ON THE GENUS OF A RANDOM GRAPH

Vojtěch Rödl*
Department of Mathematics and Computer Science
Emory University
Atlanta, GA 30332

and

Robin Thomas**
School of Mathematics
Georgia Institute of Technology
Atlanta, GA 30332–0160

October 1992, revised 8 February 1994.
Published in *Random Structures Algorithms* **6** (1995), no. 1, 1–12.

* Supported in part by NSF under Grant No. DMS-9011850
** Supported in part by NSF under Grant No. DMS-9103480

ABSTRACT

Let $p = p(n)$ be a function of n with $0 < p < 1$. We consider the random graph model $\mathcal{G}(n,p)$; that is, the probability space of simple graphs with vertex–set $\{1, 2, \ldots, n\}$ where two distinct vertices are adjacent with probability p and for distinct pairs these events are mutually independent. Archdeacon and Grable have shown that if $p^2(1-p^2) \geq 8(\log n)^4/n$, then the (orientable) genus of a random graph in $\mathcal{G}(n,p)$ is $(1+o(1))pn^2/12$. We prove that for every integer $i \geq 1$, if $n^{-i/(i+1)} \ll p \ll n^{-(i-1)/i}$, then the genus of a random graph in $\mathcal{G}(n,p)$ is $(1+o(1))\frac{i}{4(i+2)}\, pn^2$. If $p = cn^{-(i-1)/i}$, where c is a constant, then the genus of a random graph in $\mathcal{G}(n,p)$ is $(1+o(1))g(i,c,n)pn^2$ for some function $g(i,c,n)$ with $\frac{1}{12} \leq g(i,c,n) \leq 1$, but for $i > 1$ we were unable to compute this function.

1. INTRODUCTION

In this paper all graphs are simple, circuit means simple circuit, and $p = p(n)$ is a function of n with $0 < p < 1$. The genus of a graph, denoted by $\gamma(G)$, is the smallest integer g such that G has an embedding in an orientable surface (= compact 2–manifold) of genus g. We shall give a detailed combinatorial definition of this concept in the next section, but let us state the results first. Let Π be a property of graphs. We say that $G \in \mathcal{G}(n,p)$ *almost surely* satisfies Π if the probability that G satisfies Π tends to 1 as $n \to \infty$. The following is a result of [2].

(1.1) *If $\epsilon > 0$ and $p^2(1-p^2) \geq 8(\log n)^4/n$, then $G \in \mathcal{G}(n,p)$ almost surely satisfies $(1-\epsilon)\frac{1}{12}\,pn^2 \leq \gamma(G) \leq (1+\epsilon)\frac{1}{12}\,pn^2$.*

If $f, g : \mathbb{Z} \to (0,1)$ we write $f \ll g$ to mean that $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$. Our main result is the following.

(1.2) *Let $\epsilon > 0$, let $i \geq 1$ be an integer and assume that $n^{-i/(i+1)} \ll p \ll n^{-(i-1)/i}$. Then $G \in \mathcal{G}(n,p)$ almost surely satisfies $(1-\epsilon)\frac{i}{4(i+2)}\,pn^2 \leq \gamma(G) \leq (1+\epsilon)\frac{i}{4(i+2)}\,pn^2$.*

Let $k \geq 3$ be an integer, let $\epsilon > 0$ and consider an embedding of G in some surface. Let f_k be the number of faces in this embedding that are bounded by circuits of G of length k. We say that the embedding is an *ϵ–near k–gon embedding* if $kf_k \geq 2(1-\epsilon)|E(G)|$. We also prove the following, which implies Conjecture 1.4 of [2].

(1.3) *Let $\epsilon > 0$, let $i \geq 1$ and let $p \gg n^{-i/(i+1)}$. Then $G \in \mathcal{G}(n,p)$ almost surely has an ϵ–near $(i+2)$–gon embedding in some orientable surface.*

In fact, (1.2) is an easy consequence of (1.3), but we postpone this derivation until after we discuss the genus of a graph. Since $\gamma(G)$ satisfies the edge–Lipschitz property (cf. [1]), we deduce the following.

(1.4) *Let $c > 0$, let $\epsilon > 0$, let $i \geq 1$ be an integer, and let $p = cn^{-(i-1)/i}$. Then there exists a number $g(i,c,n)$ such that $\max\left(\frac{1}{12}\,, \frac{i-1}{4(i+1)}\right) \leq g(i,c,n) \leq \frac{i}{4(i+2)}$ and*

$$(1-\epsilon)g(i,c,n)pn^2 \leq \gamma(G) \leq (1+\epsilon)g(i,c,n)pn^2$$

almost surely for G in $\mathcal{G}(n,p)$.

It seems to be very difficult to compute $g(i,c,n)$ for $i > 1$, but (1.4) itself is an easy consequence of [1, Chapter 7] and (1.2), and so we omit the details.

A word about the non-orientable genus. Let $\kappa(G)$ denote the *cross-cap number* of G, that is, the smallest integer k such that G admits an embedding into a non-orientable surface of Euler characteristic $2-k$ (see [4]). Our proof of (1.2) can be easily adapted to yield the following result. We omit further details.

(1.5) *Let $\epsilon > 0$, let $i \geq 1$ be an integer and assume that $n^{-i/(i+1)} \ll p \ll n^{-(i-1)/i}$. Then $G \in \mathcal{G}(n,p)$ almost surely satisfies $(1-\epsilon)\frac{i}{2(i+2)}pn^2 \leq \kappa(G) \leq (1+\epsilon)\frac{i}{2(i+2)}pn^2$.*

The paper is organized as follows. In Section 2 we introduce the genus of a graph, and deduce (1.2) from (1.3). This is elementary. We also prove Lemma (2.1) which reduces finding a near k-gon embedding to finding a set of circuits with certain properties. In Section 3 we introduce our main tool, the near perfect matching theorems [3] and [5], and derive a technical lemma from them. In Section 4 we use this lemma and (2.1) to prove (1.3).

2. THE GENUS

A digraph is a pair $D = (V, A)$, where V, the set of *vertices*, is a finite set, and A, the set of *arcs*, is a set of ordered pairs of elements of V. We write $V(D) = V$ and $A(D) = A$. If G is a graph we define its *associated digraph* to be the digraph with vertex–set $V(G)$ and the arc–set the set of all $\overrightarrow{uv}, \overrightarrow{vu}$ for every pair u, v of adjacent vertices of G.

We shall define genus combinatorially. Let G be a graph, let D be the associated digraph and let e be an edge of G with endpoints u, v. An *orientation* of e is either $\overrightarrow{uv}$ or $\overrightarrow{vu}$. Thus every edge has two orientations. An *orientation* of G is an orientation of some edge of G and so $A(D)$ is the set of all orientations of G. Let $\Pi = (\Pi_v : v \in V(G))$ be such that Π_v is a cyclic permutation of the edges of G incident with v; then Π_v is called a *local rotation at* v and Π is called a *rotation scheme*. We say that an orientation $\overrightarrow{uv}$ Π*–follows* orientation $\overrightarrow{xy}$ if $y = u$ and the edge xy follows the edge uv in Π_u. The *faces* of Π are the equivalence classes of the transitive closure of the "Π–follows" relation. We say that an edge e of G with endpoints u, v is *incident* with a face F if either $\overrightarrow{uv} \in F$, or $\overrightarrow{vu} \in F$, or both. We denote the number of faces of Π by $f(\Pi)$. The *genus* of Π, denoted by $\gamma(\Pi)$, is defined by the formula

$$\gamma(\Pi) = \frac{1}{2}(|E(G)| - |V(G)| - f(\Pi) + k(G) + 1),$$

where $k(G)$ is the number of components of G. (This is just Euler's formula in disguise.) We define $\gamma(G)$, the *genus* of G, to be the smallest integer γ such that G has a rotation scheme of genus γ. The (fairly straightforward) proof that this definition is equivalent to the topological one mentioned earlier is omitted. We just remark that the equivalence follows from the following two facts:

(i) Every embedding of a graph in an orientable surface defines a natural rotation scheme, and the faces of this rotation scheme correpond to the usual "faces" of the embedding.

(ii) Given a rotation scheme Π of a graph G one can sew disks on the faces of Π and obtain an embedding of G in the orientable surface of genus $\gamma(\Pi)$.

Now we are ready to derive (1.2) from (1.3).

Proof of (1.2) (assuming (1.3)): To prove the lower bound let ϵ, i, p, n be as stated in (1.2), and let G be a member of $\mathcal{G}(n,p)$. The expected number of circuits G of length at most $i+1$ is

$$\sum_{j=3}^{i+1} \binom{n}{j} \frac{1}{2}(j-1)! p^j = o(n^2 p),$$

and so by Markov's inequality $G \in \mathcal{G}(n,p)$ almost surely has no more than $\frac{1}{8}\,\epsilon n^2 p$ circuits of length at most $i+1$. Also, $G \in \mathcal{G}(n,p)$ almost surely satisfies $|E(G)| \geq (1-\frac{\epsilon}{4})\frac{1}{2}\,n^2 p$. Let G be a member of $\mathcal{G}(n,p)$ with no more than $\frac{1}{8}\,\epsilon n^2 p$ circuits of length at most $i+1$ and with $|E(G)| \geq (1-\frac{\epsilon}{4})\frac{1}{2}\,n^2 p$. Let Π be a rotation scheme of G, and let f' be the number of faces of Π of cardinality at most $i+1$. Since every such face includes the edge–set of a circuit of length at most $i+1$, and a circuit is included in at most two faces we deduce that $f' \leq \frac{1}{4}\,\epsilon n^2 p$. Counting edge–face incidences in two ways we get $2|E(G)| \geq 3f' + (i+2)(f(\Pi) - f') = (i+2)f(\Pi) - (i-1)f'$. Now for sufficiently large n

$$\begin{aligned}
\gamma(\Pi) &= \frac{1}{2}(|E(G)| - |V(G)| - f(\Pi) + kG + 1) \\
&\geq \frac{1}{2}\left(\left(1 - \frac{2}{i+2}\right)|E(G)| - |V(G)| - \frac{i-1}{i+2}\,f'\right) \\
&\geq (1-\epsilon)\,\frac{i}{4(i+2)}\,n^2 p.
\end{aligned}$$

Thus $G \in \mathcal{G}(n,p)$ almost surely has genus at least $(1-\epsilon)\frac{i}{4(i+2)}\,n^2 p$.

To prove the upper bound we may assume that $\epsilon < 2$. Let G be a member of $\mathcal{G}(n,p)$ with $1 \leq |E(G)| \leq (1+\frac{\epsilon}{4})\frac{1}{2}pn^2$, let Π be the rotation scheme of an $\frac{\epsilon}{4}$–near $(i+2)$–gon embedding of G, and let f_{i+2} be the number of faces bounded by a circuit of length $i+2$. Then $(i+2)f_{i+2} \geq$

$2(1-\frac{\epsilon}{2})|E(G)|$, and hence

$$\begin{aligned}\gamma(\Pi) &= \frac{1}{2}(|E(G)| - |V(G)| - f(\Pi) + kG + 1) \\ &\le \frac{1}{2}\left(|E(G)| - n - (1-\frac{\epsilon}{4})\frac{2}{i+2}\,|E(G)| + n - 1 + 1\right) \\ &= \frac{i+\frac{\epsilon}{2}}{2(i+2)}|E(G)| \le \frac{i(1+\epsilon)}{4(i+2)}pn^2,\end{aligned}$$

as desired. □

If $k \ge 3$ is an integer and if D is a digraph, then a set $\mathcal{C}$ of directed circuits of D is called a k–*cycle* in D if every member of $\mathcal{C}$ has length k and every two distinct members of $\mathcal{C}$ are arc–disjoint. Let $l \ge 2$ be an integer, let $v, v_1, v_2, \ldots, v_l \in V(D)$, and let $\mathcal{C} = \{C_1, C_2, \ldots, C_l\}$ be a k–cycle such that $\overrightarrow{v_i v}, \overrightarrow{v v_{i+1}} \in A(C_i)$ for all $i = 1, 2, \ldots, l$ (where v_{l+1} means v_1). We say that $\mathcal{C}$ is a k–*blossom* of *length* l with *center* v and *tips* $v_1, v_2, \ldots, v_l$. We say that a k–cycle is *blossom–free* if it contains no blossom. The following proposition reduces the ϵ–near k–gon embedding problem to finding a blossom–free k–cycle.

(2.1) *Let $\epsilon > 0$, let $k \ge 3$ be an integer, let G be a graph, let D be its associated digraph, and let $\mathcal{C}$ be a blossom–free k–cycle in D. Then G has a rotation scheme Π such that every member of $\mathcal{C}$ is a face of Π. Hence if $|\mathcal{C}| \ge 2(1-\epsilon)|E(G)|/k$, then Π corresponds to an ϵ–near k–gon embedding.*

Proof. Let ϵ, k, G, D and $\mathcal{C}$ be as stated. For every $v \in V(G)$ we define an auxiliary directed graph D_v as follows. The vertex–set of D_v is the set of all neighbors of v in G, and we define $A(D_v)$ to be the set of all $\overrightarrow{xy}$ such that there exists $C \in \mathcal{C}$ with $\overrightarrow{xv}, \overrightarrow{vy} \in E(C)$. It follows that the in–degree and out–degree of every vertex of D_v is at most one, and since $\mathcal{C}$ is blossom–free we deduce that D_v contains no circuit. Thus D_v is a disjoint union of directed paths, and hence can be extended to a Hamiltonian circuit. Let $x_1, x_2, \ldots, x_l$ (in this order) be the vertices of one such Hamiltonian circuit. We define $\Pi_v = (x_1, x_2, \ldots, x_l)$ and $\Pi = (\Pi_v : v \in V(G))$. It is easy to see that $E(C)$ is a face of Π for every $C \in \mathcal{C}$. Hence Π is as desired. □

We remark that the case $k = 3$ of (2.1) is implicitly contained in [2].

3. NEAR PERFECT MATCHINGS

A *hypergraph* is a pair $\mathcal{H} = (V, E)$, where V, the set of *vertices*, is a finite set, and E, the set of *hyperedges*, is a set of subsets of V. We write $V(\mathcal{H}) = V$ and $E(\mathcal{H}) = E$. A hypergraph $\mathcal{H}$ is d–uniform, where d is an integer, if every member of $E(\mathcal{H})$ has cardinality d. The *degree* of

$x \in V(\mathcal{H})$, denoted by $\deg_{\mathcal{H}}(x)$ is the number of elements of $E(\mathcal{H})$ that contain x. The *pair–degree* of $x, y \in V(\mathcal{H})$, denoted by $\deg_{\mathcal{H}}(x,y)$, is the number of elements of $E(\mathcal{H})$ that contain both x and y. A *matching* in a hypergraph is a set of disjoint members of E. The following extension of the result of [3] is a special case of the result of [5].

(3.1) *For every $\epsilon > 0$ and for every integer $d \geq 1$ there exist a positive real number δ and an integer N_0 such that for every $N \geq N_0$ the following holds. If Δ is a real number and $\mathcal{H}$ is a d–uniform hypergraph with $|V(\mathcal{H})| = N$ and such that*
(i) $(1-\delta)\Delta \leq \deg_{\mathcal{H}}(x) \leq (1+\delta)\Delta$ *for every $x \in V(\mathcal{H})$, and*
(ii) $\deg_{\mathcal{H}}(x,y) < \delta\Delta$ *for every pair x, y of distinct elements of $V(\mathcal{H})$,*
then $\mathcal{H}$ has a matching of size at least $(1-\epsilon)N/d$.

We need the following slight strengthening.

(3.2) *For every $\epsilon > 0$ and every integer $d \geq 1$ there exist a positive real number δ and an integer N_0 such that for every $N \geq N_0$ the following holds. If Δ is a real number, and if $\mathcal{H}$ is a d–uniform hypergraph with $|V(\mathcal{H})| = N$ such that*
(i) $(1-\delta)\Delta \leq \deg_{\mathcal{H}}(x) \leq (1+\delta)\Delta$ *for every $x \in V \subseteq V(\mathcal{H})$, where $|V| \geq (1-\delta)N$,*
(ii) $\deg_{\mathcal{H}}(x,y) < \delta\Delta$ *for every two distinct elements $x, y \in V(\mathcal{H})$, and*
(iii) at most $\delta N\Delta$ members of $E(\mathcal{H})$ contain a vertex $x \in V(\mathcal{H})$ with $\deg_{\mathcal{H}}(x) > (1+\delta)\Delta$,
then $\mathcal{H}$ has a matching of size at least $(1-\epsilon)N/d$.

Proof. Let $\epsilon > 0$, let $d \geq 1$ be an integer, and let δ and N_0 be as in (3.1). Let $\delta' = \delta^2/(d+3)$; notice that $\delta \geq \delta'$, since obviously $\delta < 1$. We shall prove that if $N \geq N_0$ is an integer, Δ is a real number and $\mathcal{H}$ is a hypergraph with $|V(\mathcal{H})| = N$ such that (i), (ii), (iii) are satisfied with δ' in place of δ, then $\mathcal{H}$ has a matching of size at least $(1-\epsilon-\delta)N/d$. The theorem then follows, because δ can be assumed to be arbitrarily small. So let N, Δ and $\mathcal{H}$ be as stated, and let $\mathcal{H}'$ be the hypergraph obtained from $\mathcal{H}$ by deleting all members of $E(\mathcal{H})$ that contain a vertex $x \in V(\mathcal{H})$ with $\deg_{\mathcal{H}}(x) > (1+\delta')\Delta$. Then $|E(\mathcal{H})| \leq |E(\mathcal{H}')| + \delta' N\Delta$. Let M be the set of all vertices x of $\mathcal{H}'$ with $\deg_{\mathcal{H}'}(x) < (1-\delta)\Delta$, and let $m = |M|$. We have

$$\begin{aligned}(1-\delta')^2\Delta N &\leq \sum_{x \in V(\mathcal{H})} \deg_{\mathcal{H}}(x) = d|E(\mathcal{H})| \leq d|E(\mathcal{H}')| + d\delta' N\Delta \\ &= \sum_{x \in V(\mathcal{H}')} \deg_{\mathcal{H}'}(x) + d\delta' N\Delta \leq m(1-\delta)\Delta + (N-m)(1+\delta')\Delta + d\delta' N\Delta,\end{aligned}$$

and hence

$$m \leq \frac{(d+3)\delta'}{\delta + \delta'} N \leq \delta N.$$

Let k be an integer with $(1-\delta)\Delta \leq k \leq (1+\delta)\Delta$, let $\mathcal{H}''$ be a hypergraph obtained by taking d^k disjoint copies of $\mathcal{H}'$, and let $\mathcal{H}'''$ be a hypergraph obtained from $\mathcal{H}''$ by adding hyperedges incident only with vertices of $\mathcal{H}''$ that correspond to elements of M in such a way that $(1-\delta)\Delta \leq \deg_{\mathcal{H}'''}(x) \leq (1+\delta)\Delta$ for every $x \in V(\mathcal{H}''')$, and $\deg_{\mathcal{H}'''}(x,y) < \delta\Delta$ for every two distinct vertices $x, y \in V(\mathcal{H}''')$. This is clearly possible. By (3.1) there exists a matching $\mathcal{M}'''$ in $\mathcal{H}'''$ of size at least $(1-\epsilon)d^k N/d$. By construction, $\mathcal{M}'''$ has a subset $\mathcal{M}'' \subseteq E(\mathcal{H}'')$ of size at least $(1-\epsilon)d^k N/d - md^k/d \geq (1-\epsilon-\delta)d^{k-1}N$. Since $\mathcal{H}''$ consists of d^k disjoint copies of $\mathcal{H}'$, there is a subset $\mathcal{M}$ of $\mathcal{M}''$ of size at least $(1-\epsilon-\delta)N/d$ such that all members of $\mathcal{M}$ are contained in the same copy of $\mathcal{H}'$ in $\mathcal{H}''$. Now $\mathcal{M}$ is a desired matching in $\mathcal{H}$. □

For our application we need the following corollary of (3.2).

(3.3) *For every $\epsilon > 0$ and for every integer $d \geq 2$ there exist a positive real number δ and an integer N_0 such that for every $N \geq N_0$ the following holds. If Δ is a real number, and if $\mathcal{H}$ is a d–uniform hypergraph on N vertices such that*

(i) $(1-\delta)\Delta \leq \deg_{\mathcal{H}}(x) \leq (1+\delta)\Delta$ *for every* $x \in V \subseteq V(\mathcal{H})$*, where* $|V| \geq (1-\delta)N$,

(ii) $\deg_{\mathcal{H}}(x,y) < \delta\Delta$ *for every two distinct elements* $x, y \in V(\mathcal{H})$*, and*

(iii) at most $\delta N\Delta$ *members of* $E(\mathcal{H})$ *contain a vertex* $x \in V(\mathcal{H})$ *with* $\deg_{\mathcal{H}}(x) > (1+\delta)\Delta$,

then for every matching $\mathcal{M}$ in $\mathcal{H}$ there exists a matching $\mathcal{M}'$ in $\mathcal{H}$ with $\mathcal{M} \cap \mathcal{M}' = \emptyset$ and with $|\mathcal{M}'| \geq (1-\epsilon)N/d$.

Proof. Let $\epsilon > 0$, let $d \geq 2$ be an integer, and let δ, N_0 be as in (3.2). We claim that $\delta/2$ and N_0 satisfy (3.3). To prove this let $N \geq N_0$, let Δ be a real number and let $\mathcal{H}$ be a d–uniform hypergraph with $|V(\mathcal{H})| = N$ such that (i), (ii), (iii) are satisfied with $\delta/2$ in place of δ. From (i) and (ii) we deduce that $\delta\Delta > 2$. Let $\mathcal{M}$ be a matching in $\mathcal{H}$, and let $\mathcal{H}'$ be the hypergraph with $V(\mathcal{H}') = V(\mathcal{H})$ and $E(\mathcal{H}') = E(\mathcal{H}) - \mathcal{M}$. Then $\mathcal{H}'$ satisfies (i), (ii) and (iii) of (3.2), and hence there exists a matching $\mathcal{M}'$ in $\mathcal{H}'$ of size at least $(1-\epsilon)N/d$. Then $\mathcal{M}'$ is as desired. □

4. PROOF OF (1.3)

We need to work with random digraphs. We denote by $\mathcal{D}(n,p)$ the probability space of all digraphs D with vertex–set $\{1, 2, \ldots, n\}$ where for distinct vertices $u, v \in \{1, 2, \ldots, n\}$, $\overrightarrow{uv} \in A(D)$ with probability $p/2$, $\overrightarrow{vu} \in A(D)$ with probability $p/2$, and these two events are exclusive, and for distinct pairs u, v the the events that either $\overrightarrow{uv} \in A(D)$ or $\overrightarrow{vu} \in A(D)$ are mutually independent.

Let D be a member of $\mathcal{D}(n,p)$, let $u,v \in V(D)$, let $i \geq 1$ be an integer, and let $\epsilon > 0$. We say that $\overrightarrow{uv}$ is *(ϵ,i)–balanced* if the number of directed paths in D from v to u of length $i+1$ (that is, containing exactly $i+1$ arcs) is at least $(1-\epsilon)n^i(p/2)^{i+1}$ but no more than $(1+\epsilon)n^i(p/2)^{i+1}$; otherwise we say that $\overrightarrow{uv}$ is *(ϵ,i)–unbalanced*.

(4.1) *Let $i \geq 1$ be an integer, let $\epsilon > 0$, let $p \gg n^{-i/(i+1)}$, and let $u,v \in \{1,2,\dots,n\}$. Then $D \in \mathcal{D}(n,p)$ almost surely satisfies the following two conditions:*

(i) $\overrightarrow{uv}$ is (ϵ,i)–balanced, and

(ii) no more than ϵpn^2 arcs of D are (ϵ,i)–unbalanced.

Proof. Let $u,v \in \{1,2,\dots,n\}$, and let X be a random variable on $\mathcal{D}(n,p)$ defined by saying that $X(D)$ is the number of distinct directed paths of length $i+1$ between u and v in D. Then

$$EX = \binom{n-2}{i} i! \left(\frac{p}{2}\right)^{i+1} = n^i \left(\frac{p}{2}\right)^{i+1} + O(n^{i-1}p^{i+1}),$$

and

$$\begin{aligned} EX^2 &= \binom{n-2}{i} i! \left(\frac{p}{2}\right)^{i+1} \binom{n-i-2}{i} i! \left(\frac{p}{2}\right)^{i+1} + \sum_{j=1}^{i} O(n^{2i-j}p^{2i+2-j}) \\ &\leq n^{2i}\left(\frac{p}{2}\right)^{2i+2} + O(n^{2i-1}p^{2i+1}). \end{aligned}$$

Hence by Chebyshev's inequality for sufficiently large n

$$\begin{aligned} P\left[\left|X - n^i(p/2)^{i+1}\right| > \epsilon n^i (p/2)^{i+1}\right] &\leq P[|X-EX| \geq \frac{1}{2}\epsilon EX] \\ &\leq \frac{EX^2 - E^2X}{(\epsilon/2)^2E^2X} = O\left(\frac{1}{np}\right) = o(1). \end{aligned}$$

This proves (i). Now let a random variable Y be defined by saying that $Y(D)$ is the number of arcs of D that are not (ϵ,i)–balanced. Then for sufficiently large n

$$EY = \frac{1}{2}p\binom{n}{2}P\left[\left|X - n^i(p/2)^{i+1}\right| > \epsilon n^i(p/2)^{i+1}\right] \leq \frac{1}{4}pn^2 P\left[|X-EX| \geq \frac{\epsilon}{2}EX\right] = O(n),$$

and so by Markov's inequality

$$P[Y > \epsilon pn^2] \leq O\left(\frac{n}{\epsilon pn^2}\right) = O\left(\frac{1}{np}\right) = o(1),$$

as desired for (ii). $\square$

Let D be a directed graph, and let $\mathcal{P}$ be a set of directed paths in D. We say that the paths in $\mathcal{P}$ are *internally disjoint* if every vertex that belongs to two distinct paths in $\mathcal{P}$ is an endpoint of both.

(4.2) *Let $i \geq 1$, let $n^{-i/(i+1)} \ll p \ll n^{-(2i-1)/(2i+1)}$, and let $j \in \{0, 1, \ldots, i-1\}$. Then a digraph D in $\mathcal{D}(n,p)$ almost surely has the property that for every two vertices $u, v \in V(D)$ there do not exist $4i+2$ internally disjoint directed paths from u to v of length $j+1$.*

Proof. Let i, j, n, p be as stated in the lemma, and let D' be a digraph that consists of two vertices x, y and $4i+2$ internally disjoint directed paths from x to y of length $j+1$. The expected number of subdigraphs of D isomorphic to D' is at most

$$n^{2+j(4i+2)}(p/2)^{(j+1)(4i+2)} = o(1),$$

and so the result follows by Markov's inequality. □

The following fact is elementary and the proof is left to the reader.

(4.3) *For every integer $i \geq 1$ there exists an integer K such that for any digraph D and every two arcs $e, f \in A(D)$ the following holds. If D has K distinct directed circuits of length $i+2$ each containing both e and f, then there are vertices $u, v \in V(D)$ and an integer j such that $1 \leq j \leq i-1$ and D has $4i+2$ internally disjoint directed paths of length $j+1$ between u and v.*

(4.4) *For every integer $i \geq 1$ there exists an integer K such that if $n^{-i/(i+1)} \ll p \ll n^{-(2i-1)/(2i+1)}$ then $D \in \mathcal{D}(n,p)$ almost surely has the property that for every $e, f \in A(D)$ there are no more than K directed circuits of length $i+2$ in D containing both e and f.*

Proof. This follows immediately from (4.2) and (4.3). □

Let D be a digraph in $\mathcal{D}(n,p)$, let $\epsilon > 0$, let $i \geq 1$ be an integer, and let $u, v \in \{1, 2, \ldots, n\}$ (not necessarily $\overrightarrow{uv} \in A(D)$). We say that $\overrightarrow{uv}$ is (ϵ, i)–*overfull* if there are more than $(1+\epsilon)n^i \left(\frac{p}{2}\right)^{i+1}$ directed paths in D from v to u of length $i+1$. Thus if $\overrightarrow{uv}$ is (ϵ, i)–overfull, then it is (ϵ, i)–unbalanced.

(4.5) *Let $i \geq 1$ be an integer, let $\epsilon > 0$ and let $n^{-i/(i+1)} \ll p$. Then $D \in \mathcal{D}(n,p)$ almost surely has at most $\epsilon n^{i+2} p^{i+2}$ directed circuits of length $i+2$ that contain an (ϵ, i)–overfull arc.*

Proof. Let i, ϵ and p be as stated, let $D \in \mathcal{D}(n,p)$, let $u, v \in V(D)$, and let X be defined as in (4.1). Then for sufficiently large n

$$P[\overrightarrow{uv} \text{ is } (\epsilon, i)\text{-overfull} \mid A \subseteq A(D)] = P[\overrightarrow{uv} \text{ is } (\epsilon, i)\text{-overfull}] + o(1) = o(1),$$

because the former equality follows by an elementary (but unpleasant) calculation, and the latter follows from (4.1i).

Now let $Q = (v_1, v_2, \ldots, v_{i+2})$ be a cyclic permutation of a set $\{v_1, v_2, \ldots, v_{i+2}\} \subseteq \{1, 2, \ldots, n\}$, let A be the set of arcs $\{\overrightarrow{v_1v_2}, \overrightarrow{v_2v_3}, \ldots, \overrightarrow{v_{i+1}v_{i+2}}, \overrightarrow{v_{i+2}v_1}\}$, and let $B(Q)$ be the event that $A \subseteq A(D)$ and that at least one member of A is (ϵ, i)–overfull. Then (the summation is over all cyclic permutations of subsets of $\{1, 2, \ldots, n\}$ of size $i + 2$)

$$
\begin{aligned}
EX &= \sum_Q P[B(Q)] \\
&\leq \sum_Q P[A \subseteq A(D)](i+2)P[\overrightarrow{v_1v_2} \text{ is } (\epsilon, i)\text{–overfull} \mid A \subseteq A(D)] \\
&= o\left(n^{i+2}(\frac{p}{2})^{i+2}\right),
\end{aligned}
$$

and so the result follows by Markov's inequality. □

If A is the set of arcs of a digraph, then A^{-1} is the set of all arcs $\overrightarrow{uv}$ such that $\overrightarrow{vu} \in A$. If $D = (V, A)$ is a digraph we denote by D^{-1} the digraph (V, A^{-1}), and by $D \cup D^{-1}$ we denote the digraph with vertex–set $V(D) = V(D^{-1})$ and arc–set $A(D) \cup A(D^{-1})$. We recall that k–blossom was defined prior to (2.1). A k–blossom $\{C_1, C_2, \ldots, C_l\}$ in a digraph D is called *simple* if either $l \geq 3$, or $l = 2$ and $C_1 \neq C_2^{-1}$.

(4.6) *Let $i \geq 1$ be an integer, let $\epsilon > 0$, and let $n^{-i/(i+1)} \ll p \ll n^{-(i-\epsilon)/(i+1-\epsilon)}$. Then $D \in \mathcal{D}(n, p)$ almost surely has the property that the number of simple $(i+2)$–blossoms of length at most $1/\epsilon$ in $D \cup D^{-1}$ does not exceed ϵpn^2.*

Proof. Let j be an integer with $2 \leq j \leq 1/\epsilon$. The expected number of simple $(i + 2)$–blossoms of length j is

$$O(n^{1+j+j(i-1)}p^{j+ji}) = o(pn^2),$$

and so the lemma follows by Markov's inequality. □

(4.7) *Let $i \geq 1$, let $0 < \epsilon < 1/2$, and let $n^{-i/(i+1)} \ll p \ll n^{-(i-\epsilon)/(i+1-\epsilon)}$. Then for $D \in \mathcal{D}(n, p)$, $D \cup D^{-1}$ almost surely has a blossom–free $(i+2)$–cycle of size at least $2\left(1 - \frac{2i+5}{1-\epsilon}\epsilon\right)|A(D)|/(i+2)$.*

Proof. Let i, ϵ, p, α be as stated, let δ and N_0 be as in (3.3) with $d = i + 2$, and let K be as in (4.4). Since $\epsilon < 1/2$ we see that $p \ll n^{-(2i-1)/(2i+1)}$. By (4.1ii), (4.4), (4.5) and (4.6), $D \in \mathcal{D}(n, p)$ almost surely satisfies the following five conditions:

(1) the digraph D has no more than $\delta(1 - \epsilon)n^2p/2$ arcs that are (ϵ, i)–unbalanced,

(2) for every two arcs, $e, f \in A(D)$ there are at most K directed circuits of length $i + 2$ in D containing both e and f,

(3) no more than $\delta(1-\epsilon)(np/2)^{i+2}$ directed circuits of length $i+2$ in D contain a (δ, i)–overfull arc,

(4) $D \cup D^{-1}$ has no more than ϵpn^2 simple $(i+2)$–blossoms of length at most $1/\epsilon$, and

(5) $(1-\epsilon)\frac{1}{2}pn^2 \le |A(D)| \le (1+\epsilon)\frac{1}{2}pn^2$.

To complete the proof we will show that if $D \in \mathcal{D}(n,p)$ satisfies (1)-(5), then $D \cup D^{-1}$ has a blossom-free (i+2)-cycle of size at least $2\left(1-\frac{2i+5}{1-\epsilon}\epsilon\right)|A(D)|/(i+2)$. So let D be a digraph satisfying (1)-(5). Let $\mathcal{H}$ be the hypergraph with vertex–set $A(D)$ and edge–set the arc–sets of all directed circuits of D of length $i+2$. It follows from (1), (2), (3) and (5) that for sufficiently large n, $\mathcal{H}$ satisfies the hypotheses of (3.3) with $\Delta = n^i(p/2)^{i+1}$ and $d = i+2$, and so there exists a matching $\mathcal{M}$ in $\mathcal{H}$ with $|\mathcal{M}| \ge (1-\epsilon)|A(D)|/(i+2)$. Let $\mathcal{M}^{-1} = \{M^{-1} : M \in \mathcal{M}\}$, and let $\mathcal{H}^{-1}$ be defined as $\mathcal{H}$ with D^{-1} in place of D. Again, for sufficiently large n, $\mathcal{H}^{-1}$ satisfies the hypotheses of (3.3) and so there exists a matching $\mathcal{M}'$ in $\mathcal{H}^{-1}$ disjoint from $\mathcal{M}^{-1}$ with $|\mathcal{M}'| \ge (1-\epsilon)|A(D)|/(i+2)$. Let $\mathcal{M}_1 = \mathcal{M} \cup \mathcal{M}'$. Then $\mathcal{M}_1$ is an $(i+2)$–cycle in $D \cup D^{-1}$ with no blossoms of length 2 that are not simple. By (4) $\mathcal{M}_1$ contains a subset $\mathcal{M}_2$ with $|\mathcal{M}_2| \ge 2(1-\epsilon)|A(D)|/(i+2) - \epsilon pn^2 \ge 2\left(1-\frac{i+3}{1-\epsilon}\epsilon\right)|A(D)|/(i+2)$ that has no blossoms of length at most $1/\epsilon$. Since every arc of $D \cup D^{-1}$ is contained in at most one circuit of $\mathcal{M}_2$, every two blossoms in $\mathcal{M}_2$ with the same center have disjoint tips and thus it follows that if $v \in V(D)$ is incident with d arcs in D then there are at most ϵd blossoms of length exceeding $1/\epsilon$ in $\mathcal{M}_2$ with center v. Thus there are at most $2\epsilon|A(D)|$ blossoms of length exceeding $1/\epsilon$ in $\mathcal{M}_2$, and hence $\mathcal{M}_2$ has a blossom–free subset $\mathcal{M}_0$ of size at least $2\left(1-\frac{2i+5}{1-\epsilon}\epsilon\right)|A(D)|/(i+2)$, and so $\mathcal{M}_0$ is a desired blossom–free $(i+2)$–cycle. □

To prove (1.3) we need to reduce to the case when $p \ll n^{-(i-\epsilon)/(i+1-\epsilon)}$. We do that as follows. Let $t \ge 1$ be an integer. A t–labeled digraph is a t–tuple $L = (D_1, D_2, \dots, D_t)$ of arc–disjoint digraphs with $V(D_1) = V(D_2) = \cdots = V(D_t)$. We write $V(L) = V(D_1)$, $A(L) = A(D_1) \cup A(D_2) \cup \cdots \cup A(D_t)$. With every t–labeled digraph L we associate a graph $G(L)$ with vertex–set $V(L)$ and edge–set all pairs uv of distinct elements of $V(L)$ such that either $\overrightarrow{uv}$ or $\overrightarrow{vu}$ belongs to $A(L)$. We need to consider random t–labeled digraphs, as follows. Let $\mathcal{L}(n,p,t)$ be the probability space of all t–labeled digraphs $L = (D_1, D_2, \dots, D_t)$ with $V(L) = \{1, 2, \dots, n\}$ in which for distinct $u, v \in V(L)$ and $j = 1, 2, \dots, t$, $\overrightarrow{uv} \in A(D_j)$ with probability $\frac{p}{2t}$, $\overrightarrow{vu} \in A(D_j)$ with probability $\frac{p}{2t}$ and not both, and for different pairs u, v of vertices the events that either $\overrightarrow{uv} \in A(D_j)$ or $\overrightarrow{vu} \in A(D_j)$ are mutually independent. The following is straightforward.

(4.8) *Let Π be a property of graphs such that for $L \in \mathcal{L}(n,p,t)$, $G(L)$ almost surely has Π. Then $G \in \mathcal{G}(n,p)$ almost surely has Π.*

Now we are ready to prove (1.3), which we restate.

(4.9) *Let $\epsilon > 0$, let $i \geq 1$ and let $p \gg n^{-i/(i+1)}$. Then $G \in \mathcal{G}(n,p)$ almost surely has an ϵ-near $(i+2)$-gon embedding.*

Proof. Let i, ϵ, p be as stated, let $\epsilon_0, \epsilon_1 > 0$ be such that $\epsilon_1 < 1/2$, $\epsilon_0 = \frac{2i+5}{1-\epsilon_1}\epsilon_1$, and $2\epsilon_0 \leq \epsilon(1-\epsilon_0)$. We choose $t = t(n)$ such that t is an integer and $n^{-i/(i+1)} \ll p/t \ll n^{-(i-\epsilon_1)/(i+1-\epsilon_1)}$. By (2.1) and (4.8) it suffices to show that for $L \in \mathcal{L}(n,p,t)$, the digraph $(V(L), A(L) \cup A^{-1}(L))$ almost surely has a blossom-free $(i+2)$-cycle of size at least $2(1-\epsilon)|A(L)|/(i+2)$.

To this end let $L = (D_1, D_2, \ldots, D_t)$ be a member of $\mathcal{L}(n,p,t)$. We say that $j \in \{1, 2, \ldots, t\}$ is *L–successful* if $D_j \cup D_j^{-1}$ has a blossom-free $(i+2)$-cycle of size at least $2(1-\epsilon_0)|A(D_j)|/(i+2)$; otherwise we say that j is *L–unsuccessful*. By (4.7) there exists a function $q = q(n)$ with $q \to 0$ as $n \to \infty$ such that for $D \in \mathcal{D}(n, p/t)$, $D \cup D^{-1}$ has a blossom-free $(i+2)$-cycle of size at least $2(1-\epsilon_0)|A(D)|/(i+2)$ with probability at least $1-q$. The probability that for $L \in \mathcal{L}(n,p,t)$ there are more than $t\sqrt{q}$ integers that are L–unsuccessful is, by Markov's inequality, at most $\sqrt{q}$. Hence $L = (D_1, D_2, \ldots, D_t) \in \mathcal{L}(n,p,t)$ almost surely satisfies

(i) there are no more than $\epsilon_0 t$ integers $j \in \{1, 2, \ldots, t\}$ that are L–unsuccessful,

and (by standard results on binomial distributions)

(ii) $(1-\epsilon_0)\frac{1}{2}n^2p/t \leq |A(D_j)| \leq (1+\epsilon_0)\frac{1}{2}n^2p/t$ for all $j = 1, 2, \ldots, t$.

To complete the proof we will show that if a member L of $\mathcal{L}(n,p,t)$ satisfies (i) and (ii), then the digraph $(V(L), A(L) \cup A^{-1}(L))$ has a blossom–free $(i+2)$–cycle of size at least $2(1-\epsilon)|A(L)|/(i+2)$. So let $L = (D_1, D_2, \ldots, D_t) \in \mathcal{L}(n,p,t)$ satisfy (i) and (ii), and let J be the set of all L-successful integers $j \in \{1, 2, \ldots, t\}$. For $j \in J$ let $\mathcal{M}_j$ be a blossom-free $(i+2)$-cycle in $D_j \cup D_j^{-1}$ of size at least $2(1-\epsilon_0)|A(D_j)|/(i+2)$, and let $\mathcal{M} = \bigcup_{j \in J} \mathcal{M}_j$. Since $\overrightarrow{uv} \in A(D_j \cup D_j^{-1})$ if and only if $\overrightarrow{vu} \in A(D_j \cup D_j^{-1})$ we deduce that $\mathcal{M}$ is a blossom-free $(i+2)$-cycle in the digraph $(V(L), A(L) \cup A^{-1}(L))$. Moreover,

$$\begin{aligned}
(i+2)|\mathcal{M}| &\geq 2(1-\epsilon_0)\sum_{j \in J}|A(D_j)| = 2(1-\epsilon_0)|A(L)| - \sum_{j \notin J}|A(D_j)| \\
&\geq 2(1-\epsilon_0)|A(L)| - 2\epsilon_0 t(1+\epsilon_0)\frac{1}{2}n^2p/t \\
&\geq 2(1-\epsilon_0)|A(L)| - \frac{1+\epsilon_0}{1-\epsilon_0}\epsilon_0|A(L)| \\
&\geq 2\left(1 - \frac{2\epsilon_0}{1-\epsilon_0}\right)|A(L)| \geq 2(1-\epsilon)|A(L)|,
\end{aligned}$$

as required. □

References

1. N. Alon and J. Spencer, The probabilistic method, John Wiley & Sons, New York, 1992.
2. D. Archdeacon and D. A. Grable, The genus of a random graph, manuscript 1991.
3. P. Frankl and V. Rödl, Near perfect coverings of graphs and hypergraphs, *Europ. J. Combinatorics* **6** (1985), 317–326.
4. J. L. Gross and T. W. Tucker, Topological graph theory, John Wiley & Sons, New York, 1987.
5. N. Pippenger and J. Spencer, Asymptotic behavior of the chromatic index for hypergraphs, *J. Combin. Theory Ser. A* **51** (1989), 24–42.