in R, then $\rho(v) \geq \rho(v')$, and the inequality is strict unless $t(v) = t(v')$,*

(3) *The set $\{v \in V(R) : t(v) = t$ and $\rho(v) \leq i\}$ is finite for every $t \in V(T)$ and every $i \in \{1, 2, \dots\}$,*

(4) *For every $v \in V(R)$ with $t(v) \in A_i$, $\rho(v) \geq i$.*

We first prove the following:

(5) *If (S, R, ρ) is a sprout, $t \in V(T)$, and $i \geq 1$ is an integer, then $\{v \in W_t \cap V(R) : \rho(v) \leq i\}$ is finite.*

We prove (5) by transfinite induction. The statement follows from (3) if $t = t_0$, so let $t \in V(T) - \{t_0\}$, let $i \geq 1$ be an integer, and assume that (5) holds for all $t' < t$. There exists $t_1 \in V(T)$ with $t_1 < t$ such that $t' \notin A_1 \cup \cdots \cup A_i$ for every $t' \in V(T)$ with $t_1 < t' < t$. Hence

$$\{v \in W_t \cap V(R) : \rho(v) \leq i\} \subseteq \{v \in W_t \cap V(R) : \rho(v) \leq i,\ t(v) \leq t_1\} \cup \{v \in W_t \cap V(R) : \rho(v) \leq i,\ t(v) = t\},$$

by (4) and (W2). The first set is a subset of $\{v \in W_{t_1} \cap V(R) : \rho(v) \leq i\}$, and hence is finite by the induction hypothesis, and the second one is finite by (3). This proves (5).

(6) *There exists at least one sprout.*

For let $S = \{t_0\}$, and let $R = P_1 \cup P_2 \cup \cdots$ be the spanning ω-star of the subgraph of G induced by W_{t_0} with center v_0, which exists by (1.9), and (iii) of (3.4). Let $\rho : V(R) \to \{1, 2, \dots\}$ be such that if $\rho(v) = i$, then $v \in V(P_i)$. Then (S, R, ρ) is a sprout, as desired.

We order sprouts by saying that $(S, R, \rho) \leq (S', R', \rho')$ if $S \subseteq S'$, R is a subtree of R', and $\rho(r) = \rho'(r)$ for every $r \in V(R)$. By Zorn's lemma, there exists a maximal sprout (S, R, ρ).

(7) $S = V(T)$.

For suppose not; then there exists $t \in V(T) - S$, minimal. Let k be such that $t \in A_k$, let $F = \bigcup_{s \in S} W_s$, and let $u_k, u_{k+1}, \dots$ be all the vertices of $W_t - F$. Since $W_t \cap V(R) = W_t \cap F$, and since $W_t \cap F$ is infinite (because $W_t \cap F$ separates the infinite sets W_t and W_{t_0} and G is infinitely-connected) we deduce by repeated application of (5) and (iii) of (3.4) that there exist mutually disjoint finite paths $P_k, P_{k+1}, \dots$ such that for every $i \geq k$, $V(P_i) \subseteq W_t$, P_i

has as one end some $w \in W_t \cap F$ with $\rho(w) > i$, and has no other vertex in F, and $u_i \in \bigcup_{k\leq i'\leq i} V(P_{i'})$. In particular,

$$W_t - F \subseteq \bigcup_{i\geq k} V(P_i) \subseteq W_t.$$

Let $S' = S \cup \{t\}$, let $R' = R \cup P_k \cup P_{k+1} \cup \cdots$, and let ρ' be defined by

$$\rho'(r) = \begin{cases} \rho(r) & \text{if } r \in V(R) \\ i & \text{if } r \in V(P_i) - V(R). \end{cases}$$

Then (S', R', ρ') is a sprout, contradicting the maximality of (S, R, ρ). This proves (7).

From (7) and (W1) we deduce that R is a spanning tree of G, and from (2) and (3) it follows that R is rayless. □

We deduce that both (3.1) and its negation are relatively consistent with ZFC, and hence (3.1) is independent of ZFC.

4. Excluding the $\aleph_1$-tree

Our final objective is to prove (1.7), which we restate:

(4.1). *Every infinitely-connected graph which does not contain $T_{\aleph_1}$ has a rayless spanning tree.*

If $X \subseteq V(G)$, an *X-flap* is the vertex set of a component of $G \backslash X$. We shall need the following, a consequence of [7, Theorem (2.3)].

(4.2). *Let G be a graph that does not contain $T_{\aleph_1}$. For each $X \subseteq V(G)$ with $|X| \leq \aleph_0$, let $\beta(X)$ be a union of X-flaps such that if $X \subseteq Y \subseteq V(G)$ and $|Y| \leq \aleph_0$, then $\beta(X)$ includes precisely those X-flaps which meet $\beta(Y)$. Then $\beta(\emptyset) = \emptyset$.*

Proof of (4.1). Let us say that $X \subseteq V(G)$ is *good* if $X \subseteq V(T)$ for some rayless tree T of G (not necessarily a spanning tree).

(1) *Every countable subset of $V(G)$ is good.*

For let $X \subseteq V(G)$ be countable. Then $X \subseteq V(H)$ for some countable subgraph H of G which is infinitely-connected, and by (1.9) H has a spanning ω-star. Hence X is good.

(2) *If $X \subseteq V(G)$ is good and C_i $(i \in I)$ are good X-flaps, then $X \cup \bigcup(C_i : i \in I)$ is good.*

For let T be a rayless tree of G with $X \subseteq V(T)$, and for each $i \in I$ let T_i be a rayless tree with $C_i \subseteq V(T_i)$. Since G is connected, we may assume that each T_i intersects T. For each $i \in I$, choose $S_i \subseteq T_i$ minimal such that $S_i \cup T$ is connected and $C_i \subseteq V(S_i \cup T)$. Then each component of S_i has exactly one vertex in $V(T)$, and $V(S_i \cup T) = C_i \cup V(T)$. Thus, $T \cup \bigcup(S_i : i \in I)$ is a rayless tree of G, and its vertex set includes $X \cup \bigcup(C_i : i \in I)$, as required.

For each $X \subseteq V(G)$ with $|X| \leq \aleph_0$, let $\beta(X)$ be the union of all X-flaps that are not good.

(3) *If* $X \subseteq Y \subseteq V(G)$ *and* $|Y| \leq \aleph_0$, *then* $\beta(X)$ *includes just those* X*-flaps that meet* $\beta(Y)$.

For, if an X-flap C meets a Y-flap $D \subseteq \beta(Y)$, then $D \subseteq C$; and so C is not good, since D is not good, and hence $C \subseteq \beta(X)$. Conversely, let C be an X-flap with $C \cap \beta(Y) = \varnothing$. Let the Y-flaps included in C be C_i $(i \in I)$. Since $C \cap \beta(Y) = \varnothing$, each C_i is good, and so by (1) and (2) $Y \cup \bigcup(C_i : i \in I)$ is good. Hence C is good, since $C \subseteq Y \cup \bigcup(C_i : i \in I)$, and so $C \not\subseteq \beta(X)$. This proves (3).

From (3) and (4.2), we deduce that $\beta(\varnothing) = \varnothing$. Hence G has a rayless spanning tree, as required. □

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