# Excluding a countable clique

by

Reinhard Diestel
Faculty of Mathematics
Chemnitz University
D-09107 Chemnitz
Germany

and*

Robin Thomas**
School of Mathematics
Georgia Institute of Technology
Atlanta, Georgia 30332
USA

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**Abstract**

We extend the excluded $K_n$ minor theorem of Robertson and Seymour to infinite graphs, and deduce a structural characterization of the infinite graphs that have no $K_{\aleph_0}$ minor. The latter is a refinement of an earlier characterization of Robertson, Seymour and the second author.

## 1. Introduction

In both finite and infinite graph theory, there are numerous so-called *excluded minor theorems*, theorems that describe the structure of the graphs not containing one or several given graphs as a minor. A classic example is the theorem of Wagner [15] that a graph has no minor isomorphic to $K_5$ if and only if it can be constructed by piecing together copies of planar graphs and of two specific non-planar graphs $W$ and $K$ in a certain tree-like fashion. (In modern terms: a graph has no $K_5$ minor if and only if it has a certain tree-decomposition into planar graphs and copies of $W$ and $K$.) Such characterizations can be useful: we often need to exclude certain minors when they are obvious obstructions to some desired property, but knowledge of the structure which their exclusion forces may enable us to establish that property for the remaining graphs. Surveys of excluded minor theorems are given in [2] (for finite minors) and [8] (for infinite minors).

Recently, Robertson and Seymour [7] found an excluded minor theorem for excluding $K_n$ ($n$ fixed). In a sense, this is the most comprehensive of all finite excluded minor theorems: since every finite graph $H$ is trivially a minor of $K_n$ for $n = |H|$, and the minor relation is transitive, this theorem offers a structural description of the graphs without an $H$ minor for every finite graph $H$. This result is the cornerstone in Robertson and Seymour's proof of their *Graph Minor Theorem* (or 'Wagner's conjecture', as they call it): if $G_0, G_1, \dots$ is an infinite sequence of finite graphs, then there are indices $i < j$ such that $G_i$ is a minor of $G_j$. In its proof, Robertson and Seymour use their $K_n$ minor theorem as follows. Assume, as we may, that $G_0$ is not a minor of any other $G_i$. Then each of the graphs $G_1, G_2, \dots$ has the structure forced by the exclusion of $G_0$ (or $K_{|G_0|}$), which helps to prove the existence of the desired indices $i$ and $j$.

In this paper, we shall first extend the $K_n$ minor theorem of Robertson and Seymour to infinite graphs, though still with $n$ finite. We then use this result to deduce an excluded minor theorem for $K_{\aleph_0}$, a theorem describing the structure of all graphs without a $K_{\aleph_0}$ minor. Unlike the (finite or infinite) $K_n$ minor theorem, this result will be sharp: a graph has the structure described if and only if it has no $K_{\aleph_0}$ minor.

Our motivation for considering the exclusion of $K_{\aleph_0}$, among other possible candidates, is twofold. First, $K_{\aleph_0}$ is in a sense the most general countable minor to exclude (as explained above for $K_n$), and is therefore a natural first choice. Second, there is the challenge to extend the Graph Minor Theorem to infinite graphs: it is known to be false in general [11], but was conjectured in [12] to extend to countable graphs. In a possible proof along the lines of the finite version, our $K_{\aleph_0}$ minor theorem might assume the role played there by the $K_n$ minor theorem: given a sequence $G_0, G_1, \dots$ of countable graphs, we may assume that $G_1, G_2, \dots$ all have the structure forced by the exclusion of $K_{\aleph_0} \supseteq G_0$. We remark that the structure of the graphs without a topological $K_{\aleph_0}$ minor (i.e. the graphs not containing a subdivision of $K_{\aleph_0}$) is much simpler and easier to characterize [4] [9].

The paper is organized as follows. Section 2 gives definitions and background facts. In Section 3 we state our results. Sections 4 and 5 provide some lemmas about surfaces and about tree-decompositions. In Sections 6 and 7, respectively, we prove the infinite $K_n$ minor theorem and our $K_{\aleph_0}$ minor theorem.

## 2. Terminology and background

The graphs we consider are simple and undirected, and they may be infinite. Our terminology follows [1]; any terms not defined below are explained there.

A graph $H$ is a *minor* of a graph $G$ if $G$ contains a family $(V_x)_{x \in V(H)}$ of disjoint connected vertex sets, possibly infinite, such that $G$ has a $V_x$–$V_y$ edge whenever $xy$ is an edge of $H$.

Let $G$ be a graph, $T$ a tree, and let $\mathcal{V} = (V_t)_{t \in T}$ be a family of vertex sets $V_t$ in $G$ indexed by the vertices $t$ of $T$. The pair $\mathcal{D} = (T, \mathcal{V})$ is called a *tree-decomposition* of $G$ if it satisfies the following three conditions:

(T1) $V(G) = \bigcup_{t \in T} V_t$;

(T2) for every edge $e \in G$ there exists a $t \in T$ such that both ends of $e$ lie in $V_t$;

(T3) $V_{t_1} \cap V_{t_3} \subseteq V_{t_2}$ whenever $t_2$ lies on the $t_1$–$t_3$ path in $T$.

The tree $T$ is the *decomposition tree* of the decomposition $\mathcal{D}$. If $T$ is a path, then $\mathcal{D}$ is a *path decomposition*. The subgraphs $G[V_t]$ of $G$ induced by the sets $V_t$ are the *parts* of $\mathcal{D}$.

If an integer $k$ is minimal such that $|V_t| \leqslant k+1$ for all $t \in T$, then $\mathcal{D}$ has *width* $k$. If the values of $|V_{t_1} \cap V_{t_2}|$ for edges $t_1 t_2 \in T$ and of $\liminf_{i \to \infty} |V_{t_i} \cap V_{t_{i+1}}|$ for infinite paths $t_1 t_2 \dots$ in $T$ are always finite, we say that $\mathcal{D}$ has *finite adhesion*.

Given $t \in T$, the *torso* of $G$ at $t$ is the graph on $V_t$ in which two vertices $u, v$ are adjacent if they are adjacent in $G$ or if $\{u, v\} \subseteq V_{t'}$ for some neighbour $t'$ of $t$ in $T$. If all the torsos in $\mathcal{D}$ are subgraphs of $G$, the tree-decomposition $(T, \mathcal{V})$ is called *simplicial*. We say that $\mathcal{D}$ is a tree-decomposition *over* a given class $\mathcal{G}$ of graphs if all its torsos (not just its parts) belong to $\mathcal{G}$.

The following characterization of the graphs without an infinite complete minor was obtained in [10]:

**Theorem 2.1.** *A graph has no $K_{\aleph_0}$ minor if and only if it has a tree-decomposition of finite adhesion in which each torso fails to have a $K_n$ minor, for some integer $n$ depending on the torso.*

The 'if' part of this result is not difficult. Its substance lies in the 'only if' part, in its description of the structure of the graphs without a $K_{\aleph_0}$ minor. This structure is expressed in terms of the graphs without a $K_n$ minor. But what is the structure of those graphs?

For finite graphs, the answer is given by the $K_n$ minor theorem of Robertson and Seymour mentioned in the Introduction (Theorem 2.2 below). For infinite graphs, it is given by our generalization of that result, Theorem 3.1. In Theorem 3.2, we shall characterize the graphs without a $K_{\aleph_0}$ minor comprehensively by combining their coarse structure as in Theorem 2.1 with their fine structure given by Theorem 3.1.

In order to state both the finite and the infinite $K_n$ minor theorem precisely, we need some more definitions. We start by adapting the notion of path-decomposition to the infinite case.

Let $G$ be a graph, and let $(\mathcal{X}, \leqslant)$ be a linearly ordered family of subsets of $V(G)$. We say that $(\mathcal{X}, \leqslant)$ is a *linear decomposition* of $G$ if the following conditions are satisfied:

(L1) $V(G) = \bigcup \mathcal{X}$;

(L2) for every edge $e \in G$ there exists an $X \in \mathcal{X}$ containing both ends of $e$;

(L3) $X_1 \cap X_3 \subseteq X_2$ whenever $X_1 \leqslant X_2 \leqslant X_3$ in $\mathcal{X}$.

If $\mathcal{X}$ is finite, this is just a path-decomposition of $G$. In general, however, the linear ordering on $\mathcal{X}$ need not be discrete, and in particular need not correspond to a (finite or infinite) path. If an integer $k$ is minimal such that $|X| \leqslant k+1$ for all $X \in \mathcal{X}$, we say that $(\mathcal{X}, \leqslant)$ has *width* $k$. (Linear decompositions of 'infinite width' will not be needed.)

A *surface* in this paper is a compact connected 2-manifold with (possibly empty) boundary; the surface is *closed* if its boundary is empty. The unique surface obtained from a closed surface $S$ by removing the interiors of $k$ disjoint closed discs will be denoted by $S-k$ and called, informally, a (copy of) '*$S$ with $k$ holes*'.

The components of the boundary of a surface $S$ are the *cuffs* of $S$. Each cuff $C$ of a surface $S$ is homeomorphic to the unit circle $S^1$, so it is the image of a continuous map $f\colon [0,1] \to S$ that is 1–1 except for $f(0) = f(1)$. For every surface we consider, we shall assume that each of its cuffs $C$ comes equipped with some fixed such mapping $f$, and call $f(0)$ the *root* of $C$. The other points of $C$ then inherit the linear ordering of $(0,1)$ through $f$, so any subset of $C \setminus \{ f(0) \}$ (such as the sets $U_i$ in (N3) below) carries a natural linear ordering.

Let $G$ be a graph, and $S$ a surface with cuffs $C_1, \dots, C_k$. By an *embedding* of $G$ in $S$ we mean a continuous 1–1 function $f$ from $G$ (viewed as a CW-complex) to $S$, such that $f(G)$ meets the boundary of $S$ only in vertices and does not contain the root of any cuff. We shall not normally distinguish $f(G)$ notationally from $G$.

We say that $G$ *can be nearly embedded* in $S$ if $G$ has a set $X$ of at most $k$ vertices (where $k$ is the number of cuffs of $S$; see above) such that $G - X$ can be written as $G_0 \cup G_1 \cup \ldots \cup G_k$, where

(N1) $G_0$ has an embedding in $S$;

(N2) the graphs $G_i$ $(i = 1, \dots, k)$ are pairwise disjoint, and $U_i := V(G_0) \cap V(G_i) = V(G_0) \cap C_i$ for each $i$;

(N3) for each $i = 1, \dots, k$, the graph $G_i$ has a linear decomposition $(X_u)_{u \in U_i}$ of width less than $k$, such that $u \in X_u$ for all $u \in U_i$. (The sets $X_u$ are ordered by the ordering of their indices $u$ as points on $C_i$.)

This entire set-up—the choice of $X$, of $G_0, \dots, G_k$, of the embedding of $G_0$, and of the linear decompositions for $G_1, \dots, G_k$—will be called a *near-embedding* of $G$ in $S$, with *deleted set* $X$. The class of all countable graphs that can be nearly embedded in a surface $S$ will be denoted by $\mathcal{F}(S)$.

The important thing about near-embeddings of graphs in a surface $S$ is that they differ from proper embeddings in a standard closed surface (obtained by sewing discs on to the cuffs of $S$) only 'in a bounded way': there is a bounded number of cuffs, a bounded number of vertices that may be disregarded (those in $X$), and along each cuff an 'outgrowth of bounded width' from the surface. The fact that all these bounds were chosen the same, as the number $k$ given implicitly with $S$, will simplify matters but is otherwise inessential. We remark that subgraphs (let alone minors) of nearly embedded graphs do not necessarily inherit near-embeddings from their supergraphs: when vertices $u$ in (N3) get deleted, the remainder of their sets $X_u$ may no longer have a place in the linear decomposition formed by the other sets $X_u$.

We can now state the excluded $K_n$ minor theorem of Robertson and Seymour which we will extend (verbatim) to infinite graphs. Given a positive integer $n$, we denote by $S_n$ the orientable closed surface of highest genus in which $K_n$ cannot be embedded. Similarly, let $S'_n$ denote the non-orientable closed surface of highest genus in which $K_n$ cannot be embedded.

**Theorem 2.2.** [7] *For every $n \geqslant 0$ there exists a $k \geqslant 0$ such that every finite graph with no $K_n$ minor has a tree-decomposition over $\mathcal{F}(S_n - k) \cup \mathcal{F}(S'_n - k)$.*

Theorem 2.2 is not best possible: graphs in $\mathcal{F}(S_n - k)$ and in $\mathcal{F}(S'_n - k)$ may well have a $K_n$ minor. However, as Robertson and Seymour [7] point out, they cannot have arbitrarily large complete minors:

**Lemma 2.3.** *For every surface $S$ there exists an integer $n$ such that no graph that can be nearly embedded in $S$ has a $K_n$ minor.*

Since no proof of Lemma 2.3 is currently available in the literature, we shall sketch one in Section 8.

## 3. Statements of results

Our first result will be the extension of the $K_n$ excluded minor theorem of Robertson and Seymour to infinite graphs. Given integers $n$ and $k$, let the surfaces $S_n$ and $S'_n$ be defined as for Theorem 2.2.

**Theorem 3.1.** *For every $n \geqslant 0$ there exists a $k \geqslant 0$ such that every (finite or infinite) graph with no $K_n$ minor has a tree-decomposition over $\mathcal{F}(S_n - k) \cup \mathcal{F}(S'_n - k)$.*

By Lemma 2.3, the tree-decompositions in Theorem 3.1 have finite adhesion: since the overlaps between two adjacent parts induce complete subgraphs in their torsos, these overlaps cannot contain more than $n - 1$ vertices.

To state our second result, a characterization of the structure of infinite graphs without a $K_{\aleph_0}$ minor refining Theorem 2.1, we need some more terminology. Given a graph $G$, a closed surface $S$ and an integer $k \geqslant 0$, let us say that $G$ is *embedded in $S$ with $k$ vortices* if there exists a $k' \geqslant k$ such that $G$ is nearly embedded in $S - k'$, say as $G - X = G_0 \cup G_1 \cup \ldots \cup G_{k'}$, with $G_i \neq \emptyset$ if and only if $i \leqslant k$. (The motivation for this terminology is that, in Section 7, it will be convenient to divorce with number of holes used for the near-embedding from $|X|$ and from the maximal width of the linear decompositions around these holes: if $G$ is embedded in $S$ with $k$ vortices, we use only $k$ holes for linear decompositions but these may have widths up to some larger bound $k'$.) If $S$ is the sphere and $k \leqslant 1$, we simply say that $G$ is *plane with at most one vortex*.

**Theorem 3.2.** *A graph has no $K_{\aleph_0}$ minor if and only if it has a tree-decomposition of finite adhesion over plane graphs with at most one vortex.*

## 4. Surfaces

In this section we briefly recall the few standard facts about surfaces and graph embeddings that we shall need in our proofs.

By a standard result in topological graph theory (see [ 6 ]), a finite graph $G$ can be embedded in a given closed surface $S$ in only finitely many ways, up to homeomorphisms of $S$. The same is true when $S$ has $k \geqslant 1$ cuffs: we may simply think of each cuff as a cycle added to $G$, and apply the result for closed surfaces to the finitely many graphs that can be obtained from $G$ by adding (the edges of) $k$ disjoint cycles. In fact, the homeomorphisms may even be chosen so that they fix every root of a cuff in $S$ (and hence map every cuff onto itself), as well as the 'orientations' (that is, the linear orderings) of the cuffs; we shall call such homeomorphisms *rooted homeomorphisms* of $S$:

**Lemma 4.1.** *Given a finite graph $G$ and a surface $S$, there is a finite set $\{ f_1, \dots, f_n \}$ of embeddings of $G$ in $S$ such that, for every embedding $f\colon G \to S$, there exists a rooted homeomorphism $\phi\colon S \to S$ such that $f = \phi \circ f_i$ for some $i = 1, \dots, n$.*

Let $S$ be a closed surface, and let $C \subseteq S$ be a simple closed curve that is not homotopic to a 'point' (i.e., to a constant map). Such a curve $C$ will be called *genus-reducing* (on $S$), for the following reason. Let $S/C$ denote the space obtained by cutting along $C$ and sewing a disc on to each of the one or two new boundary circles arising from the cut; $S/C$ is either a surface or the disjoint union of two surfaces. (To see that there are indeed exactly one or two new boundary circles, and that $S/C$ has at most two components, consider a strip neighbourhood of $C$ on $S$: if the strip is twisted, we get one boundary circle, otherwise two.) Since $C$ did not bound a disc on $S$ but each of the one or two boundary circles arising from the cut bounds a disc on $S/C$, an easy calculation yields the following well-known fact:

**Lemma 4.2.** *Each of the (one or two) components of $S/C$ is a surface of strictly greater Euler characteristic than $S$.*

Lemma 4.2 is a useful tool for induction proofs based on the invariant $2 - \chi(S)$ of a surface $S$, its *Euler genus*. To perform the induction step, however, one first has to find a suitable genus-reducing curve. In the context of graph embeddings, the following lemma often provides such a curve:

**Lemma 4.3.** [ 6 ] *Let $S \neq S^2$ be a closed surface, and let $G$ be a finite graph embedded in $S$. Then $S$ contains a genus-reducing curve $C$ such that either $C \subseteq G$ or $C \cap G = \emptyset$.*

## 5. Tree-decompositions

In this section, we collect some facts about tree-decompositions that will be used later. We start with two lemmas that are standard for finite tree-decompositions, but whose proofs (e.g. as in [1, Lemmas 12.3.2 and 12.3.3]) cover the following more general cases too.

**Lemma 5.1.** *If $(T, (V_t)_{t\in T})$ is a tree-decomposition of $G$ and $t \in T$ is a vertex on the path in $T$ between two other vertices $t_1, t_2 \in T$, then $V_t$ separates $V_{t_1}$ from $V_{t_2}$ in $G$.*

**Lemma 5.2.** *If $\mathcal{D}$ is a tree-decomposition of a graph $G$ and $K \subseteq G$ is a finite complete subgraph, then $K$ is contained in one of the parts of $\mathcal{D}$.*

Short proofs of the next lemma, first proved in [13], were given by Thomassen [14] and in [5]:

**Lemma 5.3.** *Let $k \geqslant 0$ be an integer. If every finite subgraph of a graph $G$ has a tree-decomposition of width at most $k$, then so does $G$.*

Recall that a tree-decomposition of a graph $G$ is called *simplicial* if all its torsos are subgraphs of $G$; thus whenever $t_1t_2$ is an edge of the decomposition tree and $V_{t_1}, V_{t_2}$ are the corresponding parts, then $V_{t_1} \cap V_{t_2}$ spans a complete subgraph in $G$. There is also a more general concept of a *simplicial decomposition* that is not necessarily a tree-decomposition. We shall not need that concept here, but we shall use a couple of lemmas claiming the existence of simplicial tree-decompositions, where the source we cite states them only as providing simplicial decompositions. This difference, however, is a trivial technicality: since all those decompositions have finite adhesion, they will automatically be simplicial tree-decompositions [2, Cor. 1.1.8 (i)].

A graph is called *prime* if it is not separated by any complete subgraph. The following lemma follows from a well-known result of Halin (1964):

**Lemma 5.4.** [2, Thm. 2.1.6] *Every graph with no infinite complete subgraph has a simplicial tree-decomposition into prime parts.*

For simplicial tree-decompositions, Lemma 5.2 extends as follows:

**Lemma 5.5.** *If $\mathcal{D}$ is a simplicial tree-decomposition of a graph $G$ and $H \subseteq G$ is a finite and prime induced subgraph, then $H$ is contained in one of the parts of $\mathcal{D}$.*

The following easy lemma follows at once from [5, Prop. 3.2].

**Lemma 5.6.** *Let $G$ be a prime graph such that $K_n \not\subseteq G$ for some $n$, and let $H \subseteq G$ be finite. Then $G$ has a prime induced finite subgraph $H'$ such that $H \subseteq H'$.*

By a standard application of compactness, one can easily show that the assumption of $K_n \not\subseteq G$ in Lemma 5.6 is not in fact needed [3]. But it simplifies the lemma's proof and will hold when we apply it below.

A *graph property* is a class of graphs closed under isomorphism. Let us call a graph property $\mathcal{G}$ *normal* if the following conditions hold:

(i) if $H \subseteq G \in \mathcal{G}$, then $H \in \mathcal{G}$;

(ii) if every finite subgraph of a graph $G$ belongs to $\mathcal{G}$, then $G \in \mathcal{G}$;

(iii) $\mathcal{G}$ does not contain all finite graphs.

Our next lemma follows from [5, Thms. 3.9 & 3.5]; we include an independent proof for the reader's convenience.

**Lemma 5.7.** *Let $\mathcal{G}$ be a normal graph property, and let $G$ be a countable graph whose finite subgraphs each have a tree-decomposition over $\mathcal{G}$. Then $G$ has a tree-decomposition over $\mathcal{G}$.*

**Proof.** For every finite subgraph $G' \subseteq G$ consider a tree-decomposition $\mathcal{D}$ of $G'$ over $\mathcal{G}$. Let $\overline{G'}$ be the union of the torsos of $\mathcal{D}$; thus, $\overline{G'}$ arises from $G'$ by adding any missing edges in the overlaps between adjacent parts of $\mathcal{D}$. Then $\mathcal{D}$ is a simplicial tree-decomposition of $\overline{G'}$. By Lemma 5.5, any prime induced subgraph $H$ of $\overline{G'}$ is contained in one of its parts. Since these parts are the torsos of $G'$, they lie in $\mathcal{G}$, and hence by (i) so do their subgraphs $H$. Therefore $\overline{G'}$ has the property that

$$\textit{all its finite prime induced subgraphs lie in } \mathcal{G}. \qquad (*)$$

Let us extend this property to a suitable supergraph $\overline{G}$ of $G$ by compactness. Let $V(G) = \{ v_1, v_2, \dots \}$. We apply König's infinity lemma [1] to the graph $K$ whose vertices are the graphs $\overline{G_n}$ that are obtained from the graphs $G_n = G[v_1, \dots, v_n]$ by adding edges and satisfy $(*)$. For each integer $n$, there are only finitely many such graphs $\overline{G_n}$, and deleting $v_n$ from any such $\overline{G_n}$ results in a graph of the form $\overline{G_{n-1}}$; we then join these graphs $\overline{G_n}$ and $\overline{G_{n-1}}$ as vertices in $K$. By the infinity lemma, $K$ has an infinite path of the form $\overline{G_1}\,\overline{G_2}\dots$, and we take $\overline{G}$ to be the union of these graphs. Since every finite subgraph of $\overline{G}$ is contained in some $\overline{G_n}$, it is clear that $\overline{G}$ inherits $(*)$ from its subgraphs $\overline{G_n}$; by (i) and (iii), this implies in particular that $K_n \not\subseteq \overline{G}$ for some $n$.

By Lemma 5.4, therefore, $\overline{G}$ has a simplicial tree-decomposition $\mathcal{D}^*$ into prime parts. The finite ones of these lie in $\mathcal{G}$ by $(*)$. But also the infinite parts $P$ of $\mathcal{D}^*$ lie in $\mathcal{G}$. Indeed every finite $H \subseteq P$ extends to some finite induced prime subgraph $H'$ of $P$ (Lemma 5.6), and $H' \in \mathcal{G}$ by $(*)$; so $H \in \mathcal{G}$ by (i), and $P \in \mathcal{G}$ by (ii).

Since $\overline{G}$ is obtained from $G$ just by adding edges, $\mathcal{D}^*$ is a tree-decomposition of $G$, whose torsos are subgraphs of the parts of $\mathcal{D}^*$ in $\overline{G}$. As we have seen, these parts lie in $\mathcal{G}$, so by (i) the torsos of $\mathcal{D}^*$ in $G$ lie in $\mathcal{G}$ too. Therefore, $\mathcal{D}^*$ is a tree-decomposition of $G$ over $\mathcal{G}$. □

**Lemma 5.8.** *Let $\mathcal{G}$ be a class of graphs, $G$ a graph, and $\mathcal{D}$ a tree-decomposition of $G$, of finite adhesion and such that every torso has a tree-decomposition over $\mathcal{G}$. Then $G$ has a tree-decomposition $\mathcal{D}'$ over $\mathcal{G}$ whose parts are each contained in a part of $\mathcal{D}$. If all the tree-decompositions of the torsos in $\mathcal{D}$ have finite adhesion, then so does $\mathcal{D}'$.*

**Proof.** Let $\mathcal{D} =: (T, (V_t)_{t \in T})$, with torsos $G_t$ $(t \in T)$. Our plan is to refine $\mathcal{D}$ at each $t \in T$ by a tree-decomposition $(S^t, (U_s^t)_{s \in S^t})$ of $G_t$ over $\mathcal{G}$. Let $S$ be the tree obtained from the disjoint union of all the trees $S^t$ $(t \in T)$ by adding, for every edge $t_1 t_2$ of $T$, an edge joining a vertex $s_1 \in S^{t_1}$ to a vertex $s_2 \in S^{t_2}$, where each $s_i$ is chosen so that $V_{t_1} \cap V_{t_2} \subseteq U_{s_i}^{t_i}$; such $s_i$ can be found by Lemma 5.2, because $V_{t_1} \cap V_{t_2}$ induces a finite complete graph in both torsos $G_{t_i}$. For each $s \in S$, let $X_s := U_s^t$, where $t$ is such that $s \in S^t$. It is straightforward to check that $\mathcal{D}' := (S, (X_s)_{s \in S})$ is indeed a tree-decomposition of $G$ with the required properties. □

A graph $H$ is said to be a *topological minor* of a graph $G$ if $G$ contains a subdivision of $H$ as a subgraph. Note that every topological minor of $G$ is also its minor.

**Lemma 5.9.** *Let $G$ be a graph not containing $K_{\aleph_0}$ as a topological minor. Then $G$ has a tree-decomposition into countable parts whose torsos are topological minors of $G$.*

**Proof.** Let $G'$ be the graph obtained from $G$ by joining any two non-adjacent vertices that are linked in $G$ by uncountably many independent paths. By a result of Halin (1967) (see [2, Thm. 5.2.1]), $G'$ has a simplicial tree-decomposition $\mathcal{D}$ into countable parts. But $\mathcal{D}$ is also a tree-decomposition of $G$, whose torsos $G_t$ are subgraphs of $G'$. By definition of $G'$, we may replace the edges in $E(G_t) \setminus E(G)$ with independent paths avoiding the (countable) set $V(G_t)$ in their interior, and thus obtain a subdivision of $G_t$ in $G$. Hence every torso $G_t$ is a topological minor of $G$, as required. □

## 6. Excluding a finite graph

The following will be our key lemma in the proof of Theorem 3.1, and it will be used again in the proof of Theorem 3.2. The lemma says that while near-embeddability may not lend itself directly to an extension from the finite to the infinite by compactness, it does so 'up to tree-decomposition':

**Lemma 6.1.** *Let $k \geqslant 0$ be an integer, $S$ a closed surface, and $G$ a countable graph. Assume that for every finite subgraph $H \subseteq G$ there exists a finite subgraph $\hat{H} \subseteq G$ such that $H \subseteq \hat{H} \in \mathcal{F}(S - k)$. Then $G$ has a tree-decomposition of finite adhesion over $\mathcal{F}(S - 5k^2)$.*

Before we prove this lemma, let us deduce Theorem 3.1 from it.

**Proof of Theorem 3.1** (assuming Lemma 6.1). Let $n$ be given, let $S_n$ and $S'_n$ be defined as before Theorem 2.2, and let $\ell$ be the integer $k$ supplied by Theorem 2.2. We claim that $k := 5\ell^2$ satisfies Theorem 3.1. Let $\mathcal{F}$ denote the class of graphs whose finite subgraphs are all in $\mathcal{F}(S_n - \ell) \cup \mathcal{F}(S'_n - \ell)$. By Lemma 2.3, $\mathcal{F}$ does not contain all finite graphs, so $\mathcal{F}$ is a normal graph property.

Let $G$ be a graph without a $K_n$ minor. We first prove the following.

*G has a tree-decomposition over $\mathcal{F}$, of finite adhesion and into countable parts.* (1)

To prove (1), consider a tree-decomposition $\mathcal{D}$ of $G$ as in Lemma 5.9. Since $G$ has no $K_n$ minor but contains its torsos in $\mathcal{D}$ as minors, none of these torsos has a $K_n$ minor. In particular, $\mathcal{D}$ has finite adhesion, so it suffices by Lemma 5.8 to show that every torso $H$ in $\mathcal{D}$ has a tree-decomposition of finite adhesion over $\mathcal{F}$. (Note that $H$ is countable, by the choice of $\mathcal{D}$.) By Theorem 2.2, every finite subgraph of $H$ has a tree-decomposition over $\mathcal{F}$. By Lemma 5.7, therefore, $H$ has a tree-decomposition over $\mathcal{F}$, which has finite adhesion by Lemma 2.3.

Now let $\mathcal{D}$ be a tree-decomposition of $G$ as in (1). Let $H = H[v_1, v_2, \dots]$ be a torso in $\mathcal{D}$. By definition of $\mathcal{F}$, each of its finite subgraphs $H[v_1, \dots, v_i]$ lies in $\mathcal{F}(S_n - \ell)$ or in $\mathcal{F}(S'_n - \ell)$. Choose $S \in \{S_n, S'_n\}$ so that $H[v_1, \dots, v_i] \in \mathcal{F}(S - \ell)$ for infinitely many $i$. By Lemma 6.1, $H$ has a tree-decomposition over $\mathcal{F}(S - 5\ell^2)$. The result now follows by Lemma 5.8, applied to $\mathcal{D}$ with $\mathcal{G} := \mathcal{F}(S_n - 5\ell^2) \cup \mathcal{F}(S'_n - 5\ell^2)$. □

The remainder of this section is devoted to proving Lemma 6.1.

**Proof of Lemma 6.1**. Denote the cuffs of $S - k$ by $C_1, \dots, C_k$, let $V(G) = \{v_1, v_2, \dots\}$, and put $G^n := G[v_1, \dots, v_n]$. By assumption, every $G^n$ is contained in a larger finite subgraph $\hat{G^n} \in \mathcal{F}(S - k)$ of $G$. Our aim is to choose these graphs $\hat{G^n}$ and their near-embeddings in $\mathcal{F}(S - k)$ in such a way that the embedding information they induce on their subgraphs $G^n$ tends to a near-embedding in $S - k$ of most of $G$ as $n \to \infty$. The rest of $G$ will be attached in a tree-like fashion, yielding a tree-decomposition of $G$ in which the 'main' torso is nearly embedded in $S - 5k^2$ and each of the other parts has at most $5k^2$ vertices.

In order to encode formally the near-embeddings of the graphs $\hat{G^n}$, we define the following functions associated with near-embeddings. Consider a fixed near-embedding of some subgraph $H \subseteq G$ in $S - k$, with the terminology of Section 2. Thus, there is a set $X$ of at most $k$ vertices of $H$ such that $H - X$ can be written as $H_0 \cup H_1 \cup \dots H_k$ satisfying (N1)–(N3). For all vertices $v \in H$, we set

$$\alpha(v) := \begin{cases} -1 & \text{if } v \in X; \\ 0 & \text{if } v \in H_0; \\ i & \text{if } 1 \leqslant i \leqslant k \text{ and } v \in H_i - H_0. \end{cases}$$

For edges $vw \in H$ with one end in $X$ we put $\alpha(vw) := \max\{\alpha(v), \alpha(w)\}$. For edges $e \in H - X$ we set

$$\alpha(e) := \begin{cases} 0 & \text{if } e \in E(H_0); \\ i & \text{if } 1 \leqslant i \leqslant k \text{ and } e \in E(H_i) \setminus E(H_0). \end{cases}$$

For pairs $(x, u)$, where $u \in H$ is a vertex and $x \in H$ is a vertex or edge, we set

$$\beta(x, u) := \begin{cases} 1 & \text{if } \exists\, i \geqslant 1 : u \in U_i \text{ and } x \in H_i[X_u]; \\ 0 & \text{otherwise.} \end{cases}$$

(See (N2) and (N3) for the definitions of $U_i$ and $X_u$.) Note that, by (N3), we have $\beta(u, u) = 1$ for every vertex $u \in H$ on the boundary of $S - k$.

Now consider any subgraph $H = (V, E)$ of $G$ (independently of embeddings), together with two functions $\alpha \colon V \cup E \to \{-1, 0, 1, \dots, k\}$ and $\beta \colon (V \cup E) \times V \to \{0, 1\}$.

Denote by $H_0$ the subgraph of $H$ formed by the vertices $v$ with $\alpha(v) = 0$ and those edges $e$ between these vertices that satisfy $\alpha(e) = 0$. (This definition of $H_0$ coincides with the earlier one if $\alpha$ happens to come from a near-embedding of $H$.) Given any embedding $f\colon H_0 \to S - k$, we shall call the triple $(\alpha, \beta, f)$ an *encoding* of $H$. If $H \subseteq H' \subseteq G$ with encodings $\gamma = (\alpha, \beta, f)$ and $\gamma' = (\alpha', \beta', f')$, respectively, we say that $\gamma'$ *extends* $\gamma$ if $\alpha' \restriction H = \alpha$ and $\beta' \restriction H = \beta$, and $f' \restriction H = \phi \circ f$ for some rooted homeomorphism $\phi$ of $S - k$; here, $\restriction H$ denotes the restriction to $V \cup E$, to $(V \cup E) \times V$, or to $H_0$ as appropriate.

An encoding of $H \subseteq G$ that arises from a near-embedding of $H$ in $S - k$ will be called *authentic*. In our limit construction of a near-embedding of (most of) $G$, it will be vital to allow non-authentic encodings of the graphs $G^n$. Indeed, if we consider the encoding of $G^n$ induced by the near-embedding of $G$ we are looking for, it may happen that some vertices of a set $X_u$ in the linear decomposition of one of the subgraphs $G_i \subseteq G$ already lie in $G^n$, but $u$ itself does not; in such a case, the sets $X_u \cap \{ v_1, \dots, v_n \}$ do not form a linear decomposition of $G_i \cap G^n$, and setting $G_i^n := G_i \cap G^n$ would not satisfy (N3) for $G^n$. Thus if many of these vertices $u \in G$ happen to appear late in our enumeration $v_1, v_2, \dots$ of $V(G)$, then some or all of the encodings that our desired near-embedding of $G$ induces on the graphs $G^n$ may be non-authentic.

For every $n$ and every function $\alpha\colon V(G^n) \cup E(G^n) \to \{ -1, 0, 1, \dots, k \}$, consider the set $F_\alpha^n$ of all embeddings $f\colon G_0^n \to S - k$, where $G_0^n$ is again the subgraph of $G^n$ on which $\alpha$ is zero. By Lemma 4.1, $F_\alpha^n$ has a (minimal) finite subset $\tilde{F_\alpha^n}$ with the property that for every $f \in F_\alpha^n$ there exists an $\tilde{f} \in \tilde{F_\alpha^n}$ such that $f = \phi \circ \tilde{f}$ for some rooted homeomorphism $\phi$ of $S - k$. Let $\Gamma^n$ be the set of all encodings $\gamma = (\alpha, \beta, f)$ of $G^n$ that satisfy $f \in \tilde{F_\alpha^n}$ and extend to an authentic encoding of some finite subgraph $\hat{G^n} \supseteq G^n$ of $G$. By assumption, $\Gamma^n$ is a non-empty (and finite) set of encodings of $G^n$.

Let $K$ be the graph on $\Gamma^1 \cup \Gamma^2 \cup \dots$ obtained by joining, for every $n \geqslant 2$, every encoding $\gamma \in \Gamma^n$ to the encoding of $G^{n-1}$ it extends; such an encoding of $G^{n-1}$ exists (and is unique) by the definition of $F_\alpha^{n-1}$. By the infinity lemma [1], $K$ contains an infinite path $\gamma_1 \gamma_2 \dots$ with $\gamma_n \in \Gamma^n$ for all $n$. The encodings $\gamma_n =: (\alpha_n, \beta_n, f_n)$ define an encoding of $G$, as follows. By definition of $K$, we have $\alpha_1 \subseteq \alpha_2 \subseteq \dots$ and $\beta_1 \subseteq \beta_2 \subseteq \dots$; thus $\alpha := \alpha_1 \cup \alpha_2 \cup \dots$ and $\beta := \beta_1 \cup \beta_2 \cup \dots$ are well-defined functions on $V(G) \cup E(G)$ and $\big(V(G) \cup E(G)\big) \times V(G)$, respectively. The embeddings $f_n$ give rise to an embedding $f\colon G_0 \to S - k$ in a similar way; since each $f_n$ extends $f_{n-1}$ only up to a homeomorphism of $S - k$, however, we have to define $f$ inductively, as follows. Let $\hat{f}_1 := f_1$, and let $\psi_1$ be the identity map on $S - k$. Now let $n \geqslant 2$, and assume that an embedding $\hat{f}_{n-1}\colon G_0^{n-1} \to S - k$ and a rooted homeomorphism $\psi_{n-1}$ of $S - k$ have been defined so that $f_{n-1} = \psi_{n-1} \circ \hat{f}_{n-1}$. Since $\gamma_n$ extends $\gamma_{n-1}$, there exists a rooted homeomorphism $\phi$ of $S - k$ such that $f_n \restriction G^{n-1} = \phi \circ f_{n-1}$. Then $\psi_n := \phi \circ \psi_{n-1}$ is again a rooted homeomorphism of $S - k$, and we set $\hat{f}_n := \psi_n^{-1} \circ f_n$. Then $f_n = \psi_n \circ \hat{f}_n$ (as assumed before for $n - 1$), and

$$\hat{f}_n \restriction G^{n-1} = \psi_n^{-1} f_n \restriction G^{n-1} = \psi_n^{-1} \phi f_{n-1} = \psi_n^{-1} \phi\, \psi_{n-1} \hat{f}_{n-1} = \hat{f}_{n-1}\,.$$

Having defined $\hat{f}_n$ for all $n$, we may thus define $f := \hat{f}_1 \cup \hat{f}_2 \cup \dots$. Since $G_0$, the subgraph of $G$ where $\alpha$ is zero, is the union of the graphs $G_0^n$, and $G_0^n$ is the domain of $\hat{f}_n$, the function $f$ is clearly an embedding of $G_0$. So $(\alpha, \beta, f)$ is an encoding of $G$.

Let $G'$ be the subgraph of $G$ formed by the vertices $v$ such that $\alpha(v) \leqslant 0$ or $\beta(v, u) = 1$ for some $u \in G$, and those edges $e$ between these vertices that satisfy $\alpha(e) \leqslant 0$ or $\beta(e, u) = 1$ for some $u \in G$. Let us show that $(\alpha, \beta, f)$ defines a near-embedding of $G'$ in $S - k$. Let

$$X := \{ v \in G \mid \alpha(v) = -1 \}.$$

This is a set of at most $k$ vertices of $G'$: if $X$ contained more than $k$ vertices, there would be an $n$ with $\alpha_n(v) = -1$ for $k+1$ vertices $v \in G^n$, which would contradict the fact that $\gamma_n \in \Gamma^n$, i.e. that $(\alpha_n, \beta_n, f_n)$ is induced by a near-embedding of some graph $\hat{G^n}$. Moreover, since $G'_0 = G_0$, our embedding $f$ of $G_0$ satisfies (N1) for $G'_0$.

For $i = 1, \ldots, k$, let $G'_i$ be the subgraph of $G'$ whose vertex set is the union of

$$V_i := \{ v \mid \alpha(v) = i \} \quad \text{and} \quad U_i := V(G_0 \cap C_i),$$

and whose edges are all those edges of $G'$ between these vertices that are not edges of $G'_0$. These subgraphs $G'_i$ clearly satisfy (N2).

The sets $U_i$ are linearly ordered by the cuffs $C_i$ containing them. For every $u \in U_i$ (and each $i$), we set

$$X_u := \{ v \in V(G') \mid \beta(v, u) = 1 \}.$$

Then $|X_u| \leqslant k$, with the same proof as for $|X| \leqslant k$ above. To show that $u \in X_u$, choose $n$ large enough that $u \in G^n$, and let $\hat{G^n}$ be a finite subgraph of $G$ with an authentic encoding $(\hat{\alpha}, \hat{\beta}, \hat{f})$ that extends $\gamma_n$. Then $\beta(u, u) = \beta_n(u, u) = \hat{\beta}(u, u) = 1$, so $u \in X_u$. To complete the verification of (N3) for $G'_i$, let us now check that $(X_u)_{u \in U_i}$ is a linear decomposition of $G'_i$.

For (L1), we have already seen that $U_i \subseteq \bigcup_{u \in U_i} X_u$. To show that $V_i \subseteq \bigcup_{u \in U_i} X_u$, let $v \in V_i$ be given. Since $v \in G'$, we have $\beta(v, u) = 1$ for some $u \in G$; we have to show that $u \in U_i$. As before, choose $n$ large enough that $v, u \in \{ v_1, \ldots, v_n \}$, and let $(\hat{\alpha}, \hat{\beta}, \hat{f})$ be some authentic encoding extending $\gamma_n$. Then $\hat{\alpha}(v) = \alpha_n(v) = \alpha(v) = i$, so $\hat{\beta}(v, u) = \beta_n(v, u) = \beta(v, u) = 1$ implies that $u \in C_i$ under $\hat{f}$, and hence also under $f_n$, $\hat{f}_n$ and $f$. (Recall that these embeddings differ on $G_0^n$ only by rooted homeomorphisms of $S - k$, and these fix $C_i$.) Therefore $u \in U_i$ as required. The reverse inclusion in (L1), and the conditions (L2) and (L3), are checked analogously: any violation of these conditions would hinge upon a finite subgraph of $G'_i$, and could therefore be traced back to a similar violation of the near-embedding of some $\hat{G^n}$, with a contradiction.

To complete our proof that $G'_0 \cup G'_1 \cup \ldots \cup G'_k$ is a near-embedding of $G'$ in $S - k$, we still have to show that this union equals $G' - X$, i.e. that every edge $e = vw$ of $G' - X$ lies in some $G'_i$. By definition of $G'$, either $\alpha(e) \leqslant 0$ or $\beta(e, u) = 1$ for some vertex $u \in G$. Choose $n$ large enough that $vw \in G^n$ and, in the second case above, also $u \in G^n$. Let $(\hat{\alpha}, \hat{\beta}, \hat{f})$ be an authentic encoding extending $\gamma_n$. If $\alpha(e) \leqslant 0$, then also $\hat{\alpha}(e) \leqslant 0$, and hence $\alpha(v) = \hat{\alpha}(v) \leqslant 0$ and $\alpha(w) = \hat{\alpha}(w) \leqslant 0$. Since neither $v$ nor $w$ lies in $X$, this implies $\alpha(v) = \alpha(w) = \alpha(e) = 0$. Therefore $e \in E(G'_0)$ by definition of $G'_0$ (i.e. of $H_0$ for arbitrary graphs $H$ with an $\alpha$ function). In the case of $\beta(e, u) = 1$, say with $u \in C_i$, we similarly have $\hat{\beta}(e, u) = 1$, so $\alpha(v) = \hat{\alpha}(v) = i$ or $v \in C_i$ (under both $\hat{f}$ and $f$), and likewise for $w$. Hence $v, w \in V(G'_i)$, and so $e \in E(G'_i) \cup E(G'_0)$ by definition of $E(G'_i)$.

Our aim is to construct a tree-decomposition of $G$ over $\mathcal{F}(S-5k^2)$. So far, we have shown that $G' \in \mathcal{F}(S-k)$. As we shall see later, every component of $G-G'$ has a tree-decomposition of width $< k$, which is trivially a tree-decomposition over $\mathcal{F}(S-k)$. Our aim, then, will be to attach all these tree-decompositions to the 'singleton' tree-decomposition $(V(G'))$ of $G'$, so as to form one large tree-decomposition of $G$. This final contruction, however, will make it necessary to relax $k$ to $5k^2$ (or some similar function of $k$), for two reasons. First, recall that it is the torso on $V(G')$, not just $G'$ itself, that has to have the required near-embedding. And although $G'$ lies in $\mathcal{F}(S-k)$, its torso may not. The torso will, however, lie in $\mathcal{F}(S-3k^2)$: keeping our embedding of $G'_0$ in $S-k$, we shall simply enlarge the sets $X_u$ in the various linear decompositions of the graphs $G'_1,\dots,G'_k$ from size at most $k$ to size at most $3k^2$, to accommodate the additional edges (including the edges of $G[G']$, those induced by $G'$ in $G$, that are not edges of $G'$). Secondly, to ensure that the tree-decomposition of a component $C$ of $G-G'$ continues to satisfy the axioms (T1)–(T3) when it is attached to $G'$, we shall add the (at most $4k^2$) neighbours of $C$ in $G'$ to every part of the decomposition of $C$, increasing its number of vertices from at most $k$ to at most $5k^2$.

Let us call a subgraph of $G$ a *bridge* if it consists either of a single edge, together with its ends, that lies in $G[G']$ but not in $G'$, or of a component $C$ of $G-G'$ together with the neighbours of $C$ in $G'$ and all the edges between $C$ and these neighbours. Thus, $G$ is the union of $G'$ and all its bridges. The vertices of a bridge that lie in $G'-X$ will be called its *feet*. Note that distinct bridges have no edges in common, and if they share a vertex then this lies in $X$ or is a common foot.

Let us show the following:

$$\textit{Every bridge has all its feet in some graph } G'_i \textit{ with } i \geqslant 1. \tag{1}$$

(A bridge $B$ as in (1) will be called a *bridge on* $G'_i$.) If $B$ consists of a single edge $e$, let $n$ be large enough that $e \in G^n$, and consider an authentic encoding $(\hat\alpha,\hat\beta,\hat f)$ that extends $\gamma_n$. Then $i := \hat\alpha(e) = \alpha_n(e) = \alpha(e) \geqslant 1$ (since $e \notin G'$). If $e$ has an end in $X$, then its other end $v$ satisfies $\alpha(v) = \hat\alpha(v) = \hat\alpha(e) = i$ and we are done. If not, then $\hat\alpha(e) = i \geqslant 1$ implies that each end $v$ of $e$ either satisfies $\alpha(v) = \hat\alpha(v) = i$ or lies on $C_i$ (under both $\hat f$ and $f$), giving $v \in V_i \cup U_i = V(G'_i)$ as required. Now let $B$ consist of a component $C$ of $G-G'$, together with the neigbours of $C$ in $G'$ and the edges between $C$ and these neighbours. Since $C$ is connected, and every $v \in C$ satisfies $\alpha(v) \geqslant 1$ by definition of $G'$, it suffices to show the following:

$$\textit{If } vw \textit{ is any edge of } G \textit{ with } i := \alpha(v) \geqslant 1, \textit{ then } \alpha(vw) = i, \textit{ and either } \alpha(w) = i \textit{ or } w \in U_i \cup X. \tag{2}$$

To prove (2), let $n$ be large enough that $vw \in G^n$, and let $(\hat\alpha,\hat\beta,\hat f)$ be an authentic encoding extending $\gamma_n$. Then $\hat\alpha(v)$ and $\alpha$ agree on $v$, $w$, and $vw$, and $\hat f$ and $f$ agree on $G^n_0$ up to a rooted homeomorphism (which fixes $C_i$). In particular, $\hat\alpha(v) = i$, so $\alpha(vw) = \hat\alpha(vw) = i$, and either $\hat\alpha(w) = i$ or $\hat f(w) \in C_i$ or $\hat\alpha(w) = -1$. In the first case we have $\alpha(w) = i$; in the second $f(w) \in C_i$ and hence $w \in U_i$; in the third $\alpha(w) = -1$ and hence $w \in X$. This completes the proof of (2), and hence also of (1).

Our next aim is to extend the sets $X_u$ in the linear decompositions of the graphs $G'_1,\dots,G'_k$ to larger sets $Y_u$, in such a way that these $Y_u$ still form linear decompositions of their respective $G'_i$, and every bridge on $G'_i$ has all its feet in a single such set $Y_u$. (This will ensure that the resulting near-embedding of $G'$ is even a near-embedding of the torso on $V(G')$ in the tree-decomposition to be constructed.)

Let $i \in \{1, \ldots, k\}$ be given, and consider again our linear decomposition

$$\mathcal{X}_i = (X_u)_{u \in U_i}$$

of $G'_i$. For each $v \in G'_i$, let

$$I_v := \{ u \in U_i \mid v \in X_u \}.$$

By (L3), $I_v$ is an interval in $U_i$, i.e. $u \in I_v$ whenever $u' < u < u''$ and $u', u'' \in I_v$. If $I, I'$ are disjoint intervals in $U_i$ such that $u < u'$ for some, and hence every, choice of vertices $u \in I$ and $u' \in I'$, we write $I < I'$; for singleton intervals $\{u\}$ we abbreviate $I < \{u\}$ to $I < u$, etc. Let us call two vertices $v, w \in G'_i$ (and their intervals $I_v, I_w$) *close* if $I_v \cup I_w$ is again an interval in $U_i$, i.e. if $I_v$ and $I_w$ meet or are adjacent. We may now extend (1) as follows:

$$\textit{Any two feet of the same bridge on } G'_i \textit{ are close.} \tag{3}$$

To prove (4), let $v, w$ be distinct feet of a common bridge. Let $P \subseteq G$ be a $v$–$w$ path whose only vertices in $G'$ are $v$ and $w$. To check that $I_v \cup I_w$ is an interval in $U_i$, let $u_1, u_3 \in I_v \cup I_w$ be given (with $u_1 < u_3$, say); we have to show that every $u_2 \in U_i$ between $u_1$ and $u_3$ also lies in $I_v \cup I_w$. Since $I_v$ and $I_w$ are intervals, we may assume that $u_1 \in I_v$ and $u_3 \in I_w$, i.e. that $v \in X_{u_1}$ and $w \in X_{u_3}$. We shall prove that $P$ meets $X_{u_2}$; since $v$ and $w$ are the only vertices of $P$ that lie in $G'$, this will imply $u_2 \in I_v \cup I_w$ as required.

To show that $P$ meets $X_{u_2}$, let $n$ be large enough that $G^n$ contains $u_1$, $u_2$, $u_3$ and $P$, and let $\hat{G^n} \subseteq G$ be a finite supergraph of $G^n$ that has an authentic encoding $(\hat\alpha, \hat\beta, \hat f)$ extending $\gamma_n$. Let $(\hat X_u)_{u \in \hat U_i}$ be the linear decomposition of $\hat{G^n_i}$ from the corresponding near-embedding of $\hat{G^n}$. Since $\hat U_i$ is finite, this is in fact a path-decomposition. Since $\beta$ and $\hat\beta$ agree on $(v, u_1)$ and $(w, u_3)$, we have $v \in \hat X_{u_1}$ and $w \in \hat X_{u_3}$. So $P$ meets $\hat X_{u_2}$ by Lemma 5.1, i.e. some vertex $x \in P$ satisfies $\beta(x, u_2) = \hat\beta(x, u_2) = 1$. Then $x \in X_{u_2}$, completing the proof of (3).

Given vertices $v, w \in G'_i$, let us say that $v$ *touches* $w$ if $v$ and $w$ are close but $I_v$ is not a proper subset of $I_w$. Note that if two vertices of $G'_i$ are close, then at least one of them touches the other. For every $u \in U_i$ (and every $i \geqslant 1$), let

$$Y_u := \{ v \in V(G'_i) \mid v \text{ touches a vertex in } X_u \}.$$

Since every vertex in $X_u$ touches itself, clearly $X_u \subseteq Y_u$. We claim that

$$\mathcal{Y}_i := (Y_u)_{u \in U_i}$$

is a linear decomposition of $G'_i$, of width at most $3k^2$. Axioms (L1) and (L2) are clearly satisfied, because they are satisfied for $\mathcal{X}_i$. To verify (L3), let $u_1 \leqslant u_2 \leqslant u_3$ be vertices of $U_i$, and let $v \in Y_{u_1} \cap Y_{u_3}$; we have to show that $v \in Y_{u_2}$. By assumption, $v$ touches a vertex $x_1 \in X_{u_1}$ and a vertex $x_3 \in X_{u_3}$. If $x_1$ or $x_3$ lies in $X_{u_2}$, then $v$ touches this element of $X_{u_2}$ and we are done. If not, then $I_{x_1} < u_2 < I_{x_3}$. Since $I_v$ is close to both $I_{x_1}$ and $I_{x_3}$, this implies $u_2 \in I_v$, so $v \in X_{u_2} \subseteq Y_{u_2}$ as required.

In order to show that $\mathcal{Y}_i$ has width at most $3k^2$, we now show that, for every $u \in U_i$, every $x \in X_u$ is touched by at most $2k$ vertices outside $X_u$; since $|X_u| \leqslant k$, this implies $|Y_u| \leqslant 2k^2 + k \leqslant 3k^2$. If $v \in G'_i$ is a vertex outside $X_u$ touching $x$, then $I_v \not\subseteq I_x$: the two intervals cannot be equal, because $u \in I_x \setminus I_v$. So $I_v \setminus I_x$ contains a vertex $u(v)$, and either $u(v) < I_x$ or $I_x < u(v)$. We assume the latter, and show that there are at most $k$ such vertices $v$. (Similarly, there will be at most $k$ vertices $v$ of the other type.) Suppose there are $k+1$ such vertices, say $v_1, \dots, v_{k+1}$ with $u(v_1) \leqslant \dots \leqslant u(v_{k+1})$. Now every $I_{v_j}$ contains $u(v_j)$ and is close to $I_x$, but $I_x < u(v_1) \leqslant u(v_j)$. Therefore $u(v_1) \in I_{v_j}$ for every $j$, i.e. $v_1, \dots, v_{k+1} \in X_{u(v_1)}$. This contradicts $|X_{u(v_1)}| \leqslant k$.

Let $G''$ be the graph on $V(G')$ in which two vertices are adjacent if they are adjacent in $G$ or if they are feet of the same bridge. (The graph $G''$ will be the 'central torso' in our final tree-decomposition of $G$.) Our near-embedding of $G'$ in $S-k$ induces a near-embedding of $G''$ in $S-3k^2$, as follows. Let $G''_0 := G'_0 = G_0$, and choose an embedding of $G''_0$ in $S-3k^2$ that differs from $f$ only by a rooted homeomorphism of $S$ but avoids $C_{k+1}, \dots, C_{3k^2}$. For $i = 1, \dots, k$ let $G''_i$ be the subgraph of $G''$ induced by $V(G'_i)$; for $i = k+1, \dots, 3k^2$ let $G''_i = \emptyset$. Clearly, the graphs $G''_0, \dots, G''_{3k^2}$ satisfy (N1) and (N2). Now consider any edge $e = vw$ of $G'' - X$ that is not an edge of $G'$. Then $e$ either itself forms a bridge (if $e \in G$) or else joins two feet of some other bridge. In either case, (1) says that $v$ and $w$ lie in the same $G'_i$ with $1 \leqslant i \leqslant k$, so $e \in G''_i$ by definition of $G''_i$. Hence $G'' - X = G''_0 \cup G''_1 \cup \dots \cup G''_k$. Let us finally check that, for $i = 1, \dots, k$, our linear decomposition $\mathcal{Y}_i$ witnesses (N3) even for $G''_i$. (L1) and (L3) hold as for $G'_i$. To verify (L2), let again $e = vw$ be an edge of $G''_i$ that is not an edge of $G'_i$. Then $v, w$ are feet of a common bridge. By (3), $v$ and $w$ are close, so we may assume that $v$ touches $w$. Then $v, w \in Y_u$ for every $u \in I_w$. This completes the proof that $G'' \in \mathcal{F}(S-3k^2)$.

Before we construct our final tree-decomposition of $G$, let us show that every component $C$ of $G - G'$ has a tree-decomposition of width $< k$. By Lemma 5.3, it suffices to prove this for every finite subgraph $C^n := C \cap G^n$. Let $\hat{G}^n$ be a finite subgraph of $G$ with a near-embedding $\hat{G}^n_0 \cup \hat{G}^n_1 \cup \dots \cup \hat{G}^n_k$ in $S-k$ whose encoding extends $\gamma_n$. By definition of $G'$ and (2), $\alpha$ takes some constant value $i \geqslant 1$ on $C$, so $\alpha_n(x) = \alpha(x) = i$ for every vertex or edge $x \in C^n$. Thus, $C^n \subseteq \hat{G}^n_i$. The linear decomposition of $\hat{G}^n_i$ thus induces a path-decomposition of width $< k$ on $C^n$.

For every component $C$ of $G - G'$, pick a tree-decomposition of width $< k$, and add to every part of this decomposition all the neighbours of $C$ in $G'$. There are at most $k$ such neighbours in $X$, and at most $3k^2$ outside $X$: since the latter are the feet of the bridge $B$ corresponding to $C$, (3) implies that they all lie in every set $Y_u$ with $u \in I_x$, where $x$ is a foot of $B$ with $I_x$ minimal. (Such a foot $x$ exists, even if $B$ has infinitely many feet: if $x_1, \dots, x_{k+1}$ were feet of $B$ with $I_{x_1} \supsetneq \dots \supsetneq I_{x_{k+1}}$, then every $u \in I_{x_{k+1}}$ were such that $x_1, \dots, x_{k+1} \in X_u$, a contradiction.) We thus obtain a tree-decomposition of $B$ of width at most $k + k + 3k^2 \leqslant 5k^2$. Its torsos, trivially, lie in $\mathcal{F}(S-5k^2)$. Since $G$ is the union of $G[G']$ and all bridges, and different bridges meet only in $V(G')$, the union of all the above tree-decompositions with the singleton tree-decomposition $(V(G'))$ of $G[G']$ is a tree-decomposition of $G$, whose torso on $V(G')$ is precisely $G''$. Since $G''$, too, lies in $\mathcal{F}(S-5k^2)$, this tree-decomposition of $G$ satisfies the assertion of the Lemma. □

## 7. Excluding the infinite clique

We now prove Theorem 3.2. We have to show that a graph $G$ has no $K_{\aleph_0}$ minor if and only if it has a tree-decomposition of finite adhesion over plane graphs with at most one vortex. The 'if' part of this follows at once from the 'if' part of Theorem 2.1 and Lemma 2.3. For the 'only if' part, we need a couple of lemmas.

**Lemma 7.1.** *Let $S$ be a closed surface, $k \geqslant 2$ an integer, and $G$ a graph embedded in $S$ with $k$ vortices. Assume that $G \cap S$ (the embedded part of $G$) contains a path $P$ joining distinct cuffs of $S$. Then $G$ can be embedded in $S$ with $k-1$ vortices.*

**Proof.** Let $G - X = G_0 \cup G_1 \cup \ldots \cup G_{k'}$ be a near-embedding of $G$ in $S - k'$, where $k' \geqslant k$ and $G_i \neq \emptyset$ if and only if $i \leqslant k$. Without loss of generality, assume that $P$ joins a vertex $u_1$ on a cuff $C_1$ to a vertex $u_2$ on a different cuff $C_2$, and that these are the only vertices of $P$ on the boundary of $S - k'$. We may assume further that $u_1$ and $u_2$ are not the only vertices of $G$ on $C_1$ and $C_2$: otherwise we could obtain our embedding with $k-1$ vortices simply by adding the (at most $k'$) vertices of $G_1$ or of $G_2$ to the deleted set $X$.

Cutting $S - k'$ open along $P$, we merge the first two holes of $S - k'$ into one, obtaining a copy of $S - (k'-1)$ whose first cuff $C$ is the union of $C_1 - u_1$ with $C_2 - u_2$ and two copies of $P$ (Fig. 1). Our plan is to merge the linear decompositions $\mathcal{X}_1 := (X_u)_{u \in U_1}$ of $G_1$ and $\mathcal{X}_2 := (X_u)_{u \in U_2}$ of $G_2$ into one linear decomposition of

$$H := (G_1 \cup G_2) - \{ u_1, u_2 \};$$

adding $V(P)$ to $X$, we shall then have an embedding of $G$ in $S$ with $k-1$ vortices.

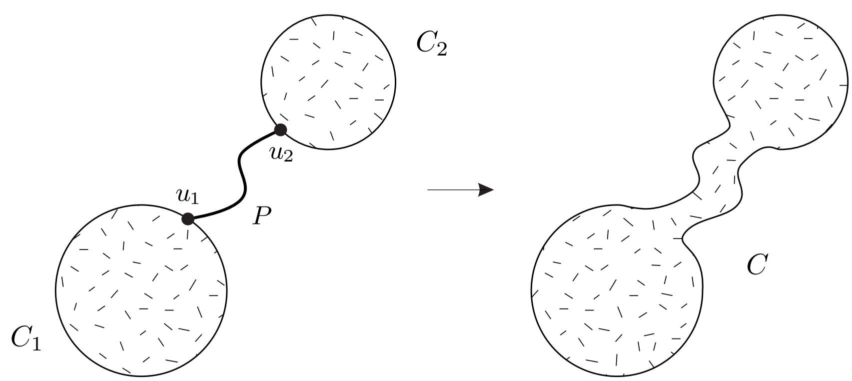


FIGURE 1. A cut joining two holes

Choose the root $r$ of $C_1$ as the root of $C$, and pick one of the two natural linear orderings of $C - r$. Note the following:

*Every interval of $C_1$ or of $C_2$ is either again an interval on $C$ (possibly with its ordering reversed) or contains $u_1$ or $u_2$.* (1)

Now let

$$U := (U_1 \cup U_2) \smallsetminus \{ u_1, u_2 \}$$

and

$$W := X_{u_1} \cup X_{u_2}\,.$$

For each $u \in U$ let

$$Y_u := (X_u \cup W) \setminus \{ u_1, u_2 \}\,.$$

We show that

$$\mathcal{Y} := (Y_u)_{u \in U}$$

is a linear decomposition of $H$; since $|Y_u|$ is bounded by $k' + |W|$, this will establish

$$G - (X \cup V(P)) = (G_0 - P) \cup H \cup G_3 \cup \ldots \cup G_k$$

as an embedding of $G$ in $S$ with $k-1$ vortices.

Conditions (L1) and (L2) are easily verified. For (L3), let $u \leqslant v \leqslant w$ be vertices in $U$, and let $x \in Y_u \cap Y_w$; we have to show that $x \in Y_v$. This is trivial if $x \in W$, so we assume that $x \notin W$ and hence $x \in X_u \cap X_w$. Since $G_1 \cap G_2 = \emptyset$ by (N2), this implies that $u$ and $w$ lie on a common cuff $C_i$ ($i \in \{1, 2\}$). If the (closed) interval $I \subseteq U_i$ between $u$ and $w$ contains $u_i$, then $x \in X_{u_i} \subseteq W$ by (L3) for $\mathcal{X}_i$. So $I$ contains neither $u_1$ nor $u_2$, and is thus by (1) an interval in $U$. Therefore $v \in I$, and hence $x \in X_v \setminus \{ u_1, u_2 \} \subseteq Y_v$ by (L3) for $\mathcal{X}_i$. □

Let $\mathcal{P}$ denote the class of all countable graphs that can be nearly embedded in a sphere $S^2$ with finitely many holes.

**Lemma 7.2.** *Every countable graph nearly embedded in a surface has a tree-decomposition of finite adhesion over $\mathcal{P}$.*

**Proof.** Let $G$ be a countable graph with a near-embedding $G - X = G_0 \cup G_1 \cup \ldots \cup G_k$ in a surface; this surface may be written as $S - k$, where $S$ is a closed surface. Our plan is to transform $S$ into a union of spheres, by a number of simple surgery operations disturbing only a finite part of $G$. Every finite subgraph $H$ of $G$ will then (after a trivial extension) be nearly embedded in $S^2 - \ell$, for some integer $\ell$ which depends on the operations performed but is independent of the choice of $H$. We may then use Lemma 6.1 to obtain the desired tree-decomposition of $G$.

We first describe in turn the three surgery operations used below. The first of these will remove paths in $G$ between different cuffs, the other two will reduce the Euler genus of the surfaces considered. Two subgraphs of $G$, each nearly embedded in some surface, will be called *almost disjoint* if they meet only inside their deleted sets.

By assumption, $G$ is embedded in $S$ with at most $k$ vortices. If $G \cap S$ contains a path linking distinct cuffs, we use Lemma 7.1 to merge these cuffs into one, obtaining an embedding of $G$ in $S$ with at most $k-1$ vortices. We repeat this step as often as possible, which is at most $k$ times. This gives us a near-embedding of $G$ in $S - k'$ (for some $k'$) in which no path in $G \cap S$ joins two distinct cuffs.

Second, suppose that $G_0$ contains a cycle $C$ that is genus-reducing on $S$. Cutting $S$ open along $C$, and sewing a disc on to each of the one or two boundary circles arising from the cut, we obtain one or two closed surfaces of smaller Euler genus than $S$ (Lemma 4.2). Each of the $k$ holes of $S - k$ is an open disc on $S \setminus C$ (and hence connected), so it lies on one of the new surfaces and does not meet the other. Hence $G$ is the almost disjoint union of one or two graphs $G'$ each nearly embedded in some surface $S' - k'$, where $S'$ is a closed surface satisfying $2 - \chi(S') < 2 - \chi(S)$: for each $G'$, let $X \cup V(C)$ be the deleted set, and replace each linear decomposition $(X_u)_{u \in U}$ with $(Y_u)_{u \in U'}$, where

$$U' := U \setminus V(C)$$

and

$$Y_u := (X_u \cup W) \setminus V(C)\,,$$

with

$$W := \bigcup_{u \in U \cap V(C)} X_u$$

(as in the proof of Lemma 7.1).

Finally, suppose that $G_0$ contains a path $P$ joining two vertices $v, w$ on the same cuff $C_i$, such that whenever $Q$ is a $v$–$w$ arc on $S$ through the corresponding hole $D_i$, the closed curve $C := P \cup Q$ is genus-reducing on $S$. As before, we cut $S$ open along $C$ and obtain one or two simpler closed surfaces $S'$. This time, $D_i$ splits into two holes $D_i'$ and $D_i''$, each on one of the surfaces $S'$, and each bounded by one of the two $v$–$w$ arcs on $C_i$ together with one of the two copies of $Q$ resulting from the cut.

Let $C_i'$ and $C_i''$ be the boundaries of $D_i'$ and $D_i''$, respectively, and assume that $C_i'$ contains the root of $C_i$. Choosing this as the root of $C_i'$, and picking any point of $Q$ as the root of $C_i''$, one easily checks that the linear decomposition $(X_u)_{u \in U_i}$ of $G_i$ around $C_i$ splits into linear decompositions

$$(Y_u)_{u \in U_i'} \quad \text{and} \quad (Y_u)_{u \in U_i''}$$

around $C_i'$ and $C_i''$, where

$$U_i' := (U_i \cap C_i') \setminus \{\, v, w \,\}$$
$$U_i'' := (U_i \cap C_i'') \setminus \{\, v, w \,\}$$

and

$$Y_u := X_u \setminus \big((X_v \cup X_w) \setminus U_i'\big) \ \text{ for } u \in U_i'$$
$$Y_u := X_u \setminus \big((X_v \cup X_w) \setminus U_i''\big) \ \text{ for } u \in U_i''.$$

Thus, as before, $G$ is the almost disjoint union of one or two graphs $G'$ each nearly embedded in some surface $S' - k'$, where $S'$ is a closed surface with $2 - \chi(S') < 2 - \chi(S)$.

Starting with $G$ itself, we apply the three reductions described above iteratively, until $G$ is given as an almost disjoint union of graphs $G'$ each nearly embedded in $S' - k'$ for some closed surface $S'$ and some integer $k'$, and none of the three reductions can be applied to any of these nearly embedded graphs $G'$. Note that this state will be attained after finitely many steps: there can be no more than $2 - \chi(S)$ successive genus-reducing operations, and between any two of these there will be at most as many vortex-joining reductions as there are vortices to join.

Let $\ell$ be the maximum of the following numbers: the total number of holes in all the surfaces $S'$ together (those used for the near-embeddings of the corresponding graphs $G'$); 1+ the maximum width of any linear decomposition around these holes; the total number of vertices deleted (those in $X$, and those deleted in any reduction).

We claim that for every finite subgraph $H$ of $G$ there exists a finite subgraph $\hat{H} \subseteq G$ such that $H \subseteq \hat{H} \in \mathcal{F}(S^2 - \ell)$; by Lemma 6.1, this will prove the assertion of the Lemma.

For each of the subgraphs $G'$ of $G$ considered above, consider the graph $H' := G' \cap H$. Since $G'$ is nearly embedded in $S' - k'$ for some closed surface $S'$ and some integer $k'$, so is $H'$—almost: the near-embedding of $G'$ induces one of $H'$, except that some boundary vertices $u$ that are needed as indices for sets $X_u$ meeting $H'$ may be missing from $H'$. Let $\hat{H}'$ be the graph obtained from $H'$ by putting them back, as isolated vertices, and let $\hat{H}$ be obtained from $H$ by adding all the isolated vertices added to the various $\hat{H}'$. Then $\hat{H}$ is a finite subgraph of $G$ extending $H$. Moreover, $\hat{H}$ is the almost disjoint union of the graphs $\hat{H}'$ (each nearly embedded in their surface $S' - k'$): distinct graphs $H'$ meet only in their deleted sets (because the $G'$ are almost disjoint), and their extensions $\hat{H}'$ have none of the additional vertices $u$ in common, because each of these $u$ had its unique place on one of the cuffs of $S$, which was passed on in each cut to a unique position in exactly one of the surfaces arising from the cut.

We now show that each of these near-embeddings $\hat{H}' - X' = \hat{H}'_0 \cup \hat{H}'_1 \cup \ldots \cup \hat{H}'_{k'}$ in $S' - k'$ is in fact a near-embedding in $S^2 - k'$, by modifying $S'$ into copies of $S^2$ without disturbing $\hat{H}'$; by the choice of $\ell$, and the fact that distinct $\hat{H}'$ are almost disjoint, these near-embeddings will combine to a near-embedding of $\hat{H}$ in $S^2 - \ell$.

If $S' = S^2$, there is nothing to show. If not, consider the graph $J$ obtained from $\hat{H}'_0$ by adding as cycles all the cuffs of $S' - k'$ used in the near-embedding. By Lemma 4.3, $S'$ contains a genus-reducing curve $C$ that either lies in $J$ or avoids it. Let us show that $C$ must avoid $J$: we may then cut along $C$ to obtain one or two simpler closed surfaces, with $\hat{H}'$ nearly embedded in their union, and reapply the reduction to these surfaces until all the surfaces are spheres.

So we have to show that $C$ cannot lie in $J$. Suppose it does. Since $S'$ is minimal with respect to $G'$ and the second of our three reductions, $C$ does not lie in $H'_0 \subseteq G'$. So $C$ contains one of the new edges of $J$ on a cuff $C_i$ of $S' - k'$. Since $G'$ is minimal with respect to the first reduction, $C$ cannot meet any other cuff, i.e. $C$ has the form

$$C = v_1 P_1 w_1 J_1 v_2 P_2 w_2 \ \ldots \ v_r P_r w_r J_r v_1 \,,$$

where each $P_j = v_j \ldots w_j$ is a path in $H'_0$ whose only vertices on $C_i$ are its ends, each $J_j = w_j \ldots v_{j+1}$ is a path on $C_i$, and $v_{r+1} = v_1$ (Fig. 2).

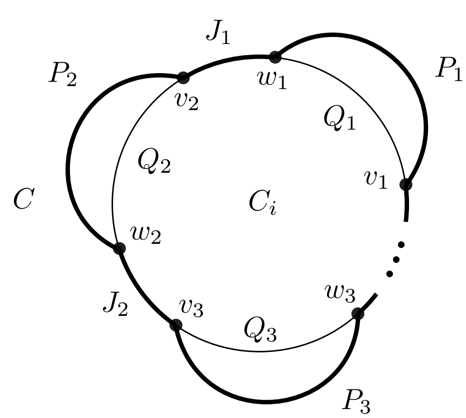


FIGURE 2. The cycle $C$ in $J$

For every $j \leqslant r$, let $Q_j$ be one of the two $v_j$–$w_j$ paths on $C_i$. Since $C$ is genus-reducing, $P_j$ cannot be homotopic to $Q_j$ for every $j$: otherwise, $C$ would be homotopic to a closed walk in $C_i$, and hence (like $C_i$) homotopic to a point. Pick $j$ so that $P_j$ is not homotopic to $Q_j$. Let $Q'_j$ be an arc from $v_j$ to $w_j$ whose interior lies in the hole $D_i$. Then $P_j$ is again not homotopic to $Q'_j$, so the curve $P_j \cup Q'_j$ is not homotopic to a point: it must be genus-reducing. Since $P_j \subseteq H' \subseteq G'$, this contradicts our assumption that $S'$ was minimal with respect to $G'$ and our third reduction. □

**Proof of Theorem 3.2** ('only if'). Let again $\mathcal{P}$ denote the class of all countable graphs that can be nearly embedded in a sphere $S^2$ with finitely many holes. We first show that every graph $G$ without a $K_{\aleph_0}$ minor has a tree-decomposition of finite adhesion over $\mathcal{P}$. By Theorem 2.1, $G$ has a tree-decomposition $\mathcal{D}$ of finite adhesion, in which each torso fails to have some $K_n$ minor. By Theorem 3.1, each of these torsos has a tree-decomposition $\mathcal{D}'$ over countable graphs each nearly embedded in some surface. By Lemma 2.3, $\mathcal{D}'$ has finite adhesion. Each of the torsos in $\mathcal{D}'$, finally, has a tree-decomposition of finite adhesion over $\mathcal{P}$, by Lemma 7.2. Applying Lemma 5.8 twice, we deduce that $G$ has a tree-decomposition of finite adhesion over $\mathcal{P}$.

To complete the proof of the Theorem, it will suffice by Lemma 5.8 to show that every torso $H$ in the above decomposition has a tree-decomposition of finite adhesion over plane graphs with at most one vortex. By assumption, $H$ can be embedded in $S^2$ with $k$ vortices for some $k$; let us choose such an embedding with $k$ minimal, and let $H - X = H_0 \cup H_1 \cup \ldots \cup H_{k'}$ be the corresponding near-embedding in a surface $S^2 - k'$. By Lemma 7.1 and (N2), no component of $H - X$ contains vertices on two different cuffs. Therefore $H - X$ can be written as a disjoint union $D_0 \cup D_1 \cup \ldots \cup D_k$, where each $D_i$ is a union of components of $H - X$ and $\hat{D}_i := H[\,V(D_i) \cup X\,]$ is plane with at most one vortex, with $X$ as deleted set. Now let $\mathcal{D}$ be the following tree-decomposition of $H$: its decomposition tree is a star whose central node corresponds to the part $H[\,X\,]$, and the other parts are the graphs $\hat{D}_i \subseteq H$ formed from the components of $H - X$ as above. The central torso in this decomposition is a finite graph on $X$, and is hence trivially plane with at most one vortex. Every other torso is a graph $\hat{D}_i$ with some additional edges inside $X$, so these too are plane with one vortex. □

## 8. Appendix: Proof of Lemma 2.3

Here is a sketch of a proof of Lemma 2.3. We start by bounding the *tree-width* (the least width of any tree-decomposition) of certain graphs that are plane with one vortex.

**Lemma 8.1.** *Let $G$ be a graph that is plane with one vortex, say $G - X = G_0 \cup G_1$. Assume that $X = \emptyset$, and that $U := V(G_0) \subseteq V(G_1)$. Let $\mathcal{X} = (X_u)_{u \in U}$ be a linear decomposition of $G_1$ as in (N3), of width less than $w$, say. Then $G$ has tree-width less than $3w$.*

**Proof.** By assumption, $G_0$ is embedded in $S^2 - 1$ with all its vertices on the boundary of the hole $D$. Let $f$ be the face of $G_0$ in $S^2$ that contains $D$. Every vertex of $G_0$ lies on the boundary of $f$, and we may assume (by adding edges to $G_0$ in $S^2 - 1$) that $G_0$ is maximal with this property. Then every face of $G_0$ in $S^2$ other than $f$ is bounded by three edges. Let $T$ be the graph obtained from the geometric dual of $G_0$ in $S^2$ by deleting the vertex $v(f)$ that corresponds to $f$. We claim that $T$ is a tree. Indeed, suppose that $C$ is a cycle in $T$. Then the face of $C$ not containing $v(f)$ includes a face $g$ of $T$, which corresponds to a vertex of $G_0$. Since $g$ is not incident with $v(f)$ in the dual, this vertex is not incident with $f$ in $G_0$, i.e. does not lie on the boundary of $f$. This contradicts the choice of $f$, so $T$ is indeed a tree.

For each vertex $t$ of $T$, consider the vertices $u_1, u_2, u_3$ of $G_0$ that are incident with the face of $G_0$ corresponding to $t$, and put $Y_t = X_{u_1} \cup X_{u_2} \cup X_{u_3}$. We claim that $\mathcal{Y} = (Y_t)_{t \in T}$ is a tree-decomposition of $G$. This will complete the proof, since clearly its width is less than $3w$.

It is clear that $\mathcal{Y}$ satisfies (T1) and (T2), and so it remains to verify (T3). Consider a vertex $x \in G$; we show that the set $S := \{ t \in T \mid x \in Y_t \}$ spans a connected subgraph in $T$. By (L3), the set $U_x := \{ u \in U \mid x \in X_u \}$ is an interval in $U$. By the maximality of $G_0$, vertices of $U_x$ that are adjacent on the cuff are also adjacent in $G_0$, so $U_x$ is the vertex set of a path in $G_0$ on the boundary of $f$. The faces other than $f$ that meet every strip neighbourhood of this path then form a walk in $T$ (taken in the order in which they meet the strip neighbourhoods). But these faces are precisely the elements of $S$, so $S$ spans a connected subgraph in $T$. □

**Proof of Lemma 2.3** (sketch). Let us rewrite the surface $S$ of the Lemma as $S - k$, where $S$ is now a closed surface. Applying induction on the Euler genus of $S$, and for fixed $S$ induction on the number $c \leqslant k$ of vortices actually used by the near-embedding, we show that no graph $G$ nearly embedded in $S - k$ and using at most $c$ vortices contains a $K_n$ minor with $n$ larger than some constant $n(S, k, c)$. With $c := k$, this yields $n(S, k, k)$ as the required bound depending only on the surface.

For the induction start, let $G = G_0 \cup G_1$ be plane with one vortex; by assumption, the deleted set $X \subseteq V(G)$ has size at most $k$, and the linear decomposition of $G_1$ has width less than $k$. We show that $G$ has no $K_n$ minor with $n \geqslant 4k + 5$. Suppose it does. At most $k$ of its *branch sets* $V_x \subseteq V(G)$ (the sets to be contracted) meet $X$; let us delete them from $G$. Of the remaining branch sets, at most four avoid the cuff $C_1$; otherwise, they would form a plane graph with a $K_5$ minor. Let us delete these too, and contract every remaining edge of $G_0$ whose ends lie in a common branch set but not both on $C_1$. We obtain a minor $G'$ of $G$ that inherits the near-embedding of $G$ but uses only vertices of $G_1$. Moreover, $G'$ still has a $K_{3k+1}$ minor. By Lemma 8.1, however, $G'$ has tree-width less than $3k$, and hence so do all its minors [1, Prop. 12.4.2]. This contradicts Lemma 5.2, which implies that $K_{3k+1}$ has tree-width at least $3k$.

For the induction step, suppose that $G$ is nearly embedded in $S - k$, with deleted set $X$ and using $c \leqslant k$ vortices. Suppose further that $G$ has a $K_n$ minor, $K$ say. We show that, by deleting vertices of $G$ that meet at most some bounded number $m(S, k, c)$ of the branch sets of $K$, and contracting some edges within the branch sets, we can transform $G$ into a minor $G'$ of $G$ nearly embedded in a surface $S' - k'$ of smaller Euler genus than $S$, or in $S - k'$ but with fewer vortices used, where in either case $k'$ is bounded by some function $k'(S, k, c)$. Thus if $n(S, k, c)$ is chosen as the sum of $m(S,k,c)$ and all the values of $n(S',k',c')$ for triples $(S',k',c')$ covered by the induction hypothesis and with $k' \leqslant k'(S,k,c)$ (which is a bounded number of triples, since $c \leqslant k$), we may deduce that $n \leqslant n(S,k,c)$.

As before, we start by deleting any branch sets of $K$ that meet $X$ or avoid the boundary of $S - k$; since we cannot embed arbitrarily large complete graphs (or minors) in $S$, this is a bounded number of branch sets. Let $K'$ be the subgraph of $K$ spanned by the remaining branch sets. We now contract every edge of $G$ that is embedded in $S$ and whose ends lie in the same branch set of $K'$ but not both on the boundary of $S - k$. This yields a minor $G'$ of $G$ which inherits the original near-embedding of $G$, uses only vertices on the boundary of $S - k$, and contains $K'$ as a minor. To simplify notation, we rename $G'$ as $G$ and $K'$ as $K$.

We now proceed as in Lemma 7.2. If $G$ contains an edge $e = u_1u_2$ joining different cuffs, we delete $e$ (and its ends) and merge the corresponding two vortices; this reduces $c$, and increases the bound $k$ on the width of the linear decompositions used to no more than some $k' \leqslant 3k$ (see the proof of Lemma 7.1).

We assume now that $G$ contains no such edge $u_1u_2$. Then $X$ pairwise separates the vortices in $G$; recall that $G$ has no vertices in the interior of $S - k$, so any path in $G - X$ between different vortices would have to be a single edge. If $c > 1$, we let $G'$ be the unique component of $G - X$ that meets the branch sets of $K$. (This component is unique, because branch sets are pairwise adjacent, and all branch sets meeting $X$ were deleted.) Then $G'$ is embedded in $S$ with at most one vortex, i.e. we have again reduced $c$.

So we may assume that $c = 1$ (and hence $S \neq S^2$, by the induction start); let $D$ be the hole of $S$ and denote the boundary of $D$ by $C$. Let $J$ be the graph obtained from $G$ by adding $C$ as a cycle. By Lemma 4.3, $S$ contains a genus-reducing curve $C'$ that either lies in $J$ or avoids $J$. In the latter case, cutting $S$ open along $C'$ reduces the Euler genus of $S$ without affecting $G$, and we are home with $G' := G$. We thus assume that $C' \subseteq J$. As in the final paragraph of the proof of Lemma 7.2, the fact that $C'$ is genus-reducing implies that one of the embedded $C$-paths $v \dots w$ in $C'$, which are now single edges, combines with an arc through $D$ to a genus-reducing curve. Cutting along this curve, we obtain one or two surfaces $S'$ of smaller Euler genus than $S$. One of these contains, nearly embedded after the deletion of a suitable subset of $X_v \cup X_w$, a subgraph $G'$ of $G$ that meets all but at most $|X_v \cup X_w| \leqslant 2k$ of the branch sets of $K$. Without disturbing more than a bounded number of branch sets of $K$, we have thus reduced the Euler genus of $S$, as planned. □

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