Efficiently four-coloring planar graphs

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Abstract

An outline of a quadratic algorithm to 4-color planar graphs is presented, based upon a new proof of the Four Color Theorem. This algorithm improves a quartic algorithm of Appel and Haken.

1. Introduction.

Within the class of planar graphs, there is an interesting variation in complexity for the problem of coloring the graph in k colors, for different values of k. It is trivial to k-color the vertices of a planar graph in linear time (if possible) for $k \leq 2$ and $k \geq 6$. Garey, Johnson, and Stockmeyer [5] showed that it is NP-hard to 3-color the vertices of a planar graph (if possible), even if the problem is restricted to graphs where the maximum degree of a vertex is four.

An unsophisticated application of Heawood's proof of the Five Color Theorem [6] gives a quadratic algorithm to 5-color a planar graph. Linear algorithms to 5-color planar graphs can be found in [4, 9]. It seems worthy to digress for a moment to show how a modification of Heawood's original proof together with a 1904 theorem of Wernicke [10] can be used to obtain a simple linear-time 5-coloring algorithm.

For this paper, let a graph possibly contain multiple edges, but not loops. Let the degree of a vertex be the number of edges that are incident with it. Let the degree of a face be the number of edge-face incidences involving it (a cut-edge produces two such incidences with the same face). Let a plane graph be a *triangulation* if each of its faces has degree three. Let a plane graph be *normal* if each of its faces has degree at least three.

Wernicke, interested in the Four Color Theorem, proved (in the dual) that every normal plane graph of minimum degree five has a vertex of degree five which is adjacent to a vertex

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1. Research partially performed under a consulting agreement with Bellcore, partially supported by DIMACS Center, Rutgers University, New Brunswick, New Jersey 08903, USA, and partially supported by NSF grant no. DMS-8903132 and ONR grant no. N00014-92-J-1965.

2. Research supported by DIMACS and ONR grant no. N00014-93-1-0325.

3. Research partially performed under a consulting agreement with Bellcore, partially supported by DIMACS, and partially supported by NSF grant no. DMS-9303761 and ONR grant no. N00014-93-1-0325.

of degree at most six. Now consider the following linear 5-coloring algorithm. The input for the algorithm is a normal, plane graph G.

By Wernicke's Theorem, G has a vertex x which is either of degree at most 4, of degree 5, but adjacent to at most 4 vertices, or of degree 5, and adjacent to 5 vertices, one of which has degree at most 6.

If x has degree at most 4, or degree 5, but adjacent to at most 4 vertices, then let $H := G - x$. Otherwise, x has degree 5, and if $v_1, \ldots, v_5$ is the clockwise cyclic neighborhood of x, then these five vertices are pairwise distinct, and without loss of generality, $\deg(v_1) \leq 6$. Check to see if v_1 is adjacent to v_3 (which takes constant time since $\deg(v_1) \leq 6$). If not, let H be the plane graph obtained from $G - x$ by identifying v_1 and v_3 (the edge lists are just merged where the edges to x were; this takes constant time, noting that multiple edges are allowed); otherwise, by planarity, v_2 is not adjacent to v_4, and let H be the plane graph obtained from $G - x$ by identifying v_2 and v_4.

Apply the algorithm recursively to H, after deleting an edge from any face of degree 2 created (this takes constant time, as the only edges that could be involved are together with x in faces of degree at most four in G). A 5-coloring of H easily gives a 5-coloring of G in constant time.

Each iteration can be performed in constant time then, with the exception of finding the appropriate x, which can be done in amortized linear time by maintaining a stack of vertices of degree at most five.

The final coloring problem for planar graphs is 4-coloring. In 1989, Appel and Haken were able to devise a quartic algorithm to 4-color planar graphs from their proof of the Four Color Theorem. A quadratic algorithm to 4-color planar graphs has been obtained from the new proof of the Four Color Theorem by the authors. The improvement in complexity occurs by the avoidance of reducing bending 6-rings. This is described in the following section.

2. A Quadratic Four-Coloring Algorithm.

Define $V(G), E(G), F(G)$ to be the sets of vertices, edges, and faces of a plane graph G.

A normal plane triangulation G is said to be *nearly 7-connected* if

1. G has no *k-ring* for $k \in \{2, 3, 4\}$, where a k-ring is a cycle on k edges with at least 1 vertex in each of its interior and exterior;
2. G has no *non-trivial 5-ring*, which is a cycle on 5 vertices with at least 2 vertices in each of its interior and exterior;
3. G has no *bending 6-ring*, which is a cycle C on 6 vertices with at least 4 vertices in each of its interior and exterior, and there is a vertex not on C which is adjacent to three consecutive vertices of C.

If G satisfies condition 1 it is called *5-connected*, and if it satisfies conditions 1 and 2, it is called *internally 6-connected*. A graph is said to be *minimal* if it is a planar graph with the fewest vertices requiring five colors. In 1913, Birkhoff [3] showed that every minimal graph is internally 6-connected. In 1948, Bernhart [2] further showed that every minimal graph is nearly 7-connected.

Appel and Haken proved the Four Color Theorem by showing that every nearly 7-connected plane triangulation contains one of a set U_Z of 1528 reducible configurations (52 extra configurations not appearing on the published list are necessary for their 4-color algorithm to avoid immersion problems [see p. 255 in 1]). Their algorithm is, then, in an arbitrary plane triangulation G, to find either a k-ring for $k \leq 4$, a non-trivial 5-ring, a bending 6-ring, or a configuration in U_Z. Once finding one of these structures, their algorithm recurses on at most three smaller graphs, whose colorings are combined to give a coloring of G.

The structure that causes the most trouble algorithmically is the bending 6-ring; it is the only structure that requires the coloring of three smaller graphs. The new proof of the Four Color Theorem by the authors [8] shows that every internally 6-connected plane triangulation contains one of a set U of 633 reducible configurations. Thus the new algorithm finds either a k-ring for $k \leq 4$, a non-trivial 5-ring, or a configuration of U in a plane triangulation and recurses on at most two smaller graphs. Avoiding the other small rings would not improve the complexity of the algorithm.

In more detail, let a *near-triangulation* be a connected plane graph where every finite face has degree three. Then K is a *configuration* if K is a near triangulation $G(K)$ together with a function γ_K from $V(G(K))$ to the non-negative integers such that properties 1, 2, and 3 below hold. The property of a configuration being reducible will be discussed later. Let the *ring-size* of K be $\sum_v (\gamma_K(v) - \deg(v) - 1)$, summed over all vertices v incident with the infinite face such that $G(K) - v$ is connected.

1. for every vertex v, $G(K) - v$ has at most two components, and if there are two, then $\gamma_K(v) = \deg(v) + 2$;
2. for every vertex v, if v is not incident with the infinite region, then $\gamma_K(v) = \deg(v)$, and otherwise $\gamma_K(v) > \deg(v)$; and in either case, $\gamma_K(v) \geq 5$;
3. the ring-size of K is at least 2.

In a figure, for a vertex v, the value of $\gamma_K(v)$ will be indicated by a shape in the figure. The standard shapes are a solid dot if $\gamma_K(v) = 5$, a circle if $\gamma_K(v) = 7$, a square if $\gamma_K(v) = 8$, a triangle if $\gamma_K(v) = 9$, a pentagon if $\gamma_K(v) = 10$, and no shape if $\gamma_K(v) = 6$. An example of a configuration of ring size 14 appears in Figure 1.

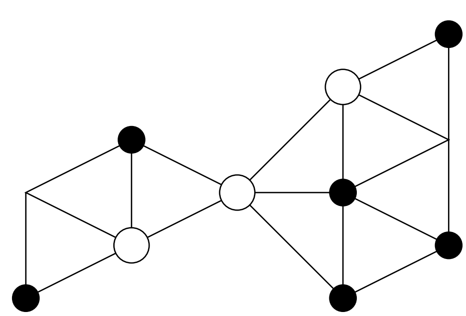

Figure 1. A configuration.

Let a configuration K be *weakly contained* in a plane graph G if there is a function f from $V(G(K))$ to $V(G)$ and a function g from the interior faces of $G(K)$ to $F(G)$ such that

1. for every $v \in V(G(K))$, $\deg(f(v)) = \gamma_K(v)$;

2. for every $v, w \in V(G(K))$, if v is adjacent to w in $G(K)$, then $f(v)$ is adjacent to $f(w)$ in G.
3. for every interior face $F = uvw$, $g(F) = f(u)f(v)f(w)$.

Also, let a configuration K be *strongly contained* in a plane graph G if K is weakly contained in G by means of functions f, g which also satisfy that

4. for every $v, w \in V(G(K))$, if $f(v) = f(w)$, then $v = w$.
5. for every $v, w \in V(G(K))$, if $f(v)$ is adjacent to $f(w)$ in G, then v is adjacent to w in $G(K)$.

Figure 2 shows the configuration of Figure 1 strongly contained in a plane graph, and also weakly, but not strongly contained in a plane graph.

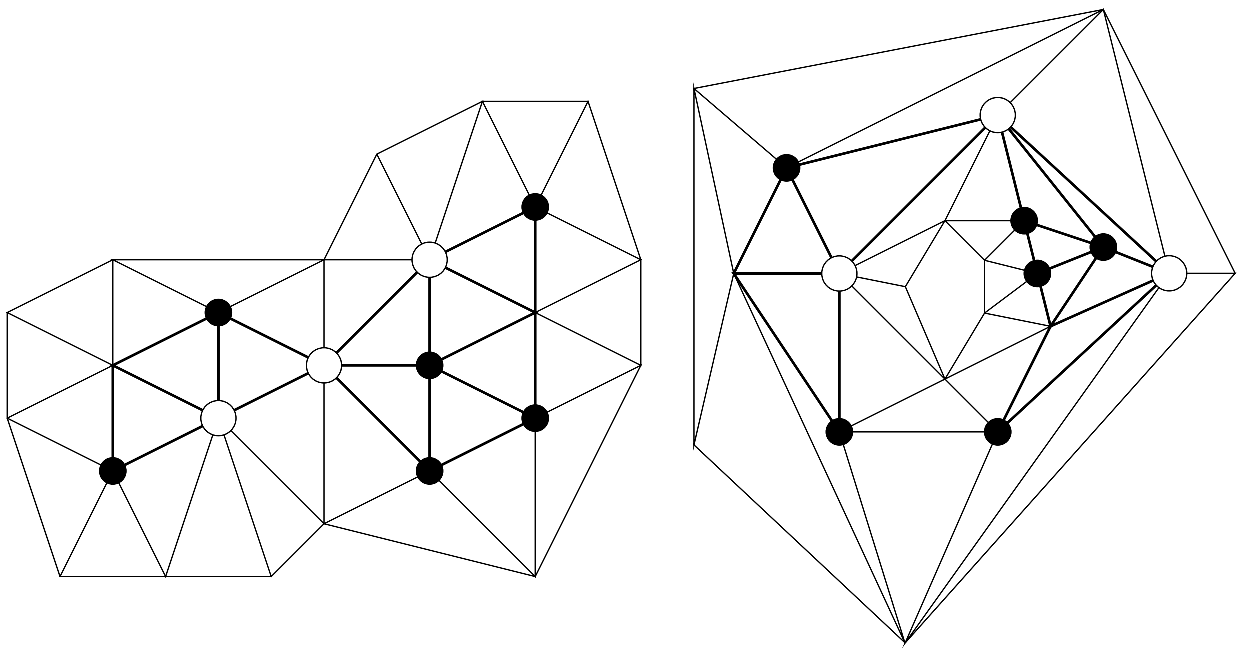

Figure 2. Examples of strong and weak containment.

The authors used the discharging method with 32 discharging rules to show that every plane triangulation G of minimum degree five weakly contains a configuration in U. This is formally stated as Theorem 1 below, which is followed by a sketch of the proof. For more details, see [8]. The proof of Theorem 1 is a very lengthy case analysis. It is written formally so that it can be verified by a computer in a few minutes or by a conscientious reader in a few months. The proof, as well as a paper explaining it and the computer program to verify it, is available by anonymous ftp from `ftp.math.gatech.edu` in the directory `pub/users/thomas/four`.

Theorem 1. *Every plane triangulation of minimum degree five weakly contains a configuration in U.*

For each vertex x of G, let charge$(x) = 10(6 - \deg(x))$. Since G is a triangulation, a simple manipulation of Euler's formula gives $\sum_{x \in V(G)}$ charge$(x) = 120$. The particular value 120 is not important, just that it is positive. Without modifying the charges, this simply says that every plane triangulation has a vertex of degree at most five. The discharging method works by sending the positive charge away from these vertices, and then, as can be seen by recounting the modified charges, there must still remain positive charge somewhere. Examining the rules that are used to move the charge shows that in each possibility of a vertex x having positive modified charge, the graph has one of the configurations of U present within the second neighborhood of x (meaning the vertices distance at most two from x). Three of the 32 rules that the authors used are as follows:

1. For each edge st such that $\deg(s) = 5$, send a charge of 2 from s to t.
2. For each triangle stu such that $\deg(s) \leq 6$, $\deg(t) \geq 7$, and $\deg(u) = 5$, send a charge of 1 from s to t.
3. For each pair of triangles stu, suv such that $\deg(s) \leq 6$, $\deg(t) \geq 6$, $\deg(u) \leq 6$, and $\deg(v) = 5$, send a charge of 1 from s to t.

Each of the 32 rules sends a charge of 1 (or 2 in just Rule 1) from a source s to a sink t along the edge st dependent only upon the degrees of certain vertices distance at most two from each of s and t. In the 1960s, when it was as yet uncertain that the method of reducible configurations would be able to solve the Four Color Problem, Heesch [7] conjectured that the discharging method could be used to solve it, and further, that the modified charge of a vertex x could be determined by only examining vertices distance at most two from x. The authors have verified this conjecture. This property of the rules is the one that forces any weakly contained configuration K found via discharging to be entirely within the second neighborhood of the vertex whose charge is positive. This in turn shows that if K is not strongly contained in the graph, then either a k-ring for $k \leq 4$ or a non-trivial 5-ring is present in the graph.

This is more easily seen using a property of U that was accidental. It turns out that every configuration K of U has diameter at most four; i.e. that each pair of vertices of $G(K)$ have distance at most four in $G(K)$. Using this bound on the diameter, it is easily seen that if K is weakly, but not strongly contained in G, then G has a k-ring for $k \leq 4$ or a non-trivial 5-ring.

Now the reducibility of configurations will be discussed. Instead of giving a formal definition of reducibility (which can be found in [8]), the properties of reducibility that are needed for the algorithm will be stated.

Given a coloring Ψ of a graph G, a pair P of colors, and a vertex x of G colored a color in P, let $H(P)$ be the graph induced by the vertices colored a color in P, and let $H(P, x)$ be the component of $H(P)$ containing x. Let the coloring Ψ' of G obtained from Ψ by recoloring each vertex v in $H(P, x)$ with the color in $P \setminus \{\Psi(v)\}$ be called the coloring which is *Kemped from* Ψ *at* x *by* P. This process of recoloring to get from Ψ to Ψ' is called a *Kempe*.

Let K be a configuration of ring-size r strongly contained in a triangulation G. Let H be the graph obtained from G by deleting the image of $G(K)$; H has a face with a facial walk of length r where the copy of $G(K)$ used to be; call R the vertices incident with this face. Given a coloring Ψ of H with colors in $\{1, 2, 3, 4\}$, then Ψ *can be Kemped into* a coloring Φ of H if there are an integer $j \geq 0$ and colorings $\Phi_0, \Phi_1, \ldots, \Phi_j$ of H such that

$\Phi_0 = \Psi$, $\Phi_j = \Phi$, and for $i \in \{1, \ldots, j\}$, Φ_i is Kemped from Φ_{i-1} at some vertex of R by some pair of colors in $\{1, 2, 3, 4\}$.

Finally, a configuration K of ring-size r is *reducible* if for every triangulation G strongly containing K, there is a set T of at most four edges of G, each incident with an image of a vertex of $G(K)$, such that if J is obtained from G by contracting the edges of T, then J is loopless, and for every coloring Ψ of J, the coloring of $G \setminus G(K)$ induced by Ψ can be Kemped into a coloring Φ which is extendable into a coloring of G. Moreover, if Ψ is given, Φ can be found by performing at most 3^{2r} Kempes.

Remark: The restriction on the size of T is not important for the algorithm. Also, it was easier for the presentation here to let T depend on G. This does not appear in the true definition of reducibility; the edges are chosen from looking at K only, not G. In fact, the true definition includes no reference to a graph G at all. Checking whether a configuration is reducible only requires the manipulation of colorings of R (whose size is at most r, regardless of G) that do not extend into $G(K)$. This is a finite problem.

It has been claimed that the 633 configurations of U previously mentioned were all reducible. Many configurations of small ring-size have been shown to be reducible by hand [see, e.g. 3], but most configurations of large ring-size have been shown to be reducible by means of a computer program. The C program that the authors used, as well as the configurations in appropriate form for input and a paper explaining the details of the program, is available by anonymous ftp from `ftp.math.gatech.edu` in the directory `pub/users/thomas/four`.

Now the algorithm can be described.

Algorithm 1.

The input to the algorithm will be a normal plane graph G with n vertices. The output will be a coloring of the vertices of G with four colors.

If $n \leq 4$, just color each vertex a different color. Clearly this uses only constant time.

If G has a face F of degree at least four, then by planarity, there are two non-adjacent vertices x, y incident with F. Create H from G by identifying x and y into a vertex z, and recurse on H. Color G by coloring each of x, y the color of z, and the other vertices receive the colors they received in H.

If G has a vertex x of degree $k \leq 4$, then the circuit C surrounding it is a k-ring. Go to the k-ring analysis below.

Otherwise G has minimum degree five. By Theorem 1, G must weakly contain a configuration in U. Find a configuration K in U that is weakly contained in G. There are several linear algorithms to perform this, as each vertex of a configuration in U has degree at most 11. If K is not strongly contained in G, then let C be a k-ring for $k \leq 4$ or a non-trivial 5-ring in G. Go to the k-ring analysis below.

Otherwise K is strongly contained in G. Suppose K is a configuration of ring-size r. Every configuration in U has $r \leq 14$. Let H be the graph obtained from G by deleting the image of $G(K)$. Let R be the set of vertices, and T be a set of edges as in the definition of reducibility, and then let J be the graph obtained from G by contracting T. Recurse on J. This induces a coloring Ψ of H. Kempe the vertices of R until a coloring Φ is found that extends into K. Since K is reducible, Φ can be found by performing at most 3^{2r} Kempes, each of which takes linear time.

Given a ring R which is either a k-ring for $k \leq 4$ or a non-trivial 5-ring, a procedure developed by Birkhoff [3] can be used. Let E, I be the subgraphs of G on the exterior and interior of R. First form a suitable graph H_1 from G by deleting E, and performing simple operations to triangulate it (see one of [1, 3, 8] for details). Recurse on H_1. This induces a coloring of $G \setminus E$. Then form a suitable graph H_2 from G by deleting I, and performing simple operations (which in this case depend on the coloring of H_1) to triangulate it. Recurse on H_2. This induces a coloring of $G \setminus I$. Birkhoff proved that the colorings of $G \setminus E$ and $G \setminus I$ can be Kemped to match on R. For the analysis, it is important to know that the simple operations performed to create H_1 and H_2 are such that $|V(H_1)| < n$, $|V(H_2)| < n$, and $|V(H_1)| + |V(H_2)| \leq n + 6$.

Theorem 2. *Algorithm* 1 4*-colors a plane graph in quadratic time.*

Indeed, the algorithm calls itself at most $O(n)$ times and as shown within the description, each iteration takes linear time. Thus the overall running time is $O(n^2)$. □

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