

# EXCLUDING SUBDIVISIONS OF INFINITE CLIQUES

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## ABSTRACT

For every infinite cardinal  $\kappa$  we characterize graphs not containing a subdivision of  $K_\kappa$ .

## 1. INTRODUCTION

In this paper graphs may be infinite, and may have loops and multiple edges. A graph  $G$  is a *subdivision* of a graph  $H$  if  $G$  can be obtained from  $H$  by replacing the edges of  $H$  by internally disjoint paths joining the same ends. Let  $G$  be a graph. A *tree-decomposition* of  $G$  is a pair  $(T, W)$ , where  $T$  is a tree (a connected graph with no circuits) and  $W = (W_t : t \in V(T))$  is such that

(W1)  $\bigcup_{t \in V(T)} W_t = V(G)$ , and every edge of  $G$  has both ends in some  $W_t$ , and

(W2) if  $t'$  lies on the path of  $T$  between  $t$  and  $t''$ , then  $W_t \cap W_{t''} \subseteq W_{t'}$ .

If  $\kappa$  is a cardinal, we say that  $(T, W)$  has *width*  $< \kappa$  if  $|W_t| < \kappa$  for every  $t \in V(T)$ , and

$$\left| \bigcup_{i=1}^{\infty} \bigcap_{j \geq i} W_{t_j} \right| < \kappa$$

for every infinite path  $t_1, t_2, \dots$  in  $T$ . We shall prove the following result for excluding a subdivision of  $K_{\aleph_0}$ , the countable clique.

(1.1) *A graph  $G$  contains no subgraph isomorphic to a subdivision of  $K_{\aleph_0}$  if and only if  $G$  admits a tree-decomposition of width  $< \aleph_0$ .*

One cannot hope for an analogous theorem for uncountable cardinals because of the following [4].

(1.2) *For every cardinal  $\kappa$  there exists a graph  $G$  with no subgraph isomorphic to a subdivision of  $K_{\aleph_1}$  such that for every tree-decomposition  $(T, W)$  of  $G$  there exists  $t \in V(T)$  with  $|W_t| \geq \kappa$ .*

However, the next best weakening works, namely “well-founded tree-decomposition”, which we now introduce. A *well-founded tree* is a nonempty partially ordered set  $T = (V, \leq)$  such that for every two elements  $t_1, t_2 \in V$  their infimum exists and such that the set  $\{t' \in V : t' < t\}$  is well-ordered for every  $t \in V$ . It follows that  $T$  has a minimum element, called the *root* and denoted by  $\text{root}(T)$ .

We write  $V(T) = V$  and call the elements of  $V(T)$  the *vertices* of  $T$ . If  $\lambda$  is an ordinal we say that  $T$  is  $< \lambda$  *high* if every chain in  $T$  has order type  $< \lambda$ . For  $t_1, t_2 \in V(T)$  we define  $T[t_1, t_2]$  to be the set of all  $t \in V(T)$  such that  $\text{inf}(t_1, t_2) \leq t$ , and either  $t \leq t_1$  or  $t \leq t_2$ . For  $t, t' \in V(T)$  we say that  $t'$  is a *successor* of  $t$ , and that  $t$  is a *predecessor* of  $t'$ , if  $t < t'$  and there is no  $t'' \in V(T) - \{t, t'\}$  with  $t \leq t'' \leq t'$ .

A *well-founded tree-decomposition* of a graph  $G$  is a pair  $(T, W)$ , where  $T$  is a well-founded tree and  $W = (W_t : t \in V(T))$  satisfies

(W1)  $\bigcup_{t \in V(T)} W_t = V(G)$ , and every edge of  $G$  has both endpoints in some  $W_t$ ,

(W2) if  $t' \in T[t, t'']$  then  $W_t \cap W_{t''} \subseteq W_{t'}$ , and

(W3) if  $C \subseteq V(T)$  is a chain and  $c = \sup C \in V(T)$ , then  $\bigcap_{t \in C} W_t \subseteq W_c$ .

We say that  $(T, W)$  has *width*  $< \kappa$  if  $|\bigcup_{t \in C} \{W_{t'} : t' \in C, t' \geq t\}| < \kappa$  for every chain  $C \subseteq V(T)$ .

It follows that if  $(T, W)$  has width  $< \kappa$  then  $|W_t| < \kappa$  for every  $t \in V(T)$ . We say that  $(T, W)$  is  $< \kappa$  *high* if  $T$  is  $< \kappa$  high.

We say that a well-founded tree  $T$  is *graph-theoretic* if every chain in  $T$  has order type  $\leq \omega$ . Let  $R$  be a tree and let  $r \in V(R)$ . We define  $t_1 \leq t_2$  for  $t_1, t_2 \in V(R)$  to mean that  $t_1$  lies on the path between  $r$  and  $t_2$ . It is easily seen that  $T = (V(R), \leq)$  is a graph-theoretic well-founded tree and that every graph-theoretic well-founded tree arises this way. Moreover,  $T[t_1, t_2]$  is the set of all vertices of  $R$  which lie on the path between  $t_1$  and  $t_2$  in  $R$ . We say that a well-founded tree-decomposition  $(T, W)$  is *graph-theoretic* if  $T$  is graph-theoretic. Thus we have proved

(1.3) *Let  $\kappa$  be a cardinal. A graph  $G$  admits a graph-theoretic well-founded tree-decomposition of width  $< \kappa$  if and only if  $G$  admits a tree-decomposition of width  $< \kappa$ .*

The true version of (1.1) for larger cardinals is the following.

(1.4) *Let  $\kappa$  be an infinite cardinal and let  $G$  be a graph. Then the following two conditions are equivalent:*

- (i)  *$G$  contains no subgraph isomorphic to a subdivision of  $K_\kappa$ ,*
- (ii)  *$G$  admits a well-founded tree-decomposition of width  $< \kappa$ .*

*If  $\kappa = \aleph_0$  then the tree-decomposition in (ii) can be chosen graph-theoretic. If  $\kappa$  is regular and uncountable, then (i) and (ii) are equivalent to*

- (iii)  *$G$  admits a well-founded tree-decomposition of width  $< \kappa$  which is  $< \kappa$  high.*

By (1.3), (1.4) implies (1.1). Notice that if  $\kappa$  is regular and  $(T, W)$  is a tree-decomposition of a graph  $G$  which is  $< \kappa$  high, then  $(T, W)$  has width  $< \kappa$  if and only if  $|W_t| < \kappa$  for every  $t \in V(T)$ .

The equivalence of (i) and (iii) for regular uncountable cardinals is similar to (and interderivable with) a result independently obtained by Diestel [2]. Diestel's theorem generalizes a theorem of Halin [3].

There are other conditions that are equivalent to the conditions of (1.4) when  $\kappa$  is regular uncountable. We now introduce two of them; others can be found in [7].

A *linear decomposition* of a graph  $G$  is a pair  $(L, X)$ , where  $L$  is a (Dedekind) complete linearly ordered set and  $X = (X_\ell : \ell \in L)$  is such that

- (L1)  $\bigcup_{\ell \in L} X_\ell = V(G)$ , and every edge of  $G$  has both ends in some  $X_\ell$ ,
- (L2) if  $\ell \leq \ell' \leq \ell''$ , then  $X_\ell \cap X_{\ell''} \subseteq X_{\ell'}$ , and
- (L3)  $\bigcap_{i \in I} X_i \subseteq X_{\inf(I)} \cap X_{\sup(I)}$  for every nonempty interval  $I \subseteq L$ .

We say that  $(L, X)$  has *width*  $< \kappa$  if  $|X_\ell| < \kappa$  for every  $\ell \in L$ . Let us remark that the requirement that  $L$  be complete is not restrictive, because any "incomplete" decomposition can be completed in the obvious way. Linear decompositions are motivated by path-decompositions from [5], and their relation to excluding infinite trees is studied in [8].

Finally, we introduce the following generalization of stoppages from [1]. Let  $G$  be a graph. A *cut* in  $G$  is an ordered pair  $(A, B)$  of subsets of  $V(G)$  such that  $A \cup B = V(G)$  and there is no edge between  $A - B$  and  $B - A$ . The *order* of  $(A, B)$  is  $|A \cap B|$ . Now let  $\kappa$  be a cardinal. A *stoppage of order  $\kappa$*  in a graph  $G$  is a set  $\mathcal{S}$  of cuts, all of order  $< \kappa$ , such that

- (i) if  $(A, B)$  is a cut in  $G$  of order  $< \kappa$ , then  $\mathcal{S}$  contains one of  $(A, B), (B, A)$ ,
- (ii) if  $(A_1, B_1), (A_2, B_2) \in \mathcal{S}$ , then  $(G|A_1) \cup (G|A_2) \neq G$  (where  $G|A$  is the restriction of  $G$  to  $A$ ),  
and
- (iii) if  $\mathcal{M} \subseteq \mathcal{S}$  is a chain of cuts (that is, for  $(A_1, B_1), (A_2, B_2) \in \mathcal{M}$  either  $A_1 \subseteq A_2$  and  $B_1 \supseteq B_2$ ,  
or  $A_1 \supseteq A_2$  and  $B_1 \subseteq B_2$ ) and for

$$A = \bigcup \{A' : (A', B') \in \mathcal{M}\}, \quad B = \bigcap \{B' : (A', B') \in \mathcal{M}\}$$

the order of  $(A, B)$  is  $< \kappa$ , then  $(A, B) \in \mathcal{S}$ .

We will refer to (i), (ii), (iii) as the stoppage axioms.

The following result extends (1.4) for  $\kappa$  regular uncountable.

(1.5) *Let  $G$  be a graph and let  $\kappa$  be a regular uncountable cardinal. Then the following conditions are equivalent:*

- (i)  *$G$  has no subgraph isomorphic to a subdivision of  $K_\kappa$ ,*
- (ii)  *$G$  admits a  $< \kappa$  high well-founded tree-decomposition of width  $< \kappa$ ,*
- (iii)  *$G$  admits a linear decomposition of width  $< \kappa$ ,*
- (iv)  *$G$  has no stoppage of order  $\geq \kappa$ .*

Theorem (1.5) is false for  $\kappa = \aleph_0$ , because it is shown in [8] that if  $\kappa = \aleph_0$  then (iii) is equivalent to not containing a subgraph isomorphic to a subdivision of the  $\aleph_0$ -branching tree. The assumption that  $\kappa$  is regular cannot be dropped either, because of the following (we do not know whether there is a similar counterexample when  $\text{cf}(\kappa) > \omega$ ).

(1.6) *There exists a graph which contains no subgraph isomorphic to a subdivision of  $K_{\aleph_\omega}$  and which has no linear decomposition of width  $< \aleph_\omega$ .*

The paper is organized as follows: In Section 2 we prove (1.4) and in Section 3 we prove (1.5) and (1.6). We end this section with the following lemma, a relative of (3.4) from [6].

(1.7) *Let  $G$  be a graph, let  $F$  be the vertex-set of a connected subgraph of  $G$ , let  $(T, W)$  be a well-founded tree-decomposition of  $G$  and let  $t_1, t_2 \in V(T)$  be such that  $W_{t_1} \cap F \neq \emptyset \neq W_{t_2} \cap F$ . If  $t, t' \in T[t_1, t_2]$  are such that  $t \leq t'$  and there is no  $t'' \in V(T)$  with  $t < t'' < t'$  then  $W_t \cap W_{t'} \cap F \neq \emptyset$ . In particular, if  $t \in T[t_1, t_2]$  then  $W_t \cap F \neq \emptyset$ .*

*Proof.* We proceed by induction on  $|V(P)|$ , where  $P$  is the shortest path connecting  $W_{t_1}$  and  $W_{t_2}$  with  $V(P) \subseteq F$ . If  $|V(P)| = 1$  the result follows from (W2). If  $|V(P)| \geq 2$  let  $u \in V(P) \cap W_{t_1}$  be an endvertex of  $P$ , and let  $v \in V(P)$  be the neighbor of  $u$  in  $P$ . By (W1) there is  $r \in V(T)$  such that  $u, v \in W_r$ . It is easy to see that either  $t, t' \in T[t_1, r]$  or  $t, t' \in T[r, t_2]$ . In the former case  $u \in W_t \cap W_{t'} \cap F$  by (W2), and in the latter one the result follows from the induction hypothesis applied to  $r, t_2$  and  $F$ . □

## 2. GREEDY METHOD

In this section we prove (1.4). First we shall prove that (ii) implies (i).

(2.1) *Let  $\kappa$  be a cardinal, let  $G$  be a graph and let  $(T, W)$  be a well-founded tree-decomposition of  $G$  of width  $< \kappa$ . Then  $G$  contains no subgraph isomorphic to a subdivision of  $K_\kappa$ .*

*Proof.* Let  $(T, W)$  be as stated in (2.1). Suppose for a contradiction that  $G$  contains a subgraph  $H$  isomorphic to a subdivision of  $K_\kappa$ , and let  $V \subseteq V(H)$  be the set of vertices corresponding to vertices of  $K_\kappa$ . For each  $v \in V$ , let  $t(v)$  be the minimal  $t \in V(T)$  with  $v \in W_t$  (it is unique by (W2)).

(1) *If  $v_1, v_2 \in V$  then either  $t(v_1) \leq t(v_2)$  or  $t(v_2) \leq t(v_1)$ .*

For let  $t = \inf(t(v_1), t(v_2))$ . There are  $\kappa$  paths of  $H$  and hence of  $G$ , between  $v_1$  and  $v_2$ , mutually disjoint except for  $v_1$  and  $v_2$ , and all passing through  $W_t$  by (1.7). Since  $|W_t| < \kappa$  it follows that one of  $v_1, v_2 \in W_t$ , and so either  $t = t(v_1)$  or  $t = t(v_2)$ . The claim follows.

(2) *If  $v_1, v_2 \in V$  and  $t(v_1) \leq t(v_2)$  then  $v_1 \in W_t$  for all  $t$  with  $t(v_1) \leq t \leq t(v_2)$ .*

For suppose not. Choose  $t_2$  with  $t(v_1) \leq t_2 \leq t(v_2)$  minimal such that  $v_1 \notin W_{t_2}$ . Since  $v_1 \in W_t$  for all  $t$  with  $t(v_1) \leq t < t_2$ , it follows that  $t_2 \neq \sup\{t : t(v_1) \leq t < t_2\}$  by (W3), and so there exists a predecessor  $t_1$  of  $t_2$ . By (1.7), each of the  $\kappa$  internally disjoint paths of  $H$  between  $v_1$  and  $v_2$  passes through  $W_{t_1} \cap W_{t_2}$ , and so one of  $v_1, v_2$  belongs to  $W_{t_1} \cap W_{t_2}$ . But  $v_1 \notin W_{t_2}$ , and  $v_2 \notin W_{t_1}$  since  $t_1 < t_2 \leq t(v_2)$ . This is a contradiction, and (2) follows.

Let  $C = \{t \in V(T) : t \leq t(v) \text{ for some } v \in V\}$ .

(3)  *$C$  is a chain, and  $V \subseteq \bigcup_{t \in C} \bigcap \{W_{t'} : t' \in C, t' \geq t\}$ .*

For  $C$  is a chain by (1). Let  $v \in V$ , and let  $t' \in C, t' \geq t(v)$ . Choose  $v' \in V$  with  $t' \leq t(v')$ . Then  $t(v) \leq t(v')$  and by (2),  $v \in W_{t'}$ . Hence  $v \in \bigcap \{W_{t'} : t' \in C, t' \geq t(v)\}$  and the claim follows.

But  $|V| = \kappa$ , and so (3) contradicts the fact that  $(T, W)$  has width  $< \kappa$ . □

Next we prove the rest of (1.4). We first prove that every graph  $G$  admits a “standard decomposition” and then prove that if  $G$  is as in (1.4i) then this decomposition satisfies the conclusion

of (1.4ii). If  $X$  is a set we put  $\|X\| = 0$  if  $X$  is finite, and  $\|X\| = |X|$  otherwise. If  $G$  is a graph and  $K \subseteq V(G)$ , then  $N(K)$  is the set of all vertices in  $V(G) - K$  which are adjacent to a vertex in  $K$ ,  $G \setminus K$  is the graph obtained from  $G$  by deleting the vertices of  $K$  and all edges incident with these vertices, and a  $K$ -flap is the set of vertices of a component of  $G \setminus K$ .

Let  $(T, W)$  be a well-founded tree-decomposition of a graph  $G$ . We say that  $(T, W)$  is a *standard decomposition* of  $G$  if

- (S1)  $W_{\text{root}(T)} = \emptyset$ ,
- (S2) if  $t'$  is a successor of  $t$  in  $T$ , then  $\|W_{t'}\| \leq \|W_t\|$  and  $W_{t'} - W_t \neq \emptyset$ ,
- (S3) for every chain  $C$  in  $T$  which does not have a maximal element, any two non-adjacent vertices in  $W_C := \bigcup_{t \in C} \bigcap \{W_{t'} : t' \in C, t' \geq t\}$  are joined in  $G$  by  $\|W_C\|$  internally disjoint paths, and if  $C$  has a supremum, say  $t$ , then  $W_t = W_C$  and  $t$  has at least one successor,
- (S4) for every  $t \in V(T)$ , every  $W_t$ -flap of  $G$  which intersects  $\bigcup_{t \leq t'} W_{t'}$  is intersected by  $W_{t'}$  for some successor  $t'$  of  $t$ .

(2.2) *Every graph admits a standard decomposition.*

*Proof.* Let  $G$  be a graph. Let  $u, v \in X \subseteq V(G)$  and let  $\mu$  be a cardinal. If there exists a set  $Y \subseteq X - \{u, v\}$  with  $\|Y\| \leq \mu$  which meets every path  $P$  joining  $u, v$  with  $V(P) \subseteq X$ , then we let  $\Theta(u, v, X, \mu)$  be one such set  $Y$ , and if not we define  $\Theta(u, v, X, \mu) = \emptyset$ .

For some ordinal  $\lambda$ , we shall construct a transfinite sequence  $W_\alpha$  ( $\alpha \leq \lambda$ ) of subsets of  $V(G)$  satisfying

$$(1) \quad \emptyset = W_0 \subseteq W_1 \subseteq \cdots \subseteq W_\alpha \subseteq \cdots \subseteq W_\lambda = V(G).$$

Also, for each  $\alpha < \lambda$  we shall construct a partition  $\mathcal{K}_\alpha$  of  $V(G) - W_\alpha$  into non-empty sets satisfying

- (2) for all  $\alpha < \lambda$  if  $K, K' \in \mathcal{K}_\alpha$  are distinct then no vertex of  $K$  has a neighbor in  $K'$ , and
- (3) for  $\alpha \leq \beta < \lambda$ , every member of  $\mathcal{K}_\beta$  is a subset of a member of  $\mathcal{K}_\alpha$ .

Third, for each  $\alpha < \lambda$  and each  $K \in \mathcal{K}_\alpha$  we shall construct a subset  $W_{(K, \alpha)}$  of  $K \cup W_\alpha$ .



The inductive definition is as follows. Let  $W_0 = \emptyset$ ,  $\mathcal{K}_0 = \{V(G)\}$ , and  $W_{(V(G),0)} = \emptyset$ . Suppose that for some ordinal  $\alpha$  we have defined  $W_\beta, \mathcal{K}_\beta$  and  $W_{(K,\beta)}$  ( $K \in \mathcal{K}_\beta$ ) for all  $\beta < \alpha$ . Now we wish to define  $W_\alpha, \mathcal{K}_\alpha, W_{(K,\alpha)}$  ( $K \in \mathcal{K}_\alpha$ ). Let

$$(4) \quad W_\alpha = \bigcup \{W_{(K,\beta)} : \beta < \alpha, K \in \mathcal{K}_\beta\}.$$

If  $W_\alpha = V(G)$  we set  $\lambda = \alpha$  and stop. Otherwise, there are two cases.

If  $\alpha$  is a successor ordinal, say  $\alpha = \beta + 1$ , let  $\mathcal{K}_\alpha$  be the set of all  $W_\alpha$ -flaps of  $G$ . For each  $K \in \mathcal{K}_\alpha$  let  $L \in \mathcal{K}_\beta$  include  $K$ , let  $w \in K$  be arbitrary, let  $Z = \{w\} \cup N(K)$ , and let

$$W_{(K,\alpha)} = Z \cup \bigcup \{\Theta(u, v, K \cup N(K), \|W_{(L,\beta)}\|) : u, v \in Z\}.$$

Now, let  $\alpha$  be a limit ordinal. For each  $u \in V(G) - W_\alpha$  and each  $\beta < \alpha$ , let  $K_\beta(u)$  be the member of  $\mathcal{K}_\beta$  containing  $u$ ; then, by (3),

$$K_0(u) \supseteq K_1(u) \supseteq \cdots \supseteq K_\beta(u) \supseteq \cdots.$$

Let  $K(u) = \bigcap_{\beta < \alpha} K_\beta(u)$ , and let  $\mathcal{K}_\alpha = \{K(u) : u \in V(G) - W_\alpha\}$ . For  $u \in V(G) - W_\alpha$ , let

$$W_{(K(u),\alpha)} = \bigcup_{\beta < \alpha} \bigcap_{\beta < \gamma < \alpha} W_{(K_\gamma(u),\gamma)}.$$

This completes the inductive definition. It is easy to see that the ordinal  $\lambda$  indeed exists, that  $W_{(K,\alpha)} \subseteq K \cup W_\alpha$  for every  $\alpha < \lambda$  and every  $K \in \mathcal{K}_\alpha$ , and that (1), (2), (3) are satisfied.

We put  $V(T) = \{(K, \alpha) : K \in \mathcal{K}_\alpha, \alpha \leq \lambda\}$  and define  $(K, \alpha) \leq (K', \alpha')$  if  $K \supseteq K'$  and  $\alpha \leq \alpha'$ . We put  $T = (V(T), \leq)$ . It follows easily that  $T$  is a well-founded tree. Let  $W = (W_{(K,\alpha)} : (K, \alpha) \in V(T))$ . We shall show that  $(T, W)$  is the desired standard decomposition, but we need several observations before we do so.

$$(5) \quad N(K) \subseteq W_{(K,\alpha)} \text{ for every } K \in \mathcal{K}_\alpha \text{ and every ordinal } \alpha < \lambda.$$

We prove this by transfinite induction on  $\alpha$ . The statement is obviously satisfied when  $\alpha = 0$ , so let  $\alpha$  be an ordinal and assume that (5) holds for all  $\beta < \alpha$ . Let  $K \in \mathcal{K}_\alpha$  and let  $v \in N(K)$ . If  $\alpha$  is a successor ordinal then  $v \in W_{(K,\alpha)}$  by the definition of  $W_{(K,\alpha)}$  so let  $\alpha$  be a limit ordinal.

Let  $u$  be a neighbor of  $v$  in  $K$ . Define  $K_\beta(u)$  ( $\beta < \alpha$ ) as before; then  $K = \bigcap_{\beta < \alpha} K_\beta(u)$ . Since  $v \notin K$ , there exists  $\beta < \alpha$  with  $v \notin K_\beta(u)$ , and hence with  $v \notin K_\gamma(u)$  for  $\beta < \gamma < \alpha$ . Since  $u, v$  are adjacent and  $u \in K_\gamma(u)$ , it follows that  $v \in N(K_\gamma(u)) \subseteq W_{(K_\gamma(u), \gamma)}$  for  $\beta < \gamma < \alpha$ , by the inductive hypothesis; and thus  $v \in W_{(K, \alpha)}$  by the definition of  $W_{(K, \alpha)}$ . This proves (5).

(6) For  $\alpha \leq \beta < \lambda$ , if  $K \in \mathcal{K}_\alpha$  and  $L \in \mathcal{K}_\beta$  and  $L \subseteq K$  then  $W_{(L, \beta)} \subseteq W_{(K, \alpha)} \cup K$ .

We prove this by transfinite induction on  $\beta$ . We may assume that  $\alpha < \beta$ . If  $\beta$  is a successor,  $\beta = \gamma + 1$  say, let  $(L, \beta)$  be a successor of  $(M, \gamma)$ . Then

$$W_{(L, \beta)} \subseteq L \cup N(L) \subseteq M \cup N(M) \subseteq M \cup W_{(M, \gamma)}$$

by (5), and  $W_{(M, \gamma)} \subseteq W_{(K, \alpha)} \cup K$  by the induction hypothesis. It follows that  $W_{(L, \beta)} \subseteq M \cup W_{(K, \alpha)} \cup K \subseteq W_{(K, \alpha)} \cup K$ , as required. Now assume that  $\beta$  is a limit ordinal. Let  $v \in W_{(L, \beta)}$ . By definition of  $W_{(L, \beta)}$  there exists  $(M, \gamma)$  with  $(K, \alpha) \leq (M, \gamma) < (L, \beta)$  with  $v \in W_{(M, \gamma)}$ . By the induction hypothesis,  $W_{(M, \gamma)} \subseteq W_{(K, \alpha)} \cup K$ , and so  $v \in W_{(K, \alpha)} \cup K$ . The result follows.

Let  $v \in V(G)$ . A pair  $(K, \alpha) \in V(T)$  is called a *nest* of  $v$  if  $v \in W_{(K, \alpha)}$  and for every ordinal  $\beta < \alpha$  there is no  $L \in \mathcal{K}_\beta$  such that  $v \in W_{(L, \beta)}$ . Since  $W_\lambda = V(G)$ , it follows from (4) with  $\alpha = \lambda$  that there is a nest of  $v$ , for every  $v \in V(G)$ .

(7) Let  $(K, \alpha)$  be a nest of  $v \in V(G)$ . Then  $v \in K$ .

For it follows from the minimality of  $\alpha$  and (4) that  $v \notin W_\alpha$ . Since  $v \in W_{(K, \alpha)} \subseteq K \cup W_\alpha$ , the claim follows.

(8) If  $(K, \alpha)$  is a nest of  $v \in W_{(L, \beta)}$ , then  $(K, \alpha) \leq (L, \beta)$ .

For  $\alpha \leq \beta$  by the minimality of  $\alpha$ . Let  $K' \in \mathcal{K}_\alpha$  with  $L \subseteq K'$ . Since  $v \in K$  by (7), and  $v \in W_{(L, \beta)} \subseteq W_{(K', \alpha)} \cup K' \subseteq W_\alpha \cup K'$  by (6), we deduce that  $K \cap (W_\alpha \cup K') \neq \emptyset$ , and so  $K = K'$ . The result follows.

From (8) we deduce that the nest of  $v$  is unique.

(9)  $(T, W)$  is a well-founded tree-decomposition.

We must verify (W1)-(W3). We start with (W1). Certainly  $\bigcup_{t \in V(T)} W_t = V(G)$ , by (4) with  $\alpha = \lambda$ . Let  $e$  be an edge of  $G$  with endpoints  $u, v$  say. Let  $(K, \alpha)$  be the nest of  $u$  and let  $(L, \beta)$

be the nest of  $v$ . We may assume that  $\alpha \leq \beta$ . It follows from (7) that  $u \in K$  and  $v \in L$ , and since they are adjacent we deduce that  $L \subseteq K$ . If  $\alpha = \beta$  then  $K = L$  and we are done, and so we assume that  $\alpha < \beta$ . Then  $u \notin L$  since  $W_{(K,\alpha)} \subseteq W_{\alpha+1} \subseteq W_\beta$  and  $L \cap W_\beta = \emptyset$ . Hence  $u \in N(L)$  and so  $u \in W_{(L,\beta)}$  by (5), as desired. This proves (W1).

To verify (W2) let  $v \in W_{(K,\alpha)} \cap W_{(L,\beta)}$ . First let  $(K, \alpha) < (M, \gamma) < (L, \beta)$ . Then  $L \subseteq M \subseteq K$  and  $v \notin M$  because  $M$  is disjoint from  $W_{(K,\alpha)}$ . Therefore,  $v \in W_{(L,\beta)} - M \subseteq W_{(M,\gamma)}$  by (6). Hence (W2) holds for  $t, t', t''$  if  $t < t' < t''$ . Now let  $K \cap L = \emptyset$ . Let  $(N, \delta)$  be the nest of  $v$ ; then  $(N, \delta) \leq (K, \alpha)$  and  $(N, \delta) \leq (L, \beta)$  by (8). This together with the above shows that  $v \in W_{(M,\gamma)}$  for every  $(M, \gamma) \in T[(K, \alpha), (L, \beta)]$ .

Finally, condition (W3) follows directly from the construction. This completes the proof of (9).

(10) *If  $(K, \alpha), (L, \alpha + 1) \in V(T)$  and  $(L, \alpha + 1)$  is a successor of  $(K, \alpha)$ , then  $N(L) \subseteq W_{(K,\alpha)}$ .*

For let  $v \in N(L)$  and let  $(M, \beta)$  be the nest of  $v$ . Since  $v \in N(L) \subseteq W_{\alpha+1}$  by (2), it follows from (4) that  $\beta \leq \alpha$ . If  $\beta = \alpha$  then  $K = M$  and  $v \in W_{(M,\beta)} = W_{(K,\alpha)}$ , while if  $\beta < \alpha$  then  $v \notin K$  because  $K \cap W_{(M,\beta)} = \emptyset$ , and hence  $v \in N(K) \subseteq W_{(K,\alpha)}$  by (5).

(11)  *$(T, W)$  is a standard decomposition.*

We must verify (S1)-(S4). Condition (S1) is clear. To prove (S2) let  $(K, \alpha + 1)$  be a successor of  $(L, \alpha)$ . Then  $N(K) \subseteq W_{(L,\alpha)}$  by (10), and by the construction

$$W_{(K,\alpha+1)} = Z \cup \bigcup \{ \Theta(u, v, K \cup N(K), \|W_{(L,\alpha)}\|) : u, v \in Z \},$$

where  $Z = \{w\} \cup N(K)$  and  $w \in K$ . Thus  $w \in W_{(K,\alpha+1)} - W_{(L,\alpha)} \neq \emptyset$ , and

$$\|W_{(K,\alpha+1)}\| \leq \|Z\| + \|Z\|^2 \cdot \|W_{(L,\alpha)}\| \leq \|W_{(L,\alpha)}\|,$$

because  $\|Z\| \leq \|W_{(L,\alpha)}\|$ . To prove (S3) let  $C$  be a chain in  $T$  which does not have a maximal element, and let  $W_C = \bigcup_{t \in C} \bigcap \{W_{t'} : t' \in C, t' \geq t\}$ . Suppose for a contradiction that  $u, v \in W_C$  are separated by a cutset, say  $A$ , of cardinality  $< \|W_C\|$ . It follows that  $\|W_C\| = |W_C| \geq \aleph_0$ . Then there exists  $(K, \alpha) \in C$  such that  $|W_{(K,\alpha)}| \geq |A|$  and  $u, v \in W_t$  for every  $t \in C$  with  $t \geq (K, \alpha)$ .

Let  $(L, \alpha + 1) \in C$  be the successor of  $(K, \alpha)$ , and let  $(M, \alpha + 2) \in C$  be the successor of  $(L, \alpha + 1)$ . It follows that  $u, v \in N(L) \cap N(M)$ . By construction  $W_{(L, \alpha + 1)} - \{u, v\}$  meets every path  $P$  joining  $u$  and  $v$  with  $V(P) \subseteq L \cup N(L)$ , but that contradicts the fact that  $u, v \in N(M)$  since  $M \cap W_{(L, \alpha + 1)} = \emptyset$  and  $M$  is a  $W_{\alpha + 2}$ -flap. Hence  $u, v$  are not separated by a cutset of size  $< \|W_C\|$ , and the first part of (S3) follows. Now if  $C$  has a supremum, say  $t = (K, \beta)$ , then  $W_t = W_C$  follows directly from the construction. It also follows that  $\beta$  is a limit ordinal, and  $\beta < \lambda$ . Since  $W_{(M, \beta)} \subseteq W_\beta$  for all  $M \in \mathcal{K}_\beta$  from the construction, and hence  $W_{\beta + 1} = W_\beta$ , it follows that  $K$  includes a  $W_{\beta + 1}$ -flap  $L$  since  $K \neq \emptyset$ , and hence  $(L, \beta + 1)$  is a successor of  $(K, \beta)$ . This proves (S3).

To prove (S4) let  $t = (K, \alpha)$  and let  $F$  be a  $W_t$ -flap which intersects  $\bigcup_{t \leq t'} W_{t'}$ . We claim that  $F \cap W_{\alpha + 1} = \emptyset$ . For if, say,  $F \cap W_{(L, \beta)} \neq \emptyset$ , where  $\beta \leq \alpha$  then  $t \not\prec (L, \beta)$  and so  $F \cap W_t \neq \emptyset$  by (1.7), a contradiction. Hence  $F \cap W_{\alpha + 1} = \emptyset$ . It follows that  $F$  is a  $W_{\alpha + 1}$ -flap and thus  $F \in \mathcal{K}_{\alpha + 1}$ . But  $F \cap W_{(F, \alpha + 1)} \neq \emptyset$  by the construction, as desired.  $\square$

The following is an easy lemma and the proof is left to the reader.

(2.3) *Let  $G$  be a graph and let  $\kappa$  be an infinite cardinal. If there exists a set  $X \subseteq V(G)$  with  $|X| \geq \kappa$  such that every two non-adjacent vertices of  $X$  are joined by  $\kappa$  internally disjoint paths, then  $G$  contains a subgraph  $H$  isomorphic to a subdivision of  $K_\kappa$  in such a way that every vertex of  $H$  which corresponds to a vertex of  $K_\kappa$  belongs to  $X$ .*

The following result implies (i)  $\Rightarrow$  (ii) of (1.4), because (1.4) obviously holds for finite graphs.

(2.4) *Let  $\kappa$  be an infinite cardinal, let  $G$  be an infinite graph which contains no subgraph isomorphic to a subdivision of  $K_\kappa$  and let  $(T, W)$  be a standard decomposition of  $G$ . Then  $|V(T)| \leq |V(G)|$  and  $(T, W)$  has width  $< \kappa$ .*

*Proof.* Let  $f : V(T) - \{\text{root}(T)\} \rightarrow V(G)$  be a mapping satisfying

- (i) if  $t'$  is a successor of  $t$  then  $f(t') \in W_{t'} - W_t$ , and
- (ii) if  $t \neq \text{root}(T)$  does not have a predecessor then  $f(t) \in W_{t'} - W_t$ , where  $t'$  is a successor of  $t$ .

Such a mapping exists by (S2) and (S3). It follows from (W2) that  $|f^{-1}(v)| \leq 2$  for every  $v \in V(G)$ , and hence  $|V(T)| \leq |V(G)|$  because  $G$  is infinite.

To prove the assertion about width we must prove that

- (1) for every chain  $C \subseteq V(T)$ , the set  $W_C = \bigcup_{t \in C} \bigcap \{W_{t'} : t' \in C, t' \geq t\}$  has cardinality  $< \kappa$ .

We prove (1) by transfinite induction on the ordinal type of the set  $\{t' \in V(T) : t' \leq t \text{ for some } t \in C\}$ , which we denote by  $h(C)$ . From (S1) we deduce that (1) is true if  $h(C) \leq 1$ . So assume that (1) is true for all  $C$  with  $h(C) < \alpha$  and let  $C$  with  $h(C) = \alpha > 1$  be given. If  $C$  has a maximal element, say  $t$ , and  $t$  does not have a predecessor, then  $W_C = W_t = W_{C-\{t\}}$  by (S3), which has cardinality  $< \kappa$  by the induction hypothesis. If  $t$  has a predecessor, say  $t'$ , then  $W_C = W_t$ ,  $W_{C-\{t\}} = W_{t'}$  and  $|W_t| < \kappa$  by (S2) and the induction hypothesis. Finally, if  $C$  does not have a maximal element, then  $|W_C| < \kappa$  by (S3) and (2.3).

This proves (1) and hence (2.4). □

If  $T = (V, \leq)$  is a well-founded tree and  $t \in V$ , then the order type of the set  $\{t' \in V : t' < t\}$  is called the *height* of  $t$  and is denoted by  $ht(t)$ . The following implies that if  $\kappa = \aleph_0$  then the well-founded tree-decomposition in (ii) of (1.4) can be chosen graph-theoretic.

- (2.5) Let  $\kappa$  be an infinite regular cardinal, and let  $G$  and  $(T, W)$  be as in (2.4). Then every chain in  $T$  has order type  $\leq \kappa$ .

*Proof.* Suppose not. Then there is a vertex  $t_0 \in V(T)$  with  $ht(t_0) = \kappa$ , by (S3) there exists a successor  $t'_0$  of  $t_0$ , and by (S2) there exists a  $W_{t_0}$ -flap  $F$  with  $F \cap W_{t'_0} \neq \emptyset$ . For every  $v \in W_{t_0}$  there exists by (S3)  $t_v < t_0$  such that  $v \in W_t$  for all  $t$  with  $t_v \leq t \leq t_0$ . Let  $t_1 = \sup\{t_v : v \in W_{t_0}\}$ . Since  $|W_{t_0}| < \kappa$  by (2.4) and since  $\kappa$  is regular, we deduce that  $t_1$  is well-defined and that  $t_1 < t_0$ . Since  $W_{t_0} \subseteq W_{t_1}$ , it follows that either  $F \cap W_{t_1} \neq \emptyset$ , or  $F$  is a  $W_{t_1}$ -flap in which case  $F \cap W_{t'_1} \neq \emptyset$  for some successor  $t'_1$  of  $t_1$ , by (S4). In either case,  $F \cap W_{t_0} \neq \emptyset$  by (1.7), a contradiction. □

Our next result implies (i)  $\Rightarrow$  (iii) of (1.4) for regular uncountable cardinals and thus completes the proof of (1.4).

(2.6) Let  $\kappa$  be a regular uncountable cardinal, and let  $G$  and  $(T, W)$  be as in (2.4). Then  $T$  is  $< \kappa$  high.

*Proof.* Suppose for a contradiction that  $C$  is a chain in  $T$  of order type  $\kappa$ . We may assume that  $C$  has the property that if  $t \in C$  and  $t' \leq t$  then  $t' \in C$ . Let  $X = \bigcup_{t \in C} \bigcap \{W_{t'} : t' \in C, t' \geq t\}$ . From (2.4)  $|X| < \kappa$ , and since  $\kappa$  is regular it follows that there exists  $t_0 \in C$  such that  $X \subseteq W_t$  for every  $t \in C$  with  $t_0 \leq t$ .

(1) For every  $t_1 \in C$  with  $t_1 \geq t_0$  there exists  $t \in C$  with  $t \geq t_1$  such that  $X = W_t$ .

To prove (1) we shall construct a sequence  $t_1 < t_2 < \dots$  of elements of  $C$  as follows. Assume that  $n \geq 1$  and that  $t_n$  has already been constructed. We may assume that  $W_{t_n} - X \neq \emptyset$ , for otherwise we are done. For every  $v \in W_{t_n} - X$  there exists  $t_v \in C$  with  $t_v \geq t_n$  and such that  $v \notin W_{t_v}$ . Since  $|W_{t_n}| < \kappa$  and  $\kappa$  is regular, it follows that  $t_{n+1} := \sup\{t_v : v \in W_{t_n} - X\}$  exists and belongs to  $C$ , and that  $(W_{t_n} - X) \cap W_{t_{n+1}} = \emptyset$ . Since  $W_{t_n} - X \neq \emptyset$  we deduce that  $t_{n+1} > t_n$ . Since  $\kappa$  is regular uncountable it follows that  $t = \sup t_n$  exists and belongs to  $C$ . We claim that  $t$  is as desired. For suppose for a contradiction that  $v \in W_t - X$ . Then  $v \in W_{t'}$  for some  $t' < t$  by (S3), and hence  $v \in W_{t_n}$  for some  $n$ . But then  $v \notin W_{t_{n+1}}$ , contrary to (W2). Hence such a choice of  $v$  is impossible, which proves (1).

From (1) we deduce that there exist  $t_1, t_2 \in C$  with  $t_0 \leq t_1 < t_2$  such that  $W_{t_1} = W_{t_2} = X$ . Let  $t'_2 \in C$  be the successor of  $t_2$  in  $C$ . By (S2)  $W_{t'_2} \cap F \neq \emptyset$  for some  $X$ -flap  $F$  of  $G$ . By (S4)  $W_{t'_1} \cap F \neq \emptyset$  for some successor  $t'_1$  of  $t_1$ . But  $X \cap F = W_{t_2} \cap F \neq \emptyset$  by (1.7), a contradiction.  $\square$

A natural question arises whether a highly connected graph can be decomposed into highly connected pieces. This turns out to be true, as follows. If  $\mu$  is a cardinal we say that a graph  $G$  is  $\mu$ -connected if  $|V(G)| \geq \mu$  and  $G \setminus X$  is connected for every  $X \subseteq V(G)$  with  $|X| < \mu$ . We say that a subset  $A \subseteq V(G)$  is  $\mu$ -connected if the graph induced by  $A$  is  $\mu$ -connected.

(2.7) Let  $\kappa$  be a regular uncountable cardinal, let  $\mu < \kappa$  be an infinite cardinal, and let  $G$  be a  $\mu$ -connected graph which contains no subgraph isomorphic to a subdivision of  $K_\kappa$ . Then there exists a well-founded tree-decomposition  $(T, W)$  of  $G$  such that

- (i)  $|V(T)| \leq |V(G)|$ ,
- (ii) every chain of  $T$  has order type  $< \kappa$ ,
- (iii)  $|W_t| < \kappa$  and  $W_t$  is  $\mu$ -connected for every  $t \in V(T)$ , and
- (iv) for  $t, t' \in V(T)$ , if  $t' \leq t$  then  $W_{t'} \subseteq W_t$ , and if  $ht(t)$  is a limit ordinal then  $W_t = \bigcup_{t' < t} W_{t'}$ .

*Proof.* Let  $(T, W'')$  be a standard decomposition of  $G$ . Then by (2.4),  $|V(T)| \leq |V(G)|$  and  $(T, W'')$  has width  $< \kappa$ , and by (2.6) every chain in  $T$  has order type  $< \kappa$ . Since  $G$  is  $\mu$ -connected there exists a  $\mu$ -connected set  $M \subseteq V(G)$  with  $|M| = \mu$ . For  $t \in V(T)$  let  $W_t = W_t'' \cup M$ , and let  $W = (W_t : t \in V(T))$ . Then  $(T, W)$  is a well-founded tree-decomposition of  $G$  of width  $< \kappa$ .

- (1) Let  $t_1 \in V(T)$  be a successor of  $t_0 \in V(T)$ . There is a subset  $D_{t_1} \subseteq \bigcup_{t \geq t_1} W_t$  such that
  - (i)  $W_{t_1} \subseteq D_{t_1}$  and  $|D_{t_1}| < \kappa$ , and
  - (ii) there is no cut  $(A, B)$  of the restriction of  $G$  to  $D_{t_1}$  such that  $|A \cap B| < \mu$ ,  $A \not\subseteq B \not\subseteq A$ , and  $W_{t_0} \cap W_{t_1} \subseteq A$ .

For since  $G$  is  $\mu$ -connected there is a  $\mu$ -connected  $F \subseteq V(G)$  with  $W_{t_1} \subseteq F$  and with  $|F| = \max(\mu, |W_{t_1}|)$ . Set  $D_{t_1} = F \cap \bigcup_{t \geq t_1} W_t$ ; then (i) is clearly satisfied. To prove that  $D_{t_1}$  satisfies (ii) suppose for a contradiction that  $(A, B)$  is a cut as in (ii). Since the restriction of  $G$  to  $F$  is  $\mu$ -connected there exists a path  $P$  in  $G$  joining a vertex of  $A - B$  to a vertex of  $B - A$  with  $V(P) \subseteq F - (A \cap B)$ . Let us choose such a path with  $|V(P)|$  minimum. Since  $(A, B)$  is a cut of the restriction of  $G$  to  $D_{t_1}$  it follows that  $V(P) - \bigcup_{t \geq t_1} W_t \neq \emptyset$ . From (W1) and (1.7) it follows that  $P$  contains a proper subpath joining a vertex  $u \in W_{t_0} \cap W_{t_1} \subseteq A$  to a vertex in  $B - A$ . But  $u \notin B$  and this subpath is shorter than  $P$ , a contradiction. Thus  $D_{t_1}$  satisfies both (i) and (ii) and the proof of (1) is complete.

Let  $D_{\text{root}(T)} = M$ , and for each  $t \in V(T) - \{\text{root}(T)\}$  with no predecessor, let  $D_t = \emptyset$ . For each  $t \in V(T)$  let  $W'_t = \bigcup_{t' \leq t} D_{t'}$ .

- (2) For each  $t \in V(T)$ ,  $|W'_t| < \kappa$  and  $W'_t$  is  $\mu$ -connected.

For since every chain of  $T$  has order type  $< \kappa$  and each  $|D_{t'}| < \kappa$  and  $\kappa$  is regular, it follows that each  $|W'_t| < \kappa$ . We prove that  $W'_t$  is  $\mu$ -connected by transfinite induction on  $ht(t)$ . Certainly

$|W'_t| \geq \mu$  since  $D_{\text{root}(T)} \subseteq W'_t$ . If  $t$  is a successor of some  $t_0$  then  $W'_t = W'_{t_0} \cup D_t$ . Suppose that  $(A, B)$  is a cut of the restriction of  $G$  to  $W'_t$  with  $|A \cap B| < \mu$  and  $A - B, B - A \neq \emptyset$ . Now from the inductive hypothesis,  $W'_{t_0}$  is  $\mu$ -connected, and so not both  $(A - B) \cap W'_{t_0}, (B - A) \cap W'_{t_0} \neq \emptyset$ . We assume that  $W'_{t_0} \subseteq A$ . Now  $(A \cap D_t, B \cap D_t)$  is a cut of the restriction of  $G$  to  $D_t$ , and certainly  $B \cap D_t \not\subseteq A \cap D_t$  since  $B \not\subseteq A$  and  $W'_{t_0} \subseteq A$ . From the choice of  $D_t$  it follows that  $A \cap D_t \subseteq B \cap D_t$ , and so  $D_t \subseteq B$ . Thus  $W_{t_0} \cap W_t \subseteq W'_{t_0} \cap D_t \subseteq A \cap B$ , and so  $|W_{t_0} \cap W_t| < \mu$ . But  $M \subseteq W_{t_0} \cap W_t$ , a contradiction. Finally, suppose that  $t$  has no predecessor. Since  $W'_{\text{root}(T)}$  is  $\mu$ -connected we may assume that  $t \neq \text{root}(T)$ . Let  $C = \{t' \in V(T) : t' < t\}$ . Then  $W'_t = \bigcup_{t' \in C} W'_{t'}$  since  $D_t = \emptyset$ , and so  $W'_t$  is  $\mu$ -connected since it is the union of a nested sequence of  $\mu$ -connected subsets. This proves (2).

Let  $W' = (W'_t : t \in V(T))$ .

(3)  $(T, W')$  is a well-founded tree-decomposition.

We observe first that  $W_t \subseteq W'_t$  for each  $t \in V(T)$ . This is clear if  $t = \text{root}(T)$  or  $t$  is a successor. Otherwise let  $C$  be the set of all  $t' < t$  which are successors. Then  $W_t \subseteq \bigcup_{t' \in C} W_{t'}$  by (S3); each such  $W_{t'} \subseteq W'_{t'}$ , and  $\bigcup_{t' \in C} W'_{t'} \subseteq W'_t$  by the definition of  $W'$ . This proves that  $W_t \subseteq W'_t$  for each  $t \in V(T)$  and (W1) follows. To see (W2), let  $t_1, t_2, t_3 \in V(T)$  with  $t_2 \in T[t_1, t_3]$ ; and let  $v \in W'_{t_1} \cap W'_{t_3}$ . Choose  $t'_1 \leq t_1$  with  $v \in D_{t'_1}$ , and  $t'_3 \leq t_3$  similarly. If  $t'_1 \leq t_2$  or  $t'_3 \leq t_2$  then  $v \in W'_{t_2}$  as required, and so we assume not. Hence  $t_2 \in T[t'_1, t'_3]$ . Now  $v \in D_{t'_1} \subseteq \bigcup_{t \geq t'_1} W_t$ , and similarly  $v \in \bigcup_{t \geq t'_3} W_t$ . Since  $t_2 \in T[t'_1, t'_3]$  and  $t_2 \leq t'_1, t'_3$  it follows from (W2) (for  $(T, W)$ ) that  $v \in W_{t_2} \subseteq W'_{t_2}$  as required. Condition (W3) is clear because  $W'_t \subseteq W'_{t'}$  for  $t \leq t'$ .

Finally, we observe that if  $t \in V(T) - \{\text{root } T\}$  has no predecessor (i.e.,  $ht(t)$  is a limit ordinal) then since  $D_t = \emptyset$ ,

$$W'_t = \bigcup_{t' < t} D_{t'} = \bigcup_{t' < t} W_{t'}.$$

The result follows. □

For an application of (2.7) see [9].



### 3. LINEAR DECOMPOSITIONS AND STOPPAGES

Now we prove (1.5). By (1.4), (i)  $\Rightarrow$  (ii). We shall prove (ii)  $\Rightarrow$  (iii)  $\Rightarrow$  (iv)  $\Rightarrow$  (i). We start with (ii)  $\Rightarrow$  (iii).

(3.1) *Let  $\kappa$  be an uncountable regular cardinal, let  $G$  be a graph and let  $(T, W)$  be a  $< \kappa$  high well-founded tree-decomposition of  $G$  of width  $< \kappa$ . Then  $G$  admits a linear decomposition of width  $< \kappa$ .*

*Proof.* Let  $\leq'$  be a well-ordering on  $V(T)$  with the property that

(1) *for every  $t \in V(T)$ , the order type of the set of successors of  $t$  is not a limit ordinal.*

Let  $L$  be the set of maximal chains of  $T$  and let  $\leq$  be defined on  $L$  as follows. For  $\ell_1, \ell_2 \in L$  we say that  $\ell_1 \leq \ell_2$  if either  $\ell_1 = \ell_2$ , or  $\ell_1 \neq \ell_2$  and  $t_1 \leq' t_2$ , where  $t_i = \min\{t \in \ell_i : t \notin \ell_1 \cap \ell_2\}$ . It follows that  $L$  is linearly ordered by  $\leq$ .

(2)  *$(L, \leq)$  is complete.*

We must verify that every non-empty subset of  $L$  has a supremum and an infimum. We shall do it for supremum, for the infimum case is analogous and in fact easier. So let  $C \subseteq L$ . Let  $t_0 = \text{root}(T)$  and assume that we have already constructed  $t_0, \dots, t_\beta, \dots$  ( $\beta < \alpha$ ), where  $ht(t_\beta) = \beta$  and  $t_0 \leq t_1 \leq \dots \leq t_\beta \leq \dots$ . If  $\{t_\beta\}_{\beta < \alpha}$  is a maximal chain in  $T$  we stop, and otherwise we choose  $t_\alpha \in V(T)$  such that

- (i)  $ht(t_\alpha) = \alpha$ ,
- (ii)  $t_\beta \leq t_\alpha$  for every  $\beta < \alpha$ ,
- (iii)  $t \leq' t_\alpha$  for every  $t$  such that  $ht(t) = \alpha$ ,  $t_\beta \leq t$  for every  $\beta < \alpha$ , and  $t \in \ell$  for some  $\ell \in C$ ,
- (iv)  $t_\alpha$  is  $\leq'$ -minimal subject to (i)-(iii).

Such a choice is possible, because it follows from (1) that there exists at least one vertex satisfying (i)-(iii). (We remark that this vertex is unique when  $\alpha$  is a limit ordinal, because any two vertices of  $T$  have an infimum.) Let  $\alpha$  be the least ordinal such that  $t_\alpha$  is undefined, and let  $\ell = \{t_\beta : \beta < \alpha\}$ .

Then  $\ell = \sup C$ , as is easily seen. This proves (2).

We define, for  $\ell \in L$ ,  $X_\ell = \bigcup_{t \in \ell} W_t$ , and put  $X = (X_\ell : \ell \in L)$ . We claim that  $(L, X)$  is a linear decomposition of  $G$  of width  $< \kappa$ . For condition (L1) is obviously satisfied. To prove (L2) let  $\ell_1 < \ell < \ell_2$  and let  $v \in X_{\ell_1} \cap X_{\ell_2}$ . Then  $v \in W_{t_1} \cap W_{t_2}$  for some  $t_1 \in \ell_1$  and some  $t_2 \in \ell_2$ . Let  $t = \inf(t_1, t_2)$ ; then  $t \in \ell$  and so  $W_t \subseteq X_\ell$ , but  $v \in W_t$  by (W2) and so  $v \in X_\ell$ , as required. To prove (L3) let  $I \subseteq L$  be a nonempty interval. Again, it is enough to show that  $\bigcap_{i \in I} X_i \subseteq X_{\sup(I)}$ . So let  $\ell = \sup I$  and let  $i_1 \in I$ ; we may assume that  $i_1 \neq \ell$ . Then there exists  $i_2 \in I$  such that  $i_1 \cap i_2 \subseteq \ell$ . Let  $t_j = \inf\{t \in V(T) : t \in i_j - i_{3-j}\}$  ( $j = 1, 2$ ) and let  $t = \inf(t_1, t_2)$ . Then  $\bigcap_{i \in I} X_i \subseteq X_{i_1} \cap X_{i_2} \subseteq \bigcup_{t' \leq t} W_{t'} \subseteq X_\ell$  using (W2), as desired.

Hence  $(L, X)$  is a linear decomposition. The statement about width follows easily, because  $\kappa$  is regular and each  $X_\ell$  is the union of  $< \kappa$  sets, each of cardinality  $< \kappa$ .  $\square$

(3.2) *Let  $\kappa$  be a cardinal and let  $G$  be a graph. If  $G$  admits a linear decomposition of width  $< \kappa$ , then  $G$  has no stoppage of order  $\geq \kappa$ .*

*Proof.* Suppose for a contradiction that  $(L, X)$  is a linear decomposition of  $G$  of width  $< \kappa$  and that  $\mathcal{S}$  is a stoppage in  $G$  of order  $\geq \kappa$ . For  $\ell \in L$  let  $C_\ell = \left( \bigcup_{\ell' \leq \ell} X_{\ell'}, \bigcup_{\ell' \geq \ell} X_{\ell'} \right)$ . We observe that  $C_\ell$  is a cut in  $G$  of order  $< \kappa$ . Let  $A$  be the set of all  $\ell \in L$  such that  $C_\ell \in \mathcal{S}$ . We deduce from the second stoppage axiom that if  $\ell \in A$  and  $\ell' \leq \ell$ , then  $\ell' \in A$ .

(1)  *$A$  has a maximum element.*

For let  $a = \sup A$ . We deduce from the third stoppage axiom that  $\left( \bigcup_{\ell < a} X_\ell, \bigcup_{\ell \geq a} X_\ell \right) \in \mathcal{S}$ . If  $\left( \bigcup_{\ell \leq a} X_\ell, \bigcup_{\ell \geq a} X_\ell \right) \notin \mathcal{S}$  then  $\left( \bigcup_{\ell \geq a} X_\ell, \bigcup_{\ell \leq a} X_\ell \right) \in \mathcal{S}$  by the first stoppage axiom, and yet  $\left( G \upharpoonright \bigcup_{\ell < a} X_\ell \right) \cup \left( G \upharpoonright \bigcup_{\ell \geq a} X_\ell \right) = G$ , contrary to the second axiom. Hence  $\left( \bigcup_{\ell \leq a} X_\ell, \bigcup_{\ell \geq a} X_\ell \right) \in \mathcal{S}$  and thus  $a \in A$ .

Similarly let  $B$  be the set of all  $\ell \in L$  such that  $\left( \bigcup_{\ell' \geq \ell} X_{\ell'}, \bigcup_{\ell' \leq \ell} X_{\ell'} \right) \in \mathcal{S}$ . Analogously,  $B$  has a minimum element, say  $b$ . It follows that  $a < b$  and that there is no  $\ell \in L$  with  $a < \ell < b$ . Hence  $\left( G \upharpoonright \bigcup_{\ell \leq a} X_\ell \right) \cup \left( G \upharpoonright \bigcup_{\ell \geq b} X_\ell \right) = G$ , contrary to the second stoppage axiom.  $\square$

(3.3) *Let  $\kappa$  be an infinite cardinal, let  $G$  be a graph and let  $H$  be a subgraph of  $G$  isomorphic to a subdivision of  $K_\kappa$ . Then  $G$  has a stoppage of order  $\kappa$ .*

*Proof.* Let  $V$  be the set of vertices of  $H$  which correspond to a vertex of  $K_\kappa$ . If  $(A, B)$  is a cut in  $G$  of order  $< \kappa$ , then exactly one of  $A, B$  contains  $V$ . Let  $\mathcal{S}$  be the set of all cuts  $(A, B)$  in  $G$  of order  $< \kappa$  such that  $V \subseteq B$ . Then it is easy to verify that the stoppage axioms are satisfied.  $\square$

Thus we have proved (1.5). Now we prove (1.6) which we restate.

(3.4) *There exists a graph which contains no subgraph isomorphic to a subdivision of  $K_{\aleph_\omega}$  and which has no linear decomposition of width  $< \aleph_\omega$ .*

*Proof.* Let  $T$  be the well-founded tree with every chain of order type  $\omega$  in which every element  $t \in V(T)$  has  $\aleph_\omega$  successors. For every  $t \in V(T)$  we choose a set  $M_t$  and an element  $m_t \in M_t$  in such a way that  $|M_t| = \aleph_{ht(t)}$  and  $M_t \cap M_{t'} = \emptyset$  for distinct  $t, t' \in V(T)$ . Let  $G$  be the simple graph with vertex set  $\bigcup_{t \in V(T)} M_t$  and such that for  $u, v \in V(G)$  there exists an edge in  $G$  with endpoints  $u, v$  if and only if  $u \neq v$ , and either  $u, v \in M_t$  for some  $t \in V(T)$ , or  $u \in M_t$  and  $v = m_{t'}$  for a successor  $t' \in V(T)$  of  $t \in V(T)$ , or  $v \in M_t$  and  $u = m_{t'}$  for a successor  $t' \in V(T)$  of  $t \in V(T)$ . For  $t \in V(T)$  let  $G_t$  be the subgraph of  $G$  induced by the set  $\bigcup_{t' \geq t} M_{t'}$ . Let  $W_{\text{root}(T)} = M_{\text{root}(T)}$ , and if  $t' \in V(T)$  is a successor of  $t \in V(T)$  let  $W_{t'} = M_t \cup M_{t'}$ . Let  $W = (W_t : t \in V(T))$ . It is easy to see that  $(T, W)$  is a well-founded tree-decomposition of  $G$  of width  $< \aleph_\omega$ , and hence  $G$  has no subgraph isomorphic to a subdivision of  $K_{\aleph_\omega}$  by (1.4).

It remains to be shown that  $G$  has no linear decomposition of width  $< \aleph_\omega$  either. So suppose for a contradiction that  $G$  has a linear decomposition  $(L, X)$  of width  $< \aleph_\omega$ . A *clique* in  $G$  is a subset  $M \subseteq V(G)$  such that every pair of distinct members of  $M$  are adjacent.

(1) *For every clique  $M$ , the set  $I(M) = \{\ell \in L : M \subseteq X_\ell\}$  is a non-empty closed interval in  $L$ .*

For  $v \in M$ ,  $I(\{v\})$  is a closed interval by (L2) and (L3), and by (L1) any two of these intervals meet. We deduce that  $\{\ell \in L : M \subseteq X_\ell\} = \bigcap_{v \in M} I(\{v\})$  is as desired.

For  $t \in V(T)$  let  $a(t) \leq b(t)$  be the endpoints of  $I(M_t)$ . Let  $t_0 = \text{root}(T)$  and assume that we have already constructed  $t_0, \dots, t_n$ . Let  $t_{n+1}$  be a successor of  $t_n$  such that  $M_{t_{n+1}} \cap (X_{a(t_n)} \cup X_{b(t_n)}) = \emptyset$ . Such a choice is possible since  $|X_{a(t_n)} \cup X_{b(t_n)}| < \aleph_\omega$  and there are  $\aleph_\omega$  such successors. This completes the inductive definition of  $t_0, t_1, \dots$

Now for each  $n \geq 0$ ,  $M = M_{t_n} \cup \{m_{t_{n+1}}\}$  is a clique, and so  $\emptyset \neq I(M) = I(M_{t_n}) \cap I(\{m_{t_{n+1}}\})$ . Yet  $a(t_n), b(t_n) \notin I(\{m_{t_{n+1}}\})$ , and hence  $I(\{m_{t_{n+1}}\}) \subseteq I(M_{t_n})$ . But  $I(M_{t_{n+1}}) \subseteq I(\{m_{t_{n+1}}\})$ , and so  $I(M_{t_{n+1}}) \subseteq I(M_{t_n})$ . By the completeness of  $L$  there exists  $\ell \in L$  such that  $\ell \in \bigcap_{n \geq 0} I(M_{t_n})$ , that is,  $M_{t_n} \subseteq X_\ell$  for all  $n \geq 0$ . But  $|M_{t_0} \cup M_{t_1} \cup \dots| = \aleph_\omega > |X_\ell|$ , a contradiction.  $\square$

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