4–CONNECTED PROJECTIVE-PLANAR GRAPHS ARE HAMILTONIAN

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May 1991, revised 23 October 1993.
Published in *J. Combin. Theory Ser. B* **62** (1994), no. 1, 114–132.

* Preparation of this paper was supported in part by NSF under Grant No. DMS-9103480.
** Preparation of this paper was supported in part by NSF under Grant No. DMS-9105173.

ABSTRACT

We prove the result stated in the title (conjectured by Grünbaum), and a conjecture of Plummer that every graph which can be obtained from a 4–connected planar graph by deleting two vertices is Hamiltonian. The proofs are constructive and give rise to polynomial–time algorithms.

1. INTRODUCTION

Whitney [7] proved that every 4–connected planar triangulation has a Hamiltonian circuit, and Tutte [6] extended this to all 4–connected planar graphs. Thomassen [5] further strengthened this result by proving that every 4–connected planar graph is Hamiltonian–connected, that is, has a Hamiltonian path connecting any two prescribed vertices. It in fact follows from Tutte's result that the deletion of any vertex from a 4–connected planar graph results in a Hamiltonian graph. Plummer [3] conjectured that the same is true if two vertices are deleted. We prove Plummer's conjecture, as follows.

(1.1) *Let G be a graph obtained from a 4–connected planar graph by deleting at most two vertices. Then G is Hamiltonian.*

Grünbaum [2] conjectured that every 4–connected projective planar graph is Hamiltonian. A proof of this conjecture is our second result.

(1.2) *Let G be a 4–connected graph which admits an embedding in the projective plane. Then G is Hamiltonian. In fact, G has a Hamiltonian circuit containing any specified edge.*

Dean [1] conjectured a strengthening of (1.2) that 4-connected projective-planar graphs are in fact Hamiltonian–connected.

Our proof method relies on Thomassen's result [5] and a generalization which is apparently new. We state this result and prove its generalization in Section 2. In Section 3 we prove (1.1), and in Section 4 we prove (1.2).

We end this section with some terminology. *Graphs* are finite, undirected and simple, that is, without loops and multiple edges. *Paths* and *circuits* have no "repeated" vertices. Every path has two *endpoints*, which are equal for the one–vertex path. If a path P has endpoints u and v, we say that P is a *path between u and v*. A *block* is a 2–connected graph or a graph isomorphic to K_2, the complete graph on 2 vertices; a *block of a graph* is a maximal subgraph which is a block. A vertex v of a graph G is a *cut–vertex* if its deletion results in a disconnected graph. Let C be a path or a circuit, and let $u, v \in V(C)$. By a *uv–segment* of C we mean a path between u and v which is a subgraph of C. Thus if C is a path, then the uv–segment of C is unique. If C is a circuit, and a clockwise orientation of C is specified, then by the *clockwise uv–segment* of C we mean the uv–segment of C which is encountered when moving on C from u to v in clockwise direction. Thus if $u = v$ then the clockwise uv–segment is the one–vertex path with vertex–set $\{u\}$.

If G is a graph and u, v are two distinct vertices of G, then by $G + uv$ we mean G if u, v are adjacent, and the graph obtained from G by adding an edge joining u and v otherwise. If W is a set, then by $G + W$ we mean the graph with vertex–set $V(G) \cup W$, edge–set $E(G)$, and the incidences of G. If G is a graph and X is a vertex, a set of vertices, an edge, or a set of edges, then $G \backslash X$ is the graph obtained from G by deleting X. If G_1, G_2 are graphs, then $G_1 \cup G_2$ is the graph with vertex–set $V(G_1) \cup V(G_2)$, edge–set $E(G_1) \cup E(G_2)$ and the obvious incidences. The graph $G_1 \cap G_2$ is defined similarly. A *separation* of a graph G is a pair of subgraphs (G_1, G_2) with $G_1 \cup G_2 = G$ and $E(G_1) \cap E(G_2) = \emptyset$.

A *plane graph* is a graph G embedded in the plane (without crossings). A *face* of G is a component of the complement of the embedding of G. Thus every face of a plane graph is an open subset of the plane. We assume that a system of coordinates is set–up in the plane, so that every circuit of G has a natural clockwise orientation. If G is moreover 2–connected, then every face is bounded by a circuit, called its *facial circuit*. The facial circuit bounding the infinite face is called the *outer circuit*. If G is not 2–connected, then every face is bounded by a walk, called the *facial walk*. An edge e is *incident* with a face F if e belongs to the facial walk of F. Let G be a plane block. We wish to extend the definition of outer circuit to include the case when G is isomorphic to K_2; in this case we define the *outer circuit* of G to be G itself.

Let G be a graph, and let H be a subgraph of G. An *H–bridge* in G is either a subgraph of G isomorphic to K_2 with both vertices, but not its edge in H, or a connected component of $G \backslash V(H)$ together with all edges from this component to H and all endpoints of these edges. If K is an H–bridge, then the vertices in $V(K) \cap V(H)$ are called the *attachments* of K.

2. A THEOREM OF THOMASSEN

Let G be a graph, let $F \subseteq E(G)$, and let H be a subgraph of G. We say that H is F–*admissible* if every H–bridge has at most three attachments, and every H–bridge containing an edge of F has at most two attachments. An F–*snake* is an F–admissible path, and an F–*sling* is an F–admissible circuit. Thus if G is 4–connected, then every $\emptyset$–sling is a Hamiltonian circuit. With this terminology, Thomassen's result [5] can be stated as follows.

(2.1) *Let G be a 2–connected plane graph with outer circuit C. Let $v \in V(C)$, let $e \in E(C)$, and let $u \in V(G) - \{v\}$. Then there is an $E(C)$–snake P between u and v with $e \in E(P)$.*

The following variation is implicit in [5]. We omit the proof which is easy.

(2.2) *Let G be a connected plane graph, let F be the set of edges of G which are incident with the infinite face, and let $x, y, z \in V(G)$ be incident with the infinite face. If there exists a path in G between x and z containing y, then there exists an F–snake between x and z containing y.*

We shall prove our results by generalizing (2.1). We shall always prove the existence of an F–snake in a graph G by pasting together suitable paths in certain subgraphs of G. These paths will have to satisfy a stronger condition than just being a snake, the following. Let G be a graph, let H be a subgraph of G, let $F \subseteq E(G)$ and let $W \subseteq V(G)$. We say that H is (F, W)–admissible if $H + W$ is a F–admissible. Thus H is $(F, \emptyset)$–*admissible* if and only if it is F–admissible. An (F, W)–admissible path will be called an (F, W)–*snake*.

(2.3) *Let G be a graph, let $F \subseteq E(G)$, and let $G = G_1 \cup G_2 \cup \cdots \cup G_n$. Let W be the set of vertices which belong to more than one G_i. For each $i = 1, 2, \ldots, n$ let P_i be an $(F \cap E(G_i), W \cap V(G_i))$–admissible subgraph of G. Let $P = P_1 \cup P_2 \cup \cdots \cup P_n$, and assume that $W \subseteq V(P)$. Then P is F–admissible.*

Proof. Let $P, P_1, P_2, \ldots, P_n$ be as stated in the lemma, and let K be a P–bridge in G. Then K is contained in some G_i, and is a P_i'–bridge in G_i, where $P_i' = P_i + (W \cap V(G_i))$, and has the same number of attachments whether regarded as a P–bridge in G, or as a P_i'–bridge in G_i. The result now follows. □

(2.4) *Let G be a plane graph, let P be a path in G with endpoints α, β such that every vertex and every edge of P is incident with the infinite face. Let $S \subseteq V(G) - V(P)$ be a set of at most two vertices, all incident with the infinite face. Then there exists an $(E(P), S)$–snake Q between α and β in G with $S \cap V(Q) = \emptyset$.*

Proof. We shall assume that S consists of two distinct vertices x and y, leaving the other two easier cases to the reader. The situation is depicted in Figure 1. We may assume that when tracing the boundary of the infinite face of G, starting from α walking along P towards β and then back to α, we encounter y before x. Let $v_0 = \alpha$, and let $v_1, v_2, \ldots, v_{n-1}$ be all the cutvertices of $G \backslash \{x, y\}$ on P which separate α from β, and assume that they occur on P in the order listed. Let $v_n = \beta$. There are blocks $B_1, B_2, \ldots, B_n$ of $G \backslash \{x, y\}$ such that $v_{i-1}, v_i \in V(B_i)$ $(i = 1, 2, \ldots, n)$ and such that the union $B = B_1 \cup B_2 \cup \cdots \cup B_n$ contains P. For $j = 1, 2, \ldots, n$ let D_j be the outer circuit of B_j, and let D be obtained from $D_1 \cup D_2 \cup \cdots \cup D_n$ by deleting $V(P) - \{v_1, v_2, \ldots, v_n\}$. Then D is a path between α and β in B. Let H be $B + \{x, y\}$. It follows that every H–bridge has at most one attachment in B, and that this attachment belongs to $V(D)$. Next we want to choose a suitable vertex $z \in V(D)$. Since G is a plane graph there is at most one bridge of H with an attachment in each of $\{x\}, \{y\}$ and $V(D)$. If there is such a bridge we choose $z \in V(D)$ to be the attachment of this bridge. Otherwise we choose $z \in V(D)$ in such a way that every H–bridge with an attachment in $\{x\}$ has its attachment in $V(B)$ on the αz segment of D, and such that every H–bridge with an attachment in $\{y\}$ has its attachment in $V(B)$ on the βz–segment of D. Such a choice is possible because G is a plane graph and because of our assumption about the position of x and y. This completes the discussion of the choice of z. We choose an edge $e \in E(D)$ incident with z; let, say $e \in E(B_i)$. For $j = 1, 2, \ldots, n$ we define $e_j = e$ if $j = i$, and otherwise we choose $e_j \in E(B_j) \cap E(D)$ arbitrarily. By (2.1) there exists, for $j = 1, 2, \ldots, n$, an $E(D_j)$–snake Q_j between v_{j-1} and v_j in B_j with $e_j \in E(Q_j)$. It is easy to see that $Q_1 \cup Q_2 \cup \cdots \cup Q_n$ is as desired. □

Let G be a graph. If $X \subseteq V(G)$ then by an X–bridge we mean a J–bridge, where J is the graph with $V(J) = X$ and $E(J) = \emptyset$. Let C be a circuit or a path in G. A *C–flap* is either the null graph, or an $\{a, b, c\}$–bridge H such that

(i) $a, b \in V(C) \cap V(H), a \neq b$, and $c \in V(H) - V(C)$,

(ii) H contains an ab–segment S of C, and

(iii) H has a plane representation with S and c on the boundary of the infinite face.

If H is as above and non–null, we say that a, b, c are its *attachments*. If H is null, we say that a, b, c are its *attachments* if $a = b = c \in V(C)$.

It is easy to see that there need not be an $E(C)$–snake as in (2.1) which contains two specified edges of $E(C)$. We overcome this difficulty as follows (see Figure 2).

(2.5) *Let G be a 2–connected plane graph with outer circuit C, let $x, y \in V(C)$ be distinct, let $e, f \in E(C)$, and assume that x, y, e, f occur on C in this clockwise order. Then there exist a C–flap H with attachments a, b, c and an $(E(C) - E(H))$–snake P between b and x in $G \backslash (V(H) - \{a, b, c\})$ such that x, a, y, b, e, f occur on C in this clockwise order, $x \neq a$, $y \in (V(H) - \{a\}) \cup \{b\}$, $e, f \in E(P)$ and $a, c \in V(P)$. Also, there exists an F–snake Q between x and y in G with $e, f \in E(Q)$, where F is the set of edges of the clockwise yx–segment of C.*

Proof. See Figure 3. We proceed by induction on $|V(G)|$. Let G be as stated and assume that the lemma holds for all graphs on less than $|V(G)|$ vertices. Let v_1, v_2 be the endpoints of e, let u_1, u_2 be the endpoints of f, and assume that y, v_1, v_2, u_1, u_2, x occur on C in this clockwise order.

A separation (G_1, G_2) of G is said to be *friendly* if $|V(G_1) \cap V(G_2)| = 2$, $V(G_1) \cap V(G_2) \subseteq V(C)$, $x, y \in V(G_1)$, $V(G_1) - V(G_2) \neq \emptyset$ and $V(G_2) - V(G_1) \neq \emptyset$. We first assume that there exists a friendly separation (G_1, G_2) of G; let $V(G_1) \cap V(G_2) = \{z, z'\}$. For $i = 1, 2$ let G'_i be $G_i + zz'$, let e_i denote the edge zz' of G'_i, and let C_i be the outer circuit of G'_i. Clearly $|V(G_i)| < |V(G)|$. We distinguish four cases depending on which of G_1, G_2 contains e, f.

If $e, f \in E(G_2)$ we apply the induction hypothesis to G_2' to get an $(E(C) \cap E(G_2))$–snake P_2 between z and z' with $e, f \in E(P_2)$. By (2.1) there exists an $(E(C) \cap E(G_1))$–snake P_1 between x and y in G_1' with $e_1 \in E(P_1)$. It is easy to see that $(P_1 \backslash e_1) \cup P_2$ is as desired for G.

If $f \in E(G_1)$, $e \in E(G_2)$, we apply the induction hypothesis to G_1' to get a C_1–flap H with attachments a, b, c and an $(E(C_1) - E(H))$–snake P_1 between b and x in $G_1' \backslash (V(H) - \{a, b, c\})$ such that $x \neq a$, $y \in (V(H) - \{a\}) \cup \{b\}$, x, a, y, b, e_1, f occur on C_1 in this order, $e_1, f \in E(P_1)$ and $a, c \in V(P_1)$. By (2.1) there exists an $E(C) \cap E(G_2)$–snake P_2 between z and z' in G_2' with $e \in E(P_2)$. It is easy to see that $(P_1 \backslash e_1) \cup P_2$ and H are as desired for G.

The case when $e \in E(G_1), f \in E(G_2)$ is symmetric to the one just described. It remains to discuss the case when $e, f \in E(G_1)$. We apply the induction hypothesis to G_1' to get a C_1–flap H with attachments a, b, c and an $(E(C_1) - E(H))$–snake P_1 between b and x in $G_1' \backslash (V(H) - \{a, b, c\})$ such that $x \neq a$, $y \in (V(H) - \{a\}) \cup \{b\}$, x, a, y, b, e, f occur on C_1 in this order, $a, c \in V(P_1)$ and $e, f \in E(P_1)$. If $e_1 \notin E(P_1)$ then P_1 and H are as desired for G, and so we assume that $e_1 \in E(P_1)$. By (2.1) there exists an $(E(C) \cap E(G_2))$–snake P_2 between z and z' in G_2' with $e_2 \notin E(P_2)$. It is easy to see that $(P_1 \backslash e_1) \cup P_2$ and H are as desired.

We have thus shown that if G has a friendly separation, then the lemma holds. We may therefore assume that

(1) *G has no friendly separation.*

Let P be the clockwise yv_1–segment of C, and let H be the graph $G \backslash V(P)$. It follows from (1) by exactly the same argument as in [5] that v_2 belongs to a unique block B of H, and that the clockwise v_2x–segment of C is a subgraph of B. Since B is a block it follows that every $(B \cup P)$–bridge of G has at most one attachment in B, and from (1) it in fact follows that

(2) *every $(B \cup P)$–bridge in G has exactly one attachment in B.*

Let D be the outer circuit of B. From (2.1) there there exists an $E(D)$–snake Q between v_2 and x in B with $f \in E(Q)$. We say that a vertex $w \in V(B)$ is a *tip* if it is an attachment of a $(B \cup P)$–bridge of G. We say that two tips $v, v' \in V(B)$ are *equivalent* if either $v = v'$, or

$v, v' \notin V(Q)$ and v and v' belong to the same Q–bridge in B. Let Θ denote the set of equivalence classes.

Let $\theta \in \Theta$. We wish to define $B_\theta, J_\theta, \alpha_\theta, \beta_\theta$. If $\theta = \{v\}$, where $v \in V(Q)$ we define B_θ to be the null graph, otherwise we define B_θ to be the Q–bridge of B with $\theta \subseteq V(B_\theta)$. It follows that in the latter case B_θ contains an edge of the outer circuit of B, and therefore B_θ has exactly two attachments. Let J'_θ be the union of all $(B \cup P)$–bridges with an attachment in θ, let α_θ be the vertex of $P \cap J'_\theta$ with the shortest distance (measured on P) from y, and let β_θ be the vertex of $P \cap J'_\theta$ with the longest distance from y. Let P_θ be the $\alpha_\theta\beta_\theta$ segment of P, and let $J_\theta = B_\theta \cup J'_\theta \cup P_\theta$. It follows from planarity that for distinct $\theta, \theta' \in \Theta$, $J_\theta, J_{\theta'}$ are either vertex–disjoint, or have at most one vertex in common, in which case this vertex is either $\alpha_\theta = \beta_{\theta'}$, or $\alpha_{\theta'} = \beta_\theta$.

Let J be obtained from B by deleting all vertices in $\bigcup_{\theta\in\Theta} V(B_\theta) - V(Q)$, and let J' be the graph induced by e and all edges of P which do not belong to $\bigcup_{\theta\in\Theta} E(J_\theta)$. Since G has no friendly separation it follows from (1) that $G = J \cup J' \cup \bigcup_{\theta\in\Theta} J_\theta$ and that, for some $\theta_1 \in \Theta$, $\theta_1 = \{v_2\}$ and $E(J_{\theta_1}) = \{e\}$. Let W be the set of all vertices which are in more than one of J, J', J_θ $(\theta \in \Theta)$; then Q is an $(F \cap E(J), W \cap V(J))$–snake in J. Let θ_0 be the unique element of Θ with the property that J_{θ_0} contains an edge of the clockwise xy–segment of C. By (2.4) there exists, for $\theta \in \Theta - \{\theta_0\}$, an $(E(C) \cap E(J_\theta), W \cap V(J_\theta))$–snake Q_θ between α_θ and β_θ in J_θ with $Q_\theta \cap W \cap V(B_\theta) = \emptyset$. If $\theta_0 = \{v\}$ and $v \in V(Q)$ we define H to be the null graph, put $a = b = c = y$, let Q'_{θ_0} be an $E(C \cap J_{\theta_0})$–snake in $J_{\theta_0} + v\beta_{\theta_0}$ between y and v with $v\beta_{\theta_0} \in V(Q'_{\theta_0})$ and put $Q_{\theta_0} = Q'_{\theta_0} \backslash v\beta_{\theta_0}$. Otherwise we put $H = J_{\theta_0}$, let a, b, c be such that H is an $\{a, b, c\}$–bridge of G, and let Q_{θ_0} be the null graph. Now by (2.3), $Q \cup J' \cup \bigcup_{\theta\in\Theta} Q_\theta$ is as desired.

It remains to prove the last assertion of (2.5). If H is the null graph then it is clearly true, so we may assume that H is not null and that P is a snake between b and x as in the first part of the lemma. By (2.4) applied to H there exists an $(F \cap E(H), \{a, c\})$–snake P' in H between b and y with $a, c \notin V(P')$. Then $P \cup P'$ is as desired. $\square$

We restate the last assertion of (2.5) as a separate theorem.

(2.6) *Let G be a 2–connected plane graph with outer circuit C, let $u, v \in V(C)$ be distinct, let $e, f \in E(C)$, assume that u, v, e, f occur on C in this clockwise order, and let F be the set of edges of the clockwise vu–segment of C. Then there exists an F–snake P between u and v in G with $e, f \in E(P)$.*

The reader will notice that we do not need the full strength of (2.5) in this paper; in fact, (2.6) would suffice. However, the proof of (2.6) is not much easier, and since we shall need (2.5) in a later paper we decided to prove it here.

(2.7) *Let G be a 2–connected plane graph with outer circuit C, and let $e_1, e_2, e_3 \in E(C)$. Then there exists an $E(C)$–sling D in G with $e_1, e_2, e_3 \in E(D)$.*

Proof. Let u, v be the endpoints of e_1. By (2.6) there exists an $(E(C) - \{e_1\})$–snake P between u and v in G with $e_2, e_3 \in E(P)$. Then $P + e_1$ is as desired. □

Theorem (2.7) has the following corollary.

(2.8) *Let G be a 4–connected plane graph, let R be a face of G incident with at least four edges of G, and let e_1, e_2, e_3 be edges of G incident with R. Then G has a Hamiltonian circuit C with $e_1, e_2, e_3 \in E(C)$.*

3. SLINGS IN PLANAR GRAPHS

(3.1) *Let G be a 2–connected plane graph with outer circuit C_1 and another facial circuit C_2 with $V(C_1) \cap V(C_2) \neq \emptyset$, and let $e \in E(C_1)$. Then there exists an $(E(C_1) \cup E(C_2))$–sling C in G with $e \in E(C)$ and $V(C_1) \cap V(C_2) \subseteq V(C)$ and such that no C–bridge contains edges of both C_1 and C_2.*

Proof. There exist an integer $k \geq 1$, graphs $G_1, G_2, \ldots, G_k$ and vertices $a_i, b_i \in V(G_i)$ $(i = 1, 2, \ldots k)$ with the following properties:

(i) G is obtained from $G_1, G_2, \ldots, G_k$ by identifying b_i and a_{i+1} for $i = 1, 2, \ldots, k$, where a_{k+1} means a_1.

(ii) Each G_i is either an edge of $E(C_1) \cap E(C_2)$ with endpoints a_i, b_i, or a 2–connected plane graph with outer circuit D_i with $a_i, b_i \in V(D_i)$ and such that one of the $a_i b_i$–segments of D_i is a segment of C_1 and the other is a segment of C_2.

For $i = 1, 2, \ldots, k$ we put $P_i = G_i$ if G_i is isomorphic to K_2. Otherwise there exists, by (2.1), an $E(D_i)$–snake P_i between a_i and b_i in G_i such that $e \in E(P_i)$ if $e \in E(G_i)$. It follows that after the identifications of (i) are made, $P_1 \cup P_2 \cup \cdots \cup P_k$ is as desired. □

(3.2) *Let G be a 2–connected plane graph with outer circuit C_1 and another facial circuit C_3, and let $e \in E(C_1)$. Then there exists an $(E(C_1) \cup E(C_3))$–sling C such that $e \in E(C)$ and such that no C–bridge contains edges of both C_1 and C_3.*

Proof. We may assume that $V(C_1) \cap V(C_3) = \emptyset$, for otherwise the theorem follows from (3.1). We proceed by induction on $|V(G)|$. If $|V(G)| \leq 5$, then there are no two disjoint circuits, and so we may and shall assume that $|V(G)| \geq 6$ and that the theorem holds for graphs on fewer than $|V(G)|$ vertices. Let B be the block of $G \backslash V(C_1)$ containing C_3, and let C_2 be the outer circuit of B. We claim that we may assume that

(1) *there is no separation (G_1, G_2) of G with $V(G_1), V(G_2) \neq V(G)$, $|V(G_1) \cap V(G_2)| = 2$ and with $V(G_1) \cap V(G_2) \cap V(C_1) \neq \emptyset$.*

To prove this assume that (G_1, G_2) is such a separation with $V(B) \subseteq V(G_2)$, and let $V(G_1) \cap V(G_2) = \{z, z'\}$. Let f denote the edge zz', and let G' be $G_2 + f$. Let e' be e if $e \in E(G_2)$ and let e' be f otherwise. From the induction hypothesis there exists an F–sling C' in G' with $e' \in E(C')$ where $F = (E(C_1) \cup E(C_3) \cup \{f\}) - E(G_1)$. If $f \notin E(C')$ then C' is as desired for G. Otherwise, by (2.1) there exists an $(E(C_1) \cap E(G_1))$–snake P in G_1 between z and z' with $e \in E(P)$ if $e \in E(G_1)$. Then $(C' \backslash f) \cup P$ is as desired for G. Thus we may assume (1).

From (1) we deduce that every $(B \cup C_1)$–bridge has exactly one attachment in B; if v denotes this attachment then $v \in V(C_2)$ and we say that the bridge *attaches* at v. A vertex $u \in V(C_2)$ is a *tip* if it is an attachment of a $(B \cup C_1)$–bridge. Let $u_1, u_2, \ldots, u_n \in V(C_2)$ be all the tips and assume that they appear on C_2 in this clockwise order. For $i = 1, 2, \ldots, n$ let L_i be the union of all $(B \cup C_1)$–bridges that attach at u_i, let $v_1^i, v_2^i, \ldots, v_{l_i}^i$ be the vertices of $L_i \cap C_1$, and assume that $v_1^1, v_2^1, \ldots v_{l_1}^1, v_1^2, v_2^2, \ldots, v_{l_2}^2, \ldots, v_1^n, v_2^n, \ldots, v_{l_n}^n$ occur on C_1 in the clockwise order listed. (This can be arranged because of planarity.) We may further assume that e belongs to the clockwise $v_{l_n}^n v_{l_1}^1$–segment of C_1. Note that by (1) no L_i $(i = 1, 2, \ldots, n)$ contains a circuit separating B and C_1, and so the above quantities are well-defined.

Let k be such that $l_2 = l_3 = \cdots = l_{k-1} = 1$, $v_{l_1}^1 = v_1^2 = v_1^3 = \cdots = v_1^{k-1} \neq v_{l_k}^k$. See Figure 4. We may assume that such a k exists, for if it did not, we could just replace "clockwise" by "counterclockwise" in the above paragraph. Let H be obtained from B by adding a new vertex $w \notin V(G)$ and edges $wu_1, wu_2, \ldots, wu_k$. The embedding of H is defined by saying that B is embedded in the same way as in G, that w is embedded in the same point as $v_{l_1}^1$ in G, for $i = 1, 2, \ldots, k-1$ the edge wu_i follows some path between $v_{l_1}^1$ and u_i in some $(B \cup C_1)$–bridge containing u_i, and the edge wu_k follows C_1 clockwise and then some path between $v_{l_k}^k$ and u_k in some $(B \cup C_1)$–bridge containing u_k. Let F be the set of edges of the clockwise $u_k u_1$–segment of C_2. By the induction hypothesis there exists an $(E(C_3) \cup F)$–sling C' in H with $wu_k \in E(C')$ such that no C'–bridge of H contains edges of both F and $E(C_3)$. Let i be such that $1 \leq i \leq k-1$ and $u_i w \in E(C')$. It follows that $u_1 \in V(C')$, because otherwise the C'–bridge containing u_1 would have at least three attachments and would contain an edge of F. Let $Q = C' \backslash w$; then Q

is an $(E(C_3) \cup F)$–snake between u_i and u_k in B containing u_1 such that no Q–bridge contains edges of both $E(C_3)$ and F. We say that two tips $v, v' \in V(B)$ are *equivalent* if either $v = v'$, or $v, v' \notin V(Q)$ and they belong to the same Q–bridge of B. We denote the set of equivalence classes by Θ. Let $\theta_1 = \{u_1\}$, $\theta_i = \{u_i\}$ and $\theta_k = \{u_k\}$; then $\theta_1, \theta_i, \theta_k \in \Theta$. Let Θ_1 be the set of all $\theta \in \Theta$ with $\theta \subseteq \{u_2, u_3, \ldots, u_{k-1}\} - \{u_i\}$, and let Θ_2 be the set of all $\theta \in \Theta$ with $\theta \cap \{u_1, u_2, \ldots, u_k\} = \emptyset$. The following is easy to see.

(2) *$\Theta_1 \cup \Theta_2 \cup \{\theta_1, \theta_i, \theta_k\} = \Theta$, and if $\theta \in \Theta_1$ and K is a Q–bridge of B with $\theta \subseteq V(K)$, then K has at most two attachments and contains no edge of F.*

From planarity we further deduce that

(3) *each element of Θ is an unbroken cyclic interval in the cyclic ordering of tips.*

In view of (3), the clockwise ordering of tips induces a clockwise ordering on Θ. Let $\theta \in \Theta$. We wish to define $B_\theta, J_\theta, \alpha_\theta, \beta_\theta, P_\theta$. If $\theta = \{v\}$, where $v \in V(Q)$ we define B_θ to be the null graph, otherwise we define B_θ to be the Q–bridge of B with $\theta \subseteq V(B_\theta) - V(Q)$. Let J'_θ be the union of all $(B \cup C_1)$–bridges which attach at an element of θ. Let $\theta = \{u_i, u_{i+1}, \ldots, u_j\}$; we define $\alpha_\theta = v^i_1$ and $\beta_\theta = v^j_{l_j}$. Let P_θ be the clockwise $\alpha_\theta\beta_\theta$–segment of C_1 and let $J_\theta = B_\theta \cup J'_\theta \cup P_\theta$. This completes the definition of $B_\theta, J_\theta, \alpha_\theta, \beta_\theta, P_\theta$. From the planarity of G and from (1) we deduce the following two claims.

(4) *If $\theta, \theta' \in \Theta$ are distinct, then $J_\theta, J_{\theta'}$ are either vertex–disjoint, or have one vertex in common, in which case this vertex is either $\alpha_\theta = \beta_{\theta'}$ or $\alpha_{\theta'} = \beta_\theta$, or have two vertices in common in which case $\Theta = \{\theta, \theta'\}$ and these vertices are $\alpha_\theta = \beta_{\theta'}$ and $\alpha_{\theta'} = \beta_\theta$.*

(5) *If $\theta, \theta' \in \Theta$ and θ' immediately follows θ in the clockwise ordering on Θ, then either $\beta_\theta = \alpha_{\theta'}$, or β_θ and $\alpha_{\theta'}$ are adjacent.*

Let J be obtained from B by deleting all vertices in $\bigcup_{\theta \in \Theta} V(J_\theta) - V(Q)$, and let J' be the graph induced by e and all edges of C_1 which do not belong to $\bigcup_{\theta \in \Theta} E(J_\theta)$. From (5) and the fact that every $(B \cup C_1)$–bridge has an attachment in $V(B)$ it follows that $G = J \cup J' \cup \bigcup_{\theta \in \Theta} J_\theta$. Let

W be the set of all vertices which are in more than one of J, J', J_θ $(\theta \in \Theta)$; since $W \cap V(J) \subseteq V(Q)$ it follows that Q is an $(E(C_3), W \cap V(J))$–snake in J. Let $\theta \in \Theta$. If B_θ is the null graph we let $x = y$ be the element of θ. Otherwise B_θ is a Q–bridge of B with exactly two attachments x, y say. Moreover, $V(B_\theta)$ does not meet both $\{u_2, u_3, \dots, u_{k-1}\}$ and $\{u_{k+1}, u_{k+2}, \dots, u_n\}$ by (2). We deduce that

(6) *for every $\theta \in \Theta$ there exist $x, y \in V(C_2) \cap V(Q)$ such that $W \cap V(J_\theta) \subseteq \{\alpha_\theta, \beta_\theta, x, y\}$.*

Moreover, it follows that

(7) *for every $\theta \in \Theta$, J_θ has the structure as described in (2.4) with $P = P_\theta$ and $S = W \cap V(J_\theta) - \{\alpha_\theta, \beta_\theta\}$.*

Let $\theta_1 = \{u_1\}, \theta_i = \{u_i\}$ and $\theta_k = \{u_k\}$. Then $\theta_1, \theta_i, \theta_k \in \Theta$. On taking $S = W \cap V(J_\theta) - \{\alpha_\theta, \beta_\theta\}$ it follows from (2.4) and (7) that there exists, for $\theta \in \Theta - \{\theta_1, \theta_i, \theta_k\}$, an $(E(C_1) \cap E(J_\theta), W \cap V(J_\theta))$–snake Q_θ between $\alpha_\theta, \beta_\theta$ in J_θ.

We wish to define Q_θ for $\theta \in \{\theta_1, \theta_i, \theta_k\}$. We begin with θ_k. If $\alpha_{\theta_k} \neq \beta_{\theta_1}$ we let Q_{θ_k} be, by (2.2), an $E(P_{\theta_k})$–snake in J_{θ_k} between u_k and β_{θ_k} with $\alpha_{\theta_k} \in V(Q_{\theta_k})$. Now we may assume that $\alpha_{\theta_k} = \beta_{\theta_1}$. Then $J_{\theta_k} + \alpha_{\theta_k} u_k$ is 2–connected, because $\alpha_{\theta_k} \neq \beta_{\theta_k}$ (otherwise J_{θ_k} could be isomorphic to K_2). By (2.1) there exists an $E(P_{\theta_k})$–snake Q'_{θ_k} in $J_{\theta_k} + \alpha_{\theta_k} u_k$ between α_{θ_k} and β_{θ_k} with $\alpha_{\theta_k} u_k \in E(Q'_{\theta_k})$. We put $Q_{\theta_k} = Q'_{\theta_k} \backslash \alpha_{\theta_k} u_k$. In either case Q_{θ_k} is an $(E(P_{\theta_k}), \{\alpha_{\theta_k}\})$–snake in J_{θ_k} between u_k and β_{θ_k}.

To define Q_θ for $\theta \in \{\theta_1, \theta_i\}$ we first assume that $i = 1$. We choose an edge f on the outer circuit of J_{θ_1} incident with β_{θ_1}. If J_{θ_1} is isomorphic to K_2 we put $Q_{\theta_1} = J_{\theta_1}$. Otherwise there exists, by (2.7), an $E(P_\theta)$–sling C_{θ_1} in $J_{\theta_1} + \alpha_{\theta_1} u_1$ with $e, f, \alpha_{\theta_1} u_1 \in E(C_{\theta_1})$ if $e \in E(J_\theta)$ and with $f, \alpha_{\theta_1} u_1 \in E(C_{\theta_1})$ otherwise. We put $Q_{\theta_1} = C_{\theta_1} \backslash \alpha_{\theta_1} u_1$.

We now assume that $i > 1$. In J_{θ_i} we find an $\emptyset$–snake Q_{θ_i} between $\alpha_{\theta_i} = \beta_{\theta_i}$ and u_i. If $\alpha_{\theta_1} = \beta_{\theta_1}$ we define Q_{θ_1} to be the graph with one vertex α_{θ_1} and no edges. Otherwise we find, by (2.6), an $E(P_{\theta_1})$–snake Q'_{θ_1} in $J_{\theta_1} + \beta_{\theta_1} u_1$ between α_{θ_1} and u_1 with $\beta_{\theta_1} u_1 \in E(Q'_{\theta_1})$ and with $e \in E(Q'_{\theta_1})$ if $e \in E(J_{\theta_1})$. We put $Q_{\theta_1} = Q'_{\theta_1} \backslash \beta_{\theta_1} u_1$. We deduce that Q_{θ_1} is an $(E(P_{\theta_1}), \{u_1\})$–

snake in J_{θ_1} between α_{θ_1} and β_{θ_1}. This completes the definition of Q_θ for all $\theta \in \Theta$.

Let $J'' = J' \backslash \alpha_{\theta_k}\beta_{\theta_1}$ if $\alpha_{\theta_k} \neq \beta_{\theta_1}$ (by (5) α_{θ_k} and $\beta_{\theta_1} = \beta_{\theta_{k-1}}$ are adjacent in this case) and let $J'' = J'$ if $\alpha_{\theta_k} = \beta_{\theta_1}$. We put $C = Q \cup J'' \cup \bigcup_{\theta\in\Theta} Q_\theta$. Then C is a circuit in G containing e. It follows from (2.3) that C is an $(E(C_1) \cup E(C_3))$–sling, and since no C'–bridge of H contains edges of both F and $E(C_3)$ it is easy to see that no C–bridge contains edges of both $E(C_1)$ and $E(C_3)$. □

(3.3) *Let G be a 4–connected plane graph, let $u_1, u_2 \in V(G)$, and let $H = G\backslash\{u_1, u_2\}$. For $i = 1, 2$ let C_i be the facial circuit of H bounding the face of H containing u_i, and let $e \in E(C_1)$. Then there exists a Hamiltonian circuit C in H with $e \in E(C)$.*

Proof. We first assume that $C_1 = C_2$. We leave the case $u_1 = u_2$ to the reader, and assume that $u_1 \neq u_2$. If u_1, u_2 belong to different faces of H then $H = C_1$ and C_1 satisfies the theorem. Otherwise the vertices of $V(C_1)$ can be numbered $v_1, v_2, \ldots, v_n$ according to their cyclic order on C_1 in such a way that for some integer i with $4 \leq i \leq n-2$, u_2 is not adjacent to vertices $v_2, v_3, \ldots, v_{i-1}$ and u_1 is not adjacent to vertices $v_{i+1}, v_{i+2}, \ldots, v_n$. Let $e_1 \in E(C_1)$ be incident with v_1 and let $e_2 \in E(C_1)$ be incident with v_i. By (2.7) there exists an $E(C_1)$–sling C in H with $e, e_1, e_2 \in E(C)$. It follows that C is as desired. We now assume that $C_1 \neq C_2$. Let C be an $(E(C_1) \cup E(C_2))$–sling as in (3.2). Again, it follows that C is as desired. □

Finally we deduce (1.1) which we restate.

(3.4) *Let G be a 4–connected planar graph and let H be the graph obtained from G by deleting at most two vertices. Then H is Hamiltonian.*

Proof. If H is obtained by deleting one or two vertices then the result follows from (3.3). Otherwise the result follows directly from (2.1). □

4. SLINGS IN THE PROJECTIVE PLANE

Let G be a graph embedded in the projective plane, and let R be a face of G. (We recall that every face is an open set.) We define the *R–width* of G to be the maximum integer k such that every non null–homotopic closed curve in the projective plane that passes through R meets the graph at least k times (counting multiplicities). More precisely, the R–width of G is the maximum integer k such that the set $\{x \in S^1 : \phi(x) \in G\}$ has cardinality at least k for every non null-homotopic continuous mapping ϕ from the unit circle S^1 to the projective plane such that $\phi(x) \in R$ for some $x \in S^1$. This number is clearly finite. Moreover, we may assume (by homotopically shifting the curve) that the curve meets the graph only in vertices.

(4.1) *Let G be a 2–connected graph embedded in the projective plane, let R be a face of G, let C_1 be the graph consisting of all vertices and edges of G incident with R, and let $e \in E(C_1)$. Then there exists an $E(C_1)$–sling C in G such that*

(i) $e \in E(C)$,

(ii) every C–bridge that contains a non null–homotopic circuit is edge-disjoint from C_1 , and

(iii) if G has R–width 1, then C is non null–homotopic.

Proof. If G has S–width 0 for some face S, then G can be regarded as a plane graph with $E(C_1)$ the set of edges incident with the outer face, and therefore the result follows from (2.1).

Next we assume that G has R–width 1, and let $u \in V(G)$ be such that some non null–homotopic closed curve ϕ passes through R and meets G only in u. By cutting open along ϕ we obtain a plane graph H which has the following structure. There are an integer $n \geq 1$, distinct vertices $v_0, v_1, \ldots, v_n$ and distinct blocks $B_1, B_2, \ldots, B_n$ of H such that $v_{i-1}, v_i \in V(B_i)$ for $i = 1, 2, \ldots, n$, $H = B_1 \cup B_2 \cup \cdots \cup B_n$ and $E(C_1) = F_1 \cup F_2 \cup \cdots \cup F_n$, where F_i is the edge–set of the outer circuit of B_i. Moreover, G is obtained from H by identifying v_0 and v_n. We may assume without loss of generality that $e \in E(B_1)$. By (2.1) we find, for $i = 1, 2, \ldots, n$ an F_i–snake P_i in B_i between v_{i-1} and v_i and such that $e \in E(P_i)$ if $i = 1$. Let C be obtained from $P_1 \cup P_2 \cup \ldots \cup P_n$ by identifying v_0 and v_n; then C is clearly an $E(C_1)$–sling in G satisfying (i) and (iii). To prove that it satisfies (ii) let B be a C–bridge of G containing a non null-homotopic circuit. Then $v_0, v_1, \ldots, v_n \in V(B)$, and hence $n = 1$ and v_0 and v_1 are attachments of B in H. Since $v_0 = v_1$ in G and G is 2–connected, B has another attachment in G, and hence in H. Thus B contains no edge of C_1, and so we see that C satisfies the conclusion of the theorem.

We now assume that G has R–width 2. Then C_1 is a circuit. Let ϕ be a non null–homotopic closed curve which meets the graph in two vertices $u, v \in V(C_1)$. We assume first that $u = v$; then G has S–width 1 for some face S incident with $u = v$. Let ϕ' be a simple closed curve homotopic to ϕ which meets G only once at $u = v$. By cutting open along ϕ' we obtain a plane graph H which has the following structure. There are an integer $n \geq 1$, distinct vertices $v_0, v_1, \ldots, v_n$ and distinct blocks $B_1, B_2, \ldots, B_n$ of H such that $v_{i-1}, v_i \in V(B_i)$ for $i = 1, 2, \ldots, n$, $H = B_1 \cup B_2 \cup \cdots \cup B_n$ and G is obtained from H by identifying v_0 and v_n. We may assume without loss of generality that $u = v = v_0$, and that $V(C_1) \subseteq V(B_1)$. By (2.1) we find an $E(C_1)$–snake P_1 in B_1 between v_0 and v_1 with $e \in E(P_1)$, and for $i = 2, 3, \ldots, n$ we find, again by (2.1) an $\emptyset$–snake P_i in B_i between v_{i-1} and v_i. Let C be obtained from $P_1 \cup P_2 \cup \ldots \cup P_n$ by identifying v_0 and v_n; then C is clearly an $E(C_1)$–sling in G satisfying (i) and (iii), and it satisfies (ii) by the same argument as in the previous paragraph. This finishes the case when $u = v$, and so we turn to the case when $u \neq v$. We cut open along ϕ and obtain a plane graph G' with vertices u_1, v_1, u_2, v_2 on the boundary of the infinite face with the following property. For $i = 1, 2$ there is a path P_i in G' between u_i and v_i with every edge incident with the unbounded face and such that $V(P_1) \cap V(P_2) = \emptyset$. Moreover, G is obtained from G' by identifying u_1, u_2 into u and v_1, v_2 into v, and $P_1 \cup P_2$ becomes C_1 after these identifications are made. Let G'' be $G' + u_2v_1 + u_1v_2$. Then G'' is 2–connected. By (2.7) there exists an $E(P_1) \cup E(P_2)$–sling D in G'' with $e, u_2v_1, u_1v_2 \in E(D)$. It follows that $D \backslash \{u_2v_1, u_1v_2\}$ becomes a desired $E(C_1)$–sling after the proper identifications are made.

We may therefore assume that G has R–width at least 3. Then $G \backslash V(C_1)$ has a non null-homotopic circuit. We first deal with the case when $G \backslash V(C_1)$ has more than one block containing non null-homotopic circuits. It then follows that G has S–width 1 for some face S such that the closures of S and R have empty intersection, and so there exists a vertex $v \in V(G) - V(C_1)$ such that every non null-homotopic circuit in G contains v. Now G can be re-embedded to become a plane graph with C_1 the outer circuit. Choose $f \in E(C_1)$ in such a way that there is no separation (G_1, G_2) of G with $|V(G_1) \cap V(G_2)| = 2$, $V(G_1) \cap V(G_2) \subseteq V(C_1)$, $E(G_2) \cap E(C_1) \neq \emptyset$, $e, f \in E(G_1)$ and $v \in V(G_2)$. Such a choice is clearly possible, because G is 2–connected. (Proof. There are two paths between $\{v\}$ and $V(C_1)$, vertex-disjoint except for v. Let v_1, v_2 be the other endpoints of these paths. Choose f in such a way that e, f belong to different v_1v_2-segments of C_1.) From (2.1) there exists an $E(C_1)$–sling C with $e, f \in E(C)$; then C satisfies (i) and (iii), and we claim that it also satisfies (ii). Indeed, let B be a C–bridge of G containing an edge of C_1. Then B has exactly two attachments, and they both belong to $V(C_1)$. Thus from the non-existence of a separation as above we deduce that $v \notin V(B)$, and hence B contains no non null-homotopic circuit, as desired.

We may therefore assume that $G \backslash V(C_1)$ has a unique block B containing all non null–homotopic circuits of $G \backslash V(C_1)$; let R' be the face of this block containing R, and let C_2 be the subgraph of B consisting of all vertices and edges incident with R'. We remark that R' is homeomorphic to an open disk, a fact that will be used later. If the R–width of G is at least 4 then C_2 is a circuit. In this case we proceed exactly as we did in the proof of (3.2), interpreting C_3 as the null graph. We omit the details.

Now we may assume that G has R–width exactly 3. In this case we proceed similarly as in (3.2), again interpreting C_3 as the null graph, but an extra care is needed. Claim (1) can be proved in exactly the same way as before.

Let K be a graph embedded in the projective plane, and let $v \in V(K)$. We say that v is a *split–vertex* of K if every non null–homotopic closed curve in the projective plane meets the graph at least once, and there exists a non null–homotopic closed curve in the projective plane meeting K exactly at v. It follows that every non null–homotopic circuit in K uses every split–vertex.

We choose an orientation of C_1, referred to as the clockwise orientation (see Figure 5). Let Z be the facial walk of R'; the clockwise orientation of C_1 induces a clockwise orientation on Z. This orientation is well–defined, because R' is homeomorphic to an open disk. Let X be a cyclically ordered set obtained as follows. Every vertex in $V(C_2)$ which is a split–vertex of B can be thought of as having two sides, and $(B \cup C_1)$–bridges can be attached at either side. We wish to split each such vertex into two vertices v_+, v_- say, one corresponding to each side. Let X be the set containing v_+, v_- for each split–vertex $v \in V(C_2)$ and all non split–vertices of $V(C_2)$. Then the clockwise orientation of Z induces a cyclic ordering on X; moreover subwalks of Z are in 1–1 correspondence with cyclic intervals of X. If $v \in V(C_2)$ is not a split–vertex of B we put $\bar{v} = v$, if $v \in V(C_2)$ is a split–vertex we put $\overline{v_+} = v$, $\overline{v_-} = v$. We shall loosely speak of elements of X as if they were vertices of G, for instance if $v \in X$ then by saying the vertex v we mean $\bar{v}$. An element $v \in X$ is a *tip* if there exists a $(B \cup C_1)$–bridge K with $\bar{v} \in K$ such that K is embedded on the v–side of $\bar{v}$. We say that K *attaches* at v.

We now proceed as in (3.2). Let $u_1, u_2, \ldots, u_n \in X$ be all the tips listed according to their clockwise cyclic order. We define $L_i, v_1^i, v_2^i, \ldots, v_{l_i}^i$, k, w and H in the same way as in (3.2), and let F be the set of edges encountered when walking along Z clockwise from u_k to u_1. Let R'' be the face of H containing R. We apply the induction hypothesis to get an F–sling C' in H with $wu_k \in E(C')$; we remark that if H has R''–width 1 then C' is non null–homotopic. We define $i, \theta_1, \theta_i, \theta_k, Q, \Theta, \Theta_1, \Theta_2$ as before. This time claim (2) needs a proof.

(2) *$\Theta_1 \cup \Theta_2 \cup \{\theta_1, \theta_i, \theta_k\} = \Theta$, and if $\theta \in \Theta_1$ and K is a Q–bridge of B with $\theta \subseteq V(K)$, then K has at most two attachments and contains no edge of F.*

To prove (2) let $\theta \in \Theta$, and let K be a Q–bridge of B with $\theta \subseteq V(K) - V(Q)$, and let K' be the C'–bridge of H containing K. Then every attachment of K is an attachment of K', and if $\theta \cap \{u_2, u_3, \ldots, u_{k-1}\} - \{u_i\} \neq \emptyset$, then w is also an attachment of K'. To prove the first assertion assume that $\theta \cap \{u_{k+1}, u_{k+2}, \ldots, u_n\} \neq \emptyset$. Then K contains an edge of F, and hence $\theta \cap \{u_2, u_3, \ldots, u_{k-1}\} - \{u_i\} = \emptyset$, for otherwise K' would have three attachments, contrary to the choice of C'. To prove the second part assume that $\theta \in \Theta_1$. Then w is an attachment of K', and hence K has at most two attachments and contains no edge of F by the choice of C'.

Claim (3) needs to be modified as follows.

(3′) *Each element of $\Theta - \Theta_1$ is an unbroken cyclic interval in the cyclic ordering of tips.*

To prove ($3'$) let K be a Q–bridge of B with $V(K) \cap \{\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_k\} - V(Q) = \emptyset$, and suppose for a contradiction that v_1, v_2, v_3, v_4 are distinct tips listed in their cyclic order such that $\bar{v}_1, \bar{v}_3 \in V(K) - V(Q)$ and $\bar{v}_2, \bar{v}_4 \notin V(K) - V(Q)$. It follows that K contains an edge of F. Since $V(K) \cap \{\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_k\} - V(Q) = \emptyset$ it follows that in H both $\bar{v}_1, \bar{v}_3$ are incident with R''. Let ϕ be a closed curve passing through R'' and following a path between $\bar{v}_1$ and $\bar{v}_3$ in $K \backslash V(Q)$ and otherwise disjoint from H. Since $\bar{v}_2, \bar{v}_4 \notin V(K) - V(Q)$ it follows that ϕ is non null-homotopic. Since in the projective plane any two non null-homotopic closed curves intersect we deduce that C' is null–homotopic. Hence H has R''–width at least two by (iii), and so R'' is bounded by a circuit. Since $\bar{v}_2, \bar{v}_4, \bar{u}_1, \bar{u}_k \notin V(K) - V(Q)$ we deduce that K has at least three attachments, contrary to the fact that it contains an edge of F. This completes the proof of ($3'$).

Let $\theta \in \Theta$. We define B_θ, J'_θ, and if $\theta \notin \Theta_1$ then also $\alpha_\theta, \beta_\theta, P_\theta$ in the same way as in (3.2). If $\theta \in \Theta_1$ then $|V(J'_\theta) \cap V(C_1)| = 1$, say $V(J'_\theta) \cap V(C_1) = \{x\}$. We define $\alpha_\theta = \beta_\theta = x$ and let P_θ be the graph with vertex–set $\{x\}$ and no edges. We define J_θ as in (3.2). The proof now continues in the same way as (3.2). In claim (5) we need to replace Θ by $\Theta - \Theta_1$ (otherwise "follows" is not defined). We need to revise (7) as follows.

($7'$) *For every* $\theta \in \Theta - \Theta_1$, J_θ *has the structure as described in* (2.4) *with* $P = P_\theta$ *and* $S = W \cap V(J_\theta) - \{\alpha_\theta, \beta_\theta\}$.

To prove ($7'$) we may assume that B_θ is non-null, for otherwise the assertion follows immediately. Let x, y be the attachments of B_θ as a Q–bridge of B. Since B_θ contains an edge of C_2 it contains no non null–homotopic circuit by (ii), and so it follows that B_θ is contained in a closed disk. Let W_0 be the facial walk of the face of B_θ that contains the boundary of this disk. Since neither x nor y is a cutvertex of B_θ we deduce that x, y divide W_0 into two subwalks; let W_1, W_2 be the vertex-sets of these two subwalks. We claim that one of W_1, W_2 is disjoint from $V(C_2) - \{x, y\}$. To prove this suppose for a contradiction that $v_1 \in W_1 \cap V(C_2) - \{x, y\}$ and $v_2 \in W_2 \cap V(C_2) - \{x, y\}$. Let ψ be a closed curve passing through R'' and following a path in $B_\theta \backslash \{x, y\}$ between v_1 and v_2 and otherwise disjoint from H. Since v_1, x, v_2, y occur on W_0 in this or reverse order and $x, y \in V(Q)$ we deduce that ψ is non null–homotopic. Thus C' is null–homotopic, because C' and ψ do not meet; hence H has R''–width at least two and so R'' is bounded by a circuit. But $x, y, u_1, u_k \notin V(B_\theta) - V(Q)$ and so B_θ has at least three attachments, a contradiction. This proves our claim that one of W_1, W_2 is disjoint from $V(C_2) - \{x, y\}$, and (7′) follows.

For $\theta \in \Theta_1$ we define Q_θ to be the graph with vertex–set $\{\alpha_\theta\}$ and no edges. The rest of the construction of C is identical to the rest of the proof of (3.2). Condition (ii) follows from (7′). □

We are now ready to prove (1.2), which we restate.

(4.2) *Let G be a 4–connected graph which admits an embedding in the projective plane, and let e be an edge of G. Then G has a Hamiltonian circuit containing e.*

Proof. We take an arbitrary embedding of G in the projective plane. Let R be a face incident with e. By (4.1) there exists an $\emptyset$–sling C with $e \in E(C)$; since G is 4–connected it follows that C is a Hamiltonian circuit. □

From (4.1) we can also deduce the following.

(4.3) *Let G be 4–connected graph embedded in the projective plane, and let $v \in V(G)$. Let $H = G \backslash v$, and let $e \in E(H)$ be incident with the face of H which contains v. Then H has a Hamiltonian circuit C with $e \in E(C)$.*

Proof. By (4.1) H contains an F–sling C with $e \in E(C)$, where F is the set of edges incident with the face of H containing v. It follows that C is as desired. □

We remark that it is easy to construct a 4–connected projective planar graph G such that $G\backslash\{u, v\}$ is not Hamiltonian for some $u, v \in V(G)$. For instance, take an embedding of K_4 in the projective plane with every face bounded by a circuit of length four, and insert a vertex into each face to form a triangulation.

Acknowledgements

We are indebted to one of the referees for carefully reading the manuscript and for providing many helpful suggestions. We would also like to acknowledge that Daniel P. Sanders, a graduate student of the first author, has independently formulated and proved (2.7). He used it to prove a yet different generalization of (2.1).

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List of symbols

$+, -, \setminus, \cap, \cup, \bigcup, \subseteq, \emptyset, <, \leq, \neq, =, |.|, \in, \notin$

$\alpha, \beta, \theta, \Theta, \phi, \psi$