GENERATING INTERNALLY FOUR-CONNECTED GRAPHS

Thor Johnson[1]
Department of Mathematics
Princeton University
Princeton, NJ 08544, USA

and

Robin Thomas[2]
School of Mathematics
Georgia Institute of Technology
Atlanta, Georgia 30332, USA

ABSTRACT

A graph is a *minor* of another if the first can be obtained from a subgraph of the second by contracting edges. A graph G is *internally* 4*-connected* if it is simple, 3-connected, has at least five vertices, and if for every partition (A, B) of the edge-set of G, either $|A| \leq 3$, or $|B| \leq 3$, or at least four vertices of G are incident with an edge in A and an edge in B. We prove that if H and G are internally 4-connected graphs such that they are not isomorphic, H is a minor of G and they do not belong to a family of exceptional graphs, then there exists a graph H' such that H' is isomorphic to a minor of G and either H' is obtained from H by splitting a vertex, or H' is an internally 4-connected graph obtained from H by means of one of four possible constructions. This is a first step toward a more comprehensive theorem along the same lines.

6 March 1997, Revised 21 August 2001.
Published in *J. Combin. Theory Ser. B* **85**, 21–58 (2002).

---

[1] Supported by NSF under the program "Research Experiences for Undergraduates," Grant No. DMS-9623031.

[2] Partially supported by NSF under Grant No. DMS-9623031 and by ONR under Grant No. N00014-93-1-0325.

1. INTRODUCTION

All *graphs* in this paper are finite, and may have loops and parallel edges. The *contraction* of an edge e of a graph G is the operation of deleting e and identifying its ends; thus a contraction may produce loops or parallel edges. A graph is a *minor* of another if the first can be obtained from a subgraph of the second by contracting edges. It is a *proper minor* if the two graphs are not isomorphic. A graph is a *wheel* if it is obtained from a circuit on at least three vertices by adding a vertex joined to every vertex on the circuit. (*Paths* and *circuits* have no "repeated" vertices.) We say that a simple graph G is obtained from a simple graph H by *splitting a vertex* if H is obtained from G by contracting an edge e, where both ends of e have degree at least three in G. Let us remark that since H is simple, it follows that e belongs to no triangle of G. The remainder of this section is devoted to motivation; readers familiar with the subject matter may want to proceed directly to the next section. The starting point of our investigations is the following well-known theorem of Tutte [18].

(1.1) *Every simple* 3*-connected graph can be obtained from a wheel by repeatedly applying the operations of adding an edge between two nonadjacent vertices and splitting a vertex.*

For some applications it is desirable to have a stronger version of this theorem, proved by Seymour [17].

(1.2) *Let H be a simple* 3*-connected minor of a simple* 3*-connected graph G such that if H is a wheel, then H is the largest wheel minor of G. Then there exists a sequence $J_0, J_1, \ldots, J_k$ of simple* 3*-connected graphs such that J_0 is isomorphic to H, J_k is isomorphic to G, and for $i = 1, 2, \ldots, k$ the graph J_i is obtained from J_{i-1} either by adding an edge between two nonadjacent vertices, or by splitting a vertex.*

To illustrate the use of Seymour's theorem, let us deduce from it the following theorem of Wagner [20].

(1.3) *Every simple* 3*-connected graph is either planar, or is isomorphic to* K_5*, or has a minor isomorphic to* $K_{3,3}$.

Proof. Let G be a simple 3-connected graph. We may assume that G is not planar, and that it is not isomorphic to K_5. By Kuratowski's theorem G has a minor isomorphic to $K_{3,3}$ or K_5. In the former case the theorem holds, and so we may assume the latter. Let $J_0, J_1, \dots, J_k$ be as in (1.2) applied to $H = K_5$ and G. Since G is not isomorphic to K_5 we see that $k > 0$, and since J_0 is a complete graph, J_1 is obtained from J_0 by splitting a vertex. There is, up to isomorphism, only one possibility for J_1, and it is easy to check that this graph has a minor isomorphic to $K_{3,3}$, and hence so does G, as desired. □

There is a large collection of similar results in Graph Theory, known as excluded minor theorems (for example [1, 2, 4, 7, 10, 19, 20]). Many of them (e.g. the results of [5, 6, 19, 21]) can be deduced using (1.2) similarly as in the above proof of (1.3)—using (1.2) the proof reduces to a straightforward case checking. For other applications, however, it is desirable to have versions of (1.2) for different kinds of connectivity. Some such versions have already been studied [3, 9, 11, 13, 14, 16]. The purpose of this and a subsequent paper is to prove a variant of (1.2) for internally 4-connected graphs, which apparently has not been investigated yet. Let us briefly explain why such a result might be of interest.

Consider, for instance, one step in the proof of Robertson's excluded V_8 theorem. First we need some definitions. By V_8 we mean the graph obtained from a circuit of length eight by joining each pair of diagonally opposite vertices by an edge. (In the terminology to be introduced in the next section, V_8 is the cubic Möbius ladder on eight vertices.) A *line graph* of a graph G has vertex-set $E(G)$, and two of its vertices are adjacent if they are adjacent edges in G. We will only need the line graph of $K_{3,3}$, and we denote it by $LK_{3,3}$. Robertson [15] proved the following.

(1.4) *Let G be an internally* 4*-connected graph with no minor isomorphic to* V_8*. Then G satisfies one of the following conditions:*

(i) *G has at most seven vertices,*

(ii) *G is planar,*

(iii) *G is isomorphic to $LK_{3,3}$,*

(iv) *there is a set $X \subseteq V(G)$ of at most four vertices such that $G \backslash X$ has no edges,*

(v) *there exist two adjacent vertices $u, v \in V(G)$ such that $G \backslash u \backslash v$ is a circuit.*

One step in the proof of Robertson's theorem is to show the following.

(1.5) *If an internally 4-connected graph G has a minor isomorphic to $LK_{3,3}$, but not to V_8, then G is isomorphic to $LK_{3,3}$.*

One can try to use (1.2) similarly as in the proof of (1.3), but this time the process does not terminate. The point is that the operations used in (1.2) produce graphs which are not internally 4-connected and have no V_8-minors. Thus it is desirable to have an analogue of (1.2) for internally 4-connected graphs. In this paper we prove a first step toward that goal.

The paper is organized as follows. In Section 2 we state our main result. In Section 3 we present an extension of our main result, to be proven in a future paper. In Section 4 we reduce the main theorem to a related statement, and outline the proof of that statement.

2. STATEMENT OF RESULTS

Let H be a graph, let $k \geq 2$, and let $v_1, v_2, \ldots, v_k$ be distinct vertices of H. If $k = 2$ then we define $H + (v_1, v_2, \ldots, v_k)$ to be the graph obtained from H by adding an edge with ends v_1 and v_2; otherwise we define it to be the graph obtained from H by adding a new vertex v and an edge with ends v and v_i for all $i = 1, 2, \ldots, k$. Now let $k \geq 2$, and let $x_1, x_2, \ldots, x_k$ be a sequence of pairwise distinct elements of $V(H) \cup E(H)$. Let H' be obtained from H by subdividing every edge that belongs to $\{x_1, x_2, \ldots, x_k\}$. If x_i is a vertex let $u_i = x_i$; otherwise let u_i be the vertex that resulted from subdividing x_i. We define $H + (x_1, x_2, \ldots, x_k)$ to be the graph $H' + (u_1, u_2, \ldots, u_k)$.

Let e be an edge of a graph G, and let v be a vertex of degree three adjacent to both ends of e. We say that v is a *violating vertex*, that e is a *violating edge*, and that (v, e) is a *violating pair*. The reason for this terminology is that such a vertex or edge violates the definition of internal 4-connectivity. It may be helpful to notice that if H is an internally

4-connected graph, and $H' = H + (u, v)$, where u and v are not adjacent in H, then either H' is internally 4-connected, or the edge uv is violating in H'.

Let H be an internally 4-connected graph, let $t \geq 1$ be an integer, and let $H_0 = H, H_1, \ldots, H_t$ be a sequence of graphs such that for $i = 1, 2, \ldots, t$,

(i) $H_i = H_{i-1} + (u_i, v_i)$, where u_i, v_i are distinct nonadjacent vertices of H_{i-1},

(ii) no edge is violating in both H_{i-1} and H_i,

(iii) if $1 < i < t$, then H_i has at most one violating pair, and

(iv) H_t is internally 4-connected.

In those circumstances we say that H_t is an *addition extension* of H. We also say that H_t is a t-step addition extension of H. See Figure 1. Let us point out that in condition (iii) we do mean $i > 1$; that is, H_1 is permitted to have more than one violating pair (but it has at most one violating edge, because H is internally 4-connected).

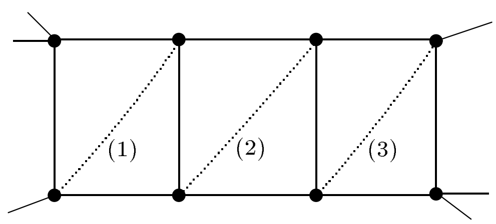

Figure 1. Addition extension.

Let H be a graph, let $\{u, v, x, y\}$ be the vertex-set of a circuit in H, where u, v, x, y all have degree three, and let $H' = H + (u, v, x, y)$. In those circumstances we say that H' is a *quadrangular extension* of H.

Let H be a graph, and let $\{v_1, v_2, v_3, v_4, v_5\}$ be the vertex-set of a circuit of H (in order). Assume that v_2 and v_5 have degree three and that v_1 is not adjacent to v_3 or v_4, and let e denote the edge of the circuit with ends v_3 and v_4. In those circumstances we say that $H + (v_1, e)$ is a *pentagonal extension* of H.

Let H be a graph, and let u, v, w be pairwise distinct, pairwise nonadjacent vertices of H. Assume further that no vertex of H of degree three has neighbors u, v, w, and that every pair of vertices from $\{u, v, w\}$ have a common neighbor of degree three. In those circumstances we say that $H+(u,v,w)$ is a *hexagonal extension* of H. See Figure 2 for a picture of a quadrangular, pentagonal and hexagonal extension.

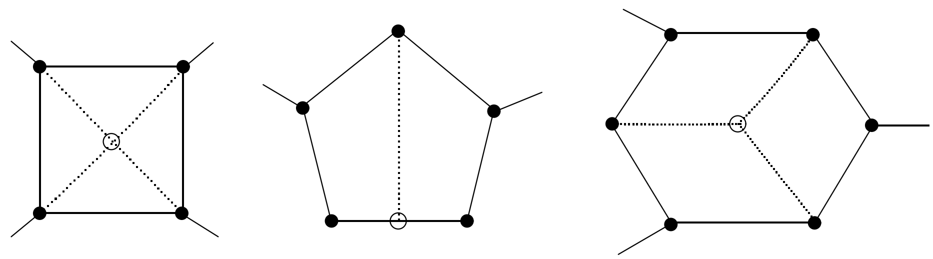

Figure 2. A quadrangular, pentagonal and hexagonal extension.

Let H and G be graphs. We say that H is *G-splittable* if there exists a graph H' such that H' is isomorphic to a minor of G, and H' satisfies one of the following conditions:

(i) H' is an addition extension of H,

(ii) H' is a quadrangular extension of H,

(iii) H' is a pentagonal extension of H,

(iv) H' is a hexagonal extension of H, or

(v) H' is obtained from H by splitting a vertex.

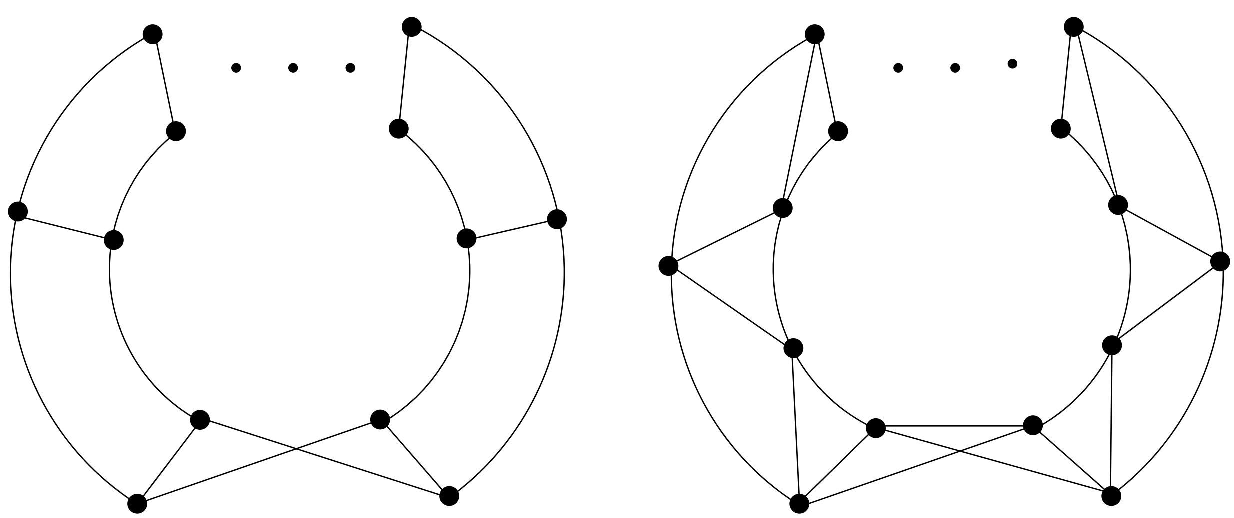

Figure 4. Möbius ladders.

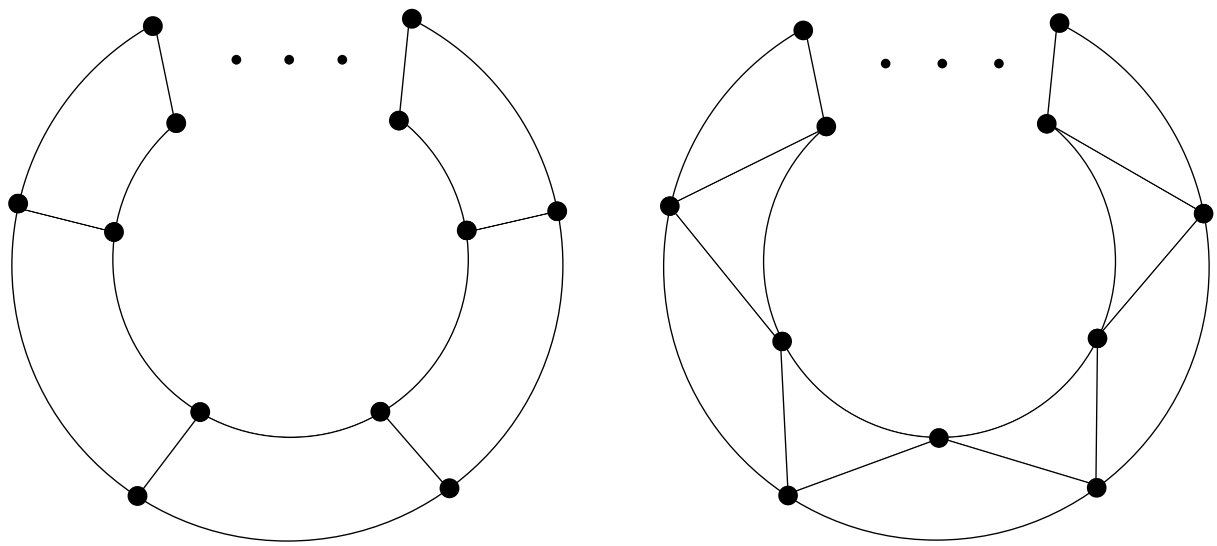

Figure 3. Planar ladders.

We need to introduce several families of graphs (see Figures 3 and 4). Let C_1 and C_2 be two vertex-disjoint circuits of length $n \geq 4$ with vertex-sets $\{u_1, u_2, \ldots, u_n\}$ and $\{v_1, v_2, \ldots, v_n\}$ (in order), respectively, and let G_1 be the graph obtained from the union of C_1 and C_2 by adding an edge joining u_i and v_i for each $i = 1, 2, \ldots, n$. We say that G_1 is a *cubic planar ladder*. Let G_2 be obtained from the union of C_1 and C_2 by adding edges joining u_i and v_i, and v_i and u_{i+1} for all $i = 1, 2, \ldots, n$, where u_{n+1} means u_1. We say that G_2 is a *quartic planar ladder*. We say that a graph is a *planar ladder* if it is a cubic planar ladder or a quartic planar ladder. Let G_3 be the graph consisting of a circuit C with vertex-set $\{u_1, u_2, \ldots, u_{2n}\}$ (in order), where $n \geq 2$ is an integer, and n edges with ends u_i and u_{n+i} for $i = 1, 2, \ldots, n$. We say that G_3 is a *cubic Möbius ladder*. Let G_4 be the graph consisting of a circuit C with vertex-set $\{u_1, u_2, \ldots, u_{2n+1}\}$ (in order), where $n \geq 2$ is an integer, and $2n + 1$ edges with ends u_i and u_{n+i}, and u_i and u_{n+i+1} for $i = 0, 1, \ldots, n$, where u_0 means u_{2n+1}. We say that G_4 is a *quartic Möbius ladder*. We say that a graph is a *Möbius ladder* if it is a cubic Möbius ladder or a quartic Möbius ladder, and we say that a graph is a *ladder* if it is planar ladder or a Möbius ladder. The cubic planar ladder on eight vertices is called the *cube*.

Let G_5 be the graph obtained from a circuit with vertex-set $\{u_1, u_2, \ldots, u_{2n}\}$ (in order), where $n \geq 2$ is an integer, by adding two vertices v and w, and edges with ends v and u_{2i}, and w and u_{2i-1} for all $i = 1, 2, \ldots, n$. We say that G_5 is a *cubic planar biwheel*. Let G_6 be obtained from G_5 by adding an edge joining v and w; we say that G_6 is a *cubic Möbius biwheel*. A graph is a *cubic biwheel* if it is either a cubic planar biwheel, or a cubic Möbius biwheel. Let G_7 be the graph obtained from a circuit with vertex-set $\{u_1, u_2, \ldots, u_n\}$ (in order), where $n \geq 3$ is an integer, by adding two vertices v and w, and edges with ends v and u_i, and w and u_i for all $i = 1, 2, \ldots, n$. We say that G_7 is a *quartic planar biwheel*. Let G_8 be obtained from G_7 by adding an edge joining v and w; we say that G_8 is a *quartic Möbius biwheel*. A graph is a *quartic biwheel* if it is either a quartic planar biwheel, or a quartic Möbius biwheel, and it is a *biwheel* if it is either a planar biwheel or a Möbius biwheel. We need to formulate the following assumptions.

(2.1) Assumptions.

(i) If H is a cubic planar ladder, then the quartic planar ladder on the same number of vertices is not isomorphic to a minor of G.

(ii) If H is a cubic Möbius ladder, then the quartic Möbius ladder on one more vertex is not isomorphic to a minor of G.

(iii) If H is a cubic planar biwheel, then the quartic planar biwheel on the same number of vertices is not isomorphic to a minor of G.

(iv) If H is a cubic Möbius biwheel, then the quartic Möbius biwheel on the same number of vertices is not isomorphic to a minor of G.

Now we can state our main result.

(2.2) *Let H and G be internally 4-connected graphs, where H is isomorphic to a proper minor of G, assume that assumptions (2.1)(i)–(iv) are satisfied, assume that H is not isomorphic to $K_{3,3}$ or the cube, and assume that G is not a cubic ladder or a cubic biwheel. Then H is G-splittable.*

We also prove the following slight variation of (2.2). It has a restrictive assumption about H, but when that assumption is satisfied this version significantly reduces the amount of case checking needed in applications. Let H and G be graphs. We say that H is *strongly G-splittable* if either G is isomorphic to an addition extension of H, or there exists a graph H' such that H' is isomorphic to a minor of G, and H' satisfies one of the following conditions:

(i) H' is a 1-step addition extension of H,

(ii) H' is a quadrangular extension of H,

(iii) H' is a pentagonal extension of H,

(iv) H' is a hexagonal extension of H, or

(v) H' is obtained from H by splitting a vertex.

(2.3) *Let H and G be internally 4-connected graphs, where H is isomorphic to a proper minor of G, assume that assumptions (2.1)(i)–(iv) are satisfied, assume that H is not isomorphic to $K_{3,3}$ or the cube, and assume that G is not a cubic ladder or a cubic biwheel. Assume further that every component of the subgraph of H induced by vertices of degree three is a tree or a circuit. Then H is strongly G-splittable.*

3. A SPLITTER THEOREM

Theorem (2.2) seems to be useful in its own right; for instance, it has been used in [8] to limit the number of possible counterexamples to Negami's planar cover conjecture [12]. However, a weakness of (2.2) is that if H' is obtained from H by splitting a vertex, then it need not be internally 4-connected. We shall remedy this in a future paper by proving a different theorem, which we now describe.

We say that a graph G is *almost 4-connected* if G is simple, 3-connected and for every partition (A, B) of $E(G)$ into disjoint sets, either $|A| \leq 4$, or $|B| \leq 4$, or at least four vertices of G are incident both with a member of A, and a member of B. Thus if a graph G is obtained from an internally 4-connected graph H by applying one of the two operations of theorem (1.2), then G is almost 4-connected, and has at most two violating edges. In our theorem we will require the stronger property that each of the intermediate graphs J_i

be almost 4-connected, and have at most one violating edge. Thus let us define a graph to be *well connected* if it is almost 4-connected, and if it has at most one violating edge. In the theorem below we will also stipulate that no edge is a violating edge of two consecutive graphs in the sequence $J_1, J_2, \ldots, J_k$. However, we need two additional operations, which we now introduce. See Figures 5 and 6. (From now on we use the following convention. When we depict a subgraph J of a graph H, a vertex v of J drawn as a solid circle indicates that all edges of H incident with v belong to J, and hence are drawn in the figure.)

Let H be a graph, let e be a violating edge in H, let v be a vertex of H such that v is not incident with or adjacent to either end of e, and let H have no violating pair (w, e) such that v is adjacent to w in H. Let G be a graph obtained from H by deleting e, and adding a new vertex and three edges joining the new vertex to v and the two ends of e. We say that G was obtained from H by a *special addition*.

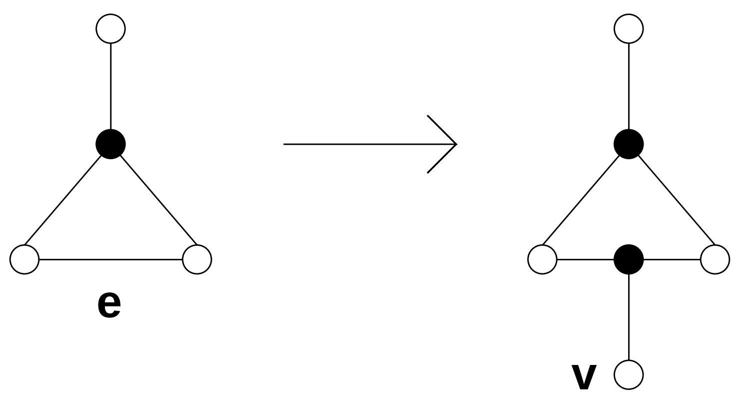

Figure 5. Special addition.

Let H be a simple graph, let (v, e) be a violating pair in H, let u be the neighbor of v that is not incident with e, and let G be obtained from H by splitting u, and then adding an edge between v and the new vertex not adjacent to u in such a way that both new vertices have degree at least four in G. We say that G was obtained from H by a *special split*.

We need to clarify a subtle but important point. Formally, a graph is a triple consisting of a set of vertices, a set of edges, and an incidence relation between them. Thus if a graph

G is obtained from a graph H by splitting a vertex, then $E(H) \subseteq E(G)$. In a future paper we will prove the following result.

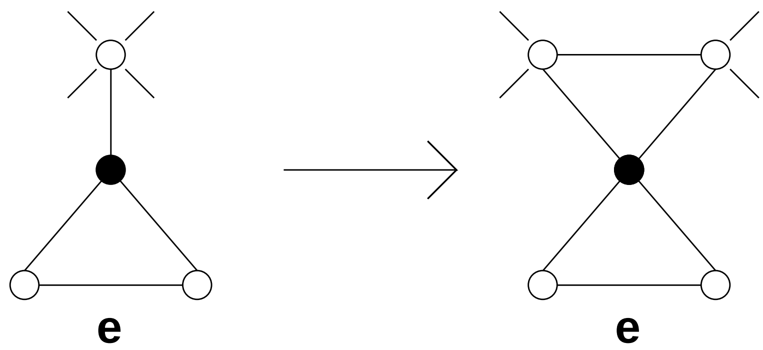

Figure 6. Special split.

(3.1) *Let H be an internally 4-connected minor of an internally 4-connected graph G such that H has at least seven vertices and is not a cubic planar ladder on eight vertices nor a quartic biwheel on eight vertices. Suppose further that if H is a ladder or biwheel, it is the largest ladder or biwheel minor of G. Then there exists a sequence $J_0, J_1, \ldots, J_k$ of well connected graphs such that J_0 is isomorphic to H, J_k is isomorphic to G, and for $i = 1, 2, \ldots, k$ the graph J_i is obtained from J_{i-1} either by adding an edge, or by splitting a vertex, or by a special addition, or by a special split. Moreover, if e is an edge of both J_{i-1} and J_i, and is violating in J_{i-1}, then it is not violating in J_i.*

4. OUTLINE OF PROOF

In this section we reduce the proof of (2.2) to a related theorem, and outline the proof of that theorem. We need to state another set of assumptions.

(4.1) Assumptions.

(i) If H is a cubic planar ladder, then no cubic planar ladder on more than $|V(H)|$ vertices is isomorphic to a minor of G.

(ii) If H is a cubic Möbius ladder, then no cubic Möbius ladder on more than $|V(H)|$ vertices is isomorphic to a minor of G.

(iii) If H is a cubic planar biwheel, then no cubic planar biwheel on more than $|V(H)|$ vertices is isomorphic to a minor of G.

(iv) If H is a cubic Möbius biwheel, then no cubic Möbius biwheel on more than $|V(H)|$ vertices is isomorphic to a minor of G.

A graph is a *subdivision* of another if the first can be obtained from the second by replacing each edge by a non-zero length path with the same ends, where the paths are disjoint, except possibly for shared ends. In order to prove (2.2) it suffices to prove the following.

(4.2) *Let H and G be internally 4-connected graphs, where H is not isomorphic to $K_{3,3}$ or the cube, and a subdivision of H is isomorphic to a proper subgraph of G. If assumptions* (4.1)(i)-(iv) *and* (2.1)(i)-(iv) *are satisfied, then H is G-splittable.*

Proof of (2.2) (assuming (4.2)). If H is a cubic planar ladder, then let H' be the largest cubic planar ladder that is a minor of G. Let H' be defined similarly if H is a cubic Möbius ladder, cubic planar biwheel, or cubic Möbius biwheel. Otherwise let $H' = H$. By (4.2) the graph H' is G-splittable, and it is fairly easy to see that this implies that H is G-splittable, as required. □

Likewise, in order to prove (2.3) it suffices to prove the following result. The proof is analogous to that of (4.2).

(4.3) *Let H and G be internally 4-connected graphs, where H is not isomorphic to $K_{3,3}$ or the cube, and a subdivision of H is isomorphic to a proper subgraph of G. Assume that assumptions* (4.1)(i)-(iv) *and* (2.1)(i)-(iv) *are satisfied, and assume that every component of the subgraph of H induced by vertices of degree three is a tree or a circuit. Then H is strongly G-splittable.*

The rest of the paper is devoted to proving (4.2) and (4.3). In the next section we formalize the concept of a subdivision by means of homeomorphic embeddings, and prove that there exists a homeomorphic embedding with particularly nice properties. We then analyze the "bridges" of this nice homeomorphic embedding. In Section 6 we prove that either our main results hold, or there is no nontrivial bridge of this homeomorphic embedding, and the trivial bridges that could occur are severely restricted. In Section 7 we prove that either our main results hold, or every edge of H is subdivided at most once. In Section 8 we prove that either (4.2) holds, or it is possible to add an edge to H in such a way that the new graph is simple, is isomorphic to a minor of G, and has at most one violating pair. Finally, in Section 9 we complete the proofs of (4.2) and (4.3).

5. HOMEOMORPHIC EMBEDDINGS

We formalize the concept of a subdivision as follows. Let H, G be graphs. A mapping η with domain $V(H) \cup E(H)$ is called a *homeomorphic embedding* of H into G if for every two vertices v, v' and every two edges e, e' of H

(i) $\eta(v)$ is a vertex of G, and if v, v' are distinct then $\eta(v), \eta(v')$ are distinct,

(ii) if e has ends v, v', then $\eta(e)$ is a path of G with ends $\eta(v), \eta(v')$, and otherwise disjoint from $\eta(V(H))$, and

(iii) if e, e' are distinct, then $\eta(e)$ and $\eta(e')$ are edge-disjoint, and if they have a vertex in common, then this vertex is an end of both.

We shall denote the fact that η is a homeomorphic embedding of H into G by writing $\eta : H \hookrightarrow G$. If K is a subgraph of H we denote by $\eta(K)$ the subgraph of G consisting of all vertices $\eta(v)$ for $v \in V(K)$, and all vertices and edges that belong to $\eta(e)$ for some $e \in E(K)$. It is easy to see that G has a subgraph isomorphic to a subdivision of H if and only if there is a homeomorphic embedding $H \hookrightarrow G$. The reader is advised to notice that $V(\eta(K))$ and $\eta(V(K))$ mean different sets. The first is the vertex-set of the graph $\eta(K)$, whereas the second is the image of the vertex-set of K under the mapping η. Thus the first set may be bigger; namely, it contains all the vertices of the paths $\eta(e)$, where $e \in E(K)$, while the second set only contains the ends of those paths.

If η is a homeomorphic embedding of H into G, an *η-bridge* is a connected subgraph B of G with $E(B) \cap E(\eta(H)) = \emptyset$, such that either

(i) $|E(B)| = 1, E(B) = \{e\}$ say, and both ends of e are in $V(\eta(H))$, or

(ii) for some component C of $G \backslash V(\eta(H))$, $E(B)$ consists of all edges of G with at least one end in $V(C)$.

Bridges satisfying (i) will be called *trivial*, and bridges satisfying (ii) will be called *non-trivial*. (We use $\backslash$ for deletion.) It follows that every edge of G not in $\eta(H)$ belongs to a unique η-bridge. We say that a vertex v of G is an *attachment* of an η-bridge B if $v \in V(\eta(H)) \cap V(B)$. We say that a vertex $u \in V(H)$ is a *foot* of a bridge B if $\eta(v)$ is an attachment of B. We say that an edge $e \in E(H)$ is a *foot* of a bridge B if some interior vertex of the path $\eta(e)$ is an attachment of B. It should be noted that the notion of a foot depends on the homeomorphic embedding η. More precisely, an η-bridge B may also be an η'-bridge for two different homeomorphic embeddings η and η', and its feet may depend on the choice of the homeomorphic embedding. When there will a danger of confusion we will indicate what homeomorphic embedding we have in mind by using language such as "a foot of an η-bridge B". We need the following simple result.

(5.1) *Let H and G be internally 4-connected graphs, let $\eta : H \hookrightarrow G$ be a homeomorphic embedding, and let C be a circuit in H of length three. If there exists an η-bridge B such that each edge of C is a foot of B, then a graph obtained from H by splitting a vertex is isomorphic to a minor of G.*

Proof. By the internal 4-connectivity of H, every vertex of C has degree at least four. Let H' be obtained from H by splitting one of the vertices of C in such a way that one of the new vertices has degree three, and is adjacent to the remaining two vertices of C. Then from the existence of B it follows that H' is isomorphic to a minor of G, as required. □

Let η be a homeomorphic embedding of H into G. A subpath P of $\eta(H)$ is an *η-segment* if $P = \eta(e)$ for some $e \in E(H)$. Let L be a subgraph of $\eta(H)$. If L is an η-segment we say that L is an *η-fragment* of type I. If L is the union of an η-segment and an isolated vertex, we say that L is an η-fragment of type J. If L is the union of two

η-segments with a common end, then we say that L is an η-fragment of type V. Assume now that L is of the form $P_1 \cup P_2 \cup P_3$, where P_1, P_2, P_3 are η-segments with a common end v, and otherwise pairwise disjoint, and v has degree three in $\eta(H)$. In those circumstances we say that L is an η-fragment of type Y. We say that a graph L is an *η-fragment* if it is an η-fragment of type I, J, V, or Y.

Let H and G be graphs, let η be a homeomorphic embedding of H into G, let $K = \eta(H)$, and let B be an η-bridge. Let L be a subgraph of K such that all the attachments of B belong to $V(L)$. If L is an η-fragment of type I, then we say that B is an η-bridge of type I. If L is an η-fragment of type J, V, or Y, respectively, and B is nontrivial, then we say that B is an η-bridge of type J, V, or Y, respectively. We say that an η-bridge is *unstable* if it is of type I, J, V, or Y, and otherwise we say that it is *stable*.

If P is a path, and $u, v \in V(P)$, we define $P[u, v]$ to be the subpath of P with ends u and v. Let H, G be graphs, and let $\eta : H \hookrightarrow G$ be a homeomorphic embedding. We say that a path P in G is an *η-path* if it has at least one edge, and its ends and only its ends belong to $\eta(H)$.

Let H, G be graphs, and let $\eta : H \hookrightarrow G$ be a homeomorphic embedding. Let $e \in E(H)$, and let P' be a path in G with both ends on $\eta(e)$, and otherwise disjoint from $\eta(H)$. Let P be the subpath of $\eta(e)$ with ends the ends of P'. Let $\eta'(e)$ be the path obtained from $\eta(e)$ by replacing P by P', and let $\eta'(x) = \eta(x)$ for all $x \in V(H) \cup E(H) - \{e\}$. Then $\eta' : H \hookrightarrow G$ is a homeomorphic embedding, and we say that η, η' are *0-close*. We also say that η' was obtained from η by *rerouting $\eta(e)$ along P'*.

Let H, G be graphs, let $\eta : H \hookrightarrow G$ be a homeomorphic embedding, let $v \in V(H)$ be a vertex of degree three, let e_1, e_2, e_3 be the three edges of H incident with v, and let their other ends be v_1, v_2, v_3, respectively. Let $x \in V(\eta(e_1)) - \{\eta(v)\}$, let y be an interior vertex of $\eta(e_2)$, and let P' be an η-path in G with ends x and y. Let $\eta'(v) = y$, let $\eta'(e_1) = \eta(e_1)[\eta(v_1), x] \cup P'$, let $\eta'(e_2) = \eta(e_2)[\eta(v_2), y]$, and let $\eta'(e_3) = \eta(e_3) \cup \eta(e_2)[\eta(v), y]$. For $x \in V(H) \cup E(H) - \{v, e_1, e_2, e_3\}$ let $\eta'(x) = \eta(x)$. Then $\eta' : H \hookrightarrow G$, and we say that η, η' are *1-close*. We also say that η' was obtained from η by *rerouting $\eta(e_1)$ along P'*.

Let H, G be graphs, and let $\eta : H \hookrightarrow G$. Let u be a vertex of H of degree three, and let e_1, e_2, e_3 be the three edges incident with u. For $i = 1, 2, 3$ let $P_i = \eta(e_i)$, and let $v = \eta(u)$

and v_i be the ends of P_i. For $i = 1, 2, 3$ let $u_i \in V(P_i) - \{v\}$, let $y \in V(G) - V(H)$, and let Q_1, Q_2, Q_3 be three paths in G such that Q_i has ends u_i and y, the paths Q_1, Q_2, Q_3 are vertex-disjoint, except for y, and each of them is vertex-disjoint from $\eta(H)$, except for u_1, u_2, u_3. For $i = 1, 2, 3$ let $\eta'(e_i) = P_i[v_i, u_i] \cup Q_i$, and let $\eta'(u) = y$. For all other $z \in V(H) \cup E(H)$ we put $\eta'(z) = \eta(z)$. Then $\eta' : H \hookrightarrow G$, and we say that η, η' are 2-*close*. We also say that η' was obtained from η by rerouting P_1, P_2, P_3 along Q_1, Q_2, Q_3.

Let $\eta, \eta' : H \hookrightarrow G$. We say that η, η' are *close* if they are i-close for some $i \in \{0, 1, 2\}$. We say that η, η' are *parallel* if for some integer $n > 0$ there exist homeomorphic embeddings $\eta_i : H \hookrightarrow G$ $(i = 1, 2, \ldots, n)$ such that $\eta_1 = \eta$, $\eta_n = \eta'$ and for $i = 2, 3, \ldots, n$, η_{i-1}, η_i are close.

Let $\eta : H \hookrightarrow G$, and let $n = |V(G)|$. For an integer $i = 1, 2, \ldots, n$ let a_{n+i} be the number of stable η-bridges B with $|V(B)| = i$, and let a_i be the number of unstable η-bridges B with $|V(B)| = i$. We say that $(a_{2n}, a_{2n-1}, \ldots, a_1)$ is the *signature* of η. We say that η is *lexicographically maximal* if there exists no homeomorphic embedding $\eta' : H \hookrightarrow G$ parallel to η with signature $(a'_{2n}, a'_{2n-1}, \ldots, a'_1)$ such that there exists an integer $i \in \{1, 2, \ldots, 2n\}$ with the property that $a_i < a'_i$ and $a_j = a'_j$ for all $j \in \{i+1, i+2, \ldots, 2n\}$.

(5.2) *Let H and G be internally 4-connected graphs, and let $\eta : H \hookrightarrow G$ be a lexicographically maximal homeomorphic embedding. If no graph obtained from H by splitting a vertex is isomorphic to a minor of G, then every η-bridge is stable.*

Proof. Suppose for a contradiction that there exists an unstable η-bridge, and choose such a bridge, say B_0, with $|V(B_0)|$ minimum. If D, D' are two η-bridges we say that D is *nicer* than D' if either D is stable and D' is not, or D, D' are both stable or both unstable and $|V(D)| > |V(D')|$. We shall define a homeomorphic embedding $\eta' : H \hookrightarrow G$ parallel to η and an η-bridge B_1 such that

(1) *B_1 is a proper subgraph of an η'-bridge B'_1, and if B_1 is stable, then so is B'_1, and*

(2) *every η-bridge B nicer than B_1 is an η'-bridge, and the feet of B as an η-bridge are the same as its feet as an η'-bridge.*

Let us assume that we have already found η' and B_1 satisfying (1) and (2), and let us derive a contradiction. Let $(a_{2n}, a_{2n-1}, \ldots, a_1)$ be the signature of η, and let $(a'_{2n}, a'_{2n-1}, \ldots, a'_1)$ be the signature of η'. Let $k = n + |V(B_1)|$ if B_1 is stable, and let $k = |V(B_1)|$ otherwise. By (2) every η-bridge B that is nicer than B_1 is also an η'-bridge and its feet as an η-bridge are the same as its feet as an η'-bridge. Thus B is a stable η-bridge if and only if it is a stable η'-bridge. It follows that $a'_j \geq a_j$ for all $j = k+1, k+2, \ldots, 2n$. Let $l = n + |V(B'_1)|$ if B'_1 is a stable η'-bridge, and let $l = |V(B'_1)|$ otherwise. Then $l > k$ by (1), and and $a'_l > a_l$ by (1) and (2). Thus η' contradicts the lexicographic maximality of η.

Thus it remains to construct η' and B_1 such that (1) and (2) hold. Since B_0 is unstable, it is of type I, J, V, or Y. Assume first that B_0 is of type I or J. Then there exists an edge $e \in E(G)$ such that all the attachments of B_0 (except possibly one if B_0 is of type J) belong to $V(\eta(e))$. Let P be the minimal subpath of $\eta(e)$ that includes all attachments of B_0 that belong to $\eta(e)$, and let u, v be the ends of P. Since H is internally 4-connected, we deduce that the set $X = V(P) - \{u, v\}$ is not empty, and hence some η-bridge other than B_0 has an attachment in X. Let B_1 be the nicest such bridge. Let Q be a subpath of B_0 with ends u, v, and otherwise disjoint from $\eta(H)$, and let η' be obtained from η by rerouting $\eta(e)$ along Q. Then $\eta' : H \hookrightarrow G$ is a homeomorphic embedding 0-close to η. To prove that (1) holds we first notice that B_1 is a proper subgraph of an η'-bridge, say B'_1. Now assume that B_1 is stable. If P is a proper subgraph of $\eta(e)$, then every foot of B_1 is a foot of B'_1, and hence B'_1 is stable. If $P = \eta(e)$, then e need not be a foot of B'_1, but the ends of e are feet of B'_1, and a simple case analysis using the internal 4-connectivity of H and (5.1) shows that B'_1 is stable. To prove (2) it suffices to notice that every η-bridge B that is nicer than B_1 has no attachment in X, and hence is an η'-bridge. This completes the construction when B_1 is of type I or J.

Assume now that B_0 is of type V, but not of type I or J. Thus all the attachments of B_0 belong to $\eta(e_1) \cup \eta(e_2)$, where $e_1, e_2 \in E(H)$ have a common end, say u. For $i = 1, 2$ let u_i be the other end of e_i. Let $x_i \in V(\eta(e_i))$ be an attachment of B_0 chosen so that there is no other attachment of B_0 closer to $\eta(u_i)$ on $\eta(e_i)$, and let $y_i \in V(\eta(e_i)) - \{\eta(u)\}$ be an attachment of B_0 chosen so that there is no other attachment of B_0 closer to $\eta(u)$ on $\eta(e_i)$. Since G is internally 4-connected and B_0 is not of type I or J, it follows that x_i

and y_i are well-defined, and that $\eta(u), x_i, y_i$ are pairwise distinct (but possibly $x_i = \eta(u_i)$). Let X be the union of the interior vertices of the paths $\eta(e_i)[x_i, \eta(u)]$ for $i = 1, 2$. By the internal 4-connectivity of G some η-bridge has at least one attachment in X; let B_1 be the nicest such bridge. From the symmetry we may assume that B_1 has an attachment in an interior vertex of the path $P = \eta(e_1)[x_1, \eta(u)]$. Let Q be a subpath of B_0 with ends x_1 and y_2, and otherwise disjoint from $\eta(H)$. It follows that u has degree three, for otherwise the graph obtained from $\eta(H) \cup Q$ by deleting the interior vertices of P witnesses that a graph obtained from H by splitting the vertex u in such a way that one of the new vertices is adjacent to u_1 and u_2 is isomorphic to a minor of G, a contradiction. Thus u has degree three. Now let η' be obtained from η by rerouting $\eta(e_1)$ along Q. Similarly as in the previous paragraph it follows that (1) and (2) hold.

Finally we assume that B_0 is of type Y, but not of type I, J, or V. Thus there exists a vertex $u \in V(H)$ of degree three such that all attachments of B belong to $\eta(e_1) \cup \eta(e_2) \cup \eta(e_3)$, where e_1, e_2, e_3 are the three edges incident with u. For $i = 1, 2, 3$ let u_i be the other end of e_i. Let $x_i \in V(\eta(e_i)) - \{\eta(u)\}$ be an attachment of B_0 as close to $\eta(u_i)$ on $\eta(e_i)$ as possible. (Such a vertex exists for all $i = 1, 2, 3$, because B_0 is not of type I, J, or V.) Let $P_i = \eta(e_i)$, and let $X = \bigcup_{i=1}^{3} V(P_i) - V(P_i[\eta(u_i), x_i])$. Since G is internally 4-connected, some η-bridge other than B_0 has an attachment in X. Let B_1 be the nicest such bridge. There exist a vertex $y \in V(B_0) - V(\eta(H))$ and three paths Q_1, Q_2, Q_3 in B_0 such that Q_i has ends x_i and y, Q_i is disjoint from $\eta(H)$, except for x_i, and the paths Q_1, Q_2, Q_3 are pairwise disjoint, except for y. Let η' be obtained from η by rerouting P_1, P_2, P_3 along Q_1, Q_2, Q_3. Again, it follows by a similar argument that (1) and (2) are satisfied.

This completes the construction of η' and B_1, and hence the proof of the theorem. $\square$

6. NONTRIVIAL BRIDGES

The main result of this section, (6.9) below, states that if $\eta : H \hookrightarrow G$ is a lexicographically maximal homeomorphic embedding, then either the conclusion of (4.3) holds, or every η-bridge is severely restricted.

(6.1) *Let H be an internally 4-connected graph, let G be a graph, and let u, v be distinct vertices of H such that $H + (u, v)$ is isomorphic to a minor of G. If H has no vertex of degree three adjacent to both u and v, then H is strongly G-splittable.*

Proof. If H has no vertex of degree three adjacent to both u and v, then $H + (u, v)$ is a 1-step addition extension of G, and hence G is strongly G-splittable, as desired. □

(6.2) *Let H be an internally 4-connected graph, let G be a graph, and let $u \in V(H)$ and $e \in E(H)$ be such that u is not an end of e, and if u is adjacent to an end v of e, then v has degree at least four. If $H + (u, e)$ is isomorphic to a minor of G, then H is strongly G-splittable.*

Proof. Let u_1 and u_2 be the ends of e. Assume first that u is not adjacent to u_1 or u_2. Let $i \in \{1, 2\}$. If no vertex in H of degree three is adjacent to both u and u_i, then $H + (u, u_i)$ is a 1-step addition extension of H, and is isomorphic to a minor of G, because it is isomorphic to a minor of $H + (u, e)$. Thus H is strongly G-splittable in this case. We may therefore assume that there exists a vertex $v_i \in V(H)$ of degree three adjacent to both u and u_i. Now $v_1 \neq v_2$ by the internal 4-connectivity of H, and hence the set $\{u, v_1, u_1, u_2, v_2\}$ establishes that $H + (u, e)$ is a pentagonal extension of H. Thus H is strongly G-splittable, as desired.

We may therefore assume that u is adjacent to an end of e, say v. Let v' be the other end of e. By hypothesis v has degree at least four. Let H' be the graph obtained from H by splitting v in such a way that one of the new vertices has degree three and is adjacent to u and v'. Then H' is isomorphic to a minor of G, because it is isomorphic to a minor of $H + (u, e)$, and hence H is strongly G-splittable, as desired. □

(6.3) *Let H and G be internally 4-connected graphs, and assume that for some distinct nonadjacent edges $e, f \in E(H)$ the graph $H + (e, f)$ is isomorphic to a minor of G. Then either H is strongly G-splittable, or $e, f \in E(C)$ for some circuit C in H of length four such that every vertex of C has degree three.*

Proof. Let us assume that H is not strongly G-splittable. Let the ends of e be v_1 and v_2, and let the ends of f be v_3 and v_4. Since $H+(v_1, f)$ is a minor of $H+(e, f)$, we deduce from (6.2) that one of v_3, v_4 has degree three and is adjacent to v_1. From the symmetry we may assume that v_4 has degree three and is adjacent to v_1. From the internal 4-connectivity of H we deduce that v_2 is not adjacent to v_4. Similarly, one of v_3, v_4 has degree three and is adjacent to v_2, and hence v_3 has degree three and is adjacent to v_2. From the symmetry it follows that v_1 and v_2 also have degree three. Thus the second alternative of the lemma holds. □

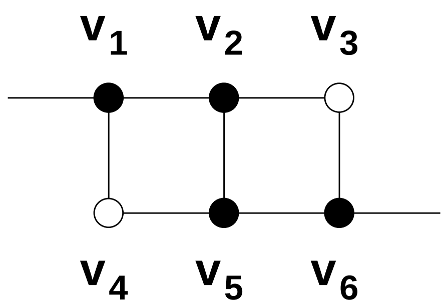

Figure 7. Cubic ladder chain.

Let H be an internally 4-connected graph, and let $n \geq 2$ be an integer. We say that the $2n$-tuple $\gamma = (v_1, v_2, \ldots, v_{2n})$ of distinct vertices of H is a *cubic ladder chain in H of length $n-1$* if v_i has degree three for all $i \in \{1, 2, \ldots, n-2, n-1, n+2, \ldots, 2n\}$, and for all $i \in \{1, 2, \ldots, 2n\} - \{n\}$ the pairs (v_i, v_{i-1}), (v_i, v_{i+1}), (v_i, v_{i-n}), and (v_i, v_{i+n}) are adjacent whenever both indices are between 1 and $2n$. See Figure 7.

(6.4) *Let H be an internally 4-connected graph, and let $\gamma = (v_1, v_2, \ldots, v_{2n})$ be a cubic ladder chain in H.*

(i) *If v_1 is adjacent to v_n, or if v_{n+1} is adjacent to v_{2n}, then H is a cubic planar ladder with vertex-set $\{v_1, v_2, \ldots, v_{2n}\}$.*

(ii) *If v_1 is adjacent to v_{2n}, or if v_n, v_{n+1} are adjacent and at least one of them has degree three, then H is a cubic Möbius ladder with vertex-set $\{v_1, v_2, \ldots, v_{2n}\}$.*

Proof. To prove (i) we may assume, by reversing the order in γ if necessary, that v_1 is adjacent to v_n. If $V(H) \neq \{v_1, v_2, \ldots, v_{2n}\}$, then $H \backslash \{v_n, v_{n+1}, v_{2n}\}$ is disconnected, and hence one component has exactly one vertex. But v_n is adjacent to v_{2n}, contrary to the internal 4-connectivity of H. Thus $V(H) = \{v_1, v_2, \ldots, v_{2n}\}$, and the internal 4-connectivity of H implies that H is a cubic planar ladder. This proves (i). We omit the proof of (ii), because it is almost identical. □

(6.5) *Let H and G be internally 4-connected graphs such that H is isomorphic to a minor of G, and such that (4.1)(i) and (4.1)(ii) are satisfied. Let e, f be distinct edges of H such that if e and f have a common end, then the common end has degree at least four. If $H + (e, f)$ is isomorphic to a minor of G, then H is strongly G-splittable.*

Proof. Suppose for a contradiction that H is not G-splittable, and let $e, f \in E(H)$ be as stated. If e and f have a common end v, then v has degree at least four, and the graph obtained from H by splitting v in such a way that one of the new vertices is incident with e and f and no other edge of H is isomorphic to a minor of G, a contradiction. Thus the edges e and f have no common end. By (6.3) there exists a cubic ladder chain (v_1, v_2, v_3, v_4) of length one such that $H + (v_1v_2, v_3v_4)$ is isomorphic to a minor of G. Let $n \geq 2$ be the maximum integer with the property that H has a cubic ladder chain $\gamma = (v_1, v_2, \ldots, v_{2n})$ such that $H + (v_1v_2, v_{n+1}v_{n+2})$ is isomorphic to a minor of G. By (6.3) the vertices $v_1, v_2, v_{n+1}, v_{n+2}$ have degree three. Let u be the neighbor of v_1 other than v_2 and v_{n+1}, and let v be the neighbor of v_{n+1} other than v_1 and v_{n+2}.

Then $u \neq v$, because H is internally 4-connected. It follows that $H + (uv_1, vv_{n+1})$ is isomorphic to a minor of G. By (6.3) and the internal 4-connectivity of H the vertices u and v are adjacent and both have degree three. From the maximality of n we deduce that one of the vertices u, v is equal to one of v_n, v_{2n}. In either case, (6.4) implies that H is a cubic planar ladder or a cubic Möbius ladder. Moreover, since $H + (v_1v_2, v_{n+1}v_{n+2})$ is isomorphic to a minor of G, it follows that the same kind of cubic ladder on two more vertices is isomorphic to a minor of G, contrary to (4.1)(i) or (4.1)(ii). □

(6.6) *Let H and G be internally 4-connected graphs, and let $\{x, y, z, w\}$ be the vertex-set of a circuit in H. If $H + (x, y, z, w)$ is isomorphic to a minor of G, then H is strongly G-splittable.*

Proof. Let x, y, z, w appear on the circuit in the order listed. If x, y, z, w all have degree three, then $H + (x, y, z, w)$ is a quadrangular extension of H, and hence H is G-splittable, as desired. We may therefore assume that x has degree at least four. Let H' be the graph obtained from H by splitting x in such a way that one of the new vertices has degree three and is adjacent to y and w. Then H' is isomorphic to a minor of G, because it is isomorphic to a minor of $H + (x, y, z, w)$. Thus H is strongly G-splittable, as desired. □

Let H be a graph. We say that a set $S \subseteq V(H) \cup E(H)$ of size three is *free* if no vertex in S is an end of an edge in S, and there exists no connected subgraph T of H such that $|E(T)| \leq 3$, $S \subseteq V(T) \cup E(T)$ and at most three vertices of H are incident with both an edge of T and an edge of $E(H) - E(T)$. Thus if H is 3-connected and T is as in the previous sentence, then T is a subgraph of K_3 or $K_{1,3}$.

(6.7) *Let H and G be internally 4-connected graphs such that assumptions* (4.1)(i) *and* (4.1)(ii) *are satisfied, and let $\{x, y, z\} \subseteq V(H) \cup E(H)$ be free. If $H + (x, y, z)$ is isomorphic to a minor of G, then H is strongly G-splittable.*

Proof. Suppose for a contradiction that H is not strongly G-splittable. We first notice that not all x, y, z are edges. Indeed, otherwise they would have to pairwise share a vertex of degree three by (6.5), contrary to the freedom of x, y, z. Next we claim that not every element of $\{x, y, z\}$ is a vertex. To prove this claim suppose for a contradiction that x, y, z are vertices. Then some two of these vertices are adjacent, for otherwise $H + (x, y)$, $H + (x, z)$, $H + (x, z)$ are isomorphic to minors of G, and hence (6.1) and the freedom of x, y, z imply that $H + (x, y, z)$ is a hexagonal extension of H, contrary to the assumption that H is not strongly G-splittable. Thus some two members of $\{x, y, z\}$ are adjacent. By symmetry we may assume that x is adjacent to y. Then by deleting the edge xy from $H + (x, y, z)$ we see that $H + (z, xy)$ is isomorphic to a minor of G. By (6.2) the vertex z

is adjacent to x or y, contrary to the freedom of $\{x, y, z\}$. This proves our claim that not all x, y, z are vertices.

We may therefore assume that x is a vertex, and that z is an edge. Since $H + (x, z)$ is isomorphic to a minor of G, (6.2) implies that one end of z, say w, has degree three and is adjacent to x. Since $H + (y, z)$ is isomorphic to a minor of G, (6.5) implies that if y is an edge, then it is adjacent to z. Since $\{x, y, z\}$ is free, we conclude that if y is an edge, then it is not incident with x or w. By deleting the edge xw from $H + (x, y, z)$ we deduce that $H + (y, xw)$ is isomorphic to a minor of G. Thus y is a vertex by (6.5), and by (6.2) the vertex y is adjacent to x or w. But y is not adjacent to w by the freedom of $\{x, y, z\}$, and hence y is adjacent to x. By deleting the edge xy from $H + (x, y, z)$ we deduce that $H + (xy, z)$ is isomorphic to a minor of G, contrary to (6.5). □

(6.8) *Let H and G be internally 4-connected graphs such that assumptions (4.1)(i) and (4.1)(ii) are satisfied, and let $\eta : H \hookrightarrow G$ be a homeomorphic embedding. If there exists a nontrivial stable η-bridge, then H is strongly G-splittable.*

Proof. Let H, G, η be as stated, let B be a nontrivial stable η-bridge, and let A be the set of all feet of B. We claim that we may assume the following:

(1) *There exist distinct elements $x, y, z \in A$ such that no vertex in $\{x, y, z\}$ is an end of an edge in $\{x, y, z\}$*

To prove that we may assume (1) we first notice that if $A \subseteq V(H)$, then $|A| \geq 3$ (because B is nontrivial) and any three elements of A satisfy (1). Thus we may assume that A includes an edge of H. Suppose that there exist distinct edges $e, f \in A$. If e, f are not adjacent, then H is strongly G-splittable by (6.5), and so we may assume that e and f are adjacent. Since B is stable there exists an element $x \in A - \{e, f\}$ that is not a vertex incident with e or f. Then $\{e, f, x\}$ satisfies (1). Thus we may assume that there is a unique edge $e \in A$. Since B is stable there exist distinct vertices $x, y \in A - \{e\}$ not incident with e. Then $\{x, y, e\}$ satisfies (1), as desired. This proves that we may assume (1).

By (5.1) we may assume the following:

(2) *Let* $x, y, z \in A$ *be as in* (1). *Then it is not the case that* $x, y, z \in E(H)$ *and* $\{x, y, z\}$ *is the edge-set of a circuit in* H.

Next we claim that we may assume that

(3) *if* $x, y, z \in A$ *are as in* (1), *then either*

(a) $x, y, z \in V(H)$, *and at least two edges of* H *have both ends in* $\{x, y, z\}$, *or*

(b) *there exists a vertex* $v \in V(H)$ *of degree three such that each of* x, y, z *is either an edge incident with* v, *or a vertex adjacent to* v.

To prove that we may assume (3) let $x, y, z \in A$ be as in (1). If $\{x, y, z\}$ is free, then (6.8) holds by (6.7), and so we may assume that $\{x, y, z\}$ is not free. It follows from (2) that (a) or (b) holds. This proves that we may assume that (3) holds.

Now let $x, y, z \in A$ be as in (1), chosen so that as many of them as possible are edges. Since B is stable and we are assuming that (3) holds, there exists an element $w \in A - \{x, y, z\}$ such that the following assertions hold:

(4) *if* (a) *holds, then* w *is not an edge of* H *with both ends in* $\{x, y, z\}$,

(5) *if* (a) *holds, then it is not the case that* w *is an edge and one of* x, y, z *has degree three, is adjacent to the other two, and is incident with* w, *and*

(6) *if* (b) *holds and* v *is as in* (b), *then* w *is not an edge incident with* v *and* w *is not a vertex adjacent to* v.

From the existence of B we deduce that

(7) $H + (x, y, z, w)$ *is isomorphic to a minor of* G.

We claim the following:

(8) $x, y, z, w \in V(H)$.

To prove (8) suppose for a contradiction that one of x, y, z, w is an edge. It follows from (3), the internal 4-connectivity of H and the choice of x, y, z that one of x, y, z, say x, is an edge. Thus the triple x, y, z satisfies (b). It follows that one of the triples x, y, w and x, z, w satisfies the conclusion of (1), but it does not satisfy (3) by the internal 4-connectivity of H, a contradiction. This proves (8).

By (8) every triple of elements of $\{x, y, z, w\}$ satisfies (1), and hence it satisfies (a) or (b) of (3). If every triple of elements of $\{x, y, z, w\}$ satisfies (a), then it is easy to see that H has a circuit with vertex-set $\{x, y, z, w\}$. In that case (6.8) follows from (6.6) and (7). We may therefore assume that the triple x, y, z satisfies (b).

Assume now that every triple of elements of $\{x, y, z, w\}$ satisfies (b). Then it follows that no edge of H has both ends in $\{x, y, z, w\}$. Let $v \in V(H)$ be the vertex of H of degree three with neighbors x, y, z. By (7) the graph $H + (v, w)$ is isomorphic to a minor of G, and it is internally 4-connected, because no edge of H has both ends in $\{x, y, z, w\}$. Thus H is strongly G-splittable.

We may therefore assume that the triple x, y, w satisfies (a). The vertices x and y are not adjacent by the internal 4-connectivity of H, and hence w is adjacent to x and y. Again, by the internal 4-connectivity of H, the triple y, z, w does not satisfy (b), and hence it satisfies (a), and so w is adjacent to z. Thus we have shown that w is adjacent to x, y, z. Since v is also adjacent to x, y, z and has degree three, the internal 4-connectivity of H implies that w has degree at least four. Let H' be obtained from H by splitting w in such a way that one of the new vertices has degree three and is adjacent to x and y; by (7) it follows that H' is isomorphic to a minor of G. Thus H is strongly G-splittable, as required. □

Let H and G be internally 4-connected graphs, and let $\eta : H \hookrightarrow G$ be a homeomorphic embedding. We say that an η-bridge B is *elusive* if it is trivial, and there exists a vertex $v \in V(H)$ of degree three, and two edges e_1, e_2 incident with v such that one attachment of B belongs to $V(\eta(e_1)) - \{\eta(v)\}$, and the other attachment belongs to $V(\eta(e_2)) - \{\eta(v)\}$.

(6.9) *Let H and G be internally 4-connected graphs such that (4.1)(i) and (4.1)(ii) are satisfied, and let $\eta : H \hookrightarrow G$ be a lexicographically maximal homeomorphic embedding. If H is not strongly G-splittable, then every η-bridge is elusive.*

Proof. Let H, G, η be as stated. By (5.2) every η-bridge is stable, and hence by (6.8) every η-bridge is trivial. By (6.1), (6.2) and (6.5) every η-bridge is elusive, as required. □

7. BOUNDING SUBDIVISIONS

The main result of this section, (7.3) below, states that if $\eta : H \hookrightarrow G$ is a lexicographically maximal homeomorphic embedding, then either the conclusion of (4.3) holds, or $\eta(e)$ has at most two edges for every $e \in E(H)$.

Let H be an internally 4-connected graph, and let $n \geq 2$ be an integer. We say that the $(n+2)$-tuple $\gamma = (u_1, u_2, v_1, ..., v_n)$ of distinct vertices of H is a *cubic biwheel chain in H of length $n-2$* if v_i has degree three for all $i = 1, 2, \ldots, n$, the vertices $v_1, v_2, \ldots, v_n$ form the vertex-set of a path in the order listed, and for $i = 1, 2, \ldots, n$, the vertex v_i is adjacent to u_1 if i is odd, and to u_2 if i is even. See Figure 8.

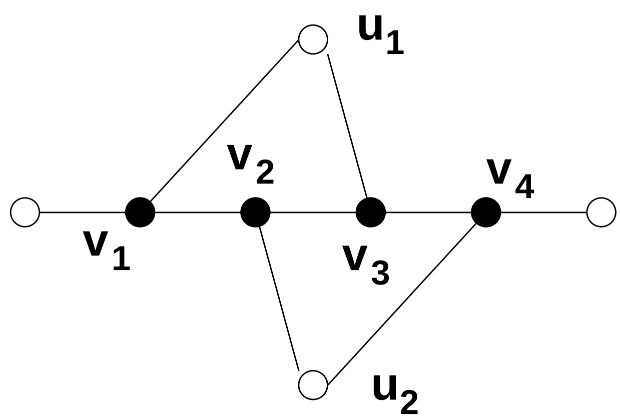

Figure 8. Cubic biwheel chain.

(7.1) *Let H and G be internally 4-connected graphs such that (4.1)(iii) and (4.1)(iv) are satisfied. Let (u_1, u_2, v_1, v_2) be a cubic biwheel chain in H of length 0, and let H' be obtained from H by deleting the edge v_1v_2 and adding two vertices v_1', v_2' and the edges $v_1'v_2'$, $v_1'u_1$, $v_2'u_2$, v_1v_2', and $v_1'v_2$. If H' is isomorphic to a minor of G, then H is strongly G-splittable.*

Proof. Suppose for a contradiction that H is not strongly G-splittable. Let $\gamma = (u_1, u_2, v_1, v_2, \ldots, v_n)$ be a cubic biwheel chain of maximum length such that the graph H' defined in the statement of (7.1) is isomorphic to a minor of G. Since H is internally 4-connected, we see that for all integers $i = 1, 2, \ldots, n$, if i is odd, then v_i is not adjacent to u_2, and if i is even, then v_i is not adjacent to u_1.

Let v_0 be the neighbor of v_1 other than u_1 and v_2. The graph $H + (u_2, v_0 v_1)$ is isomorphic to a minor of H' (to see this delete the edge $u_1 v_1$ of H' and suppress v_1), and hence $H + (u_2, v_0 v_1)$ is isomorphic to a minor of G. By (6.2) the vertex v_0 has degree three and is adjacent to u_2. If $v_0 \notin \{u_1, u_2, v_1, v_2, \ldots, v_n\}$, then the cubic biwheel chain $(u_1, u_2, v_0, v_1, \ldots, v_n)$ contradicts the maximality of γ. Thus $v_0 \in \{u_1, u_2, v_1, v_2, \ldots, v_n\}$, and hence $v_0 = v_n$. Since G is internally 4-connected, we deduce that H is a planar or Möbius cubic biwheel, and since H' is isomorphic to a minor of G we see that the same type (i.e., planar or Möbius) cubic biwheel on two more vertices is isomorphic to a minor of G, contrary to (4.1)(iii) or (4.1)(iv). □

(7.2) *Let H and G be internally 4-connected graphs, and let $\eta : H \hookrightarrow G$ be a lexicographically maximal homeomorphic embedding. Let u be a vertex of H of degree three, and let e_1, e_2 be two distinct edges of H incident with u. If there exists a trivial η-bridge B with one attachment $x \in V(\eta(e_1)) - \{\eta(u)\}$ and another attachment in an interior vertex of $\eta(e_2)$, then the path $\eta(e_1)[\eta(u), x]$ has only one edge.*

Proof. If $\eta(e_1)[\eta(u), x]$ has an interior vertex, then some η-bridge B' has an attachment at that vertex. Let $\eta' : H \hookrightarrow G$ be the homeomorphic embedding obtained from η by rerouting $\eta(e_1)$ along B. Since B is trivial, the homeomorphic embedding η' contradicts the lexicographic maximality of η, because B' is a subgraph of a nontrivial η'-bridge. □

(7.3) *Let H and G be internally 4-connected graphs such that assumptions (4.1)(i)–(iv) are satisfied, and let $\eta : H \hookrightarrow G$ be a lexicographically maximal homeomorphic embedding. If for some edge $e \in E(H)$ the path $\eta(e)$ has at least three edges, then H is strongly G-splittable.*

Proof. Suppose for a contradiction that H is not strongly G-splittable. Then every η-bridge is elusive by (6.9). Let v_1 and v_2 be the ends of e, and let x be an internal vertex of $\eta(e)$. Since G is 3-connected there exists an η-bridge B with an attachment x. Since B is elusive, its other attachment, say x', is in $V(\eta(u_1v_i)) - \{\eta(v_i)\}$, where for some $i \in \{1,2\}$ the vertex v_i has degree three and is adjacent to u_1. If possible, let us choose the integer i and bridge B in such a way that

(a) $\eta(e)[\eta(v_i), x]$ has at least two edges,

and, subject to that,

(b) the path $\eta(u_1v_i)[\eta(u_1), x']$ is as short as possible.

From the symmetry we may assume that $i = 1$. Let u_1' be the neighbor of v_1 other than u_1 and v_2.

(1) *There is no η-bridge with one attachment in $V(\eta(v_1u_1')) - \{\eta(v_1)\}$ and another attachment in an interior vertex of $\eta(e)$.*

To prove (1) suppose for a contradiction that such a bridge, say B', exists. Let η_1 be the homeomorphic embedding obtained from η by rerouting $\eta(u_1v_1)$ along B, and let η_2 be the homeomorphic embedding obtained from η_1 by rerouting $\eta_1(u_1'v_1)$ along B'. Then η_2 is parallel to η, contrary to the lexicographic maximality of η, because $\eta(v_1)$ is a vertex of a nontrivial η_2-bridge, and yet both B and B' are trivial. This proves (1).

(2) *If an η-bridge has an attachment in an interior vertex of $\eta(e)[\eta(v_1), x]$, then its other attachment belongs to $\eta(v_1u_1)$.*

To prove (2) suppose for a contradiction that there exists a bridge B'' with an attachment in an interior vertex of $\eta(e)[\eta(v_1), x]$ such that its other attachment does not belong to $\eta(v_1u_1)$. By (1) and the fact that B'' is elusive we have that the other attachment of B'' is $\eta(u_2)$ or belongs to an interior vertex of $\eta(u_2v_2)$, where u_2 is adjacent to v_2, and v_2 has degree three. Now (u_1, u_2, v_1, v_2) is a cubic biwheel chain in H of length 0, and the graph H' from (7.1) is isomorphic to a minor of G. By (7.1) the graph H is strongly G-splittable, a contradiction. This proves (2).

(3) *The path $\eta(u_1v_1)[\eta(v_1), x']$ has only one edge.*

Claim (3) follows from (7.2) applied to $e_1 = u_1v_1$ and $e_2 = e$.

(4) *The path $\eta(e)[\eta(v_1), x]$ has only one edge.*

To prove (4) suppose for a contradiction that $\eta(e)[\eta(v_1), x]$ has at least two edges. Then $x' = \eta(u_1)$ by (7.2) applied to $e_1 = e$ and $e_2 = u_1v_1$, and hence by (3) the path $\eta(u_1v_1)$ has only one edge. By the internal 4-connectivity of G there exists an η-bridge with one attachment in an internal vertex of $\eta(e)[\eta(v_1), x]$, and the other attachment not in $\eta(u_1v_1)$, contrary to (2). This proves (4).

By (3), (4) and the internal 4-connectivity of G there exists an η-bridge $B''' \neq B$ with one attachment x. The bridges B and B''' are elusive by (6.9), and hence are isomorphic to K_2. Thus the other attachment of B''' is not x', because G is simple. This, (3) and (b) imply that the other attachment of B''' does not belong to $\eta(u_1v_1)$. Since B''' is elusive, by (1) the other attachment of B''' is either $\eta(u_2)$ or belongs to an interior vertex of $\eta(u_2v_2)$, where u_2 is a neighbor of v_2, and v_2 has degree three. Since $\eta(e)$ has at least three edges, we see that $\eta(e)[\eta(v_2), x]$ has at least two edges, and hence the pair $(2, B''')$ satisfies (a), contrary to the choice of the pair $(1, B)$. □

Let H and G be internally 4-connected graphs, and let $\eta : H \hookrightarrow G$ be a homeomorphic embedding. Recall that an η-bridge B is *elusive* if it is trivial, and there exists a vertex $v \in V(H)$ of degree three, and two edges e_1, e_2 incident with v such that one attachment of B belongs to $V(\eta(e_1)) - \{\eta(v)\}$, and the other attachment belongs to $V(\eta(e_2)) - \{\eta(v)\}$. For $i = 1, 2$ let v_i be the end of e_i other than v. We say that v_1 and v_2 are the *foundations* of B, and that v is its *focus*. The foundations are unique by the internal 4-connectivity of H, but an elusive bridge can have several foci. We define the *multiplicity* of B to be the number of vertices of degree three in H that are adjacent to both foundations of B.

(7.4) *Let H and G be internally 4-connected graphs, let $\eta : H \hookrightarrow G$ be a lexicographically maximal homeomorphic embedding, let u be a vertex of H of degree three with neighbors u_1, u_2 and u_3, and let B be an elusive η-bridge with foundations u_1 and u_2 and focus u. Assume that assumptions* (4.1)(i)–(iv) *hold. If H is not strongly G-splittable, then there exist a vertex $u' \in V(H) - \{u, u_1, u_2, u_3\}$ and an elusive η-bridge B' such that one foot of B' is u or uu_1 or uu_2, and the other foot is u' or an edge incident with u'. Moreover, no edge of H is a foot of both B and B'.*

Proof. If some η-bridge has foot u, then that η-bridge satisfies the conclusion of the lemma by (6.1) and (6.2), because it is elusive by (6.9). We may therefore assume that no η-bridge has foot u. Let the attachments of B be $x_1 \in V(\eta(uu_1))$ and $x_2 \in V(\eta(uu_2))$. Let $P = \eta(uu_1)[\eta(u), x_1] \cup \eta(uu_2)[\eta(u), x_2]$. By the internal 4-connectivity of G some η-bridge has an attachment in an interior vertex of P. Let us choose such an η-bridge B' such that its attachment y that belongs to the interior of P is as close to u as possible, where the distance is measured on P. Let y' be the other attachment of B'. We claim that $y' \notin V(\eta(uu_1) \cup \eta(uu_2) \cup \eta(uu_3))$. To prove this claim suppose for a contradiction that $y' \in V(\eta(uu_j))$ for some $j \in \{1, 2, 3\}$. Then (7.2) implies that $\eta(uu_j)[\eta(u), y']$ has only one edge, and hence, by the internal 4-connectivity of G, some η-bridge has an attachment in $V(P[\eta(u), y]) - \{y\}$, contrary to the choice of B'. Thus $y' \notin V(\eta(uu_1) \cup \eta(uu_2) \cup \eta(uu_3))$. Since H is internally 4-connected, no edge of H has both ends in $\{u_1, u_2, u_3\}$, and hence there exists a vertex $u' \in V(H) - \{u, u_1, u_2, u_3\}$ such that u' and B' satisfy the first part of (7.4). Since y is an interior vertex of P (and hence $y \notin \{x_1, x_2\}$), it follows from (7.3) that no edge is a foot of both B and B', and hence u' and B' are as desired. □

8. MULTIPLICITY

The main result of this section, (8.7) below, states that if H, G are as in (4.2) and $\eta : H \hookrightarrow G$ is a lexicographically maximal homeomorphic embedding, then either the conclusion of (4.2) holds, or there is an elusive η-bridge of multiplicity one. We remark that the results of this section are about G-splittability, and not strong G-splittability.

(8.1) *Let H and G be internally 4-connected graphs, and let $\{x, y, z, w\}$ (in order) be the vertex-set of a circuit in H, where x and z have degree three. If H is not isomorphic to $K_{3,3}$ and $H + (x, z) + (y, w)$ is isomorphic to a minor of G, then H is G-splittable.*

Proof. Since x, z have degree three, and H is internally 4-connected and not isomorphic to $K_{3,3}$, we deduce that no vertex in $V(H) - \{y, w\}$ is violating in $H + (x, z)$. Thus $H + (x, z) + (y, w)$ is internally 4-connected, and hence H is G-splittable, as desired. □

(8.2) *Let H and G be internally 4-connected graphs, let $\eta : H \hookrightarrow G$ be a lexicographically maximal homeomorphic embedding, let u be a vertex of H of degree three with neighbors u_1, u_2 and u_3, and let B be an elusive η-bridge with foundations u_1 and u_2 and focus u. If every η-bridge has multiplicity at least two and H is not G-splittable, then there exist an integer $i \in \{1, 2\}$, a vertex $u' \in V(H) - \{u, u_1, u_2, u_3\}$ and an elusive η-bridge B' such that u' is adjacent to u_3 and u_i, the vertices u_3 and u_i both have degree three, one foot of B' is u or uu_i, and the other foot is u', $u'u_i$ or $u'u_3$. Moreover, the edge uu_i is not a foot of both B and B'.*

Proof. By (7.4) there exist a vertex $u' \in V(H) - \{u, u_1, u_2, u_3\}$ and an elusive η-bridge B' as in (7.4). Since B' has multiplicity at least two, at least two of the neighbors of u have degree three and are adjacent to u'. If u_1 and u_2 have that property, then H is G-splittable by (8.1) applied to the circuit with vertex-set $\{u_1, u', u_2, u\}$, a contradiction. Thus u_3 and one of u_1, u_2 have that property. Thus if the feet of B' are u and u', then the result holds. Otherwise it follows easily from (6.2). □

Let H be an internally 4-connected graph, and let $\gamma = (v_1, v_2, \ldots, v_{2n})$ be a cubic ladder chain in H. Let $\eta : H \hookrightarrow G$ be a homeomorphic embedding. We say that the sequence $(B_1, B_2, \ldots, B_{n-1})$ is an *η-cover* of γ if for all $i = 1, 2, \ldots, n-1$

(i) B_i is an elusive η-bridge with foundations v_{i+1} and v_{n+i},

(ii) v_i or v_{n+i+1} or both are foci of B_i, and

(iii) if $i > 1$, then the edge $v_i v_{n+i}$ is not a foot of both B_{i-1} and B_i.

We say that γ is *η-covered* if it has an η-cover.

(8.3) *Let H and G be internally 4-connected graphs such that assumptions* (4.1)(i) *and* (4.1)(ii) *are satisfied, let $\eta : H \hookrightarrow G$ be a homeomorphic embedding, and let $\gamma = (v_1, v_2, \ldots, v_{2n})$ be a cubic ladder chain in H of length at least two with η-cover $(B_1, B_2, \ldots, B_{n-1})$. If v_{2n} is not a focus of B_{n-1} and v_1 is not a focus of B_1, then H is G-splittable.*

Proof. Since v_1 is not a focus of B_1 we have that either

(1) *one foot of B_1 is $v_{n+1} v_{n+2}$*

or

(2) *one foot of* B_1 *is* v_2v_{n+2}.

Since v_{2n} is not a focus of B_{n-1} we have that either

(3) *one foot of* B_{n-1} *is* $v_{n-1}v_n$

or

(4) *one foot of* B_{n-1} *is* $v_{n-1}v_{2n-1}$.

Now we have four cases to distinguish, but two of them are symmetric. Assume first that (2) and (3) hold. Let $H' = H + (v_1v_2, v_{n+1}v_{n+2})$. Then H' is isomorphic to a minor of G. (To see this delete the interior vertices and all edges of the paths $\eta(v_{n+1}v_{n+2})$ and $\eta(v_iv_{2i})$ for $i = 3, 4, \ldots, n-1$ from the graph $\eta(H) \cup B_1 \cup B_2 \cup \ldots B_{n-1}$.) This contradicts (6.5).

The case when (1) and (4) hold is symmetric to the previous case, and so we assume that (1) and (3) hold. Again, we obtain a contradiction because H' is isomorphic to a minor of G (delete the interior vertices and edges of $\eta(v_iv_{n+i})$ for $i = 2, 3, \ldots, n-1$).

The last case is when (2) and (4) hold. The last condition in the definition of a cover implies that $n \geq 3$ in this case. It follows that again H' is isomorphic to a minor of G (delete the interior vertices and edges of $\eta(v_{n+1}v_{n+2})$, $\eta(v_{n-1}v_n)$, and $\eta(v_iv_{n+i})$ for all $i = 3, 4, \ldots, n-2$), and hence we obtain a contradiction using (6.5) as before. □

(8.4) *Let* H *and* G *be internally* 4*-connected graphs such that* H *is not isomorphic to* $K_{3,3}$ *or the cube, and assumptions* (4.1)(i), (4.1)(ii), (2.1)(i), *and* (2.1)(ii) *are satisfied, let* $\eta : H \hookrightarrow G$ *be a lexicographically maximal homeomorphic embedding, and let* γ *be an* η*-covered cubic ladder chain in* H *of length at least two. If* H *has no* η*-covered cubic biwheel chain of larger length, then either* H *is* G*-splittable or some* η*-bridge has multiplicity one.*

Proof. Suppose for a contradiction that H is not G-splittable, and that every η-bridge has multiplicity at least two. Let $\gamma = (v_1, v_2, \ldots, v_{2n})$, where $n \geq 3$, and let $(B_1, B_2, \ldots, B_{n-1})$ be an η-cover of γ. We may assume that H has no covered cubic ladder chain of larger length. We start with the following claim.

(1) *If v_{2n} is a focus of B_{n-1}, then v_1 is adjacent to v_{2n} and v_n is adjacent to v_{n+1}, and there exists an elusive bridge B_n with one foot v_{n+1} or $v_n v_{n+1}$ or $v_1 v_{n+1}$, and the other foot v_{2n} or $v_n v_{2n}$. Moreover, the edge $v_n v_{2n}$ is not a foot of both B_{n-1} and B_n.*

To prove (1) let y be the neighbor of v_{2n} other than v_n and v_{2n-1}. By (8.2) there exist vertices $z \in \{v_n, v_{2n-1}\}$ and $x \in V(H) - \{v_n, v_{2n-1}, v_{2n}, y\}$, and an elusive η-bridge B_n such that y and z have degree three, x is adjacent to both y and z, and one foot of B_n is v_{2n} or $v_{2n}z$, and the other foot is x, xy, or xz. Moreover, the edge zv_{2n} is not a foot of both B_{n-1} and B_n. From (8.1) applied to the circuit with vertex-set $\{v_{n-1}, v_n, v_{2n-1}, v_{2n}\}$ we deduce that $x \neq v_{n-1}$. We now distinguish two cases.

Assume first that $z = v_{2n-1}$. Since $x \neq v_{n-1}$, it follows that $x = v_{2n-2}$. If $y = v_{n-2}$, then $n = 3$ (because the vertices $v_1, v_2, \ldots, v_{2n}$ are pairwise distinct), and hence (6.4) implies that H is isomorphic to $K_{3,3}$, a contradiction. Thus $y \neq v_{n-2}$, but x is adjacent to y, and hence either $n = 3$, or $n \geq 4$ and $y = v_{n-3}$. In the latter case it follows by the same argument that H is isomorphic to the cube, a contradiction. Thus $n = 3$. Now $\gamma' = (v_4, v_n, v_1, v_2, v_5, v_6, y)$ is a cubic biwheel chain in H of length three, and hence is not η-covered by hypothesis. Since the sequence (B_1, B_2, B_3) is not a cover of γ' it follows that B_3 has feet v_4 and v_3v_6. Thus $H + (v_4, v_3v_6)$ is isomorphic to a minor of G. By (6.2) the vertex v_4 is adjacent to v_3, and v_3 has degree three in H. By (6.4) the graph H is isomorphic to $K_{3,3}$, a contradiction. This completes the case $z = v_{2n-1}$.

We may therefore assume that $z = v_n$. If $x, y \notin \{v_1, v_2, \ldots, v_{2n}\}$, then $(B_1, B_2, \ldots, B_n)$ is a cover of the cubic ladder chain $(v_1, v_2, \ldots, v_n, x, v_{n+1}, v_{n+2}, \ldots, v_{2n}, y)$, contrary to the maximality of n. By (6.4) the graph H is a planar cubic ladder or a Möbius cubic ladder with vertex-set $\{v_1, v_2, \ldots, v_{2n}\}$. If H is a cubic planar ladder, then the bridges $B_1, B_2, \ldots, B_n$ prove that the quartic planar ladder on the same number of vertices is isomorphic to a minor of G, contrary to (2.1)(i). Thus we may assume that H is a cubic Möbius ladder; that is, $x = v_{n+1}$ and $y = v_1$. Thus (1) holds.

From the symmetry between $(v_1, v_2, \ldots, v_{2n})$ and $(v_{2n}, v_{2n-1}, \ldots, v_1)$ and from (8.3) we may assume that v_{2n} is a focus of B_{n-1}. By (1) the vertex v_1 is adjacent to v_{2n} and v_n is adjacent to v_{n+1} (and hence H is a cubic Möbius ladder with vertex-set $\{v_1, v_2, \ldots, v_{2n}\}$

by (6.4)), and there exists an elusive bridge B_n with foundations v_{n+1} and v_{2n} such that v_n or v_1 or both are the foci of B_n, and the edge $v_n v_{2n}$ is not a foot of both B_{n-1} and B_n. Now there is symmetry between B_1 and B_n. Let us assume first that v_1 is a focus of B_n. By (1) applied to the cubic ladder chain $(v_2, v_3, \ldots, v_{2n}, v_1)$ we deduce that there exists an elusive bridge B_{n+1} with one foot v_1 or $v_1 v_{n+1}$ and the other foot v_{n+2} or $v_{n+1}v_{n+2}$ or $v_2 v_{n+2}$. If B_{n+1} has no foot in common with B_1, then the graph $H+(v_1, v_{n+2})+(v_2, v_{n+1})$ is isomorphic to a minor of G, contrary to the fact that H is not G-splittable. If B_{n+1} and B_1 share a common foot, then this common foot is $v_{n+1}v_{n+2}$, $v_2 v_{n+2}$, or $v_1 v_{n+1}$. Let $J = \eta(H) \cup B_1 \cup B_2 \cup \ldots \cup B_{n+1}$. In the first case the graph J has a minor isomorphic to the quartic Möbius ladder on $|V(H)|+1$ vertices, contrary to (2.1)(ii). The second and third case are symmetric, and so we may assume that the second case holds. Let L be the graph obtained from H by adding the edges $v_1 v_3$, $v_2 v_{n+1}$, $v_3 v_{n+2}, \ldots, v_{n+1} v_{2n}$. By adding the edges in the order listed we see that L is an addition extension of H. Let $\eta' : H \hookrightarrow G$ be the homeomorphic embedding obtained from η by rerouting $\eta(v_1 v_2)$ along B_{n+1}. By contracting all edges of the path $\eta(v_2 v_3)$ and considering the bridges $B_1, B_2, \ldots, B_n$ we see that L is isomorphic to a minor of G, contrary to the fact that H is not G-splittable. This completes the case when v_1 is a focus of B_n.

We may therefore assume that v_1 is not a focus of B_n. Thus either

(2) *one foot of* B_n *is* $v_n v_{n+1}$,

or

(3) *one foot of* B_n *is* $v_n v_{2n}$.

From the symmetry between B_1 and B_n we may also assume that v_1 is not a focus of B_1. Thus either

(4) *one foot of* B_1 *is* $v_{n+1} v_{n+2}$,

or

(5) *one foot of* B_1 *is* $v_2 v_{n+2}$.

We claim that the cubic Möbius ladder on $|V(H)|+2$ vertices is isomorphic to a minor of G. If (2) and (4) hold, then it follows by considering the η-bridges $B_1, B_2, \ldots, B_n$ and the path $\eta(v_1 v_{n+1})$. If (2) and (5) hold, then it follows by rerouting $\eta(v_{n+1}v_{n+2})$ along B_1, and considering the η-bridges $B_2, B_3, \ldots, B_n$, the path $\eta(v_1 v_{n+1})$ and a subpath of $\eta(v_2 v_{n+2})$. The case when (3) and (4) hold is symmetric to the case when (2) and (5) hold. Finally, when (3) and (5) hold, then the containment is seen by rerouting $\eta(v_n v_{n+1})$ along B_n, rerouting $\eta(v_{n+1}v_{n+2})$ along B_1, and considering subpaths of $\eta(v_n v_{2n})$, $\eta(v_1 v_{n+1})$, and $\eta(v_2 v_{n+2})$. Thus the cubic Möbius ladder on $|V(H)|+2$ vertices is isomorphic to a minor of G, contrary to (4.1)(ii). $\square$

Let H be an internally 4-connected graph, and let $\gamma = (u_1, u_2, v_1, ..., v_n)$ be a biwheel chain in H. Let $\eta : H \hookrightarrow G$ be a homeomorphic embedding. We say that the sequence $(B_2, B_3, \ldots, B_{n-1})$ is an *η-cover* of γ if for all $i = 2, 3, \ldots, n-1$

(i) B_i is an elusive η-bridge with foundations v_i and u_1 if i is even, and v_i and u_2 if i is odd,

(ii) v_{i-1} or v_{i+1} or both are foci of B_i, and

(iii) if $i > 2$, then the edge $v_{i-1}v_i$ is not a foot of both B_{i-1} and B_i.

We say that γ is *η-covered* if it has an η-cover.

(8.5) *Let H and G be internally 4-connected graphs such that assumptions (4.1)(i)–(iv) are satisfied, let $\eta : H \hookrightarrow G$ be a lexicographically maximal homeomorphic embedding, and let $\gamma = (u_1, u_2, v_1, v_2, \ldots, v_n)$ be a cubic biwheel chain in H of length at least two with η-cover $(B_2, B_3, \ldots, B_{n-1})$. If v_n is not a focus of B_{n-1} and v_1 is not a focus of B_2, then H is G-splittable.*

Proof. Let $j_1 = 1$ and $j_2 = 2$ if n is odd, and let $j_1 = 2$ and $j_2 = 1$ otherwise. Thus v_n is adjacent to u_{j_1}. Since v_n is not a focus of B_{n-1} we deduce that either

(1) *one foot of B_{n-1} is $v_{n-2}v_{n-1}$*

or

(2) *one foot of B_{n-1} is $u_{j_1}v_{n-2}$.*

Since v_1 is not a focus of B_2 we deduce that either

(3) *one foot of* B_1 *is* u_1v_3

or

(4) *one foot of* B_1 *is* v_2v_3.

Let H' be the graph defined in (7.1). We claim that H' is isomorphic to a minor of G. Assume first that (1) and (3) hold. Let η_1 be obtained from η by rerouting $\eta(v_2v_3)$ along B_2. If $n=4$, then the attachment of B_3 other than $\eta(u_2)$ belongs to a nontrivial η'-bridge, contrary to the lexicographic maximality of η. Thus $n \geq 5$, and our claim follows by considering $\eta(v_1u_1), \eta(v_2u_2), \eta_1(v_3u_1), B_3, B_4, \ldots, B_{n-1}, \eta(v_{n-1}u_{j_2}), \eta(v_{n-1}u_{j_1})$. This completes the case when (1) and (3) hold. The case when (2) and (4) hold is symmetric. Next we consider the case when (1) and (4) hold. Then our claim follows by considering $\eta(v_1u_1), \eta(v_2u_2), B_2, B_3, \ldots, B_{n-1}, \eta(v_{n-1}u_{j_2}), \eta(v_nu_{j_1})$. Finally, we consider the case when (2) and (3) hold. Let η_2 be obtained from η by rerouting $\eta(v_2v_3)$ along B_2. If $n=4$, then B_3 is an η_2-bridge with feet v_2u_2 and v_3v_4. Thus $H+(v_2u_2, v_3v_4)$ is isomorphic to a minor of G, contrary to (6.5). We may therefore assume that $n \geq 5$. Let η_3 be obtained from η_2 by rerouting $\eta_2(v_{n-2}v_{n-1})$ along B_{n-1}. If $n=5$, then B_3 is a subgraph of a nontrivial η_3-bridge, contrary to the lexicographic maximality of η. Thus $n \geq 6$. Now our claim that H' is isomorphic to a minor of G follows by considering η_3 and the paths $\eta(v_1u_1), \eta(v_2u_2), \eta_3(v_3u_1), B_3, B_4, \ldots, B_{n-2}, \eta_3(v_{n-2}u_{j_1}), \eta(v_{n-1}u_{j_2}), \eta(v_nu_{j_1})$. The claim, however, contradicts (7.1). □

(8.6) *Let H and G be internally 4-connected graphs such that H is not isomorphic to $K_{3,3}$ or the cube, and assumptions* (2.1)(i)–(iv) *and* (4.1)(i)–(iv) *are satisfied, let $\eta : H \hookrightarrow G$ be a lexicographically maximal homeomorphic embedding, and let γ be an η-covered cubic ladder or biwheel chain in H of length at least two. Then H is G-splittable or some η-bridge has multiplicity one.*

Proof. Let γ be the longest η-covered cubic ladder or biwheel chain in H, and suppose for a contradiction that H is not G-splittable. By (8.4) γ is a biwheel chain. Let $\gamma = (u_1, u_2, v_1, v_2, \ldots, v_n)$, where $n \geq 4$, and let $(B_2, B_3, \ldots, B_{n-1})$ be an η-cover of γ. Let v_0 be the neighbor of v_1 other than v_2 and u_1. By (8.5) and by replacing γ by

$(u_1,u_2,v_n,v_{n-1},\ldots,v_1)$ if n is odd or $(u_2,u_1,v_n,v_{n-1},\ldots,v_1)$ if n is even if necessary we may assume that v_1 is a focus of B_2.

By (8.2) there exist a vertex $z\in\{v_2,u_1\}$, a vertex $z'\in V(H)-\{v_0,v_1,v_2,u_1\}$ and an elusive η-bridge B such that z' is adjacent to z and v_0, the vertices z and v_0 both have degree three, one foot of B is v_1 or v_1z and the other foot is z' or $z'z$ or $z'v_0$, and the edge v_1z is not a foot of both B_2 and B. By (8.1) $z'\neq v_3$, and hence either $z=u_1$ in which case $z'\notin\{u_1,u_2,v_0,v_1,\ldots,v_4\}$, or $z=v_2$ in which case $z'=u_2$. (In the former case $z'\neq u_2$, because $z=u_1$ has degree three and H is not isomorphic to $K_{3,3}$.) In the former case v_1v_2 is not a foot of B by (6.2), and hence $(v_0,v_1,v_2,u_2,z',u_1,v_3,v_4)$ is a cubic ladder chain of length three with cover $(B,B_2,B_3,\ldots,B_{n-1})$, contrary to the maximality of γ. Thus $z=v_2$ and $z'=u_2$. If $v_0\notin\{u_1,u_2,v_1,v_2,\ldots,v_n\}$, then $(u_1,u_2,v_0,v_1,\ldots,v_n)$ is a cubic biwheel chain with cover $(B,B_2,B_3,\ldots,B_{n-1})$, contrary to the maximality of γ. Thus $v_0\in\{u_1,u_2,v_1,v_2,\ldots,v_n\}$, and by the internal 4-connectivity of H and the fact that the vertices $u_1,u_2,v_1,v_2,\ldots,v_n$ are pairwise distinct it follows that $v_0=v_n$. By the internal 4-connectivity of H it follows that H is a biwheel with vertex-set $\{u_1,u_2,v_1,v_2,\ldots,v_n\}$. Now we disregard the fact that v_1v_n is not a foot of B, and gain symmetry between B and B_{n-1} that way. If v_n is not a focus of B or B_{n-1}, then u_2v_2 or v_1v_2 is a foot of B, and $v_{n-2}u_2$ or $v_{n-2}v_{n-1}$ is a foot of B_{n-1}. In this case we obtain contradiction similarly as in the proof of (8.5). We omit the details.

Thus it follows that v_n is a focus of B or B_{n-1}, and from the symmetry we may assume that it is a focus of B. By (8.2) applied to $u=v_n$ there exists an elusive η-bridge B' with one foundation v_n and all the properties described in (8.2). Let z be the other foundation. Then $z=v_{n-2}$ or $z=u_1$. The first case cannot hold, because (8.1) implies that u_2 has degree three, which in turn implies that H is isomorphic to $K_{3,3}$ or the cube, and in the second case the quartic biwheel of the same type (planar or Möbius) is isomorphic to a minor of G, contrary to (2.1)(iii) and(2.1)(iv). □

(8.7) *Let H and G be internally 4-connected graphs such that H is not isomorphic to $K_{3,3}$ or the cube, and assumptions (2.1)(i)–(iv) and (4.1)(i)–(iv) are satisfied, and let $\eta: H\hookrightarrow G$ be a lexicographically maximal homeomorphic embedding. Assume that every η-bridge is* *elusive, and has multiplicity at least two. If there is an η-bridge, then H is G-splittable.*

Proof. Suppose for a contradiction that H is not G-splittable. Let B_1 be an η-bridge, and let v_2 and v_4 be its foundations. Since B_1 has multiplicity at least two, there exist vertices v_1 and v_5, both of degree three and both adjacent to both v_2 and v_4. We may assume that v_5 is a focus of B_1. Let v_6 be the neighbor of v_5 other than v_2 and v_4. By (8.2) there exist an integer $i \in \{2, 4\}$, a vertex $v_3 \in V(H) - \{v_2, v_4, v_5, v_6\}$, and an elusive η-bridge B_2 with foundations v_3 and v_5 such that v_3 is adjacent to v_i and v_6, the vertices v_i and v_6 have degree three, v_i or v_6 or both are foci of B_2, and the edge v_iv_5 is not a foot of both B_1 and B_2. Moreover, $v_3 \neq v_1$, because by (8.1) v_3 is not adjacent to both v_2 and v_4. From the symmetry between v_2 and v_4 we may assume that $i = 2$; then $(v_1, v_2, \ldots, v_6)$ is a cubic ladder chain of length two in H, and (B_1, B_2) is its η-cover. By (8.6) the graph H is G-splittable, as desired. □

9. BRIDGEWORKS

In this section we complete the proofs of (4.2) and (4.3). Let H and G be internally 4-connected graphs, and let $\eta : H \hookrightarrow G$ be a homeomorphic embedding such that every η-bridge is elusive. We say that a sequence $\beta = (B_0, B_1, \ldots, B_{n-1})$ is an *η-bridgework* if $n \geq 1$ and there exist vertices $u_0, v_0, u_1, v_1, \ldots, u_{n-1}, v_{n-1}, v_n, v_{n+1} \in V(H)$ such that for all integers $i = 0, 1, \ldots, n-1$

(i) B_i is an elusive η-bridge with foundations u_i and v_i and focus v_{i+1},

(ii) if $i > 0$, then v_i is a foot of B_i,

(iii) the vertices $u_0, v_0, v_1, \ldots, v_n$ are pairwise distinct,

(iv) v_{i+1} has degree three and its neighbors are u_i, v_i, and v_{i+2}, and

(v) either B_0 has multiplicity one, or at least one foot of B_0 is an edge.

See Figure 9. We say that β is an η-bridgework *based* at $u_0, v_0, u_1, v_1, \ldots, u_{n-1}, v_{n-1}, v_n, v_{n+1}$. If B_0 has multiplicity one, then we say that β is *stationary*. If at least one foot of B_0 is an edge, then we say that β is a *sliding* bridgework.

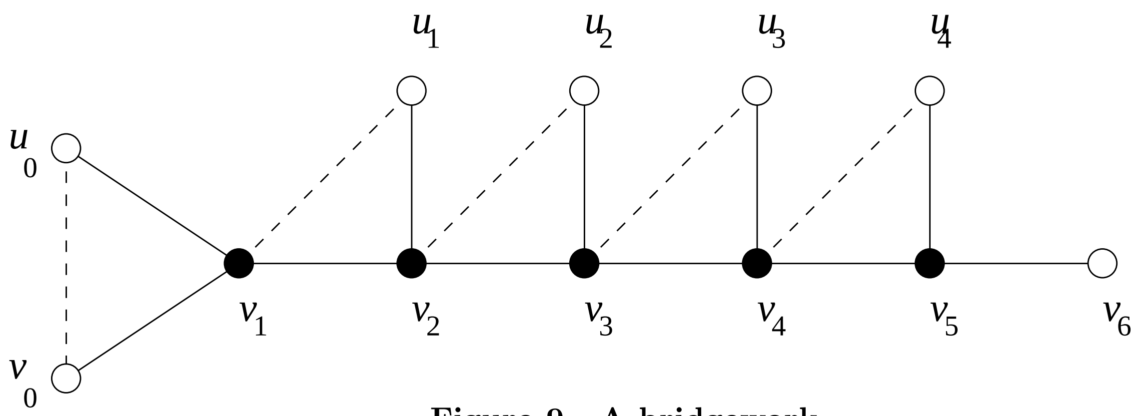

Figure 9. A bridgework.

Let $\eta : H \hookrightarrow G$ be a homeomorphic embedding, and let $\beta = (B_0, B_1, \ldots, B_{n-1})$ be an η-bridgework based at $u_0, v_0, u_1, v_1, \ldots, u_{n-1}, v_{n-1}, v_n, v_{n+1}$. We say that an elusive bridge B is an *η-extension* of β if there exists a vertex $u \in V(H) - \{u_{n-1}, v_{n-1}, v_n, v_{n+1}\}$ such that one foot of B is v_n or $v_n u_{n-1}$ or $v_n v_{n-1}$, the other foot is u or an edge incident with u, and no edge is a foot of both B_{n-1} and B. We say that B is *regressive* if either v_n is not a foot of B, or one foot of B is v_n and the other foot is an edge not incident with v_{n+1}. We say that the η-extension B is *stable* if one foot of B is v_n, and the other foot, say x, satisfies the property that if x is a vertex adjacent to v_{n+1} or an edge incident with v_{n+1}, then either v_{n+1} has degree at least four or $v_{n+1} \in \{u_0, v_0, u_1, v_1, \ldots, u_{n-1}, v_{n-1}\}$. We say that B is *strongly stable* if one foot of B is v_n, and the other foot, say x, satisfies the property that if x is a vertex adjacent to v_{n+1} or an edge incident with v_{n+1}, then v_{n+1} has degree at least four.

(9.1) *Let H and G be internally 4-connected graphs, let $\eta : H \hookrightarrow G$ be a lexicographically maximal homeomorphic embedding, and let β be an η-bridgework. If H is not strongly G-splittable, then β has an η-extension.*

Proof. The η-bridge guaranteed by (7.4) (with $u = v_n$) is an η-extension of β, as required. □

(9.2) *Let H and G be internally 4-connected graphs such that assumptions (4.1)(i)–(iv) are satisfied, let $\eta : H \hookrightarrow G$ be a lexicographically maximal homeomorphic embedding, and let β be an η-bridgework. If β has a regressive extension, then H is strongly G-splittable.*

Proof. Let $\beta = (B_0, B_1, \ldots, B_{n-1})$, let β be based at $u_0, v_0, u_1, v_1, \ldots, u_{n-1}, v_{n-1}, v_n, v_{n+1}$, and let B be a regressive η-extension of β. Suppose for a contradiction that H is not strongly G-splittable. We may assume that among all triples η, β, B as above we have chosen one with n minimum.

We wish to define $z \in V(H)$ and $e, f \in E(H)$. If v_n is a foot of B, then let $e \in E(H)$ be the other foot of B. By (6.2) one end of e, say z, has degree three and belongs to $\{u_{n-1}, v_{n-1}\}$. Let f denote the edge $v_n z$. If v_n is not a foot of B, then let $z \in \{u_{n-1}, v_{n-1}\}$ be such that $v_n z$ is a foot of B, and let x be the other foot of B. By (6.9) x is a vertex adjacent to or an edge incident with an end of $v_n z$ of degree three. It follows from the definition of an extension that this end is z; let e and f both denote the edge xz if x is a vertex, and the edge x otherwise. This completes the definition of z, e, f. Using the definition of z, e, f, the next paragraph combines five cases into one. The reader may wish to draw a separate picture for each of those cases.

Now let η' be the homeomorphic embedding obtained from η by rerouting $\eta(f)$ along B. Then B_{n-1} is an η'-bridge with one foot e', where e' is the edge of H incident with z other than e and zv_n. If $n \geq 2$, then $\beta' = (B_0, B_1, \ldots, B_{n-2})$ is an η'-bridgework. By (7.3) the η'-bridge B_{n-1} is an η'-extension of β'. It follows that either B_{n-1} is a regressive η'-extension of β', or B_{n-1} is a subset of a nontrivial η'-bridge. The first alternative contradicts the choice of η, β, B, and the second alternative contradicts the lexicographic maximality of η. Thus $n = 1$. Let z' be the member of $\{u_0, v_0\} - \{z\}$. Then the η'-bridge B_0 witnesses that $H + (z', e')$ is isomorphic to a minor of G, and hence (6.2) implies that z'' has degree three and is adjacent to z', where z'' is the end of e' other than z. But z'' and v_1 both have degree three, and both are adjacent to the foundations of B_0. Thus B_0 has multiplicity at least two, and hence, by the definition of bridgework, at least one foot of B_0 is an edge. If $v_1 z$ is a foot of B_0, then $v_1 z$ is not a foot of B, and hence v_1 is a foot of B. This contradicts (7.2) applied to $u = z$, $e_1 = f$, $e_2 = e$, and the η-bridge B. Thus $v_1 z'$ is a foot of B_0, and hence the η'-bridge B_0 witnesses that $H + (z' v_1, e')$ is isomorphic to a minor of G, contrary to (6.5). □

(9.3) *Let H and G be internally 4-connected graphs such that assumptions (4.1)(i)–(iv) are satisfied, and such that every component of the subgraph of H induced by vertices of degree three is a tree or a circuit. Let $\eta : H \hookrightarrow G$ be a lexicographically maximal homeomorphic embedding, and let β be a sliding η-bridgework. If β has a strongly stable extension, then H is strongly G-splittable.*

Proof. Let $\beta = (B_0, B_1, \ldots, B_{n-1})$, let β be based at $u_0, v_0, u_1, v_1, \ldots, u_{n-1}, v_{n-1}, v_n, v_{n+1}$, and let B_n be a strongly stable η-extension of β. Since every component of the subgraph of H induced by vertices of degree three is a tree or a circuit, we deduce that $v_i \neq u_j$ for all $i, j = 0, 1, \ldots, n$. Suppose for a contradiction that H is not strongly G-splittable. From the symmetry between u_0 and v_0 we may assume that v_0v_1 is a foot of B_0. Since B_n is stable, one of its feet is v_n; let x be the other foot. Then x is not a vertex adjacent to or an edge incident with v_n, and if x is a vertex adjacent to or incident with v_{n+1}, then v_{n+1} has degree at least four.

Let $\eta_0 = \eta$, and for $i = 1, 2, \ldots, n$ let η_i be obtained from η_{i-1} by rerouting $\eta(u_{i-1}v_i)$ along B_{i-1}. Then B_n is an η_n-bridge with one foot v_nv_{n+1}, and so by (6.9) its other foot is a vertex adjacent to or an edge incident with an end of v_nv_{n+1} of degree three. It follows that the other foot is v_{n-1} or $v_{n-1}v_n$. Thus $x = v_{n-1}$ or $x = v_{n-1}v_{n-2}$. Let η' be obtained from η_{n-1} by rerouting $\eta_{n-1}(v_{n-1}v_n)$ along B_n; then $\eta(v_{n-1})$ is a vertex of a nontrivial η'-bridge, contrary to the lexicographical maximality of η. □

(9.4) *Let H and G be internally 4-connected graphs such that assumptions (4.1)(i)–(iv) and (2.1)(iii)–(iv) are satisfied, and such that every component of the subgraph of H induced by vertices of degree three is a tree or a circuit. Let $\eta : H \hookrightarrow G$ be a lexicographically maximal homeomorphic embedding. If there exists an η-bridge B such that at least one foot of B is an edge, then H is strongly G-splittable.*

Proof. Suppose for a contradiction that H is not strongly G-splittable. Let B_0 be an η-bridge with foot v_0v_1, where v_0, v_1 are two adjacent vertices of H. By (6.9) we may assume that v_1 has degree three and that the other foot of B_0 is u_0 or u_0v_1, where u_0 is a neighbor of v_1. Let v_2 be the third neighbor of v_1.

Let η_1 be obtained from η by rerouting $\eta(u_0v_1)$ along B_0. From (7.4) applied to η_1 and the η_1-bridge $\eta(u_0v_1)$ we deduce that there exist a vertex $u \in V(H) - \{v_0, v_1, v_2, u_0\}$ and an η_1-bridge B_0' such that one foot of B_0' is v_1 and the other foot is u or an edge incident with u. (Notice that u_0v_1 cannot be a foot of B_0' because B_0 has only one edge.) Then B_0' is also an η-bridge, and as such one of its feet is v_0v_1, and the other is u or an edge incident with u. It follows from (6.9) that v_0 has degree three and is adjacent to u. Let u' be the third neighbor of v_0.

Now (B_0') is a sliding η-bridgework based at v_1, u, v_0, u'. By (9.1) it has an η-extension; by (9.2) the extension is not regressive, and by (9.3) it is not strongly stable. Thus u' has degree three.

Similarly, (B_0) is a sliding η-bridgework based at u_0, v_0, v_1, v_2. Let n be the maximum integer such that there exist η-bridges $B_1, B_2, \ldots, B_{n-1}$ and vertices $u_1, u_2, \ldots, u_{n-1}$ and $v_3, v_4, \ldots, v_{n+1}$ such that $\beta = (B_0, B_1, \ldots, B_{n-1})$ is a sliding η-bridgework based at $u_0, v_0, u_1, v_1, \ldots, u_{n-1}, v_{n-1}, v_n, v_{n+1}$. By (9.1) β has an η-extension B_n. By (9.2) B_n is not regressive, and by (9.3) it is not strongly stable. Thus the maximality of n implies that $v_{n+1} \in \{u_0, v_0, u_1, v_1, \ldots, u_{n-1}, v_{n-1}\}$. Since every component of the subgraph of H induced by vertices of degree three is a tree or a circuit and u' has degree three, we see that $v_{n+1} = v_0$. Let us put $u_n = u$. By a similar argument there exists a sliding bridgework $\beta' = (B_0', B_1', \ldots, B_{n-1}')$ based at $u_n, v_1, u_{n-1}, v_0, u_{n-2}, v_n, u_{n-3}, v_{n-1}, \ldots, u_1, v_3, v_2, v_1$. Likewise, β' has an η-extension B_n' which is neither regressive, nor stable, and hence one of its feet is v_2, and the other foot is u_0 or u_0v_1.

The homeomorphic embedding η_1 has been defined above. For $i = 2, 3, \ldots, n+1$ let η_i be obtained from η_{i-1} by rerouting $\eta(u_{i-1}v_i)$ along B_{i-1}. For $i = 1, 2, \ldots, n$ the graph B_{n+1-i}' is an η_{i+1}-bridge, and as such has one foot $v_{i+1}v_{i+2}$, and the other foot is u_{i-1} or $u_{i-1}v_i$, where we define $v_{n+2} = v_1$ and $u_{n+1} = u_0$. By (6.9) $u_{i+1} = u_{i-1}$. It follows that H is a cubic biwheel, and that the quartic biwheel on the same number vertices and of the same type (i.e., planar or Möbius) is isomorphic to a minor of G, contrary to (2.1)(iii) and (2.1)(iv). □

Let $\eta : H \hookrightarrow G$ be a lexicographically maximal homeomorphic embedding, and let

$\beta = (B_0, B_1, \dots, B_{n-1})$ be an η-bridgework based at $u_0, v_0, u_1, v_1, \dots, u_{n-1}, v_{n-1}, v_n, v_{n+1}$. We say that β is *η-optimal* if

(i) either v_0 is a foot of B_0, or there is no lexicographically maximal homeomorphic embedding $\eta' : H \hookrightarrow G$ such that $\eta'(x) = \eta(x)$ for $x \in \{u_0, v_1, u_0v_1\}$ and $\eta'(v_0v_1)$ is a proper subgraph of $\eta(v_0v_1)$, and

(ii) for all $i = 0, 1, \dots, n-1$, either u_i is a foot of B_i, or there is no lexicographically maximal homeomorphic embedding $\eta' : H \hookrightarrow G$ such that $\eta'(x) = \eta(x)$ for all vertices $x \in \{u_0, v_0, u_1, v_1, \dots, u_{i-1}, v_{i-1}, v_i, v_{i+1}\}$ and all edges x with both ends in that set, and $\eta'(u_iv_{i+1})$ is a proper subgraph of $\eta(u_iv_{i+1})$.

(9.5) *Let H and G be internally 4-connected graphs such that (4.1)(i)–(iv) and (2.1)(i)–(iv) are satisfied, H is not isomorphic to G and such that there exists a homeomorphic embedding $H \hookrightarrow G$. If H is not G-splittable, then there exists a lexicographically maximal homeomorphic embedding $\eta : H \hookrightarrow G$ and an η-optimal stationary η-bridgework.*

Proof. Let $\eta : H \hookrightarrow G$ be a lexicographically maximal homeomorphic embedding. By (8.7) there exists an η-bridge B of multiplicity one. The η-bridge B is elusive by (6.9), and hence there exists a vertex $v_1 \in V(H)$ of degree three and two neighbors u_0, v_0 of v_1 such that B has foundations u_0, v_0 and focus v_1. Let us choose η and B in such a way that as many feet of B as possible are vertices. Then the sequence with sole term B is an η-optimal stationary η-bridgework by (7.3), as desired. □

(9.6) *Let H and G be internally 4-connected graphs such that (4.1)(i)–(iv) and (2.1)(i)–(iv) are satisfied, let $\eta : H \hookrightarrow G$ be a lexicographically maximal homeomorphic embedding, and let β be an η-optimal stationary η-bridgework. If β has a stable extension, then H is G-splittable.*

Proof. Suppose for a contradiction that H is not G-splittable. Let $\beta = (B_0, B_1, \dots, B_{n-1})$ be based at $u_0, v_0, u_1, v_1, \dots, u_{n-1}, v_{n-1}, v_n, v_{n+1}$, and let B_n be a stable extension of β. Then one foot of B_n is v_n. Let $u_n \in V(H) - \{v_n, v_{n-1}, u_{n-1}, v_{n+1}\}$ be such that the other foot is u_n or an edge incident with u_n. Then if u_n is adjacent to v_{n+1}, then either v_{n+1} has degree at least least four, or $v_{n+1} \in \{u_0, v_0, u_1, v_1, \ldots, u_{n-1}, v_{n-1}\}$. Let L be the graph obtained from H by adding the edges $u_0v_0, u_1v_1, \ldots, u_nv_n$. By adding them in the order listed we see that L is an addition extension of H. For $i = 0, 1, \ldots, n$ the graph $\eta(H) \cup B_i$ proves that $H + (u_i, v_i)$ is isomorphic to a minor of G. Likewise, we would like to conclude that L is isomorphic to a minor of G. To prove that we need to show that if for some integers i, j with $0 \le i < j \le n$ we have $u_i = v_{j+1}$ and $u_j = v_{i+1}$, then the edge u_iv_{i+1} is not a foot of both B_i and B_j, and that if $v_0 = v_{j+1}$ and $u_j = v_1$, then the edge v_0v_1 is not a foot of both B_0 and B_j. From the symmetry between u_0 and v_0 it suffices to prove the former. Suppose for a contradiction that the former does happen for some integers i, j. By (6.2) applied to $H + (v_j, u_jv_{j+1})$ we deduce that v_{j+1} has degree three (for $j = n$ this does not follow from the definition of bridgework). Let η' be the homeomorphic embedding obtained from η by rerouting $\eta(v_jv_{j+1})$ along B_j. Then η' contradicts the η-optimality of β, because $\eta'(u_iv_{i+1})$ is a proper subpath of $\eta(u_iv_{i+1})$. Thus L is an addition extension of H isomorphic to a minor of H, a contradiction. □

Proof of (4.2). Let H and G be as in the statement of (4.2), and suppose for a contradiction that H is not G-splitable. By (9.5) there exists a lexicographically maximal homeomorphic embedding $\eta : H \hookrightarrow G$ and an η-optimal stationary η-bridgework $\beta = (B_0, B_1, \ldots, B_{n-1})$. Let us choose η and β with n maximum. Let β be based at $u_0, v_0, u_1, v_1, \ldots, u_{n-1}, v_{n-1}, v_n, v_{n+1}$.

By (9.1) there exists an η-extension B_n of β. By (9.2) B_n is not regressive, and by (9.6) it is not stable. Thus v_{n+1} has degree three, $v_{n+1} \notin \{u_0, v_0, u_1, v_1, \ldots, u_{n-1}, v_{n-1}\}$, one foot of B_n is v_n, and the other is u_n or u_nv_{n+1}, where u_n is a neighbor of v_{n+1} other than v_n. Then $\beta' = (B_0, B_1, \ldots, B_n)$ is an η-bridgework. By the maximality of n the bridgework β' is not η-optimal. Thus u_n is not a foot of B_n, and there exists a lexicographically maximal homeomorphic embedding $\eta' : H \hookrightarrow G$ such that $\eta'(x) = \eta(x)$ for every $x \in \{u_0, v_0, \ldots, u_{n-1}, v_{n-1}, v_n, v_{n+1}\}$ and for every edge x with both ends in that set, and $\eta'(v_{n+1}u_n)$ is a proper subset of $\eta(v_{n+1}u_n)$. Then β' is an η'-bridgework. By (7.3) $\eta'(v_{n+1}u_n)$ has only one edge, and hence β' is η'-optimal, contrary to the maximality of n. □

Proof of (4.3). Let H and G be as in the statement of (4.3). We proceed by induction on $|E(G)| - |E(H)|$. Let $\eta : H \hookrightarrow G$ be a lexicographically maximal homeomorphic embedding. Since H is isomorphic to a subdivision of a proper subgraph of G, there exists at least one η-bridge.

Suppose first that $|V(G)| = |V(H)|$. By (4.2) the graph H is G-splittable, and hence either H is strongly G-splittable, or there exists an addition extension H' of H such that H' is isomorphic to a minor of G. Then either H' is isomorphic to G, or by the induction hypothesis H' is strongly G-splittable. In the former case the result holds, and so we assume the latter. But $|V(G)| = |V(H')|$, and hence G is isomorphic to H' or to an addition extension of H'. In either case, G is isomorphic to an addition extension of H, and hence H is strongly G-extendable, as desired.

Thus we may assume that $|V(G)| > |V(H)|$. Since every η-bridge is trivial by (6.9), and G is 3-connected, it follows that for some edge e of H the path $\eta(e)$ has at least one internal vertex. Since G is 3-connected, some η-bridge B has an attachment in that internal vertex. Thus e is a foot of B, and hence the result follows from (9.4). □

References

1. D. Archdeacon, A Kuratowski theorem for the projective plane, Ph.D. thesis, Ohio State University 1980.
2. S. Arnborg and A. Proskurowski, Characterization and recognition of partial 3-trees, *SIAM J. Alg. Disc. Meth.* **7** (1986), 305–314.
3. J. Geelen and G. Whittle, Matroid 4-connectivity: a deletion-contraction theorem, manuscript.
4. R. Halin, Über einen graphentheoretischen Basisbegriff und seine Anwendung auf Färbungsprobleme, Diss. Köln, 1962.
5. R. Halin, Über einen Satz von K. Wagner zum Vierfarbenproblem, *Math. Ann.* **153** (1964), 47–62.

6. R. Halin, Zur Klassifikation der endlichen Graphen nach H. Hadwiger und K. Wagner, *Math. Ann.* **172** (1967), 46–78.
7. R. Halin and K. Wagner, Homomorphiebasen von Graphenmengen, *Math. Ann.* **147** (1962), 126–142.
8. P. Hliněný and R. Thomas, On possible counterexamples to Negami's planar cover conjecture, manuscript.
9. A. K. Kelmans, On 3-connected graphs without essential 3-cuts or triangles, *Soviet Math. Dokl.* **33** (1986), 698–703.
10. K. Kuratowski, Sur le problème des courbes gauches en topologie, *Fund. Math.* **15** (1930), 271–283.
11. W. McCuaig, Brace generation, *J. Graph Theory* **38** (2001), 124–169.
12. S. Negami, The spherical genus and virtually planar graphs, *Discrete Math.* **70** (1988), 159–168.
13. T. Politof and A. Satyanarayana, The structure of quasi 4-connected graphs, *Discrete Math.* **161** (1996), 217–228.
14. N. Robertson, Minimal cyclic-4-connected graphs, *Trans. Amer. Math. Soc.* **284** (1984), 665–687.
15. N. Robertson, The structure of graphs not topologically containing the Wagner graph, manuscript.
16. N. Robertson, P. D. Seymour and R. Thomas, Cyclically 5-connected cubic graphs, manuscript.
17. P. D. Seymour, Decomposition of regular matroids, *J. Combin. Theory Ser. B* **28** (1980), 305–359
18. W. T. Tutte, A theory of 3-connected graphs, *Indag. Math.* **23** (1961), 441-455.
19. K. Wagner, Über eine Eigenschaft der ebenen Komplexe, *Math. Ann.* **114** (1937), 570-590.
20. K. Wagner, Über eine Erweiterung des Satzes von Kuratowski, *Deutsche Math.* **2** (1937), 280–285.
21. K. Wagner, Bemerkungen zu Hadwigers Vermutung, *Math. Ann.* **141** (1960), 433-451.