THREE-COLORING KLEIN BOTTLE GRAPHS OF GIRTH FIVE

Robin Thomas[1]

and

Barrett Walls[2]

School of Mathematics
Georgia Institute of Technology
Atlanta, Georgia 30332, USA

ABSTRACT

We prove that every graph of girth at least five which admits an embedding in the Klein bottle is 3-colorable. This solves a problem raised by Woodburn, and complements a result of Thomassen who proved the same for projective planar and toroidal graphs.

15 July 1998, Revised 26 April 2004. Published in *J. Combin. Theory Ser. B* **92** (2004), 115–135.

[1] Partially supported by NSF under Grant No. DMS-9623031 and by NSA under Grant No. MDA904-98-1-0517.

[2] Partially supported by NSA under Grant No. MDA904-98-1-0517.

1. INTRODUCTION

The motivation for this paper goes back to the following classical result of Grötzsch [4].

(1.1) *Every triangle-free loopless planar graph is* 3*-colorable.*

Until the recent work of Thomassen [7], this was regarded as a very difficult theorem. Thomassen in his paper not only found an easier proof, but also extended the theorem. To discuss Thomassen's extension, we need to make two remarks. First, it is easy to show that a minimum counterexample to (1.1) has no facial cycle of length four. (*Paths* and *cycles* have no "repeated" vertices.) Indeed, suppose that G is a minimum counterexample to (1.1), and that C is a facial cycle in G with vertex-set $\{v_1, v_2, v_3, v_4\}$ (in order). Let G_{13} be the graph obtained from G by identifying v_1 and v_3, and let G_{24} be defined analogously. It is easy to see that one of G_{13}, G_{24}, say G_{13}, is loopless and triangle-free. But G_{13} has a 3-coloring by the minimality of G, and this 3-coloring gives rise to a 3-coloring of G, a contradiction. Thus every minimum counterexample to (1.1) has no facial cycle of length four. In fact, Thomassen was able to strengthen (1.1) in such a way that a minimum counterexample to the stronger result has no cycles of length less than five. The strengthening is stated below as (3.2).

By a *surface* we mean a compact 2-dimensional manifold. By a *drawing* Γ in a surface S we mean a graph drawn in S (without crossings). If G is isomorphic to Γ as an abstract graph, we say that Γ is a *drawing of G*. Our second observation is that (1.1) does not extend to other surfaces. For example, the Grötzsch's graph [1], or, more generally, every graph obtained from an odd cycle by means of the Mycielski's construction [1, Section 8.5] is loopless, triangle-free, and yet has a drawing in every surface other than the sphere.

The *girth* of a graph is the length of the shortest cycle, or infinity if the graph has no cycles. In light of the above two remarks it seems reasonable to study 3-colorability of drawings of girth at least five in nonplanar surfaces. Strengthening results of Kronk and White [6] and proving a conjecture of Kronk [5], Thomassen [7] established the following result.

(1.2) *Every drawing of girth at least five in the projective plane or in the torus is* 3-*colorable.*

The purpose of this article is to answer a question of Woodburn [11] by proving that the same result holds for drawings in the Klein Bottle, formally as follows.

(1.3) *Every drawing of girth at least five in the Klein bottle is* 3-*colorable.*

In fact, we prove a more general result, stated as (2.1) below — a characterization of 3-colorable Klein bottle drawings with no null-homotopic cycle of length three or four.

We say that a graph G is 4-*critical* if it is not 3-colorable, but $G\backslash e$ is 3-colorable for every edge $e \in E(G)$. It is clear that (1.3) does not generalize to *all* surfaces, but in a 1998 version of this paper and in [10] we conjectured an analogue of [8] that for each surface S there are only finitely many 4-critical graphs of girth at least five that embed in S. This conjecture was proved by Thomassen [9]:

(1.4) *For every surface S there are only finitely many* 4-*critical drawings in S that have girth at least five.*

Youngs [12] showed that even for the projective plane this is not true with five replaced by four. Incidentally, Youngs' result led to the following elegant characterization of 3-colorable triangle-free projective graphs by Gimbel and Thomassen [3]. We say that a drawing in a surface S is a *quadrangulation of S* if every face is bounded by a cycle of length four.

(1.5) *A projective planar graph of girth at least four is* 3-*colorable if and only if it has no subgraph isomorphic to a quadrangulation of the projective plane.*

It would be interesting to decide if there is a polynomial-time algorithm to test, for a fixed surface S, whether an input loopless triangle-free drawing in S is 3-colorable. For the projective plane such an algorithm can be derived from (1.5). The following question seems relevant.

(1.6) Problem. *Let S be an orientable surface. Does there exist an integer $q = q(S)$ such that every drawing G in S of girth at least four with the property that every nonnull-homotopic cycle in G has length at least q is 3-colorable?*

For nonorientable surfaces the answer is negative by (1.5).

The paper is organized as follows. In the next section we introduce notation needed to state the main result. In the following two sections we prove some auxiliary lemmas. In Section 5 we prove that a minimum counterexample to our main theorem has a face with some nice properties, and we use the existence of such a face in Section 6 to complete the proof.

2. MAIN RESULT

All *graphs* in this paper are finite, and may have loops and parallel edges. By a *coloring* of a graph G we mean proper coloring; that is, a function ϕ mapping $V(G)$ into some set such that $\phi(u) \neq \phi(v)$ for every pair of adjacent vertices $u, v \in V(G)$. A coloring ϕ is a 3-*coloring* if the range of ϕ is a subset of $\{1, 2, 3\}$. Throughout this paper, Σ will denote a (fixed) Klein bottle. Thus Σ can be regarded as being obtained from a sphere by removing the interiors of two disjoint closed disks, and identifying diagonally opposite points on the boundary of each of the two disks. This construction is known as "adding two crosscaps to a sphere". In our pictures the two disks will be indicated by asterisks.

A cycle in a drawing is *null-homotopic* if it bounds a disk. In [7] Thomassen proved (1.2) by showing the 3-colorability of all projective planar and toroidal loopless drawings with no null-homotopic cycle of length three or four. Thus he permitted cycles of lengths three or four, as long as they were not null-homotopic. This strengthening was crucial for his inductive proof. Unfortunately, the analogous statement for the Klein bottle is false, as shown by the drawings F_0, F_1 and F_2 depicted in Figure 1. In fact, as the three drawings suggest, there is an infinite family of counterexamples, which we now define formally.

A cycle C in a drawing in a surface S is *separating* if $S-C$ has at least two components. We wish to define a family $\mathcal{F}$ of drawings in Σ with two distinguished edges, called *the special edges*, each forming a chord of a separating cycle of length four. The definition is

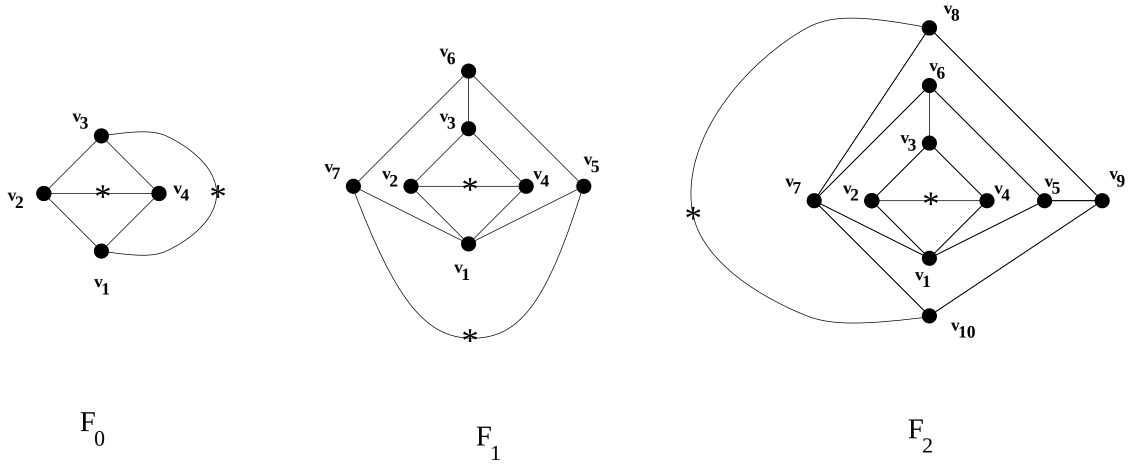

Figure 1. Three examples of drawings.

recursive. We define that F_0 defined in Figure 1 belongs to $\mathcal{F}$, and that its special edges are v_1v_3 and v_2v_4. Now suppose that a drawing F belongs to $\mathcal{F}$, and let e be a special edge with ends x_1 and x_3 forming a chord of a separating cycle C with vertex-set $\{x_1, x_2, x_3, x_4\}$ in order. Let f be the other special edge, and let F' be obtained from F by the operation indicated in Figure 2. More precisely, the portion of the drawing in the component of $\Sigma - C$ that contains e is changed as indicated in Figure 2, while the portion that belongs to the other component of $\Sigma - C$ remains unchanged. Then we define that F' with special edges uv and f belongs to $\mathcal{F}$. Thus $F_0, F_1, F_2 \in \mathcal{F}$.

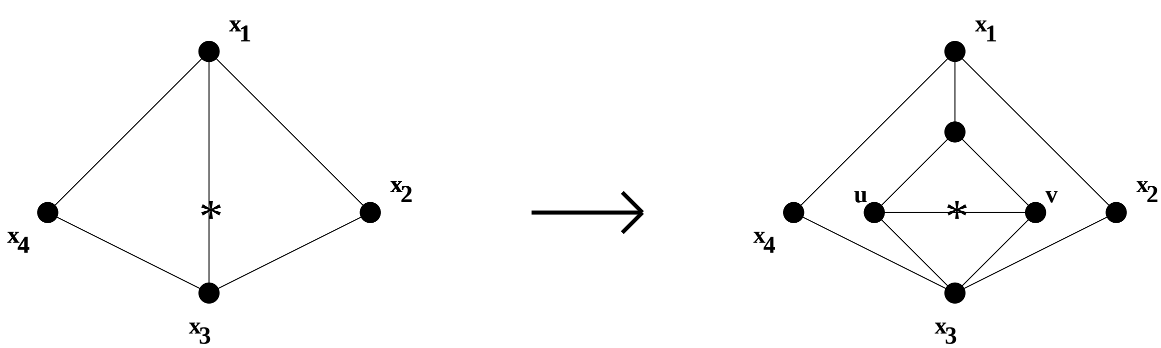

Figure 2. An operation that generates $\mathcal{F}$.

Every member of $\mathcal{F}$ has no null-homotopic cycle of length less than five, and yet is not 3-colorable. Fortunately, those are the only such drawings in Σ, as we shall see. We say that a drawing G in Σ is $\mathcal{F}$-free if there is no subdrawing G' of G and a homeomorphism of Σ mapping a member of $\mathcal{F}$ onto G'. Now we are ready to state our main result.

(2.1) *Every loopless $\mathcal{F}$-free drawing in Σ with no null-homotopic cycle of length three or four is* 3*-colorable.*

The proof of (2.1) will occupy the remainder of the paper. For future convenience we define a *minimum counterexample* to be a loopless $\mathcal{F}$-free drawing G in Σ such that G has no null-homotopic cycle of length three or four, G is not 3-colorable, and every loopless $\mathcal{F}$-free drawing H in Σ with no null-homotopic cycle of length three or four and $|V(H)| + |E(H)| < |V(G)| + |E(G)|$ is 3-colorable. Thus our objective is to show that no minimum counterexample exists.

3. LEMMAS

In this section we state results of Thomassen, and Erdös, Rubin and Taylor that we shall need later. We begin with an easy lemma.

(3.1) *Let G be a minimum counterexample. Then G has no parallel edges and no vertices of degree less than three.*

The following is a result of Thomassen [7].

(3.2) *Let G be a planar graph of girth at least* 5*, and let C be a facial cycle of length at most nine. Then any* 3*-coloring of $G\backslash(V(G) - V(C))$ can be extended to a* 3*-coloring of G, unless C has length nine and $G\backslash V(C)$ has a vertex joined to three vertices of C, which have three distinct colors.*

We deduce the following corollary.

(3.3) *Let G be a minimum counterexample, and let Δ be an open disk in Σ bounded by a walk of length $l \leq 9$. Then Δ includes at most one vertex of G. If $l \leq 8$, then Δ includes no vertices of G, and if $l \leq 7$, then Δ includes no vertices or edges of G.*

Proof. Let Δ be as stated, and assume that the result holds for all disks bounded by a walk of length $< l$. If Δ includes no vertices of G, then the result holds, because G has no null-homotopic cycles of length less than five. Thus we may assume that Δ includes a vertex of G. Let H be the drawing obtained from G by deleting all vertices that belong to Δ. Then H has a 3-coloring by the minimality of G. By (3.2) this 3-coloring extends to G, unless G has a vertex v in Δ that is joined to three vertices on the boundary of Δ. The three edges incident with v divide Δ into three disks $\Delta_1, \Delta_2, \Delta_3$. Since G has no null-homotopic cycle of length at most four, it follows that $l = 9$, and that Δ_i is bounded by a walk of length five. By the minimality of l, each Δ_i includes no vertices of G, and so the result holds. □

We need the following result of Erdös, Rubin and Taylor [2].

(3.4) *Let G be a 2-connected graph that is not complete and is not an odd cycle. For every $v \in V(G)$ let L_v be a set of size equal to the degree of v in G. Then G has a coloring ϕ such that $\phi(v) \in L_v$ for every $v \in V(G)$.*

(3.5) *Let G be a drawing of K_4 in Σ with no null-homotopic cycle. Then there is a homeomorphism of Σ mapping G onto F_0.*

Proof. Let G be a drawing of K_4 in Σ. By Euler's formula G has at least two faces that are homeomorphic to open disks. Those faces are not bounded by cycles, and hence are bounded by walks of lengths at least six. On the other hand, the sum of the sizes of all faces is twelve, and so G has precisely two faces, each bounded by a walk of length six. Since those two walks cover every edge of G precisely twice, it follows that, up to symmetry, there is only one possibility for these walks. The result follows by the Jordan-Schönflies theorem. □

(3.6) *Let G be a minimum counterexample. Then G has no cycle with an even number of vertices and all vertices with a degree of three.*

Proof. Suppose for a contradiction that C is a cycle in G of even length such that every vertex of C has degree three in G. Then $G\backslash V(C)$ has a 3-coloring by the minimality of G. For $v \in V(G)$ let L_v be the set of colors not used by the neighbors of v that do not belong to C. By (3.5) the graph $G\backslash(V(G) - V(C))$ and the lists L_v satisfy the hypothesis of (3.4), and hence the 3-coloring of $G\backslash V(C)$ can be extended to a 3-coloring of G, a contradiction. □

4. MORE LEMMAS

Let G be a drawing in Σ, and let P be a path in G with vertices $v_0, v_1, \ldots, v_k$ in order, and for $i = 1, \ldots, k$ let e_i be the edge of P with ends v_{i-1} and v_i. Let Δ be a disk in Σ containing P, and let us choose an orientation of Δ. We say that P is *2-sided* if there exist indices $i, j \in \{1, 2, \ldots, k-1\}$ and edges $e, f \in E(G) - E(P)$ such that e_{i-1}, e, e_i occur in the given order around v_i, and e_{j-1}, e_j, f occur in the given order around v_j. This definition is clearly independent of Δ and the choice of orientation of Δ.

(4.1) *Let G be a minimum counterexample, and let P be a 2-sided path in G of length three or four. Let H be the drawing obtained from G by deleting all edges of $E(G) - E(P)$ incident with internal vertices of P, and then contracting at most three edges of P. Then H has no contractible cycle of length three or four.*

Proof. If H has a contractible cycle of length three or four, then G has a contractible cycle C of length at most seven such that P is a subpath of C. Since P is 2-sided, the open disk bounded by C includes an edge e incident with an internal vertex of P, contrary to (3.3). □

(4.2) *Let G be a minimum counterexample, and let C be a separating cycle in G of length three. Then one of the components of $\Sigma - C$ contains no vertices of G.*

Proof. Let C be a separating cycle in G of length three, let Σ_1, Σ_2 be the closures of the two components of $\Sigma - C$, and for $i = 1, 2$ let G_i be the subdrawing of G induced by all vertices and edges of G that belong to Σ_i. If G_1, G_2 have fewer vertices than G, then they both have 3-colorings by the minimality of G, and those 3-colorings can be combined to yield a 3-coloring of G, a contradiction. Thus one of the components of $\Sigma - C$ contains no vertices of G, as desired. □

(4.3) *Let G be a minimum counterexample, and let C be a separating cycle in G of length four. Then one of the components of $\Sigma - C$ contains no vertices of G.*

Proof. Let C be a separating cycle in G of length four, and let Σ_1, Σ_2 be the closures of the two components of $\Sigma - C$. For $i = 1, 2$ let G_i be the subdrawing of G induced by all vertices and edges of G that belong to Σ_i, let $\{x_1, x_2, x_3, x_4\}$ be the vertex-set of C (in order), and suppose for a contradiction that both G_1 and G_2 have fewer vertices than G. Thus G_1, G_2 have 3-colorings, say ϕ_1 and ϕ_2, by the minimality of G. By permuting the colors we may assume that $\phi_1(x_1) = \phi_2(x_1) = 1$ and that $\phi_1(x_2) = \phi_2(x_2) = 2$. If $\phi_1(x_3) = \phi_2(x_3) = 1$ and $\phi_1(x_4) = \phi_2(x_4) = 2$, then ϕ_1 and ϕ_2 can be combined to yield a 3-coloring of G, a contradiction. Thus we may assume that $\phi_2(x_3) = 3$, in which case it follows that $\phi_2(x_4) = 2$.

Let G_1' be the drawing in Σ obtained from G_1 by adding an edge with ends x_1 and x_3 in such a way that $\{x_1, x_2, x_3, x_4\}$ does not include the vertex-set of a null-homotopic cycle in G_1' of length three. In other words, the new edge is inserted as the edge x_1x_3 in the left-hand side of Figure 2. Then G_1' is clearly loopless. It follows that no null-homotopic cycle in G_1' uses the new edge, and hence G_1' has no null-homotopic cycle of length less than five. We deduce that G_1' is not $\mathcal{F}$-free, for otherwise it would be 3-colorable by the minimality of G, and yet a 3-coloring of G_1' can be combined with ϕ_2 to produce a 3-coloring of G, a contradiction.

Thus we may assume that some subdrawing F_1' of G_1' belongs to $\mathcal{F}$. Since G_1' has no null-homotopic cycle of length less than five, we deduce that C is a subgraph of F_1' and that the new edge x_1x_3 belongs to F_1'. We deduce that $\phi_1(x_4) \neq 2$, because ϕ_1 is a 3-coloring of $F_1' \backslash x_1x_3$. Thus $\phi_1(x_4) = 3$, and hence $\phi_1(x_3) = 1$. Let G_2' be defined analogously by adding an edge with ends x_2 and x_4 to G_2. Similarly as above we deduce that we may assume that some subdrawing F_2' of G_2' belongs to $\mathcal{F}$, that C is a subgraph of F_2' and that the new edge x_2x_4 belongs to F_2'. Now the union of $F_1' \backslash x_1x_3$ and $F_2' \backslash x_2x_4$ is isomorphic to a member of $\mathcal{F}$, contrary to the assumption that G is $\mathcal{F}$-free. □

(4.4) *No minimum counterexample has a separating cycle of length four.*

Proof. Let G be a minimum counterexample, and suppose for a contradiction that C is a separating cycle in G of length four. Let Σ_1, Σ_2 be the closures of the two components of $\Sigma - C$, and let $\{x_1, x_2, x_3, x_4\}$ be the vertex-set of C in order. By (4.3) we may assume that the interior of Σ_2 includes no vertices of G. Since G is $\mathcal{F}$-free it follows from (3.5) that one of the pairs x_1, x_3 and x_2, x_4 are not adjacent in G, and so from the symmetry we may assume that x_2 and x_4 are not adjacent. Let G'' be the drawing obtained from G by adding an edge with ends x_2 and x_4 in such a way that $\{x_1, x_2, x_3, x_4\}$ does not include the vertex-set of a null-homotopic triangle in G''. This is possible, because the interior of Σ_2 includes at most one edge of G, and if it includes one, then that edge has ends x_1 and x_3. Let G' be obtained from $G'' \backslash x_1x_2 \backslash x_2x_3$ by contracting the edge x_2x_4. Then G' is clearly loopless.

We claim that G' has no null-homotopic cycle of length less than five. To prove this claim suppose to the contrary that it does. Since G has no such cycles we deduce that $G'' \backslash x_1x_2 \backslash x_2x_3$ has a null-homotopic cycle C'' of length at most five that uses the edge x_2x_4. Further, it follows that x_1 is adjacent to x_3, and that the edge x_1x_3 belongs to C''. Thus $C'' \backslash x_1x_3 \backslash x_2x_4$ is a disjoint union of two paths P_1 and P_2, and we may assume from the symmetry that P_1 has ends x_1 and x_2, and that P_2 has ends x_3 and x_4. Let Δ'' be the closed disk bounded by C''; it follows that the edges x_1x_2 and x_3x_4 belong to Δ''. We deduce that C'' is homotopic to the cycle obtained from P_1 by adding the edge

x_1x_2, contrary to the fact that G has no null-homotopic cycle of length less than five. This proves our claim that G' has no null-homotopic cycle of length less than five.

We claim that G' is $\mathcal{F}$-free. To see this suppose for a contradiction that G' has a subgraph $F' \in \mathcal{F}$. Let us apply the operation that produced G' from G in reverse, and let us restrict it to F'. Let the result be F. The inverse operation takes a vertex $v \in V(G')$ and splits it into two nonadjacent vertices x_2 and x_4. Since G is $\mathcal{F}$-free we see that $v \in V(F')$, and hence $x_2, x_4 \in V(F)$. Let $\tilde{F} = F \backslash x_1x_3$ if the edge x_1x_3 belongs to F, and let $\tilde{F} = F$ otherwise. The vertices x_2 and x_4 are incident with a non-simply connected face of $G \backslash x_1x_3$, and hence they are incident with a non-simply connected face of $\tilde{F}$. A simple case analysis shows that this can only happen when G has a subgraph isomorphic to a member of $\mathcal{F}$ with a special edge deleted. Since G is $\mathcal{F}$-free and has no null-homotopic cycles of length at most five it follows that G is isomorphic to a member of $\mathcal{F}$ with a special edge deleted. But the latter graph is 3-colorable, a contradiction. This proves that G' is $\mathcal{F}$-free.

Thus G' is 3-colorable by the minimality of G. But any 3-coloring of G' gives rise to a 3-coloring of G, a contradiction. □

(4.5) *Let G be a minimum counterexample, and let P be a path in G of length two, three or four. Let H be the drawing obtained from G by deleting all edges of $E(G) - E(P)$ that are incident with internal vertices of P, and then contracting at most three edges of P. Then H is $\mathcal{F}$-free.*

Proof. Suppose for a contradiction that H is not $\mathcal{F}$-free; then we may assume that H has a subdrawing $F \in \mathcal{F}$. Thus G has a subdrawing F' that is obtained from F by either

(i) subdividing an edge of F up to three times, or

(ii) splitting a vertex of F of degree four into two vertices of degree three, joined by a path of length two or three.

Since G has no separating cycle of length four by (4.4) we deduce that either $F = F_0$ and F' is obtained as in (i), or $F = F_1$ and F' is obtained as in (ii).

Let us first assume the former. Let the vertices of F_0 be numbered as in Figure 1; since G has no separating cycle of length four we may assume from the symmetry that the edge that has been subdivided is v_1v_2. Let $z_1, z_2, \ldots, z_k$ be the new vertices so that the edge v_1v_2 of F is replaced in F' by a path P with vertex-set $\{v_1, z_1, z_2, \ldots, z_k, v_2\}$ in order, where $k \leq 3$. Then $k > 0$, because G has no separating cycle of length four. Each of the vertices $z_1, z_2, \ldots, z_k$ is incident with an edge of $E(G) - E(F')$ by (3.1). Thus at least one of the faces of F' incident with P has length at least eight by (3.3), and hence $k \geq 2$. On the other hand $k \leq 2$, for otherwise at least two of the edges of $E(G) - E(F')$ incident with z_1, z_2, z_3 belong to the same face, contrary to (3.3). Thus $k = 2$, both faces of F' incident with P have length eight, and each of them includes precisely one edge of $E(G) - E(F')$ incident with $\{z_1, z_2\}$. From (3.1) and (3.3) we deduce that G is isomorphic to the drawing in Figure 3. That drawing, however, is 3-colorable, a contradiction.

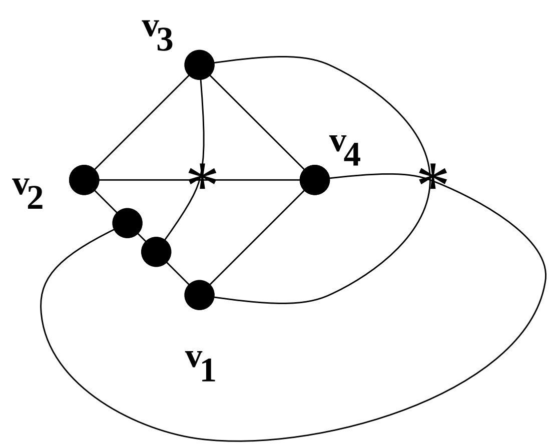

Figure 3. A 3-colorable drawing.

We may therefore assume that $F = F_1$, and that F' is obtained as in (ii). There are two ways to split a vertex of degree four, but one of them preserves both separating cycles of length four, and hence F' arises by means of the other split. By arguing in terms of edges of $E(G) - E(F')$ similarly as in above paragraph, we deduce that G is isomorphic to one of the drawings depicted in Figure 4. Each of those drawings has an even cycle with all vertices of degree three, contrary to (3.6). □

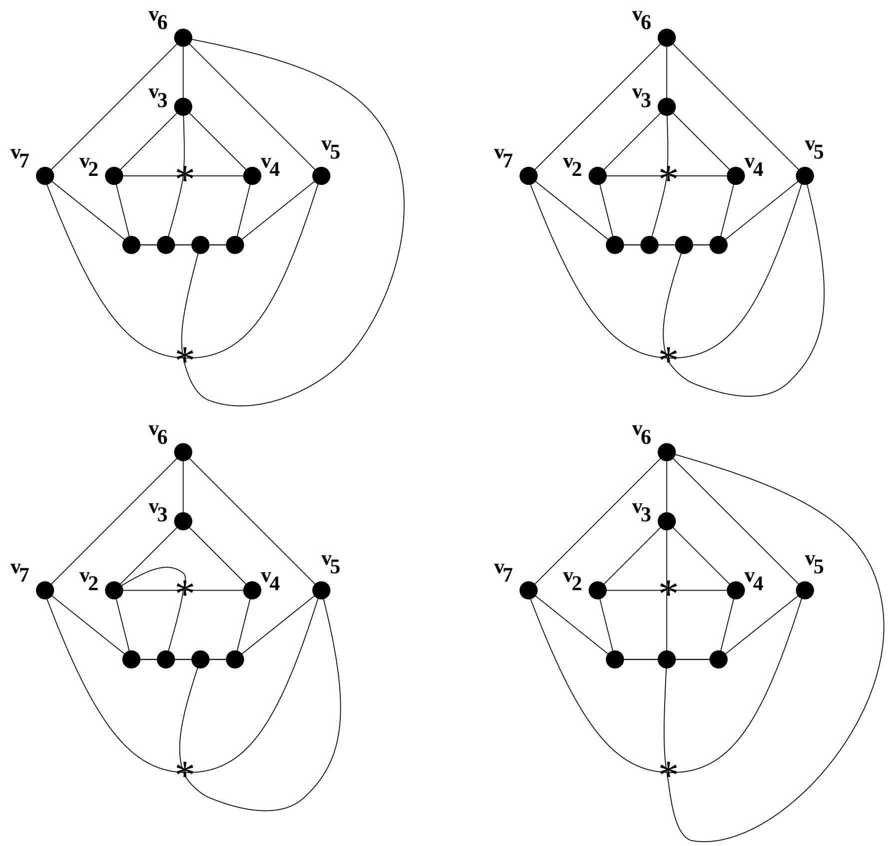

Figure 4. Four 3-colorable drawings.

(4.6) *Let G be a minimum counterexample, and let P be a path in G of length two such that the edges of P are consecutive in the walk bounding a face f of G. Let H be the drawing obtained from G by identifying the ends of P along a curve in f that is fixed-end-homotopic to P. Then H is $\mathcal{F}$-free.*

Proof. Suppose for a contradiction that H has a subdrawing $F \in \mathcal{F}$. Then F is obtained from a subdrawing F' of G by identifying two vertices $u, v \in V(F')$ such that u, v have a common neighbor w, and the edges uw, vw are consecutive in the cyclic ordering of edges of G incident with w. If $w \notin V(F')$, then the result follows from (4.5), and so we may assume that $w \in V(F')$. Since G has no separating cycles of length four, we deduce that $F = F_0$ or $F = F_1$. In either case a simple case-checking gives a contradiction to (3.1) or (3.3). □

(4.7) *Let G be a minimum counterexample, and let $u, v \in V(G)$ be adjacent vertices of degree three. Let u_1, u_2 be the other two neighbors of u, let v_1, v_2 be the other two neighbors of v, and assume that either $u_1 \neq u_2$, $u_2 \neq v_2$ and the path with vertex-set $\{u_2, u, v, v_2\}$ is 2-sided, or the vertices v_2, v, u, u_2 belong to the boundary of a face bounded by a cycle of length five. If u_1 is adjacent to v_1, then u_2 is adjacent to v_2.*

Proof. Suppose for a contradiction that u_1, v_1 are adjacent, but u_2, v_2 are not. Let H be the drawing obtained from G by deleting the edges uu_1 and vv_1, and contracting the edges uu_2, uv and vv_2. Then H is clearly loopless, it is $\mathcal{F}$-free by (4.5), and has no null-homotopic cycle of length three or four by (4.1) and (3.3). Thus H is 3-colorable by the minimality of G, and yet every 3-coloring of H extends to a 3-coloring of G, a contradiction. □

(4.8) *Let G be a minimum counterexample. If G has a face f bounded by a walk with vertices $x_1, x_2, ..., x_k$ in order, where $k > 6$, then x_1 is adjacent to x_3.*

Proof. By (3.1) $x_1 \neq x_3$. Let H be the drawing obtained by identifying x_1 and x_3 along a curve in f, fixed-end-homotopic to the path with vertex-set $\{x_1, x_2, x_3\}$. By (4.6) the drawing H is $\mathcal{F}$-free. From (3.1) and (3.3) we deduce that H has no null-homotopic cycle of length less than five. If H is loopless, then it is 3-colorable by the minimality of G, but every 3-coloring of H gives rise to a 3-coloring of G. Thus H has a loop, and hence x_1 is adjacent to x_3, as desired. □

(4.9) *Let G be a minimum counterexample, and let C be a cycle with vertex-set $\{x_1, x_2, \ldots, x_6\}$ (in order) that bounds a face f of G. Then either x_1 is adjacent to x_3 or x_4 is adjacent to x_6.*

Proof. Let G, C, f and $x_1, x_2, \ldots, x_6$ be as stated, and suppose for a contradiction that x_1 is not adjacent to x_3 and that x_4 is not adjacent to x_6. Let H be the drawing obtained by identifying x_1 and x_3 along a simple curve in f, fixed-end-homotopic to the path with vertex-set $\{x_1, x_2, x_3\}$, and by identifying x_4 and x_6 along a simple curve in f, fixed-end-homotopic to the path with vertex-set $\{x_4, x_5, x_6\}$. Then H is clearly loopless. Let u be the vertex of H that results from identifying x_1 and x_3, and let v be the vertex that results from identifying x_4 and x_6.

We claim that H has no null-homotopic cycles of length three or four. Indeed, suppose for a contradiction that H has such a cycle, say C'. Then C' is not a cycle in G. If $E(C')$ is the edge-set of a path in G with both ends in $V(C)$, then $E(C) \cup E(C')$ includes the edge-set of a null-homotopic cycle D in G of length at most six. Note that D has length at least five. By (3.1) the disk the cycle D bounds includes an edge of G incident with x_2 or x_5, contrary to (3.3). Thus $E(C')$ is the union of the edge-sets of two paths in G, both with ends in $\{x_1, x_3, x_4, x_6\}$. Then $E(C') \cup \{x_1x_2, x_2x_3, x_4x_5, x_5x_6\}$ is the edge-set of a null-homotopic cycle in G of length at most eight, bounding an open disk Δ. If $f \subseteq \Delta$, then $E(C') \cup \{x_3x_4\}$ includes the edge-set of a null-homotopic cycle in G of length at most four, a contradiction. Thus $f \not\subseteq \Delta$. On the other hand, x_2 and x_5 belong to the boundary of Δ, and hence Δ includes an edge of G incident with x_2, and an edge of G incident with x_5. But by (3.3) Δ includes no vertices and at most one edge of G. Thus x_2 and x_5 are adjacent in G, and both have degree three in G. Let J be the drawing obtained from $G\backslash x_5$ by contracting x_1x_2 and x_2x_3. Then J is clearly loopless, it is $\mathcal{F}$-free by (4.5), and has no null-homotopic cycles of length three or four by (3.3). Thus J is 3-colorable by the minimality of G, but every 3-coloring of J extends to a 3-coloring of G, a contradiction. This proves our claim that H has no null-homotopic cycles of length three or four.

Since every 3-coloring of H gives rise to a 3-coloring of G, we deduce from the minimality of G that H is not $\mathcal{F}$-free. Thus we may assume that H has a subdrawing $F \in \mathcal{F}$. We deduce from (4.6) that $u, v \in V(F)$. Since H has no null-homotopic cycles of length three or four, and every face of F has size five or six, we deduce that u and v are adjacent in F. But G has no separating cycle of length four by (4.4), and hence the set $\{u, v\}$ meets every separating cycle of length four in F. Thus F is one of F_0, F_1, F_2. Moreover, from the symmetry we may assume that $F = F_0$ and $\{u, v\} = \{v_1, v_2\}$, or $F = F_0$ and $\{u, v\} = \{v_1, v_3\}$, or $F = F_1$ and $\{u, v\} = \{v_1, v_2\}$, or $F = F_2$ and $\{u, v\} = \{v_1, v_7\}$. In each of these cases except the last we obtain a contradiction to (3.1) or (3.3). In the last case we deduce that G is isomorphic to the drawing depicted in Figure 5. That drawing is 3-colorable, a contradiction. □

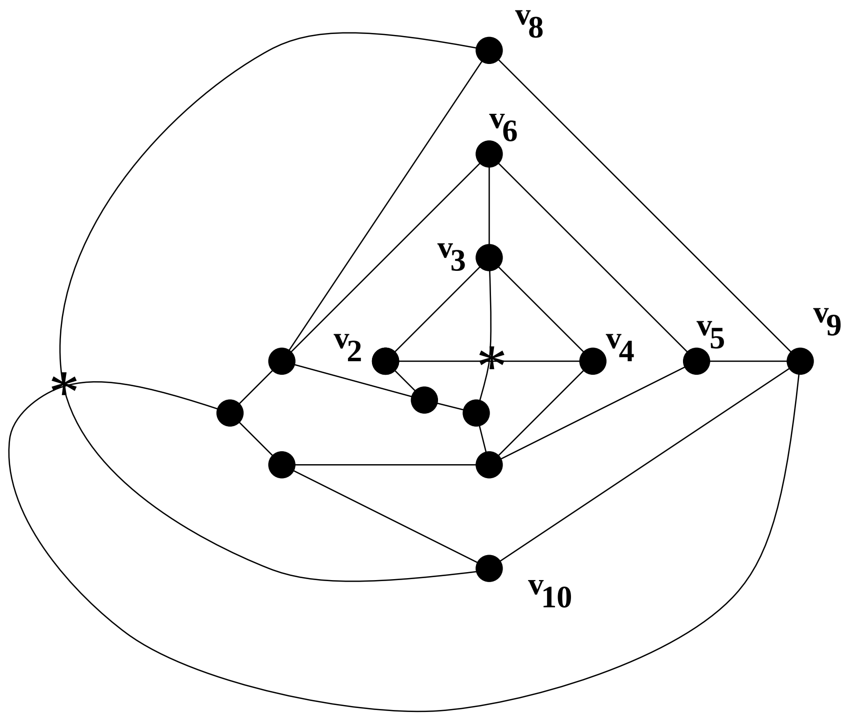

Figure 5. A 3-colorable drawing.

(4.10) *In a minimum counterexample, no face is bounded by a cycle of length six.*

Proof. Let G be a minimum counterexample, and suppose for a contradiction that C is a cycle in G with vertices $x_1, x_2, \ldots, x_6$ (in order) that bounds a face f of G. By applying (4.9) three times we deduce that the subdrawing H of G induced by $V(C)$ has at least nine edges. By Euler's formula it has at least three faces homeomorphic to open disks. Those faces have size at least five, f has size six, and every other face has size at least three. But the sum of the sizes of all faces of H is $2|E(H)|$, and so we see that H has precisely nine edges and three faces, each homeomorphic to an open disk. One of those faces is f, and the other two either have size six, or one has size five and the other has size seven. In either case (3.3) implies that $H = G$, which is a contradiction, because H is 3-colorable. □

(4.11) *Let G be a minimum counterexample, and let f be a face of G bounded by a walk of length at least five. Then f is homeomorphic to an open disk.*

Proof. Suppose for a contradiction that f is not homeomorphic to an open disk, and let Σ' be the surface obtained from Σ by replacing f by an open disk. Then Σ' is the sphere or the projective plane, and G is a drawing in Σ'. From (4.2) and (4.4) it follows that G, regarded as a drawing in Σ', has at most one null-homotopic cycle of length three or four, and if it does, then said cycle is a triangle bounding a face homeomorphic to the Möbius band. In that case G can be regarded as a drawing in the sphere with exactly one face bounded by a triangle and all other cycles of length at least five. Thus G is 3-colorable by (1.2) and (3.2), a contradiction. □

5. FINDING A SUITABLE FACE

A *pentagon* is a cycle of length five.

(5.1) *Let G be a minimum counterexample such that every face bounded by a pentagon is incident with at most three vertices of degree three. Then every face of G is bounded by a pentagon that has three vertices of degree three and two vertices of degree four.*

Proof. Let G be as stated. For every vertex v of G of degree d we define the *charge* of v to be $12 - 4d$ and for a face f of G of size l we define the *charge* of f to be $12 - 2l$ if f is simply connected and $-2l$ otherwise. By Euler's formula the sum of the charges, over all vertices and faces, is nonnegative.

Every face f bounded by a pentagon has a charge of two, and, by hypothesis, is incident with at least two vertices of degree at least four. We pick two such vertices, and send one unit of charge to each of the two vertices. We do this for all faces bounded by pentagons. Thus the new charge of every face is nonpositive. Also, since a vertex of degree $d \geq 4$ receives a charge of at most d, we see that the new charge of every vertex is nonpositive. However, the sum of the charges remained the same, and hence the new charge of every vertex and face is zero. This proves that every face of G is bounded either by a walk of length six with all six vertices of degree three, or by a pentagon with three degree three vertices, and two degree four vertices.

Suppose for a contradiction that some face is bounded by a walk W of length six with all six vertices of degree three. By (3.6) the walk has repeated vertices, and hence it has a repeated edge, because all vertices on W have degree three. It follows that W includes a separating cycle of length four, contrary to (4.4). ☐

(5.2) *Every minimum counterexample has a face bounded by a pentagon with at least four vertices of degree three.*

Proof. We use ideas from [7]. Let G be a minimum counterexample, and suppose for a contradiction that G has no face bounded by a pentagon with at least four vertices of degree three. By (5.1) every face is bounded by a pentagon with three vertices of degree three and two vertices of degree four. We claim that G has two adjacent vertices of degree four. Indeed, suppose that the claim is false. Then every vertex of degree three is adjacent to at most one vertex of degree three, because otherwise there would be a face with three consecutive degree three vertices, and the other two vertices of that face would be adjacent and of degree four, a contradiction. Thus we can color all vertices of degree four with one color, and use the remaining two colors to color vertices of degree three, contrary to the fact that G is not 3-colorable. This proves our claim that G has two adjacent vertices of degree four.

Let x_1 and x_5 be two adjacent vertices of degree four. The two faces incident with the edge x_1x_5 are pentagons with the remaining three vertices of degree three; let the vertices of those two pentagons (in order) be x_1, x_2, x_3, x_4, x_5, and x_1, x_5, x_6, x_7, x_8. For $i = 1, 2, \ldots, 8$ let y_i be the unique neighbor of x_i such that the edge x_iy_i is not incident with either of the two above-mentioned faces.

We distinguish two cases. Assume first that $y_1 \neq y_5$. Let G' be the drawing obtained from G by deleting x_i for $i = 2, 3, 4, 6, 7, 8$, and contracting the edges x_5y_5 and x_5x_1. Thus, the graph of G' is isomorphic to the graph obtained from the graph of G by deleting x_i for $i = 1, 2, \ldots, 8$, and adding the edge y_1y_5. By (4.5) the drawing G' is $\mathcal{F}$-free, it is clearly loopless, and by (4.1) it has no null-homotopic cycle of length three or four. Since G is a minimum counterexample, G' has a 3-coloring, say c. We claim that c can be extended to a 3-coloring of G. Since y_1 is adjacent to y_5 in G' we may assume that $c(y_1) = 1$ and $c(y_5) = 2$. If $x_i = x_j$ for some $i < j$ then $i \in \{2, 3, 4\}$ and $j \in \{6, 7, 8\}$. Furthermore, since x_i has degree three we see that in fact one of x_{i-1}, x_{i+1} equals one of x_{j-1}, x_{j+1}. Since G is simple, we may assume from the symmetry that the pair (x_7, x_8) equals (x_2, x_3), (x_3, x_4), or (x_4, x_3). Thus there are four subcases — either $x_i \neq x_j$ for $i \neq j$, or one of the above three cases holds. We discuss the subcases separately. Assume first that $(x_7, x_8) = (x_2, x_3)$. If $c(y_6) \neq 2$, then we set $c(x_5) = c(y_6)$, and we color the remaining vertices in the order $x_4, x_1, x_3 = x_8, x_2 = x_7, x_6$. Thus we may assume that $c(y_6) = 2$, and from the symmetry we may assume that $c(y_4) = 2$. Now we set $c(x_5) = 1$, and color the remaining vertices in the same order giving them colors $3, 3, 1, 2, 3$, respectively. For the second subcase assume that $(x_7, x_8) = (x_3, x_4)$. If $c(y_2) \neq 1$, then we set $c(x_1) = c(y_2)$, and color the remaining vertices in the order $x_5, x_4 = x_8, x_6, x_3 = x_7, x_2$. Thus we may assume that $c(y_2) = 1$, and from the symmetry we may assume that $c(y_6) = 2$. In this case we set $c(x_1) = 2$ and color the remaining vertices in the same order using the colors $1, 3, 3, 1, 3$ in order. For the third subcase we assume that $(x_7, x_8) = (x_4, x_3)$. If $c(y_2) \neq 1$ we set $c(x_1) = c(y_2)$, and color the remaining vertices in the order $x_5, x_6, x_4 = x_7, x_3 = x_8, x_2$. Thus we may assume that $c(y_2) = 1$, and from the symmetry we may assume that $c(y_6) = 2$. Now we set $c(x_1) = 2$, and color the remaining vertices in the same order giving them colors $3, 1, 2, 1, 3$. Finally, the last subcase is that the x_i are all distinct. If $c(y_8) = 2$ then we color $c(x_1) = 2$ and then color the rest of the vertices of G in the order $x_2, x_3, ..., x_8$. Thus we may assume that $c(y_8) \neq 2$, and from the symmetry we may assume that $c(y_2) \neq 2$, $c(y_4) \neq 1$, and $c(y_6) \neq 1$. If $c(y_8) = c(y_2) = 1$ and $c(y_6) = c(y_4) = 2$ then we color $c(x_1) = 2$, $c(x_5) = 1$ and $c(x_8) = c(x_2) = c(x_6) = c(x_4) = 3$; then we can color x_3 and x_7 to get a 3-coloring of G. Thus we may assume that $c(y_8) = 3$. If $c(y_2) = c(y_3) = 1$ and $c(y_4) = 2$ then we color $c(x_1) = 2$, $c(x_2) = 3$, $c(x_3) = 2$ and then color the remaining vertices in the order x_8, x_7, x_6, x_5, x_4. If y_2, y_3, and y_4 do not have this coloring then we color $c(x_1) = 3$ and $c(x_5) = 1$ and color the remaining vertices in the order $x_2, x_3, x_4, x_6, x_7, x_8$. This completes all four subcases, and hence proves our claim that c extends to a 3-coloring of G. That, however, contradicts the hypothesis that G is a minimum counterexample. This completes the case when $y_1 \neq y_5$.

We may therefore assume that $y_1 = y_5$. We claim that the edges y_1x_1 and y_1x_5 are not consecutive in the cyclic ordering of edges incident with y_1. To prove this suppose to the contrary that they are consecutive. Then they are incident with a common face f. From the symmetry we may assume that x_2 is incident with f. Since f is bounded by a pentagon, we deduce that x_2 is adjacent to x_4 or x_6, and either the edge x_4x_5 or the edge x_5x_6 is incident with f. Let f' be the face incident with the edge x_2x_3, opposite the face bounded by the pentagon with vertex-set $\{x_1, x_2, x_3, x_4, x_5\}$. Then either $x_3x_2x_6x_7$ or $x_3x_2x_4x_3$ are subwalks of the walk W bounding f'. But W is a simple walk containing at most three vertices of degree three, a contradiction. This proves our claim that the edges y_1x_1 and y_1x_5 are not consecutive in the cyclic ordering of edges incident with y_1. In particular, y_1 has degree four. Let x_1, z, x_5, w be the neighbors of y_1 with z and w chosen so that $y_2x_2x_1y_1z$ is a subwalk of a walk bounding a face of G. That face is bounded by a pentagon, and so we have accounted for all of its vertices. It follows that there is a face bounded by $y_8x_8x_1y_1w$. Similarly, there are either faces bounded by $y_4x_4x_5y_1z$ and $y_6x_6x_5y_1w$, or $y_4x_4x_5y_1w$ and $y_6x_6x_5y_1z$. In the first case let W be the closed walk $x_6x_7x_8y_8wy_6$; in the second case let W be the closed walk $x_6x_7x_8y_8wy_4x_4x_3x_2y_2zy_6$. In either case W has even length, and all its vertices have degree three in G. By (3.6) W is not a simple closed walk, and hence it has a repeated edge. But it follows that one of the faces incident with a repeated edge is incident with at least four vertices of W, contrary to the fact that every face is incident with at most three vertices of degree three. □

6. USING THE FACE

In this section we complete the proof of (2.1). Again, we use ideas from [7]. The following will be a hypothesis common to many statements, and so to simplify exposition we give it a name.

Hypothesis H. Let G be a minimum counterexample, let C_0 be a pentagon in G bounding a face f_0, let the vertices of C_0 in order be x_1, x_2, x_3, x_4, x_5, where x_1, x_2, x_3, x_4 all have degree three. For $i = 1, 2, 3, 4$ let y_i be the unique neighbor of x_i that is not a neighbor of x_i in C, and let $x_1, x_4, y_5, y_6, \ldots, y_k$ be all the neighbors of x_5. Thus $k \geq 5$ by (3.1).

(6.1) *Under Hypothesis* H, $x_i \neq y_j$ *for all* $i = 1, ..., 5$ *and* $j = 1, ..., k$.

Proof. Suppose for a contradiction that the result is false. Since G is simple we may assume by symmetry that $y_1 = x_4$ or $y_2 = x_4$ or $y_2 = x_5$. But $y_1 \neq x_4$, because $\{x_1, x_2, x_3, x_4\}$ is not the vertex-set of a cycle by (3.6). If $y_2 = x_4$ then delete x_2,x_3, x_4, and 3-color the rest of the graph. Now since x_1 and x_5 have different colors we can extend this coloring into $\{x_2, x_3, x_4\}$ which would give us a 3-coloring of G, a contradiction. Thus we may assume that $y_2 = x_5$. Let H be the drawing obtained form G by deleting the edges x_4x_5 and x_3y_3, and contracting the edges x_2x_3 and x_3x_4. Then H is clearly loopless, it is $\mathcal{F}$-free by (4.5), and it has no contractible cycle of length less than five by (4.1). Since G is a minimum counterexample, the graph of H can be 3-colored. A 3-coloring of H gives rise to a 3-coloring of $G \backslash x_3 \backslash x_4$ in which x_2 and y_4 receive different colors. This coloring can be extended to a 3-coloring of G, a contradiction. □

(6.2) *Under Hypothesis* H, *if* c *is a* 3*-coloring of* $G \backslash \{x_1, x_2, x_3, x_4\}$, *then it satisfies one of the following conditions:*

(i) $c(y_1) = c(y_2) = c(y_3) = c(y_4) \neq c(x_5)$,
(ii) $c(y_1) = c(y_4) \neq c(y_2) = c(y_3) = c(x_5)$, *or*
(iii) $c(x_5) \neq c(y_1) = c(y_2) \neq c(y_3) = c(y_4) \neq c(x_5)$.

Proof. It is easy to check that, by (6.1), if c satisfies none of the above conditions, then it extends to a 3-coloring of G, a contradiction. □

(6.3) *Under Hypothesis* H, *if* $y_1 = y_2 = y_3 = y_4$, *then this vertex is not adjacent to* x_5.

Proof. Suppose for a contradiction that $y_1 = y_2 = y_3 = y_4$, and that this vertex is adjacent to x_5. Let H be the drawing obtained from G by deleting $V(G) - \{x_1, x_2, x_3, x_4, x_5, y_1\}$. Thus the graph of H consists of a pentagon and one vertex adjacent to every vertex of the pentagon. By Euler's formula H has at least four faces that are homeomorphic to open disks, and hence each of them is incident with at least five edges of H. But the sum of the sizes of all faces is 20, and hence H has precisely four faces, each bounded by a pentagon.

On the other hand, H has a vertex y_1 of degree five, and so for some face the vertex y_1 occurs more than once in the boundary walk bounding that face. Such face cannot be bounded by a pentagon, a contradiction. □

(6.4) *Under Hypothesis* H, x_5 *has degree at least four.*

Proof. Suppose for a contradiction that x_5 has degree three. Then there is symmetry between the vertices of C_0, and so by (6.3) we may assume that $y_1 \neq y_3$. Let H be the drawing obtained from G by deleting the vertices x_4 and x_5, deleting the edge x_2y_2, and contracting the edges x_1y_1, x_1x_2 and x_3y_3. Then H is clearly loopless, it is $\mathcal{F}$-free by (4.5), and has no contractible cycle of length less than five by (4.1). On the other hand, every 3-coloring of H extends to a 3-coloring of G, a contradiction. □

(6.5) *Under Hypothesis* H, *the vertices* y_1, y_2, y_3, y_4 *are not equal.*

Proof. Suppose for a contradiction that $y_1 = y_2 = y_3 = y_4$, and let H be the drawing obtained from G by deleting x_1, x_2, x_3, x_4 and identifying y_1 and x_5 along the path $y_1x_2x_1x_5$. Then H is loopless by (6.3), it is $\mathcal{F}$-free by (4.5), and it has no null-homotopic cycle of length at most four by (4.1). Since G is a minimum counterexample, the drawing H has a 3-coloring, and this 3-coloring gives rise to a 3-coloring of G, a contradiction. □

(6.6) *Under Hypothesis* H, *if* $y_i \neq y_{i+1}$ *for some* $i \in \{1, 2, 3\}$, *then the two faces incident with the edge* x_ix_{i+1} *are pentagons.*

Proof. Let f be the face incident with x_ix_{i+1} opposite f_0, and let u, v be such that the walk bounding f has a subwalk $vy_{i+1}x_{i+1}x_iy_iu$. By (6.1) x_i is not adjacent to y_{i+1}. By (4.8) f is not bounded by a walk of length at least seven. By (4.10) f is not bounded by a cycle of length six and since the vertices $y_i, x_i, x_{i+1}, y_{i+1}$ are pairwise distinct and x_i is not adjacent to y_{i+1}, we deduce that f is not bounded by a walk of length six, either. Thus f is bounded by a pentagon, as desired. □

(6.7) *Under Hypothesis* H*, y_i is not adjacent to y_{i+1} for $i = 1, 2, 3$.*

Proof. To prove that y_i is not adjacent to y_{i+1} we may assume that $y_i \neq y_{i+1}$. Then the two faces incident with the edge $x_i x_{i+1}$ are bounded by pentagons by (6.6). Since $y_2 \neq x_4$, $y_2 \neq x_5$ and $y_3 \neq x_5$ by (6.1), (4.7) applied to $\{u, v\} = \{x_i, x_{i+1}\}$ implies that y_i is not adjacent to y_{i+1}, as desired. □

(6.8) *Under Hypothesis* H*, if y_3 is not adjacent to x_5, then $y_1 \neq y_3$.*

Proof. Suppose for a contradiction that y_3 is not adjacent to x_5, and that $y_1 = y_3$. Let H be the drawing obtained from $G \backslash x_1 \backslash x_2$ by deleting the edges $x_2 x_3$ and $x_4 y_4$, and contracting the edges $y_3 x_3, x_3 x_4, x_4 x_5$. Then H is clearly loopless, it has no null-homotopic cycle of length three or four by (4.1), and is $\mathcal{F}$-free by (4.5). On the other hand, no 3-coloring of H satisfies the conclusion of (6.2), a contradiction. □

(6.9) *Under Hypothesis* H*, if $y_2 = y_3$, then $y_1 = y_4$.*

Proof. Suppose for a contradiction that $y_2 = y_3$, and that $y_1 \neq y_4$. Let H be the drawing obtained from $G \backslash \{x_1, x_2, x_3, x_4\}$ by adding an edge with ends y_1 and y_4, placing the edge along the path with vertex-set $\{y_1, x_1, x_2, x_3, x_4, y_4\}$. Then H is loopless, and an argument similar to the one used in the proof of (4.1) shows, using (4.11), that H has no contractible cycle of length three of four. Since no 3-coloring of H satisfies the conclusion of (6.2), we deduce from the minimality of G that H is not $\mathcal{F}$-free. Thus we may assume that H has a subdrawing $F \in \mathcal{F}$. Let J be the drawing obtained from G by deleting x_2, x_3, and all edges incident with x_5 except $x_1 x_5$ and $x_4 x_5$, and then contracting $y_1 x_1$, $x_1 x_5$, and $x_4 x_5$. If $x_5 \notin V(F)$, then J has a subdrawing isomorphic to F, contrary to (4.5). Thus $x_5 \in V(F)$. It follows that G has a subdrawing F' obtained from F by subdividing one of its edges four times, creating vertices x_1, x_2, x_3, x_4. But x_1 and x_4 are adjacent to x_5 in G, which is easily seen to contradict (3.3). □

(6.10) *Under Hypothesis* H, *either y_2 or y_3 is not adjacent to x_5.*

Proof. Suppose for a contradiction that both y_2 and y_3 are adjacent to x_5. From (6.1) and (4.7) applied to $\{u, v\} = \{x_1, x_2\}$ we deduce that $y_1 = y_3$, and similarly $y_2 = y_4$. It follows from (6.5) that the drawing H induced by $\{x_1, x_2, x_3, x_4, x_5, y_1, y_2\}$ has seven vertices and at least eleven edges. Thus at least four faces of H are homeomorphic to open disks. Those faces have size at least five, but the sum of the sizes of all faces is $2|E(H)|$, and so we see that H has exactly eleven edges and exactly four faces, and each face has size at most seven. Thus (3.3) implies that $H = G$. But H is 3-colorable, a contradiction. □

(6.11) *Assume Hypothesis H, assume that y_1, y_2, y_4 are pairwise distinct and that y_4 has degree three. Then*

(i) *y_4 is not adjacent to both y_1 and y_2, and*

(ii) *if a vertex v is adjacent to y_1, y_2 and y_4, then v has degree at least four.*

Proof. Let H be the drawing obtained from $G\backslash\{x_1, x_2, x_3, x_4, y_4\}$ by identifying the vertices y_1 and y_2 along the path $y_2x_2x_1y_1$. By (6.7) the drawing H is loopless, by the argument in the proof of (4.1), using (4.8) and (4.10) and the fact that y_1 is not adjacent to x_2, it follows that H has no contractible cycles of length three or four, and by (4.5) the drawing H is $\mathcal{F}$-free. Thus H has a 3-coloring c. If y_4 is adjacent to y_1 and y_2, then c can be extended to a 3-coloring of $G\backslash\{x_1, x_2, x_3, x_4\}$ such that $c(y_4) = c(x_5)$ if $c(y_1) \neq c(x_5)$, contrary to (6.2). This proves (i).

To prove (ii) we apply the same argument to the drawing $H\backslash v$. We conclude that $H\backslash v$ has a 3-coloring, say d. Since y_4 is adjacent to only one vertex currently colored, the coloring d can be extended to y_4 in such a way that the resulting coloring does not satisfy any of the outcomes of (6.2). By (6.2) this coloring cannot be extended to v, and hence v has degree at least four, as desired. □

(6.12) *No minimum counterexample exists.*

Proof. Suppose for a contradiction that G is a minimum counterexample. By (5.2) we may assume that Hypothesis H holds. By (6.10) and the symmetry we may assume that y_3 is not adjacent to x_5. Let H be the drawing obtained from G by deleting x_1, x_2, x_3, x_4, and identifying x_5 and y_3 along the path with vertex-set $\{x_5, x_4, x_3, y_3\}$, and if $y_1 \neq y_2$, then also identifying y_1 and y_2 along the path with vertex-set $\{y_1, x_1, x_2, y_3\}$.

We claim that H is loopless. If $y_1 = y_2$, or if the vertices resulting from the two identifications are distinct, then H is loopless by (6.7) and the fact that y_3 is not adjacent to x_5. Thus we may assume that $y_1 \neq y_2$, and that the vertices resulting from the identifications are equal. By (6.1) and (6.8) it follows that $y_2 = y_3$, and hence (6.9) implies that $y_1 = y_4$. Since y_3 is not adjacent to x_5 and y_2 is not adjacent to y_1, it remains to show that x_5 is not adjacent to $y_1 = y_4$. But if it was, then the sets $\{x_2, x_3, y_3\}$, $\{x_1, x_5, y_1\}$, $\{x_4, x_5, y_1\}$ would each induce a triangle in G. It would follow that $\{x_4, x_5, x_1, y_1\}$ is the vertex-set of a separating cycle of length four, contrary to (4.4). This proves that H is loopless.

Next we claim that H has no null-homotopic cycle of length three or four. Suppose for a contradiction that it does. From (4.1) we deduce that $y_1 \neq y_2$, and that G has two disjoint paths P_1 and P_2 such that $|E(P_1)| + |E(P_2)|$ is equal to three or four, and $E(P_1) \cup E(P_2) \cup \{x_1y_1, x_1x_2, x_2y_2, y_3x_3, x_3x_4, x_4x_5\}$ is the edge-set of a null-homotopic cycle in G, say C. Thus $|E(C)| \leq 10$. Since $y_1 \neq y_2$, (6.6) implies that both faces incident with x_1x_2 are bounded by pentagons. Let $v \in V(G)$ be such that $\{y_1, x_1, x_2, y_2, v\}$ is the vertex-set of one of the pentagons. Then $v \notin V(C)$, for otherwise $V(C)$ includes the vertex-set of a null-homotopic cycle in G of length at most seven bounding an open disk that includes at least one edge of G, contrary to (3.3). Let C' be the graph obtained from $C \backslash x_1 \backslash x_2$ by adding the vertex v and the edges vy_1 and vy_2. Then $|E(C')| \leq 9$ and C' bounds an open disk Δ' by (4.11). By (3.3) the disk Δ' includes either one vertex of G of degree three and the incident edges, or no vertices and at most one edge of G. It follows that $f_0 \cap \Delta' = \emptyset$. Thus both x_4 and v are incident with an edge that belongs to Δ', and hence they both have degree three. Moreover, either x_4 is adjacent to v, in which case $v = y_4$ and vx_4 is the only edge in Δ', or y_4 is the only vertex in Δ', in which case y_4 has degree three and is adjacent to v. In either case we get a contradiction to (6.11). This completes the proof of the claim that H has no null-homotopic cycle of length three of four.

Our next and final objective will be to show that H is $\mathcal{F}$-free. That will complete the proof, because it will follow that H has a 3-coloring by the minimality of G, and yet no 3-coloring of H satisfies the conclusion of (6.2), a contradiction. Thus it remains to show that H is $\mathcal{F}$-free.

Suppose for a contradiction that H is not $\mathcal{F}$-free; then we may assume that H has a subdrawing $F \in \mathcal{F}$. Thus G has a subdrawing F' that is obtained from F by applying once or twice the following two operations:

(i) subdividing an edge three times,

(ii) splitting a vertex of degree four into two vertices of degree three, joined by a path of length three.

We deduce from (4.5) that the operations are, in fact, applied twice. By (4.4) each special edge of F is involved in at most one operation (i). Since F' is a subgraph of G it follows that every cycle of G is homeomorphic to an open disk, and hence has size at least five. We deduce that if operation (i) is applied twice, then each time it is applied to a different edge of F.

The edges that result from operations (i) and (ii) are y_1x_1, x_1x_2, x_2y_2, y_3x_3, x_3x_4, and x_4x_5. We shall refer to them as *new edges*. There is an obvious bijection between the faces of F and the faces of F'. Let us recall that the face f_0 is defined in Hypothesis H. Let f_1' be the face of F' that includes the face f_0. Then f_1' includes the edges x_1x_5 and x_2x_3. The edge x_1x_2 is incident with f_1'; let f_2' be the other face of F' it is incident with. Then possibly $f_1' = f_2'$, but, since x_1 and x_2 have degree two in F', f_2' is also incident with y_1x_1 and y_2x_2. Similarly, the edge x_3x_4 is incident with f_1'; let f_3' be the other face of F' it is incident with. Again, f_3' is also incident with y_3x_3 and x_4x_5. Since G has no separating cycles of length four by (4.4), we deduce that at least one of the triples of new edges does not arise by subdividing a special edge of F. (Special edges were defined in Section 2; they are those that pass "through" the crosscaps in our pictures.) It follows that f_1' is bounded by a walk of length 15 if one of the triples of new edges arose by subdividing a special edge, and otherwise it is bounded by a walk of length at most twelve. We claim the following.

(1) *If $f'_2 \neq f'_3$, then the edges y_3x_3, x_3x_4, x_4x_5 arise by means of operation (ii).*

To prove (1) let us assume that $f'_2 \neq f'_3$, and suppose for a contradiction that y_3x_3, x_3x_4, x_4x_5 arise by means of operation (i). Then x_4, and one of y_3, x_5 have degree two in F'. From (3.1) and (6.4) we deduce that x_4 and one of y_3, x_5 are each incident with an edge of $E(G) - E(F') - \{x_2x_3, x_1x_5\}$, say e_1 and e_2. We claim that $e_1 \neq e_2$. To prove that suppose for a contradiction that $e_1 = e_2$. Since G is simple, e_1 has ends x_4 and y_3. From the fact that the triangle $x_4x_3y_3$ is not null-homotopic we deduce that $f'_1 = f'_3$. Thus y_3x_3, x_3x_4, x_4x_5 arise by means of subdividing a special edge. This, however, is impossible, because every face of G has size at least five. This proves that $e_1 \neq e_2$.

Assume now that $f'_3 \neq f'_1$. Then f'_3 has size at most nine (every face of F has size at most six, and in this case at most one of the operations (i), (ii) affected the boundary of f'_3), and hence at most one of e_1, e_2 belongs to f'_3 by (3.3). Thus the other element of $\{e_1, e_2\}$ belongs to f'_1, in addition to the edges x_1x_5 and x_2x_3. Those three edges divide f'_1 into at least four faces, each of size at least five. Thus f'_1 has size at least 14, and hence at least one of the triples of new edges arises by subdividing a special edge. But $f'_3 \neq f'_1$, and so there is exactly one such triple, namely y_1x_1, x_1x_2, x_2y_2. But one of y_1, y_2 has degree two in F', and so is incident with an edge of $E(G) - E(F')$. Since f'_1 is bounded by a walk of length 15, we deduce that this edge is one of e_1, e_2, say e_1. A simple case-checking (using only the fact that every face of G has size at least five) reveals that this is impossible. For instance, if e_1 has ends y_2 and y_3, then $y_2y_3x_3x_2$ bounds a face in G. Thus $f'_3 = f'_1$. Since f'_1 includes the edges x_1x_5, x_2x_3, e_1, e_2 we have a contradiction, because f'_1 has size at most 15, and yet every face of G has size at least five. This completes the proof of (1).

We now claim that $f'_2 = f'_3$. To prove this claim suppose for a contradiction that $f'_2 \neq f'_3$. Then, by (1), the edges y_3x_3, x_3x_4, x_4x_5 arise by means of operation (ii). Since F_0 has no vertex of degree four and G has no separating cycle of length four, we may assume that $F = F_1$ or $F = F_2$. In either case we assume that the vertices of F are numbered as in Figure 1, and from the symmetry we may assume that the vertex being split to produce y_3x_3, x_3x_4, x_4x_5 is v_1. Since G has no separating cycles of length four, we see that one of the new vertices resulting from splitting v_1 is adjacent to v_4 and v_5, and the other is adjacent to v_2 and v_7. Let f'' be the face of F' corresponding to the face of F

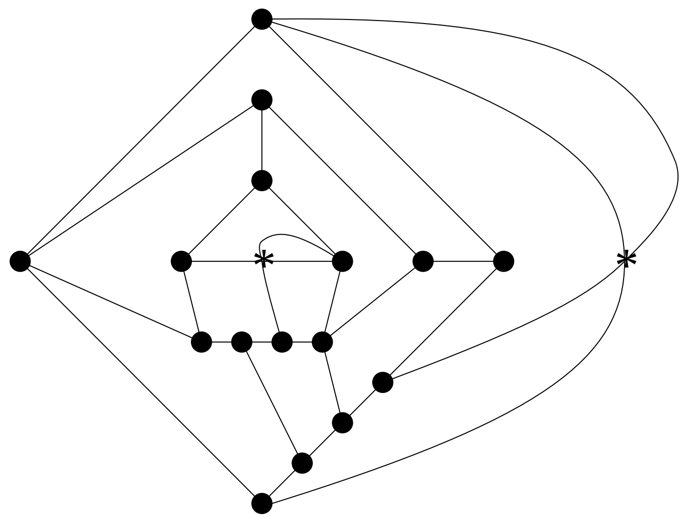

Figure 6. An example of a drawing.

bounded by the walk $v_1v_2v_4v_3v_2v_4v_1$. If $f_1' = f''$, then the fact that every face of G has size at least five implies that y_2, x_2, x_1, y_1 all have degree three in G, and hence G has a cycle of length six with all vertices of degree three (consider the cycle of F' that corresponds to the cycle of F with vertex-set $\{v_2, v_3, v_4\}$), contrary to (3.6). Thus $f_1' \neq f''$, and hence f'' has size nine. If $F = F_1$, then from the symmetry we could assume that $f_1' = f''$. Thus $F = F_2$. By (3.3), f'' includes either one vertex of G of degree three, or no vertices and at most one edge of G. It cannot be the former, for such vertex would have to be adjacent to x_4, and its other two neighbors would have to be equal, contrary to (3.1). But x_4 has degree two in F', and so f'' includes an edge of G incident with x_4. It follows that this edge is incident with a face of G of size six. By (4.10) this face is not bounded by a cycle. This fact uniquely specifies the other end of the edge of $E(G) - E(F')$ incident with x_4. Using the same arguments we deduce that G is isomorphic to the drawing depicted in Figure 6. That drawing, however, has a cycle of length ten with all vertices of degree three, contrary to (3.6). This contradiction proves that $f_2' = f_3'$.

Since $f_2' = f_3'$ and since G has no separating cycles of length four, we deduce that F

is isomorphic to F_0. Thus f'_1 has size 12, and hence it includes no edge of $G\backslash\{x_1x_5, x_2x_3\}$. The edges x_1x_5 and x_2x_3 divide f'_1 into two faces of size five and one face of size six. The face of size six is either bounded by a cycle, or its boundary includes a separating cycle of length four. The former case contradicts (4.10), and the latter contradicts (4.4). □

From (6.12) we deduce that (2.1) holds, as desired.

Acknowledgement

The results of this paper appeared in the Ph.D. dissertation of the second author [10].

References

1. J. A. Bondy and U. S. R. Murty, Graph Theory with Applications, North-Holland, New York, Amsterdam, Oxford, 1976.
2. P. Erdös, A. L. Rubin and H. Taylor, Choosability in graphs, *Congressus Numer.* **26** (1979), 125–157.
3. J. Gimbel and C. Thomassen, Coloring graphs with fixed genus and girth, *Trans. Amer. Math. Soc.* **349** (1997), 4555–4564.
4. H. Grötzsch, Ein Dreifarbensatz für dreikreisfreie Netze auf der Kugel, *Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Natur. Reihe* **8** (1959), 109–120.
5. H. V. Kronk, The chromatic number of triangle-free graphs, In: Graph Theory and Applications, Y. Alavi, D. R. Lick, and A. T. White Eds., *Lecture Notes in Mathematics* **303** (1972), 179–181.
6. H. V. Kronk and A. T. White, A 4-color theorem for torodial graphs, *Proc. Amer. Math. Soc.* **34** (1972), 83–86.
7. C. Thomassen, Grötzsch's 3-color theorem and its counterparts for the torus and the projective plane, *J. Combin. Theory Ser. B* **62** (1994), 268–279.
8. C. Thomassen, Color-critical graphs on a fixed surface, *J. Combin. Theory Ser. B* **70** (1997), 67–100.

9. C. Thomassen, The chromatic number of a graph of girth 5 on a fixed surface, *J. Combin. Theory Ser. B* **87** (2003), 38-71.
10. B. Walls, Coloring girth restricted graphs on surfaces, Ph.D. dissertation, Georgia Institute of Technology, 1999.
11. R. L. Woodburn, A 4-color theorem for the Klein bottle, *Discrete Math.* **76** (1989), 271–276.
12. D. A. Youngs, 4-chromatic projective graphs, *J. Graph Theory* **21** (1996), 219–227.

This material is based upon work supported by the National Science Foundation under Grant No. 0200595. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.