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A Menger-like Property of Tree-Width: The Finite Case*

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The notion of tree-width was introduced by Robertson and Seymour. A graph has tree-width $\leqq w$ if it admits a tree-decomposition of tree-width $\leqq w$. We prove here that if G is finite and has tree-width $\leqq w$ then it admits a tree-decomposition of tree-width $\leqq w$ which satisfies a certain Menger-like condition. This result will be used in a future paper on well-quasi-ordering infinite graphs of bounded tree-width. © 1990 Academic Press, Inc.

1. INTRODUCTION

In this paper graphs are finite unless stated otherwise, and may have loops and multiple edges. A graph is a *minor* of another if the first can be obtained by contraction from a subgraph of the second. A *tree-decomposition* of a graph G is a pair (T, X), where T is a tree and $X = (X^t : t \in V(T))$ is a family of sets such that

(W1) $\bigcup_{t \in V(T)} X^t = V(G)$,

(W2) for every edge e of G there is $t \in V(T)$ such that e has both its ends in X^t,

(W3) whenever $t, t', t'' \in V(T)$ and t' is on the path between t and t'' then $X^t \cap X^{t''} \subseteq X^{t'}$.

The *tree-width of the tree-decomposition* is

$$\max_{t \in V(T)} (|X^t| - 1).$$

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e *tree-width of a graph G* is the least w such that G has a tree-decomposi-
n of tree-width w.

It can be proved that forests have tree-width $\leqslant 1$, series-parallel graphs ve tree-width $\leqslant 2$, for $n \geqslant 1$ the complete graph K_n has tree-width $n-1$, d for $n \geqslant 2$ the $n \times n$ grid (i.e., the adjacency graph of the $n \times n$ essboard) has tree-width n. The chromatic number of G is $\leqslant w+1$ if G s tree-width w and if G is k-connected then its tree-width is at least k.

Robertson and Seymour proved the following three theorems:

THEOREM 1 [2]. *For every planar graph H there is a number w such that ery graph with no minor isomorphic to H has tree-width $\leqq w$.*

THEOREM 2 [1]. *If the tree-width of G is at most w, then G admits a e-decomposition (T, X) of tree-width $< 3.2^w$ such that*

(*) *for any $t_1, t_2 \in V(T)$ and any $k > 0$, either there are k disjoint ths, each between X^{t_1} and X^{t_2}, or else there is a t on the path between t_1 d t_2 in T such that $|X^t| < k$.*

THEOREM 3 [1]. *If $G_1, G_2, \ldots$ is an infinite sequence of graphs all with e-width $\leqq w$, where w is some integer, then there exist i, j such that $i < j$ d G_i is isomorphic to a minor of G_j.*

Theorem 2 is a technical tool for proving Theorem 3 and combining eorems 1 and 3 we get the following.

THEOREM 4 [1]. *If H is a fixed finite planar graph and $G_1, G_2, \ldots$ is an inite sequence of graphs with no minor isomorphic to H, then there exist such that $i < j$ and G_i is isomorphic to a minor of G_j.*

Our aim is to extend Theorem 4 to infinite graphs (but with H still ite). This will be done using the following theorems, which will be oved in future papers.

THEOREM 1′ [4]. *If G is an infinite graph all of whose finite subgraphs ve tree-width $\leqq w$, then G has tree-width $\leqq w$.*

THEOREM 2′ [3]. *If G is an infinite graph of tree-width at most w, then admits a tree-decomposition (T, X) of tree-width at most w, which isfies (*).*

THEOREM 3′ [5]. *If $G_1, G_2, \ldots$ is an infinite sequence of not necessarily ite graphs all with tree-width $\leqslant w$, where w is some integer, then there ist i, j such that $i < j$ and G_i is isomorphic to a minor of G_j.*

In this paper we give a new proof of Theorem 2 and we improve the upper bound for the tree-width of the desired tree-decomposition from $3.2^w - 1$ to the best possible value w. In fact we prove a stronger result which will be needed for the proof of Theorem 2′ in [3].

We shall adopt the following conventions. $\mathbb{N}$ denotes the set of positive integers. If G is a graph and $A \subseteq V(G)$ then $G \restriction A$ denotes the subgraph induced by A. A *path* is a nonempty set of vertices with the usual property, "repeated" vertices not being allowed. A path has two *ends* (which are equal for a one-vertex path), and we say that a path *connects* its ends. A set $A \subseteq V(G)$ *separates* $X, Y \subseteq V(G)$ if every path in G from X to Y uses a vertex from A. The minimum of the cardinalities of the subsets of $V(G)$ which separate X from Y will be denoted by $s(X, Y)$. If T is a tree and $t_1, t_2 \in V(T)$ then $[t_1, t_2]$ will denote the path between t_1 and t_2 (so that $\{t_1, t_2\} \subseteq [t_1, t_2] \subseteq V(T)$) and we shall write

$$(t_1, t_2\rangle = \{t \in V(T) : t_2 \in [t, t_1]\}.$$

If there might be doubt as to the tree in which these notations are interpreted, it will be indicated by a subscript, e.g., $[t_1, t_2]_T$.

2. Main Result

LEMMA. *Let (T, X) be a tree-decomposition of a graph G and let P be a path in G such that P meets both X^{t_1} and X^{t_2}, where $t_1, t_2 \in V(T)$. Then P meets each X^t for $t \in [t_1, t_2]$.*

Proof. Assume the contrary: let P meet X^{t_1} and X^{t_2}, but not X^t for some $t \in [t_1, t_2]$. Then for any $x \in P$ there is by (W3) a unique $t_x \in V(T)$ adjacent to t such that

$$x \in \bigcup \{X^s : s \in (t, t_x\rangle\}.$$

Since the ends of P have this t_x different, there is an edge e of P with ends u, v such that $t_u \neq t_v$. By (W2) e has both ends in some X^r, but

$$r \in (t, t_u\rangle \cap (t, t_v\rangle = \varnothing,$$

a contradiction which proves the lemma. ▮

Let us fix an integer w and a graph G. We are interested in tree-decompositions (T, X) of G of tree-width $\leqslant w$, i.e. in those tree-decompositions which satisfy

(W4) $|X^t| \leqslant w+1$ for any $t \in V(T)$.

e introduce the following condition, which is obviously stronger an (*).

(W5) For every quadruple $t_1 \in V(T)$, $X_1 \subseteq X^{t_1}$, $t_2 \in V(T)$, $X_2 \subseteq X^{t_2}$ ch that $s(X_1, X_2) < \min(|X_1|, |X_2|)$ there exists $t \in [t_1, t_2]$ such that $^t| \leqslant s(X_1, X_2)$.

y the Lemma, the final inequality in (W5) could in fact be replaced by equality.)

Now the main result of this paper reads as follows.

THEOREM 5. *If G is a graph of tree-width $\leqslant w$, then it admits a tree-composition satisfying* (W1)–(W5).

Proof. Let (T, X) be a tree-decomposition of G. By an (n, d)-*cell* in $, X)$ we mean any component of $T \upharpoonright \{t \in V(T): |X^t| \geqslant n\}$ which has at st d vertices. Let us remark that if K is an (n, d)-cell in (T, X) and $n = n'$, $\geqslant d'$, then K is an (n', d')-cell as well. The set of (n, d)-cells in (T, X) will denoted by $C(T, X, n, d)$. The *size* of a tree-decomposition (T, X) is the nily of numbers

$$(a_{n,d}: (n, d) \in \mathbb{N} \times \mathbb{N}), \tag{a}$$

ere $a_{n,d}$ is the number of (n, d)-cells in (T, X). Sizes are ordered icographically, i.e., if

$$(b_{n,d}: (n, d) \in \mathbb{N} \times \mathbb{N}) \tag{b}$$

he size of another tree-decomposition (R, Y) of the graph G, we say that is greater than (b) if there are n, d such that $a_{n,d} > b_{n,d}$ and $a_{m,l} = b_{m,l}$ enever $m - 1/2^l > n - 1/2^d$.

If (a) is the size of a tree-decomposition (T, X) of the graph G, then $_d \leqslant a_{n',d'}$ whenever $n \geqslant n'$ and $d \geqslant d'$. Hence the lexicographical ordering t introduced is well-founded on the set of sizes of tree-decompositions G.

Let us take a tree-decomposition (T, X) of G which satisfies (W1)–(W4) d which has the size minimal. To complete the proof of the theorem it sufficient to show that (T, X) satisfies (W5). This is done in the rest of paper.

3. PROOF OF THE FACT THAT (T, X) SATISFIES (W5)

Suppose that (T, X) does not satisfy (W5). Then at least one quadruple $, t_2, X_1, X_2)$ *violates* (W5) in the sense that $t_1, t_2 \in V(T)$, $X_1 \subseteq X^{t_1}$,

$X_2 \subseteq X^{t_2}$, $s(X_1, X_2) < \min(|X_1|, |X_2|)$, and $|X^t| > s(X_1, X_2)$ for every $t \in [t_1, t_2]$. Let m be the minimum of $s(X_1, X_2)$ over all quadruples (t_1, t_2, X_1, X_2) which violate (W5). Let a quadruple (t_1, t_2, X_1, X_2) which violates (W5) and has $s(X_1, X_2) = m$ be chosen so that

(A1) the path $[t_1, t_2]_T$ is as short as possible.

Let a set $A \subseteq V(G)$ be chosen so that

(A2) A separates X_1 and X_2,

(A3) $|A| = s(X_1, X_2) = m$,

and, subject to this,

(A4) $\sum_{x \in A} d_x$ is minimal,

where

$$d_x = \min\{\operatorname{dist}_T(t, t'): x \in X^t, t' \in [t_1, t_2]\}.$$

We can find sets F_1, F_2 such that

(F1) F_1, F_2, A are pairwise disjoint,

(F2) $F_1 \cup F_2 \cup A = V(G)$,

(F3) A separates F_1 and F_2,

(F4) $X_i \subseteq F_i \cup A$ $(i = 1, 2)$.

By Menger's theorem there are m disjoint paths $P_1, ..., P_m$, each between X_1 and X_2. Each P_i uses exactly one vertex from A, say x_i. Then $A = \{x_1, ..., x_m\}$.

Let R_1, R_2 denote two isomorphic copies of the tree T and $\xi_i: V(R_i) \to V(T)$ the corresponding isomorphisms. Let $r_{i,j} = \xi_i^{-1}(t_j)$ $(i, j = 1, 2)$. Assume that $V(R_1) \cap V(R_2) = \varnothing$ and define an ordered pair (R, Y), which we shall prove to be a tree-decomposition of G, by

$$V(R) = V(R_1) \cup V(R_2),$$
$$E(R) = E(R_1) \cup E(R_2) \cup \{\{r_{1,2}, r_{2,1}\}\},$$
$$Y = (Y^r: r \in V(R)),$$

where

$$Y^r = (F_i \cap X^{\xi_i(r)}) \cup A^r \qquad \text{for} \quad r \in V(R_i)$$

and A^r is the set of all $x_p \in A \cap \bigcup\{X^t: t \in (t_{3-i}, \xi_i(r)\rangle_T\}$ such that the corresponding path P_p includes at least one element of $(F_{3-i} \cup A) \cap X^{\xi_i(r)}$. Let us note the following:

(N1) $X_i \subseteq (F_i \cup A) \cap X^{t_i}$ $(i=1, 2)$,

(N2) $A \subseteq Y^{r_{3-i,i}}$ $(i=1, 2)$,

(N3) $X^{\xi_i(r)} \cap (F_i \cup A) \subseteq Y^r$ for $r \in V(R_i)$ $(i=1, 2)$,

(N4) $(F_i \cup A) \cap P_p$ is a subpath of P_p for $p=1, ..., m$ $(i=1, 2)$.

I1) follows from (F4), (N2) from

$$\varnothing \neq P_p \cap X_i \subseteq P_p \cap (F_i \cup A) \cap X^{t_i} \qquad (p=1, ..., m)$$

d the fact that $(t_i, \xi_{3-i}(r_{3-i,i})\rangle_T = (t_i, t_i\rangle_T = V(T)$. (N3) follows from e definition of Y. (N4) is an immediate consequence of (F3) and the fact at $P_p \cap A = \{x_p\}$.

We claim that (R, Y) satisfies (W1)–(W4). To prove (W1) we observe at

$$\bigcup_{r \in V(R)} Y^r = \bigcup_{r \in V(R_1)} Y^r \cup \bigcup_{r \in V(R_2)} Y^r$$
$$\supseteq ((F_1 \cup A) \cap \bigcup_{r \in V(R_1)} X^{\xi_1(r)}) \cup ((F_2 \cup A) \cap \bigcup_{r \in V(R_2)} X^{\xi_2(r)})$$
$$= \bigcup_{t \in V(T)} X^t = V(G)$$

ing (N3) and (F2).

To prove (W2) let e be an edge of G. Since (T, X) satisfies (W2), e has th its ends in some X^t for $t \in V(T)$. Since A separates F_1 and F_2 we may fely assume that e has both ends in $F_1 \cup A$. Then e has both ends in

$$X^t \cap (F_1 \cup A) \subseteq Y^{\xi_1^{-1}(t)}$$

(N3) and so (R, Y) satisfies (W2).

To prove (W3) let $s, s', s'' \in V(R)$, let s' be on the path between s and s'' R, and let $v \in Y^s \cap Y^{s''}$. By symmetry it suffices to discuss two cases.

Let first $s, s', s'' \in V(R_1)$. If $v \notin A$ then

$$v \in F_1 \cap X^{\xi_1(s)} \cap X^{\xi_1(s'')} \subseteq F_1 \cap X^{\xi_1(s')} \subseteq Y^{s'}$$

(W3) applied to (T, X) and (N3). If $v \in A$ then $v \in A^s \cap A^{s''}$ by (F1) and nce $v = x_p$ for some p such that

$$P_p \cap (F_2 \cup A) \cap X^{\xi_1(s)} \neq \varnothing \qquad \text{and} \qquad P_p \cap (F_2 \cup A) \cap X^{\xi_1(s'')} \neq \varnothing.$$

us $P_p \cap (F_2 \cup A) \cap X^{\xi_1(s')} \neq \varnothing$ by (N4) and the Lemma. If $\xi_1(s'') \in$ $_1(s), t_2]_T$ or $\xi_1(s) \in [\xi_1(s''), t_2]_T$ then

$$x_p \in A^s \cap A^{s''} \subseteq A \cap \bigcup\{X^t : t \in (t_2, \xi_1(s)\rangle_T\} \cap \bigcup\{X^t : t \in (t_2, \xi_1(s'')\rangle_T\}$$
$$\subseteq A \cap \bigcup\{X^t : t \in (t_2, \xi_1(s')\rangle_T\}$$

by the fact that

$$(t_2, \xi_1(s)\rangle_T \cap (t_2, \xi_1(s'')\rangle_T \subseteq (t_2, \xi_1(s')\rangle_T.$$

Hence $v = x_p \in Y^{s'}$. If $\xi_1(s'') \notin [\xi_1(s), t_2]_T$ and $\xi_1(s) \notin [\xi_1(s''), t_2]_T$ then

$$v = x_p \in A^s \cap A^{s''} \subseteq A \cap \bigcup\{X^t : t \in (t_2, \xi_1(s)\rangle_T\} \cap \bigcup(X^t : t \in (t_2, \xi_1(s'')\rangle_T\}$$
$$\subseteq A \cap X^{\xi_1(s')} \subseteq Y^{s'}$$

by (W3) applied to (T, X) and (N3).

Let second $s, s' \in V(R_1)$, $s'' \in V(R_2)$. Then $s' \in [r_{1,2}, s]_R$ and we have

$$v \in Y^s \cap Y^{s''} = A^s \cap A^{s''} \subseteq Y^s \cap A \subseteq Y^s \cap Y^{r_{1,2}} \subseteq Y^{s'}$$

by (F1), (N2), and the first case. This completes the proof of (W3).

Moreover, for $r \in V(R_i)$ the mapping $Y^r \to X^{\xi_i(r)}$ defined by

$$v \to \begin{cases} v & \text{if} \quad v \in F_i \cap X^{\xi_i(r)} \\ \text{some element of } P_p \cap (F_{3-i} \cup A) \cap X^{\xi_i(r)} & \text{if} \quad v = x_p \end{cases}$$

is 1–1, and so

(L1) $|X^{\xi_i(r)}| \geqslant |Y^r|$ for $r \in V(R_i)$ $(i=1, 2)$.

From (W4) applied to (T, X) and (L1), it follows that (R, Y) satisfies (W4). Thus (R, Y) satisfies (W1)–(W4). We claim that its size is less than that of (T, X). To prove this put

$$k := \max(\{m+1\} \cup \{|X^t| : t \in V(T), |X^t| > \max(|Y^{\xi_1^{-1}(t)}|, |Y^{\xi_2^{-1}(t)}|)\}).$$

We shall now prove that

(L2) If $t \in V(T)$, $r_1 \in V(R_1)$, and $r_2 \in V(R_2)$ are such that $t = \xi_1(r_1) = \xi_2(r_2)$ and $|Y^{r_i}| > m$ $(i=1, 2)$, then $|X^t| > |Y^{r_i}|$ $(i=1, 2)$.

(L3) $|Y^{r_{3-i,i}}| < k$ $(i=1, 2)$.

(L4) If K is an $(n, 1)$-cell in (R, Y) and $n \geqslant k$, then $V(K) \subseteq V(R_1)$ or $V(K) \subseteq V(R_2)$.

(L5) There exist d and a (k, d)-cell K in (T, X) such that for any (k, d)-cell L in (R, Y)

$$\{\xi_i(v) : v \in V(L)\} \cap V(K) = \varnothing,$$

where i is such that $V(L) \subseteq V(R_i)$.

To prove (L2) let $t = \xi_1(r_1) = \xi_2(r_2)$ and $|X^t| \leqslant |Y^{r_1}|$, hence $|X^t| = |Y^{r_1}|$ by (L1). By symmetry it is sufficient to show $|Y^{r_2}| \leqslant m$. We claim

$X^t \cap F_2 = \varnothing$. Suppose not and let $X^t \cap F_2 = \{v_1, ..., v_p\}$. It follows from $X^t| = |Y^{r_1}|$ that the mapping introduced in the proof of (L1) is onto for $= r_1$ and $i = 1$. Hence each v_i $(i = 1, ..., p)$ belongs to some P_j $(j = 1, ..., m)$ nd each P_j $(j = 1, ..., m)$ contains at most one element of $X^t \cap (F_2 \cup A)$. Ve may assume the notation is chosen in such a way that $v_i \in P_i$ $i = 1, ..., p)$. Using this and the above surjectivity we further have $_i \in Y^{r_1} - X^t$ $(i = 1, ..., p)$. Consequently $x_i \in \bigcup\{X^s : s \in (t_2, t\rangle_T\}$ $(i = 1, ..., p)$. et X_0 be either X^t or X_1. We claim that $A' = \{v_1, ..., v_p, x_{p+1}, ..., x_m\}$ eparates X_0 and X_2. So assume that P is a path connecting $v \in X_0$ and $_2 \in X_2$ which avoids A'. This P must use some $x_j (1 \leqslant j \leqslant p)$; for $X_0 = X_1$ his follows from (A2), and for $X_0 = X^t$ it follows from (F3) by the observa-ion that $v \in X^t - A' \subseteq F_1 \cup A$ and $v_2 \in X_2 \subseteq F_2 \cup A$ by (F4). Take x_j $1 \leqslant j \leqslant p)$ and a subpath P' of P such that P' joins $\{x_j\}$ and X_2 and lies n $F_2 \cup \{x_j\}$. Let $s' \in (t_2, t\rangle_T$ be such that $x_j \in X^{s'}$. Hence $P' \cap X^{s'} \neq \varnothing \neq$ $P' \cap X_2 \subseteq P' \cap X^{t_2}$, and so, by the Lemma,

$$\varnothing \neq P' \cap X^t \subseteq P \cap (F_2 \cup \{x_j\}) \cap X^t = P \cap F_2 \cap X^t = P \cap \{v_1, ..., v_p\} = \varnothing,$$

contradiction. Thus A' separates X_1 and X_2 as well as X^t and X_2. If $\in [t_1, t_2]_T - \{t_1\}$ then letting $t_1' = t$, $t_2' = t_2$, $X_1' = X^t$, $X_2' = X_2$ we get a ontradiction to (A1). If $t \notin [t_1, t_2]_T - \{t_1\}$ then using the fact that $_i \in \bigcup\{X^s : s \in (t_2, t\rangle_T\} - X^t$ we see that $d_{v_i} < d_{x_i}$ $(i = 1, ..., p)$. Hence $\sum_{x \in A'} d_x < \sum_{x \in A} d_x$, which contradicts (A4), and we conclude $X^t \cap F_2 = \varnothing$. Hence $Y^{r_2} \subseteq A$, consequently $|Y^{r_2}| \leqslant m$ and (L2) is proved.

To prove (L3) assume the contrary, i.e., $m < k \leqslant |Y^{r_{3-i,i}}|$. By (N1) and N3)

$$X_i \subseteq (F_i \cup A) \cap X^{t_i} \subseteq Y^{r_{i,i}},$$

nd therefore $m < |X_i| \leqslant |Y^{r_{i,i}}|$. From (L2) and the definition of k we get

$$k \leqslant |Y^{r_{3-i,i}}| \leqslant \max(|Y^{r_{3-i,i}}|, |Y^{r_{i,i}}|) < |X^{t_i}| \leqslant k,$$

contradiction, which proves (L3).

(L4) is an easy consequence of (L3). To prove (L5), observe that $X^t| > m$ for every $t \in [t_1, t_2]_T$ since (t_1, t_2, X_1, X_2) violates (W5), and o some $(m+1, 1)$-cell in (T, X) contains $[t_1, t_2]_T$. If $k = m+1$, we let K be this $(m+1, 1)$-cell; if $k > m+1$ let $t \in V(T)$ be such that $= |X^t| > \max(|Y^{\xi_1^{-1}(t)}|, |Y^{\xi_2^{-1}(t)}|)$, and let K be that $(k, 1)$-cell in (T, X) vhich contains t. We put $d = |V(K)|$ in both cases. Now let L be a $k, d)$-cell in (R, Y), and $i \in \{1, 2\}$ be such that $V(L) \subseteq V(R_i)$. By (L1) $L' = \{\xi_i(v) : v \in V(L)\}$ is contained in some (k, d)-cell in (T, X). By the hoice of d either $L' \cap V(K) = \varnothing$ or $L' = V(K)$. The latter case is impossible by (L3) if $k = m+1$ and by the property of t if $k > m+1$. This proves (L5).

Now we are ready to show that the size of (T, X) is greater than that of (R, Y). To this end let (a) be the size of (T, X) and (b) the size of (R, Y). We define a mapping Φ, which maps (n, d)-cells in (R, Y) $(n \geqslant k)$ to (n, d)-cells in (T, X). Let K be an (n, d)-cell in (R, Y) $(n \geqslant k)$: then $V(K) \subseteq V(R_i)$ $(i = 1$ or $2)$ by (L4). The set $\{\xi_i(v) : v \in V(K)\}$ is a subset of some (n, d)-cell L in (T, X) by (L1). We set $\Phi(K) := L$. Let now (n, d) be such that Φ does not induce a bijection of $C(R, Y, n, d)$ onto $C(T, X, n, d)$ but induces a bijection of $C(R, Y, n', d')$ onto $C(T, X, n', d')$ whenever $n' - 1/2^{d'} > n - 1/2^d$. By (L5) such an (n, d) exists and $n \geqslant k$. We claim that Φ is 1–1 on the set of (n, d)-cells in (R, Y). To prove this assume that there are distinct (n, d)-cells K_1, K_2 in (R, Y) such that $\Phi(K_1) = \Phi(K_2)$. Clearly $V(K_1) \cap V(K_2) = \varnothing$. Assume without loss of generality that K_1 is chosen in such a way that

$$|V(K_1)| = \max\{|V(K)| : K \text{ is an } (n, d)\text{-cell in } (R, Y) \text{ and } \Phi(K) = \Phi(K_1)\}.$$

If $|V(K_1)| < |V(\Phi(K_1))|$ then we put $d' = |V(\Phi(K_1))|$. Let K be an (n, d)-cell in (R, Y) such that $\Phi(K) = \Phi(K_1)$: then

$$|V(K)| \leqslant |V(K_1)| < |V(\Phi(K_1))| = d',$$

and so K cannot be an (n, d')-cell. Thus no (n, d')-cell in (R, Y) is mapped by Φ onto $\Phi(K_1)$, which contradicts the choice of (n, d). Hence $|V(K_1)| = |V(\Phi(K_1))|$. By (L4) we may assume $V(K_1) \subseteq V(R_1)$: then $V(K_2) \subseteq V(R_2)$. Let $r_2 \in V(K_2)$ be arbitrary. We have $\xi_2(r_2) \in V(\Phi(K_1)) = \{\xi_1(v) : v \in V(K_1)\}$. Hence there is $r_1 \in V(K_1)$ such that $\xi_1(r_1) = \xi_2(r_2)$. By (L2) and the definition of k

$$k \leqslant n \leqslant |Y^{r_i}| < |X^{\xi_i(r_i)}| \leqslant k \qquad (i = 1, 2),$$

a contradiction. Hence Φ is 1–1 on the set of (n, d)-cells in (R, Y), and so the choice of (n, d) implies that $a_{n,d} > b_{n,d}$ and $a_{n',d'} = b_{n',d'}$ when $n' = 1/2^{d'} > n - 1/2^d$.

Thus the size of (T, X) is greater than that of (R, Y), a contradiction to the choice of (T, X). Hence (T, X) satisfies (W5) and the theorem is proved. ∎

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References

N. Robertson and P.D. Seymour, Graph minors, IV. Tree-Width and well-quasi-ordering, *J. Combin. Theory Ser. B*, to appear.

N. Robertson and P. D. Seymour, Graph minors, V. Excluding a planar graph, *J. Combin. Theory Ser. B* **41** (1986), 92–114.

I. Kříž and R. Thomas, The Menger-like property of the tree-width of infinite graphs, submitted for publication.

R. Thomas, The tree-width compactness theorem for hypergraphs, submitted for publication.

R. Thomas, Well-quasi-ordering infinite graphs with forbidden finite planar minor, *Trans. Amer. Math. Soc.* **312**, No. 1 (1989), 279–313.

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