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The Menger-like Property of the Tree-width of Infinite Graphs*

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The tree-width of a (possibly infinite) graph G is the minimum n such that G may be decomposed into a "tree-structure" of pieces each with at most $n+1$ vertices. We prove that if G has tree-width n, then G can be decomposed in such a way that for every $k \geqslant 0$ and any two pieces P and Q of this "tree-structure" either there are k disjoint paths between P and Q, or there is a third piece R between P and Q with at most $k-1$ vertices. Our proof makes use of a finite version of this result proved by the second author and a compactness argument. © 1991 Academic Press, Inc.

1. INTRODUCTION

A graph is a pair $G=(V, E)$, where V is a (possibly infinite) set and E is a subset of $[V]^2$, the set of all 2-element subsets of V (thus we do not allow loops and multiple edges, but that is without loss of generality). We write $V(G)=V$ and $E(G)=E$. A *tree-decomposition* of a graph G is a pair (T, W), where T is a tree and $W=(W_t : t \in V(T))$ is such that

(W1) $\bigcup_{t \in V(T)} W_t = V(G)$,

(W2) every edge $e \in E(G)$ has both its endpoints in some W_t,

(W3) $W_t \cap W_{t''} \subseteq W_{t'}$ whenever t' is on the path between t and t'' in T.

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'he *width* of the tree-decomposition (T, W) is $\sup_{t \in V(T)} |W_t| - 1$, provided hat the sup is finite, otherwise it is undefined. A graph G has *tree-width* w ' w is the least non-negative integer such that G admits a tree-decomposiion of width w.

It can be shown that graphs with tree-width $\leqslant 1$ are precisely the forests, raphs with tree-width $\leqslant 2$ are the series-parallel ones. The complete graph K_n has tree-width $n-1$ and the $n \times n$ grid (the adjacency graph of the $n \times n$ hessboard) has tree-width n.

The notion of tree-width was introduced by Robertson and Seymour [5] /ho used it in the well-quasi-ordering theory and in some algorithmic ivestigations. One of us, in order to extend the results of Roberson and eymour on well-quasi-ordering finite graphs to the infinite case, proved he following two theorems (for the first one see [8], or [10] for a implified proof).

(1.1) *If each finite subgraph of a graph G has tree-width $\leqslant w$ then G has ree-width $\leqslant w$.*

(1.2) *If G has tree-width w then it admits a tree-decomposition (T, W) atisfying* (W1), (W2), (W3), *and*

(W4) $|W_t| \leqslant w+1$ *for any* $t \in V(T)$,

(W5) *for any $t_1, t_2 \in V(T)$, any $X_1 \subseteq W_{t_1}$, $X_2 \subseteq W_{t_2}$ and any $k>0$ uch that $|X_1| \geqslant k$ and $|X_2| \geqslant k$ either there are k disjoint paths in G, each etween X_1 and X_2, or there is a vertex t on the path in T between t_1 and $_2$ such that $|W_t| < k$.*

Clearly (W4) says that the width of (T, W) is $\leqslant w$. For finite graphs the ast theorem is proved in [7], but the original proof was unnecessarily omplicated in the general case (and thus will not be published). The aim f this paper is to give a simple proof of (1.2). We use a compactness arguient, but since there is a problem with direct application of compactness rguments we must find an equivalent locally verifiable property. For that ve develop further Thomassen's idea [10] to use chordal graphs. Applicaions of (1.2) can be found in [6, 9].

We need the following two lemmas about tree-decompositions. The first s a lemma from [5], the proof of the second is left to the reader.

(1.3) *Let (T, W) be a tree-decomposition of a graph G, let $t_1, t_2 \in V(T)$, et P be a path in G between W_{t_1} and W_{t_2}, and let $t \in V(T)$ lie on the path n T between t_1 and t_2. Then P uses a vertex of W_t.*

(1.4) *Let (T, W) be a tree-decomposition of a graph G and let K be a inite complete subgraph of G. Then there exists $t \in V(T)$ such that $V(K) \subseteq W_t$.*

A graph is called *chordal* if it contains no induced cycle of length $\geqslant 4$. We need the following lemma.

(1.5) *Let G be a chordal graph, let $k \geqslant 0$ be an integer, and let K, K' be complete subgraphs of G. Then the following statements are equivalent.*

(i) *There are at least k disjoint paths in G between K and K'.*

(ii) *There exists a sequence $K = K_1, ..., K_n = K'$ of complete subgraphs of G such that $|V(K_i) \cap V(K_{i+1})| \geqslant k$ for $i = 1, ..., n-1$.*

Proof. It is easily seen that if there exists a graph for which (1.5) fails, then there also exists a finite such graph. Hence it is enough to prove (1.5) for finite graphs, which is an easy exercise in chordal graphs and is left to the reader. ▮

Finally, we require Rado Selection Lemma, as follows.

(1.6) *Let E be a set and for every finite set $A \subseteq E$, let $f_A: A \to \{0, 1\}$ be a mapping. Then there exists a mapping $f: E \to \{0, 1\}$ such that for every finite $A \subseteq E$ there exists a finite set B such that $A \subseteq B \subseteq E$ and $f(a) = f_B(a)$ for every $a \in A$.*

Proof. The system (f_A) forms a net in the topological space $\{0, 1\}^E$, which is compact by Tychonoff's theorem. Hence (see [2]) (f_A) has a cluster point, say f, which is the desired mapping. See [4] for a different proof. ▮

2. Main Result

The following result can be deduced from Halin's theory of simplicial decompositions [1], but we prove it from first principles.

(2.1) *Let G be a chordal graph and $w \geqslant 0$ an integer such that G contains no isomorphic copy of K_w. Then G admits a tree-decomposition (T, W) such that for every $t \in V(T)$, W_t induces a complete graph in G.*

Proof. Let Δ be the complete graph whose vertices are the vertex-sets of maximal complete subgraphs of G. For an edge $\{K_1, K_2\}$ of Δ define its weight to be $|K_1 \cap K_2|$. We first construct a "maximum weight" spanning tree T of Δ. Let $E_{w-1} = \varnothing$ and assuming that we have already constructed $E_{w-1} \subseteq \cdots \subseteq E_{i+1}$, let E_i be such that

(a) E_i is obtained from E_{i+1} by adding a (possibly empty) set of edges of weight i,

(b) E_i does not contain a cycle, and

(c) E_i is maximal subject to (a) and (b)

ince every edge of Δ has weight $\leqslant w-2$, it follows that E_0 is the set of dges of a spanning tree T of Δ. Let $W=\{t: t \in V(T))$.

(1) *(T, W) satisfies* (W3).

Suppose for a contradiction that $K_1, ..., K_n$ $(n \geqslant 3)$ are the vertices of a ath in T such that $v \in K_1 \cap K_n - K_i$ for all $i=2, ..., n-1$ and for some $\in V(G)$. Let $k=\min\{|K_i \cap K_{i+1}|: i=1, ..., n-1\}$ and let G' be the graph btained from G by deleting v and all edges incident with v. We deduce om (1.5) applied to G' that there exist k disjoint paths in G' between $_1-\{v\}$ and $K_n-\{v\}$, and hence there exist $k+1$ disjoint paths in G etween K_1 and K_n. By (1.5) applied to G there exists a path in Δ between $_1$ and K_n with all edges of weight $\geqslant k+1$, a contradiction to the maxi- ality of T.

It follows from (1) that (T, W) is as desired. ∎

Let G be a graph of tree-width w and let G be a subgraph of a chordal raph H. We say that H is an *M-closure* of G if H contains no isomorphic opy of K_{w+2} and for any two complete subgraphs K_1 and K_2 of H and ny integer $k \geqslant 0$, if there are k disjoint paths in H between $V(K_1)$ and (K_2), then there are k disjoint paths in G between $V(K_1)$ and $V(K_2)$.

(2.2) *Let G be a graph of tree-width w. Then G has an M-closure if and nly if it admits a tree-decomposition (T, W) satisfying* (W1)–(W5).

Proof. Assume first that G has a tree-decomposition (T, W) satisfying W1)–(W5). Let H be the graph with vertex set $V(G)$ and with all edges f the form $\{u, v\}$ where $u, v \in W_t$ for some $t \in V(T)$. Then G is a subgraph f H by (W2), it is easy to see that H is chordal, and it follows from (1.4) hat H contains no K_{w+2}. Let K_1 and K_2 be two complete subgraphs of H nd $k \geqslant 0$ an integer such that there are k disjoint paths between $V(K_1)$ and (K_2) in H. By (1.4) there exist vertices $t_1, t_2 \in V(T)$ such that $V(K_1) \subseteq W_{t_1}$ nd $V(K_2) \subseteq W_{t_2}$. By (W5) there exist k disjoint paths between $V(K_1)$ and (K_2) in G, because $|W_t| \geqslant k$ for every $t \in V(T)$ between t_1 and t_2 by (1.3).

For the converse let H be an M-closure of G, let (T, W) be the tree- ecomposition of H as in (2.1). Then (T, W) is obviously a tree-decomposi- on of G and has width $\leqslant w$, because H contains no K_{w+2}. We subdivide ach edge e of T by a vertex t_e, denote the tree thus obtained by R, and ut

$$Y_r=\begin{cases} W_r & \text{if } \ r \in V(T) \\ W_t \cap W_{t'} & \text{if } \ r=t_e \text{ and } e=\{t, t'\} \end{cases}$$

and $Y=(Y_r: r \in V(R))$. Then (R, Y) is a tree-decomposition of G of width $\leqslant w$, we must verify that it satisfies (W5). Indeed, let $r_1, ..., r_n \in V(R)$ be a path in R, let $X_i=Y_{r_i}$ $(i=1, ..., n)$, let k be an integer, and let $X_0 \subseteq X_1$, $X_{n+1} \subseteq X_n$ be sets such that $|X_i| \geqslant k$ for $i=0, ..., n+1$. Then also $|X_i \cap X_{i+1}| \geqslant k$ by the definition of Y, and (since every X_i induces a complete graph in H) it follows from (1.5) that there exist k disjoint paths in H between X_0 and X_{n+1}, and hence there exist k disjoint paths in G between X_0 and X_{n+1}, as desired. ∎

(2.3) *Every finite graph has an M-closure.*

Proof. Every finite graph of tree-width w admits a tree-decomposition (T, W) satisfying (W1)–(W5) by the finite version of (1.2) proved in [7]. Hence it has an M-closure by (2.2). ∎

Let G be a graph and $A \subseteq [V(G)]^2$. We denote by $G \upharpoonright A$ the subgraph of G induced by all vertices which are elements of some set in A.

(2.4) *Every graph of finite tree-width has an M-closure.*

Proof. Let G be a graph of tree-width w, and let $E=[V(G)]^2$. For every finite set $A \subseteq E$ there exists by (2.3) an M-closure H_A of $G \upharpoonright A$. We define a mapping $f_A: A \rightarrow \{0, 1\}$ by

$$f_A(\{u, v\})=\begin{cases} 1 & \text{if } \ u, v \text{ are adjacent in } H_A \\ 0 & \text{otherwise.} \end{cases}$$

Let $f: E \rightarrow \{0, 1\}$ be as in (1.6) and let H be the graph with vertex-set $V(G)$ and edges $\{u, v\}$ such that $f(\{u, v\})=1$. Then it is easy to verify that H is an M-closure of G. For instance, let K_1 and K_2 be two complete subgraphs of H and $k \geqslant 0$ an integer such that $V(K_1)$ and $V(K_2)$ are joined by k disjoint paths $P_1, ..., P_k$ in H. Then there exists a finite set $A \subseteq E$ such that $K_1, K_2, P_1, ..., P_k$ are all subgraphs of H_A and since H_A is an M-closure of $G \upharpoonright A$ there exist k disjoint paths between $V(K_1)$ and $V(K_2)$ in $G \upharpoonright A$, and hence in G, as desired. ∎

Proof of (1.2). Let G be a graph of tree-width w. Then G has an M-closure by (2.4) and thus G has a tree-decomposition satisfying (W1)–(W5) by (2.2). ∎

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