But this is false; the graph $K_{4,4}^-$, obtained from $K_{4,4}$ by deleting one edge, provides a counterexample. It is true for graphs containing no subdivision of $K_{4,4}^-$, but we omit the proof.

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Printed by Catherine Press, Ltd., Tempelhof 41, B-8000 Brugge, Belgium

Reprinted from Journal of Combinatorial Theory, Series B
Vol. 64, No. 2, July 1995
Printed in Belgium

Kuratowski Chains

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Received January 28, 1992

We prove that if H and H' are subgraphs of a graph G, and both are isomorphic to subdivisions of K_5 or $K_{3,3}$, then the following are equivalent: (i) there is a sequence $H = H_1, H_2, ..., H_k = H'$ of subgraphs of G, each isomorphic to a subdivision of K_5 or $K_{3,3}$ and each differing only a "small amount" from its predecessor; (ii) H and H' are not "separated" in G by a vertex separation of order $\leqslant 3$. This is a lemma for use in a future paper concerning linkless embeddings of graphs in 3-space. © 1995 Academic Press, Inc.

1. Introduction

A *subdivision* of a loopless graph H is a graph obtained from H by replacing its edges by internally vertex-disjoint paths. (All graphs in this paper are finite, and we permit loops and parallel edges.) If G is a graph, a *hexad* in G is a subgraph of G isomorphic to a subdivision of $K_{3,3}$, and a *pentad* is isomorphic to a subdivision of K_5. A *Kuratowski subgraph* of G is a hexad or a pentad in G.

Before we describe the results of this paper, let us give them some motivation. Our main result is intended as a lemma for use in a future paper [4]

* This research was carried out under a consulting agreement with Bellcore, and with support from NSF Grant DMS 8903132 and from ONR Grant N00014-911-J-1905.
† This research was carried out while visiting the DIMACS center at Rutgers University and with support from NSF Grant DMS 9103480.

0095-8956/95 $12.00

on "panelled" embeddings. An *embedding* (in 3-space) of a graph G is a homeomorphism ϕ from G (regarded as a topological space in the usual way) to a subspace $\phi(G)$ of the 3-sphere S^3; and it is *panelled* if for every circuit C of G, the simple closed curve $\phi(C)$ in S^3 bounds a disc in S^3 that is disjoint from the remainder of $\phi(G)$. If we have a panelled embedding of G we can obtain others by applying homeomorphisms of S^3; and one goal of [4] is a converse, that if G is sufficiently connected (4-connected, say) then *every* panelled embedding of G can be obtained from a given one by applying homeomorphisms of S^3. How can we prove this?

Now, up to orientation-preserving homeomorphisms of S^3, any hexad or pentad has exactly two panelled embeddings (as a *labelled* graph; if it is unlabelled there is only one). Let us call these the two possible *orientations* of the hexad or pentad. Any panelled embedding of a graph G will therefore assign an orientation to each Kuratowski subgraph of G. One can also show that if two panelled embeddings ϕ_1, ϕ_2 of G give equal orientations to every Kuratowski subgraph of G, then ϕ_1 and ϕ_2 are related by an orientation-preserving homeomorphism of S^3. Consequently, we could prove that every panelled embedding of a 4-connected graph can be obtained from a given one by applying homeomorphisms of S^3, if we could prove that any two panelled embeddings of a 4-connected graph either give equal orientations to every Kuratowski subgraph of G, or that they give opposite orientations to every Kuratowski subgraph. In other words, it remains to prove that if ϕ_1, ϕ_2 are panelled embeddings of a 4-connected graph G, and H, H' are Kuratowski subgraphs of G, and ϕ_1, ϕ_2 give the same orientations to H, then they give the same orientation to H'. Now this can be proved directly if H and H' differ only slightly (we shall make this precise later), and so it would follow for a general pair H, H' if we show that there is always a sequence of Kuratowski subgraphs

$$H = H_1, H_2, ..., H_k = H',$$

each differing only slightly from its predecessor. This is true and is a corollary of the main result of this paper, which is a necessary and sufficient condition on H, H' for such a sequence to exist. (The condition is, roughly, that no $(\leqslant 3)$-separation of G "separates" H_1 and H_2.)

That then is the motivation of the paper; now let us give a precise statement of the theorem. The *valency* of a vertex of a graph is the number of edges incident with it, counting loops twice. A vertex with valency k is said to be *k-valent*. A graph is *linear* if it is isomorphic to a subdivision of a non-null loopless graph with minimum valency $\geqslant 3$. If J is a linear graph, its vertices of valency $\geqslant 3$ are called its *nodes*. An *arc* of J is a path of J (paths are non-null and do not have "repeated" vertices or edges) with distinct ends, both nodes of J, and with no internal vertex that is a node of J

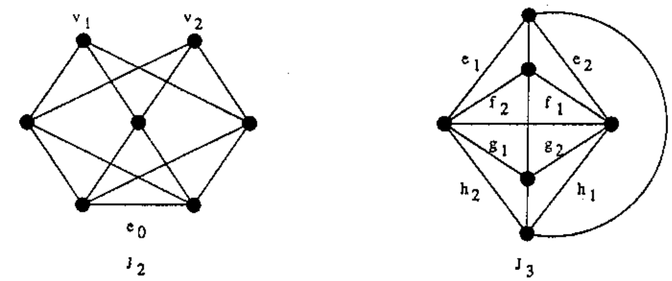

FIGURE 1

Thus, the arcs of J partition $E(J)$. (The vertex- and edge-sets of a graph G are denoted by $V(G)$ and $E(G)$.) If L is a path, its *interior* consists of all its edges, and all its vertices except its ends. (We shall need to speak of deleting the interior of arcs.) An arc L is *incident* with a node v if v is an end of L. We observe that the union of two linear subgraphs is a linear subgraph. If L is an arc of a linear subgraph J of G, and $X \subseteq E(G)$, we say that *X meets L* if $X \cap E(L) \neq \varnothing$.

If H_1 and H_2 are both hexads or both pentads of G, we say that H_1 and H_2 are *1-adjacent* if there are arcs L_1, L_2 of $H_1 \cup H_2$, such that H_i is obtained from $H_1 \cup H_2$ by deleting the interior of L_i $(i = 1, 2)$. If H_1 is a hexad and H_2 is a pentad of G, we say that H_1 and H_2 are *1-adjacent* if there is an arc L of $H_1 \cup H_2$ such that H_2 is obtained from $H_1 \cup H_2$ by deleting the interior of L. If H_1 and H_2 are both hexads, we say that H_1, H_2 are *2-adjacent* if there is a subgraph J of G isomorphic to a subdivision of the graph J_2 of Fig. 1, and for $i = 1, 2$ H_i may be obtained from J by deleting the vertex corresponding to v_i, the interiors of the arcs incident with v_i, and the interior of the arc corresponding to e_0. We say that Kuratowski subgraphs H, H' of G *communicate* if there is a sequence $H = H_1, H_2, ..., H_k = H'$ of Kuratowski subgraphs of G, such that for $1 \leqslant i \leqslant k-1$, either H_i and H_{i+1} are 1-adjacent, or they are both hexads and are 2-adjacent.

A *separation* is a pair (A, B) of subgraphs of G with $A \cup B = G$ and $E(A \cap B) = \varnothing$, and its *order* is $|V(A \cap B)|$. It is a *$(\leqslant k)$-separation* if $|V(A \cap B)| \leqslant k$ and a *k-separation* if $|V(A \cap B)| = k$. We observe that

(1.1) *If H is a Kuratowski subgraph of G, and (A, B) is a $(\leqslant 3)$-separation of G, then one (and hence exactly one) of $E(A), E(B)$ meets $\leqslant 3$ arcs of H.*

The proof is clear. If H, H' are Kuratowski subgraphs of G, a $(\leqslant 3)$-separation (A, B) of G *divides* H and H' if $E(A)$ meets $\leqslant 3$ arcs of H and $E(B)$ meets $\leqslant 3$ arcs of H', or vice versa. Our main theorem is the following.

(1.2) *Let H, H' be Kuratowski subgraphs of G. Then H and H' communicate in G if and only if no ($\leqslant 3$) separation of G divides H and H'.*

If H_1 and H_2 are both hexads in G, we say that they are 3-*adjacent* if there is a subgraph J of G isomorphic to a subdivision of the graph J_3 of Fig. 1, and for $i = 1, 2$ H_i may be obtained from J by deleting the interiors of the arcs of J corresponding to e_i, f_i, g_i and h_i.

For $t = 1, 2, 3$ we say that hexads H, H' in G *t-communicate* (in G), or *H t-communicates with H'*, if there is a sequence $H = H_1, H_2, ..., H_k = H'$ of hexads of G such that for $1 \leqslant i \leqslant k-1$, H_i and H_{i+1} are j-adjacent for some $j \leqslant t$. The main step in the proof of (1.2) consists of proving the following.

(1.3) *Let H, H' be hexads in a graph G. Then H and H' 3-communicate in G if and only if no ($\leqslant 3$)-separation of G divides H and H'.*

(1.3) has the advantage that it does not involve pentads, and this helps to reduce the amount of case analysis. The proof of (1.3) will occupy most of the paper.

Alexander Kelmans informs us that in about 1979 he proved a result like (1.3) for graphs with no circuits of length $\leqslant 3$ and with minimum valency $\geqslant 3$, using 1-communication instead of 3-communication. He did not publish the result explicitly, but tells us that it follows by applying the lemmas and proof techniques in [2].

2. Separations

In this section we prove a variety of lemmas about separations that we shall need to prove (1.3), and in particular we prove the easy "only if" part of (1.3). We begin with the following.

(2.1) *Let H be a hexad of G, and let (A, B) be a ($\leqslant 3$)-separation of G. Then $E(A)$ meets $\leqslant 3$ arcs of H if and only if $H \cap A$ is a forest; and if so, then*

(i) *there is a node v of H so that every arc of H with more than one vertex in A is incident with v, and the other five nodes of H belong to $V(B)$*

(ii) *if there are $\geqslant 3$ arcs of H with more than one vertex in A then some node of H is in $V(A) - V(B)$.*

The proof is clear.

For a graph J, we denote by $J \backslash X$ the result of deleting the members of the set X. Here the members of X may be vertices or edges or both. We write $H \subseteq G$ to denote that H is a subgraph of G, for graphs G, H.

(2.2) *If (A, B) is a ($\leqslant 3$)-separation of G, and H_1, H_2 are hexads in G such that $H_1 \cap A$ and $H_2 \cap B$ are forests, then for $1 \leqslant t \leqslant 3$ H_1 and H_2 are not t-adjacent.*

Proof. Suppose first that H_1 and H_2 are 1-adjacent, and let $J = H_1 \cup H_2$, and let L_1, L_2 be arcs of J so that for $i = 1, 2$. H_i is obtained from J by deleting the interior of L_i. For a general connected graph H, let

$$\rho(H) = |E(H)| - |V(H)| + 1;$$

then $\rho(H)$ is the minimum number of edges whose deletion from H leaves a forest. Since deleting the interior of L_1 from J leaves a hexad, it follows that $\rho(J) = 5$. Let $e_i \in E(L_i)$ $(i = 1, 2)$, and let $J' = J \backslash \{e_1, e_2\}$. Then $J' \cap A$ is a forest, since $H_1 \cap A$ is a forest, and $J' \cap B$ is a forest, similarly. Since $J' \cap A$ is a forest, by deleting at most two more edges e_3, e_4 we can destroy all paths of $J' \cap A$ joining distinct vertices in $V(A \cap B)$, since $|V(A \cap B)| \leqslant 3$. Hence $J \backslash \{e_1, e_2, e_3, e_4\}$ is a forest and so $\rho(J) \leqslant 4$, a contradiction. Thus H_1 and H_2 are not 1-adjacent.

Suppose that they are 2-adjacent. Let J be the corresponding subgraph of G isomorphic to a subdivision of the graph J_2 of Fig. 1. Let J have nodes $v_1, ..., v_7$, where v_1, v_2 are as in Fig. 1, and v_3, v_4 correspond to the ends of e_0. For $1 \leqslant i < j \leqslant 7$, let L_{ij} be the arc of J with ends v_i, v_j if there is one. Then for $i = 1, 2$ H_i is obtained from J by deleting v_i and the interiors of L_{i5}, L_{i6}, L_{i7}, and L_{34}.

Now since $H_1 \cap A$ is a forest it follows from (2.1)(i) that at most three arcs of H_1 have an edge in A, and hence at most six arcs of $H_1 \cup H_2$ have an edge in A, with equality only if all the arcs incident with v_1 have an edge in A. Similarly, at most six arcs of $H_1 \cup H_2$ have an edge in B. Since $H_1 \cup H_2$ has 12 arcs, we have equality throughout, and so no arc of $H_1 \cup H_2$ has an edge in both A and B, every arc incident with v_1 is included in A, every arc incident with v_2 is included in B, and A, B each include three of the arcs $L_{ij} (i = 3, 4, j = 5, 6, 7)$. Consequently, $v_5, v_6, v_7 \in V(A \cap B)$, and so $V(L_{34}) \cap V(A \cap B) = \emptyset$. Thus, since L_{34} is connected, we may assume from the symmetry that $V(L_{34}) \subseteq V(A) - V(B)$. For $i = 3, 4$ and $j = 5, 6, 7$, L_{ij} has one end in $V(A) - V(B)$, and no vertex in $V(A \cap B)$ except its other end, and so $L_{ij} \subseteq A$, a contradiction, since three of these arcs are included in B. Hence H_1 and H_2 are not 2-adjacent.

Finally, suppose that H_1 and H_2 are 3-adjacent, and let $H_1 \cup H_2 = J$. By (2.1)(i), since $H_1 \cap A$ is a forest, at least five nodes of H_1 (and hence of J) belong to $V(B)$, and similarly at least five belong to $V(A)$. Since J only has six nodes, at least four belong to $V(A \cap B)$, a contradiction. Thus H_1 and H_2 are not 3-adjacent, as required. ∎

From (2.2), the easy part of (1.3) follows.

(2.3) *Let H, H' be hexads in a graph G. If H and H' 3-communicate then no $(\leqslant 3)$-separation of G divides H and H'.*

Proof. Let $H = H_1, ..., H_k = H'$ be a sequence of hexads such that for $1 \leqslant i < k$, H_i and H_{i+1} are 1-, 2-, or 3-adjacent. Suppose that (A, B) is a $(\leqslant 3)$-separation of G, and $H \cap B$ is a forest. Choose $i \leqslant k$ maximum so that $H_i \cap B$ is a forest. If $i < k$, then $H_{i+1} \cap A$ is a forest by (1.1), contrary to (2.2). Thus $i = k$, and so $H' \cap B$ is a forest, and (A, B) does not divide H_1 and H_2. The result follows. ∎

If H is a subgraph of G and e is an edge of G, we denote the union of H and the subgraph consisting of e and its ends by $H + e$. If H is a hexad, a *side* of H is a set of three nodes of H which has an end from every arc. Thus, H has two sides.

(2.4) *Let H_1, H_2, H_3, H_4 be hexads of a graph G, and let e be an edge of G with ends u, v, such that $e \notin E(H_1)$, $e \notin E(H_2)$, $\{u, v\} \nsubseteq V(H_3)$, and $\{u, v\} \nsubseteq V(H_4)$. Then either*

(i) *some $(\leqslant 3)$-separation of G divides H_1 and H_2, or*

(ii) *no $(\leqslant 3)$-separation of $G \backslash e$ divides H_1 and H_2, or*

(iii) *some $(\leqslant 3)$-separation of G divides H_3 and H_4, or*

(iv) *no $(\leqslant 3)$-separation of G/e divides H_3 and H_4.*

(G/e denotes the graph obtained from G by contracting e. Here we are, loosely, regarding H_3 and H_4 as subgraphs of G/e, and we hope this will cause no confusion.)

Proof. Suppose that (i), (ii), (iii) and (iv) are all false. Let (A, B) be a $(\leqslant 3)$-separation of $G \backslash e$ such that $B \cap H_1$ and $A \cap H_2$ are forests. Let (C, D) be a separation of $G \backslash e$ such that $((C+e)/e, (D+e)/e)$ is a $(\leqslant 3)$-separation of G/e and $((D+e)/e) \cap H_3$ and $((C+e)/e) \cap H_4$ are forests. Hence $(D+e) \cap H_3$ and $(C+e) \cap H_4$ are forests, and so $(C, D+e)$ has order $\geqslant 4$, since (iii) is false. Since $((C+e)/e, (D+e)/e)$ has order $\leqslant 3$ it follows that $|V(C \cap D)| = 4$ and $u, v \in V(C \cap D)$. Not both $u, v \in V(A)$, since $(A+e, B)$ has order $\geqslant 4$ (because (i) is false), and, similarly, not both $u, v \in V(B)$. Thus we may assume that $u \in V(A) - V(B)$ and $v \in V(B) - V(A)$.

Now $(A \cap C, (B \cup D)+e)$ and $(A \cup C+e, B \cap D)$ are separations of G, and the sum of their orders equals the sum of the orders of (A, B) and (C, D), and hence it is $\leqslant 7$. From the symmetry we may assume that $(A \cap C, (B \cup D)+e)$ has order $\leqslant 3$. Similarly, $(A \cap D, B \cup C+e)$ and $(A \cup D+e, B \cap C)$ are separations of G and the sum of their orders is $\leqslant 7$, and so one of $(A \cap D, B \cup C+e)$, $(A \cup D+e, B \cap C)$ has order $\leqslant 3$.

First, let us suppose that $(A \cap D, B \cup C+e)$ has order $\leqslant 3$. Certainly $H_2 \cap A \cap D$ is a forest, and so $H_1 \cap A \cap D$ is a forest, since (i) is false. Hence at most three arcs of H_1 have edges in $A \cap D$. Similarly, from the $(\leqslant 3)$-separation $(A \cap C, (B \cup D)+e)$, it follows that $H_1 \cap A \cap C$ is a forest and at most three arcs of H_1 have edges in $A \cap C$. But (A, B) has order $\leqslant 3$ and $H_1 \cap B$ is a forest, and so at most three arcs of H_1 have edges in B. Now $(A \cap D) \cup (A \cap C) \cup B = G \backslash e$, and H_1 has nine arcs, and so we have equality throughout. In particular, $A \cap D$, $A \cap C$, B each include three arcs of H_1. Since A includes six arcs of H_1 and B includes three, and (A, B) is a $(\leqslant 3)$-separation of $G \backslash e$, it follows from (2.1) that $V(A \cap B)$ is a side S_1 of H_1 and that $V(B) - V(A)$ contains a node of H_1. Similarly, $V((A \cap D) \cap ((B \cup C)+e))$ is a side S_2 of H_1. Since $u \in S_2 - S_1$ it follows that $S_1 \neq S_2$, and so every node of H_1 belongs to $S_1 \cup S_2$; but this is impossible, since we have shown that $V(B) - V(A)$ contains a node.

We deduce that $(A \cap D, B \cup C+e)$ has order $\geqslant H$, and so $(A \cup D+e, B \cap C)$ has order $\leqslant 3$. Now $B \cap C \cap H_4$ is a forest since $C \cap H_4$ is a forest, and so $B \cap C \cap H_3$ is a forest since (iii) is false. Similarly, $A \cap C \cap H_3$ is a forest, from the $(\leqslant 3)$-separation $(A \cap C, B \cup D+e)$. Thus $B \cap C$ and $A \cap C$ each have edges from at most three arcs of H_3. But also at most three arcs of H_3 have edges in D; for $((C+e)/e, (D+e)/e)$ is a $(\leqslant 3)$-separation of G/e and $H_3 \cap ((D+e)/e)$ is a forest, and so at most three arcs of H_3 have edges in $(D+e)/e$, or, equivalently, in D. Since $(A \cap C) \cup (B \cap C) \cup D = G \backslash e$ and H_3 has nine arcs, we have equality throughout, and so $V(((A \cup D)+e) \cap B \cap C)$ is a side S_1 of H_1, and $V(A \cap C \cap ((B \cup D)+e))$ is a side S_2 of H_1. If $S_1 \neq S_2$, then $S_1 \cap S_2 = \varnothing$ and so $V(A \cap B \cap C) = \varnothing$, and

$$S_1 = V(((A \cup D)+e) \cap B \cap C) \subseteq V(C \cap D)$$

$$S_2 = V(A \cap C \cap ((B \cup D)+e)) \subseteq V(C \cap D),$$

which is impossible, since $|V(C \cap D)| \leqslant 4$. Hence $S_1 = S_2$. But

$$u \in V(A \cap C \cap D) \subseteq V(A \cap C \cap ((B \cup D+e)) = S_2$$

and

$$u \notin V(B) \supseteq V(((A \cup D)+e \cap B \cap C) = S_1,$$

a contradiction. Thus, one of (i)–(iv) is true, as required. ∎

A graph is *triangle-free* if every circuit has $\geqslant 4$ edges.

(2.5) *If G is a triangle-free graph with $n \geqslant 3$ vertices, and G has no hexad, then $|E(G)| \leqslant 2n-4$.*

Proof. We proceed by induction on n. If C, C' are different components of G, let us add an edge joining a vertex of C to a vertex of C'; this does not introduce a circuit of length $\leqslant 3$ or a hexad. By repeating this process, we may assume that G is connected.

Suppose that there is a 1-separation (A, B) of G with $|V(A)|, |V(B)| \geqslant 2$. Let $|V(A)| = n_1, |V(B)| = n_2$. If $n_1 \geqslant 3$ then $|E(A)| \leqslant 2n_1 - 4$ from the inductive hypothesis, and if $n_1 = 2$ then $|E(A)| \leqslant 1 = 2n_1 - 3$. Thus in either case $|E(A)| \leqslant 2n_1 - 3$ and, similarly, $|E(B)| \leqslant 2n_2 - 3$. Hence,

$$|E(G)| = |E(A)| + |E(B)| \leqslant 2(n_1 + n_2) - 6 = 2(n+1) - 6 = 2n - 4$$

as required. Hence we may assume that there is no such (A, B), and so G is 2-connected.

Suppose that there is a 2-separation (A, B) with $|V(A)|, |V(B)| \geqslant 3$. Let $|V(A)| = n_1, |V(B)| = n_2$. Then $|E(A)| \leqslant 2n_1 - 4$ and $|E(B)| \leqslant 2n_2 - 4$, and so

$$|E(G)| \leqslant 2(n_1 + n_2) - 8 = 2n - 4$$

as required. Hence we may assume that there is no such (A, B), and so G is 3-connected.

Now G is not isomorphic to K_5 since it is triangle-free. Since it has no hexad and is 3-connected, it follows from a theorem of Hall [1] that G is planar. Take a planar drawing of G with r regions. By Euler's formula, $|E(G)| = n + r - 2$. Since G is triangle-free and $n \geqslant 3$, it follows that $r \leqslant \frac{1}{2}|E(G)|$, and so $|E(G)| \leqslant n + \frac{1}{2}|E(G)| - 2$, as required. ∎

3. Contraction

Let H be a hexad in G, and let $e \in E(G)$ with ends u, v. We say that e is *H-contractible* if either

(i) $\{u, v\} \nsubseteq V(H)$, or

(ii) $e \in E(H)$ and not both u, v are nodes of H.

In either case, there is a hexad in G/e with edge set $E(H) - \{e\}$, and we denote it by H/e.

(3.1) *Let e be an edge of a graph G. For every hexad H of G/e there is a unique hexad H' of G such that $E(H) \subseteq E(H') \subseteq E(H) \cup \{e\}$. Furthermore, e is H'-contractible, and $H'/e = H$.*

The proof is clear. If H, H' are related as in (3.1) we call H' the *source* of H. The main result of this section is the following.

(3.2) *Let H, H' be hexads in a graph G, and let $e \in E(G)$ be H- and H'-contractible. For $t = 1, 2, 3$, if H/e and H'/e t-communicate in G/e then H and H' t-communicate in G.*

Proof. It suffices to show that if H/e and H'/e are t-adjacent in G/e then H and H' t-communicate in G. For if H/e and H'/e t-communicate in G/e, let

$$H/e = H_1, H_2, ..., H_k = H'/e$$

be the corresponding sequence of hexads in G/e. For $1 \leqslant i \leqslant k$, let H'_i be the source of H_i. Then (from the statement we need to prove) H'_i and H'_{i+1} t-communicate in G for $1 \leqslant i \leqslant k$, and hence H and H' t-communicate in G, since $H'_1 = H$ and $H'_k = H'$, as required.

We may assume that e is not a loop. Let e have ends x and y. Suppose first that H_1/e and H_2/e are 1-adjacent in G/e. Let $J = (H_1/e) \cup (H_2/e)$; then J is a subgraph of G/e. Let J' be the subgraph of G consisting of all edges in J and their ends in G, together with e if x, y are both incident in G with an edge of G in $E(J)$. Then J' is linear. Let L_1, L_2 be arcs of J, so that H_i/e is obtained from J by deleting the interior of L_i $(i = 1, 2)$. Let L'_1, L'_2 be the arcs of J' with $E(L_i) \subseteq E(L'_i)$ $(i = 1, 2)$. Then $E(L'_i) = E(L_i)$ or $E(L_i) \cup \{e\}$, for $i = 1, 2$. We claim that H_i is obtained from J' by deleting the interior of L'_i $(i = 1, 2)$. To show this, it suffices to show that $E(H_i) = E(J') - E(L_i)$, since neither H_i nor the graph obtained from J' by deleting the interior of L'_i has isolated vertices. Now

$$E(H_i) - \{e\} = E(H_i/e) = E(J) - E(L_i) = (E(J') - E(L'_i)) - \{e\}.$$

It remains to check that $e \in E(H_i)$ if and only if $e \in E(J') - E(L'_i)$. Now if $e \notin E(J')$, then at most one of x, y is incident in G with an edge in $E(J)$, and so $e \notin E(H_i)$. We may therefore assume that $e \in E(J')$, and so x, y are both incident in G with edges in $E(J)$. Now $e \in E(L'_i)$ if and only if one of x, y is incident in G with only one edge of J and that edge belongs to $E(L_i)$. But $e \in E(H_i)$ if and only if x, y are both incident in G with edges in $E(H_i/e) = E(J) - E(L_i)$. Consequently, $e \in E(L'_i)$ if and only if $e \notin E(H_i)$, and so $e \in E(H_i)$ if and only if $e \in E(J') - E(L'_i)$. This proves that H_i is obtained from J' by deleting the interior of L'_i $(i = 1, 2)$, and hence H_1, H_2 are 1-adjacent in G, and therefore 1-communicate.

Now suppose that H_1/e and H_2/e are 2-adjacent in G/e, and let $J \subseteq G/e$ be the corresponding subgraph, isomorphic to a subdivision of the graph J_2 of Fig. 1. Let the nodes of J be $u_1, ..., u_7$, where u_1, u_2 correspond to v_1, v_2 of Fig. 1, and u_3, u_4 correspond to the ends of e_0 in Fig. 1. For $1 \leqslant i < j \leqslant 7$ let L_{ij} be the arc of J with ends u_i, u_j if there is such an arc. Then for $i = 1, 2$, H_i/e is obtained from J by deleting u_i and the interiors of L_{i5}, L_{i6}, L_{i7}, and L_{34}. Let J' be the subgraph of G consisting of the edges of G in $E(J)$ and their ends, together with e if both x and y are incident in G with edges in $E(J)$. If $e \notin E(J')$ (and hence $\{x, y\} \nsubseteq V(J')$), or if $e \in E(J')$ and one of x, y has valency 2 in J', then J' is also isomorphic to

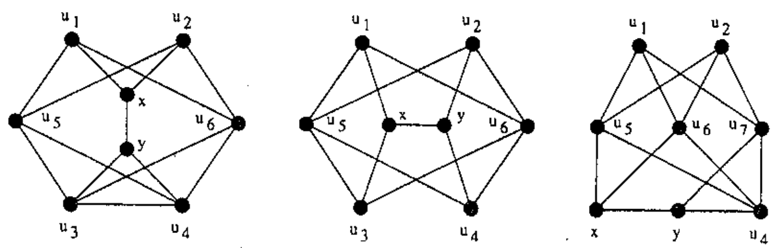

FIGURE 2

a subdivision of the graph J_2 of Fig. 1, and hence H_1 and H_2 are 2-adjacent in G. We may assume therefore that $e \in E(J')$, and x, y both have valency $\geqslant 3$ in J'. Up to symmetry, there are three possibilities for J', shown in Fig. 2. (The lines in Fig. 2 represent arcs of J', not edges, except for the line between x and y which represents the edge e.)

In each case we shall show that in fact H_1 and H_2 1-communicate in J', and, hence, they 2-communicate in G. It is convenient to label the interiors of the arcs of J' as if they were edges; thus, $J' \backslash \{u_1 u_6\}$ means the graph obtained from J' by deleting the interior of the arc with ends u_1, u_6. Also, $J' \backslash u_1$ means (temporarily) the graph obtained from J' by deleting u_1 and the interiors of all incident arcs; we hope this misuse of notation will cause no confusion.

In the first case, we have the following sequence of hexads of G, each 1-adjacent to the next,

$$H_1 = J' \backslash \{u_1, u_3 u_4\}, J' \backslash \{u_1 u_6, u_3 u_4, u_2 u_5\}, J' \backslash \{u_2, u_3 u_4\} = H_2.$$

In the second case, we have the sequence

$$\begin{aligned} H_1 = J' \backslash \{u_1, u_3 u_4\}, & J' \backslash \{u_1 u_6, u_3 u_4, u_3 u_5\}, \\ & J' \backslash u_3, J' \backslash \{u_3 u_5, u_3 u_6, y u_4\}, J' \backslash \{u_3 u_5, u_4 u_6, y u_4\}, \\ & J' \backslash u_4, J' \backslash \{u_2 u_5, u_3 u_4, u_4 u_6\}, J(\backslash \{u_2, u_3 u_4\} = H_2. \end{aligned}$$

In the third case we have

$$\begin{aligned} H_1 = J' \backslash \{u_1, y u_4\}, & J' \backslash \{u_1, u_4 u_7\}, J' \backslash \{u_1 u_6, u_2 u_5, u_4 u_7\}, \\ & J' \backslash \{u_2, u_4 u_7\}, J' \backslash \{u_2, y u_4\} = H_2. \end{aligned}$$

In each case H_1 and H_2 1-communicate, as required.

Now suppose that H_1/e and H_2/e are 3-adjacent in G/e, and let J be the corresponding subgraph of G/e isomorphic to a subdivision of the graph J_3 of Fig. 1. Let u_1, u_2 be the two 5-valent nodes of J, and let u_3, u_4, u_5, u_6 be the other four nodes, in order on the circuit of $J \backslash \{u_1, u_2\}$ through them.

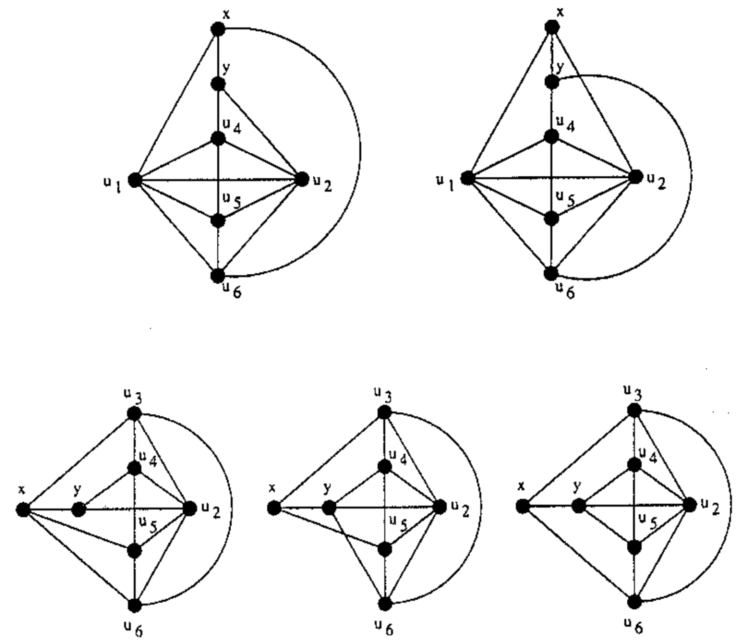

FIGURE 3

Then we may assume that H_1/e is obtained from J by deleting the interiors of the arcs with ends $u_1 u_3, u_2 u_4, u_1 u_5, u_2 u_6$, and H_2/e is obtained by deleting the interiors of the arcs with ends $u_2 u_3, u_1 u_4, u_2 u_5, u_1 u_6$. Define J' from J as in the previous case; then, as before, we may assume that $e \in E(J')$ and x, y both have valency $\geqslant 3$ in J'. Now, up to isomorphism, and exchanging H_1, H_2 if necessary, we find there are five possibilities for J' as shown in Fig. 3.

In the first case we have the following sequence of hexads of J', each 1-adjacent to the next,

$$\begin{aligned} H_1 = J' \backslash \{u_1 x, u_2 u_4, u_1 u_5, u_2 u_6\}, & J' \backslash \{u_1 u_6, u_2 u_4, u_1 u_5, u_2 u_6\}, \\ & J' \backslash \{u_1 u_6, u_2 u_4, u_1 u_5, u_2 u_5\}, J' \backslash \{u_1 u_6, u_2 u_4, u_1 u_4, u_2 u_5\}, \\ & J' \backslash \{u_1 u_6, u_2 y, u_1 u_4, u_2 u_5\} = H_2. \end{aligned}$$

In the second case, we have

$$\begin{aligned} H_1 = J' \backslash \{u_1 x, u_2 u_4, u_1 u_5, u_2 u_6\}, & J' \backslash \{u_1 u_2, u_2 u_4, u_1 u_5, u_2 u_6\}, \\ & J' \backslash \{u_1 u_2, u_4 u_5, u_1 u_5, u_2 u_6\}, J' \backslash \{u_1 u_2, u_5\}, \\ & J' \backslash \{u_1 u_2, u_1 u_4, u_2 u_5, u_5 u_6\}, J' \backslash \{u_1 u_2, u_1 u_4, u_2 u_5, u_1 u_6\}, \\ & J' \backslash \{x u_2, u_1 u_4, u_2 u_5, u_1 u_6\} = H_2. \end{aligned}$$

In the third case, we have

$$H_1 = J'\{xu_3, u_2u_4, xu_5, u_2u_6\}, J'\backslash\{u_3u_6, u_2u_4, xu_5, u_2u_6\},$$
$$J'\backslash\{u_3u_6, u_2u_4, xu_5, u_2u_5\}, J'\backslash\{u_3u_6, u_2u_4, xu_6, u_2u_5\},$$
$$J'\backslash\{u_2u_3, u_2u_4, xu_6, u_2u_5\}, J'\backslash\{u_2u_3, yu_4, xu_6, u_2u_5\} = H_2.$$

In the fourth case (the first step of this sequence is a 2-adjacency, the only one we need in all of this analysis), we have

$$H_1 = J'\backslash\{x, u_2u_4, u_2u_6\}, J'\backslash\{u_2u_4, u_6\}, J'\backslash\{yu_6, u_5u_6, u_2u_3, u_2u_4\},$$
$$J'\backslash\{yu_6, u_2u_5, u_2u_3, u_2u_4\}, J'\backslash\{yu_6, u_2u_5, u_2u_3, yu_4\} = H_2.$$

In the fifth case, we have

$$H_1 = J'\backslash\{xu_3, u_2u_4, yu_5, u_2u_6\}, J'\backslash\{u_3u_6, u_2u_4, yu_5, u_2u_6\},$$
$$J'\backslash\{u_3u_6, u_2u_4, yu_5, u_2u_5\}, J'\backslash\{u_3u_6, u_2u_4, yu_4, u_2u_5\},$$
$$J'\backslash\{u_3u_6, u_2u_3, yu_4, u_2u_5\}, J'\backslash\{xu_6, u_2u_3, yu_4, u_2u_5\} = H_2.$$

In all these five cases, H_1 and H_2 2- (and hence 3-) communicate in J' and hence in G. This completes the proof. ∎

4. Triads

A *triad* in a graph G is a subgraph G isomorphic to a subdivision of $K_{1,3}$. Its three 1-valent vertices are called its *feet.* Two triads T_1, T_2 in G are *matched* if they have the same set of feet and are otherwise disjoint. If H is a subgraph of G, an *H-path* is a path of G with distinct ends, both in $V(H)$, and with no other vertex or edge in H. The following is a preliminary form of the main result of this section.

(4.1) *Let T_1, T_2, T_3, T_4 be triads in a graph G, so that T_1 and T_2 are not matched but all the other five pairs are matched. Then the hexads $T_1 \cup T_3 \cup T_4$ and $T_2 \cup T_3 \cup T_4$ 1-communicate in G.*

Proof. We proceed by induction on $|E(T_1 \cup T_2)|$. Let the common feet be p, q, r.

(1) *We may assume that either $T \cup T_1 = T_1 \cup T_2$, or $T \cup T_2 = T_1 \cup T_2$, for every triad $T \subseteq T_1 \cup T_2$ with feet p, q, r.*

For suppose that $T \subseteq T_1 \cup T_2$ is a triad with feet p, q, r. If the hexads $T_1 \cup T_3 \cup T_4$ and $T \cup T_3 \cup T_4$ 1-communicate and the hexads $T \cup T_3 \cup T_4$ and $T_2 \cup T_3 \cup T_4$ 1-communicate then we are done, so from the symmetry we may assume that the first pair do not. From the inductive hypothesis, either T, T_1 are matched, or $T \cup T_1 = T_1 \cup T_2$. If T, T_1 are matched then $E(T) \subseteq E(T_2)$, and so $T = T_2$ and $T \cup T_1 = T_1 \cup T_2$. Thus in either case $T \cup T_1 = T_1 \cup T_2$, as required.

Let P, Q, R be the paths of T_1 between its trivalent vertex and p, q, r, respectively. We may assume that $T_1 \neq T_2$, for otherwise the result is trivial. Hence, from the symmetry, we may assume that $E(P) \nsubseteq E(T_2)$ and $V(Q \cap T_2) \neq \{q\}$, since T_1 and T_2 are not matched. Let $e \in E(P) - E(T_2)$, and let X, Y be the two components of $T_1\backslash e$, where $p \in V(X)$ and $q, r \in V(Y)$. Since $V(Y \cap T_2) \neq \{q, r\}$ (because $V(Q \cap T_2) \neq \{q\}$), there is a T_1-path $L \subseteq T_1 \cup T_2$ with one end in $V(X)$ and the other, y, in $V(Y) - \{q, r\}$. Since $y \neq q, r$, it follows that there is a triad $T \subseteq L \cup T_1$ with feet p, q, r and with $e \notin E(T)$. Since $e \notin E(T)$ and hence $T \cup T_2 \neq T_1 \cup T_2$, it follows from (1) that $T \cup T_1 = T_1 \cup T_2$. Hence $T_2 \subseteq T \cup T_1 \subseteq L \cup T_1$. Since no internal vertex of L belongs to $V(T_1)$ and $y \neq q, r$, it follows that $T_1 \cup T_3 \cup T_4$ and $T_2 \cup T_3 \cup T_4$ are 1-adjacent and, hence, 1-communicate, as required. ∎

(4.2) *Let T_1, T_2, T_3, T_4 be triads in G, pairwise matched, and with set of feet X. If there is a $(T_1 \cup T_2 \cup T_3 \cup T_4)$-path with one end in $V(T_1) - X$ and the other in $V(T_2) - X$, then the hexads $T_1 \cup T_3 \cup T_4$ and $T_2 \cup T_3 \cup T_4$ 1-communicate, and the hexads $T_1 \cup T_2 \cup T_3$ and $T_1 \cup T_2 \cup T_4$ 2-communicate.*

Proof. Let L be a $(T_1 \cup T_2 \cup T_3 \cup T_4)$-path with one end in $V(T_1) - X$ and the other in $V(T_2) - X$. There is a triad $T \subseteq T_1 \cup T_2 \cup L$ with $L \subseteq T$ and with feet X. Since neither end of L is in X, it follows that T and T_1 are not matched, and T and T_2 are not matched. By (4.1), $T \cup T_3 \cup T_4$ 1-communicates with $T_1 \cup T_3 \cup T_4$, and with $T_2 \cup T_3 \cup T_4$. This proves the first claim of the theorem.

For the second claim, let $H = T_1 \cup T_2 \cup T_3 \cup T_4 \cup L$; we proceed by induction on $|E(H)|$. Let v_i be the 3-valent vertex of T_i $(1 \leqslant i \leqslant 4)$. If some edge e of T_i incident with v_i does not have its other end in X, then e is $T_1 \cup T_2 \cup T_3$- and $T_1 \cup T_2 \cup T_4$-contractible, and the result follows from the inductive hypothesis and (3.2), by contracting e. If there is no such edge, then H is isomorphic to a subdivision of the graph J_2 of Fig. 1 and $T_1 \cup T_2 \cup T_3$ and $T_1 \cup T_2 \cup T_4$ are 2-adjacent, and hence they 2-communicate as required. ∎

The main result of this section is the following extension of (4.1) and (4.2).

(4.3) *Let T_1, T_2, T_3, T_4 be triads in G, pairwise matched except possibly for T_1, T_2. Then either*

(i) *the hexads $T_1 \cup T_3 \cup T_4$ and $T_2 \cup T_3 \cup T_4$ 2-communicate, or*

(ii) *T_1 and T_2 are matched, and there is a separation (A, B) of G so that $V(A \cap B)$ is the set of feet of each T_i, and $T_1 \subseteq A$, $T_2 \subseteq B$, and either $T_3 \subseteq A$ and $T_4 \subseteq B$ or $T_3 \subseteq B$ and $T_4 \subseteq A$.*

Proof. By (4.1), we may assume that T_1 and T_2 are matched, for otherwise (i) holds. Let $H = T_1 \cup T_2 \cup T_3 \cup T_4$, and for $1 \leqslant i \leqslant 4$ let H_i be the hexad

$$\bigcup (T_j\colon 1 \leqslant j \leqslant 4, j \neq i).$$

We wish to show that H_1 and H_2 2-communicate unless (ii) holds. Let X be the set of feet of each T_i.

We assume that H_1 and H_2 do not 2-communicate. Hence they do not both 2-communicate with H_4, and from the symmetry we may assume that H_1 and H_4 do not 2-communicate. Since H_1 and H_2 do not 2-communicate, it follows from (4.2) that there is no H-path with one end in $V(T_1) - X$ and the other in $V(T_2) - X$, and none with one end in $V(T_3) - X$ and the other in $V(T_4) - X$. But since H_1 and H_4 do not 2-communicate, it also follows from (4.2) that there is no H-path with one end in $V(T_1) - X$ and the other in $V(T_4) - X$, and none with one end in $V(T_2) - X$ and the other in $V(T_3) - X$. In summary, there is no H-path with one end in $V(T_1 \cup T_3) - X$ and the other in $V(T_2 \cup T_4) - X$. Hence there is a separation (A, B) with $V(A \cap B) = X$ and $T_1 \cup T_3 \subseteq A$ and $T_2 \cup T_4 \subseteq B$, and so (ii) holds, as required. ∎

(4.3) has the following useful corollary.

(4.4) *Let (A, B) be a $(\leqslant 3)$-separation of a graph G, and let H be a hexad in G such that $H \cap A$ is a forest. Let $T \subseteq A$ be a triad with feet $V(A \cap B)$ if $|V(A \cap B)| = 3$, a path with distinct ends both in $V(A \cap B)$ if $|V(A \cap B)| = 2$, and null if $|V(A \cap B)| \leqslant 1$. Then there is a hexad $H' \subseteq (H \cap B) \cup T$ with $H' \cap B = H \cap B$. Moreover, if either $A \backslash V(A \cap B)$ is connected, or H meets the component of $G \backslash V(A \cap B)$ containing $T \backslash V(A \cap B)$, or $B \backslash V(A \cap B)$ is connected, or $|V(A \cap B)| \leqslant 2$, then H' can be chosen so that H and H' 2-communicate.*

Proof. By (2.1) there is a node v_1 of H such that every arc of H with more than one vertex in $V(A)$ is incident with v_1, and every node of H except v_1 belongs to $V(B)$. Let T_1 be the triad formed by the three arcs of H incident with v_1, and let T_3, T_4 be the two other triads of H with the same feet as T_1. From our choice of v_1, all the feet of T_1 belong to $V(B)$, and so $T_3, T_4 \subseteq B$. Let T_2 be a triad with the same feet as T_1 and with $T_1 \cap B \subseteq T_2 \subseteq (T_1 \cap B) \cup T$. (It is easy to see that T_2 exists.) Then T_1, T_2, T_3, T_4 are mutually matched, except possibly for T_1, T_2. Now $H = T_1 \cup T_3 \cup T_4$; let $H' = T_2 \cup T_3 \cup T_4$. Then H' satisfies the first claim of the theorem. Suppose that H and H' do not 2-communicate. By (4.3), T_1 and T_2 are matched, and there is a 3-separation (C, D) of G such that $T_1 \cup T_3 \subseteq C$ and $T_2 \cup T_4 \subseteq D$ (exchanging T_3 and T_4 if necessary). Since $T_2 \subseteq (T_1 \cap B) \cup T$ and T_1, T_2 are matched, it follows that $T_2 = T$. Hence

$$V(C \cap D) = V(T_2 \cap T_3) = V(T) \cap V(T_3) \subseteq V(A) \cap V(B) = V(A \cap B),$$

and, since $|V(C \cap D)| = 3$ and $|V(A \cap B)| \leqslant 3$, we have $V(C \cap D) = V(A \cap B)$. For $2 \leqslant i \leqslant 4$, let v_i be the 3-valent vertex of T_i. Then $v_1, v_3 \in V(C) - V(D)$, $v_2, v_4 \in V(D) - V(C)$, $v_1, v_2 \in V(A) - V(B)$, and $v_3, v_4 \in V(B) - V(A)$. Since $v_1 \in V(C) - V(D)$ and $v_2 \in V(D) - V(C)$, every path of G from v_1 to v_2 meets $V(C \cap D) = V(A \cap B)$. Since $v_1, v_2 \in V(A) - V(B)$, it follows that $A \backslash V(A \cap B)$ is not connected, and indeed, the component of $G \backslash V(A \cap B)$ containing v_2 does not meet $V(H)$. Similarly, $B \backslash V(A \cap B)$ is not connected, since $v_3, v_4 \in V(B) - V(A)$, $v_3 \in V(C) - V(D)$ and $v_4 \in V(D) - V(C)$. The result follows. ∎

We conclude this section with a few easy observations about triads.

(4.5) *Let (A, B) be a $(\leqslant 3)$-separation of a 3-connected graph G. Then either*

(i) *there is a triad $T \subseteq A$ with feet $V(A \cap B)$ and with $V(A) \neq V(T)$, or*

(ii) *A can be drawn in a disc with $V(A \cap B)$ on the boundary, or*

(iii) *$V(A) = V(G)$.*

Proof. We assume that (iii) is false. If $V(B) = V(G)$ then (ii) is true, and so we may assume that $V(A), V(B) \neq V(G)$. Hence $|V(A \cap B)| = 3$, and there is a triad $T \subseteq A$ with feet $V(A \cap B)$. Choose T with $V(T)$ minimal. If $V(A) \neq V(G)$ then (i) holds, and so we may assume that $V(A) = V(T)$. Let v be the 3-valent vertex of T, and let P, Q, R be the three paths of T from v to $V(A \cap B)$ with ends $p, q, r \in V(A \cap B)$ say. We may assume that G has no loops or parallel edges, and hence from the minimality of T, every edge of A not in T has ends in two of $V(P) - \{v\}$, $V(Q) - \{v\}$, $V(R) - \{v\}$. Moreover, if e_1, e_2 are edges of A not in T, and e_i has ends $x_i \in V(P) - \{v\}$, $y_i \in V(Q) - \{v\}$ for $i = 1, 2$, say, then either $x_1 = p$ or y_1 is adjacent to v in T, by the minimality of $V(T)$. Consequently, if x_2 is on the path of P between p and x_1 and $x_2 \neq x_1$, then y_2 is on the path of Q between q and y_1. It follows that (ii) holds, as required. ∎

(4.6) *Let (A, B) be a $(\leqslant 3)$-separation of a 3-connected graph G such that $V(A) \neq V(G)$ and there is a hexad H with $H \cap B$ a forest. Then there is a triad $T \subseteq A$ with feet $V(A \cap B)$ and with $V(A) \neq V(T)$.*

Proof. If A can be drawn in a disc with $V(A \cap B)$ on the boundary, then $A \cup (H \cap B)$ is planar since $H \cap B$ is a forest, and in particular H is planar, a contradiction. The result follows from (4.5). ∎

(4.7) *Let H_1, H_2 be hexads of a 3-connected graph G, divided by some $(\leqslant 3)$-separation. Then there is a hexad H with $V(H) \neq V(G)$.*

Proof. Let (A, B) be a $(\leqslant 3)$-separation of G such that $H_1 \cap B$ and $H_2 \cap A$ are forests. Since $H_2 \cap A$ is a forest it follows from (2.1) that

$|V(B)-V(A)|\geqslant 2$, and so $V(A)\neq V(G)$. From (4.6) there is a triad $T\subseteq A$ with feet $V(A\cap B)$ and with $V(T)\neq V(A)$. By (4.4) there is a hexad $H\subseteq (H_2\cap B)\cup T$. But then $V(A)\nsubseteq V(H)$, and so $V(H)\neq V(G)$, as required. ▮

5. The Main Proof

In this section we prove (1.3). Let us say that a pair of hexads H_1, H_2 in a graph G is a *bad pair* if they do not 3-communicate, and there is no $(\leqslant 3)$-separation of G dividing them. (1.3) asserts that there are no bad pairs. Suppose (1.3) is false; then we may choose G with $|V(G)|+|E(G)|$ minimum such that there is a bad pair of hexads in G. Throughout this section, G is a fixed graph with this property, which we refer to as the "minimality" of G. We shall prove a sequence of lemmas about G, and eventually obtain a contradiction, thereby proving (1.3).

(5.1) *If H_1, H_2 are a bad pair, and for $i=1,2$ H_i' is a hexad in G 3-communicating with H_i, then H_1', H_2' is a bad pair.*

Proof. Certainly H_1' and H_2' do not 3-communicate. If (A, B) is a $(\leqslant 3)$-separation and $H_1'\cap A$ is a forest, then $H_1\cap A$ is a forest by (2.3); so $H_2\cap A$ is a forest since H_1, H_2 is a bad pair. Hence $H_2'\cap A$ is a forest by (2.3). Thus H_1', H_2' is a bad pair, as required. ▮

If H is a hexad in G, we denote by $C(H)$ the intersection of all hexads in G that 3-communicate with H.

(5.2) *If H_1, H_2 is a bad pair then $V(C(H_1)\cup C(H_2))=V(G)$.*

Proof. We show first that $V(H_1\cup H_2)=V(G)$ for every bad pair H_1, H_2. Suppose that $v\in V(G)-V(H_1\cup H_2)$. If v has valency 0, then H_1, H_2 is a bad pair in $G\backslash v$, contrary to the minimality of G. If v is incident with an edge e, then by (2.4) (taking $H_3=H_1$ and $H_4=H_2$), in one of $G\backslash e$, G/e there is no $(\leqslant 3)$-separation dividing H_1 and H_2. But H_1 and H_2 3-communicate in neither $G\backslash e$ nor G/e, by (3.2), and so H_1, H_2 is a bad pair in one of $G\backslash e$, G/e, contrary to the minimality of G. Hence, $V(H_1\cup H_2)=V(G)$ for every bad pair H_1, H_2.

Now let H_1, H_2 be a bad pair, and suppose that $v\in V(G)-V(C(H_1)\cup C(H_2))$. Since $v\notin V(C(H_i))$, there is a hexad H_i' with $v\notin V(H_i')$ that 3-communicates with $H_i (i=1,2)$. By (5.1) it follows that H_1', H_2' is a bad pair, and yet $V(H_1'\cup H_2')\neq V(G)$ contradicting the result of the previous paragraph. Thus there is no such v, as required. ▮

(5.3) *G is simple and 3-connected.*

Proof. If e is a loop or parallel edge of G, then by (5.1) we may choose a bad pair H_1, H_2 with $e\notin E(H_1\cup H_2)$; but then H_1, H_2 is a bad pair in $G\backslash e$, contrary to the minimality of G. Thus G is simple. If some vertex v of G has valency $\leqslant 2$, it has valency 2 by (5.2); let e be incident with v, then e is H_1- and H_2-contractible for any bad pair H_1, H_2 of G, and so $H_1/e, H_2/e$ is a bad pair in G/e by (3.2), contrary to the minimality of G. Thus G has minimum valency $\geqslant 3$.

Suppose that (A, B) is a $(\leqslant 2)$-separation of G with $V(A), V(B)\neq V(G)$, and let H_1, H_2 be a bad pair. Exchanging A, B if necessary, we may assume that $H_1\cap A$ is a forest. Since (A, B) does not divide H_1 and H_2 it follows that $H_2\cap A$ is a forest. By (5.2), $V(H_1)\cup V(H_2)\nsubseteq V(B)$, and so $|V(A\cap B)|=2$, $V(A\cap B)=\{s,t\}$ say, and there is a path L in A between s and t. Choose L with $|V(L)|$ minimal. Then any internal vertex of L has only two neighbours in L, and yet has at least three in G. Consequently $V(A)\nsubseteq V(L)$. By (4.4), for $i=1,2$ there is a hexad H_i' 3-communicating with H_i, with $H_i'\cap A\subseteq L$. But $V(H_1'\cup H_2')\neq V(G)$ since $V(A)\nsubseteq V(L)$, contrary to (5.2). Thus there is no such (A, B), and the result follows. ▮

(5.4) *If $X\subseteq V(G)$ with $|X|\leqslant 3$, then $G\backslash X$ has at most two connected components.*

Proof. Let the vertex sets of the components of $G\backslash X$ be $F_1, ..., F_k$, and suppose that $k\geqslant 3$. For $1\leqslant i\leqslant k$, let A_i be the subgraph with vertex set $F_i\cup X$ and edge set consisting of all edges of G with an end in F_i; and let A_0 be $G\backslash(F_1\cup ...\cup F_k)$. For $1\leqslant i\leqslant k$, let

$$B_i=\bigcup\,(A_j\colon 0\leqslant j\leqslant k, j\neq i).$$

Then $A_0, A_1, ..., A_k$ are mutually edge-disjoint and have union G, and for $1\leqslant i\leqslant k$, (A_i, B_i) is a 3-separation of G. For $1\leqslant i\leqslant k$, let $T_i\subseteq A_i$ be a triad with feet X; this exists since G is 3-connected and $V(A_i), V(B_i)\neq V(G)$.

Let H_1, H_2 be a bad pair. Suppose that for all i with $1\leqslant i\leqslant k$, $A_i\cap H_1$ is a forest. Then $A_i\cap H_2$ is a forest for $1\leqslant i\leqslant k$, since (A_i, B_i) is a 3-separation. Now $A_i\backslash V(A_i\cap B_i)$ is connected, and so by repeated application of (4.4), there is a hexad H_1' 3-communicating with H_1, such that $H_1'\cap A_i\subseteq T_i$ for $1\leqslant i\leqslant k$. Choose H_2' similarly. It follows that for $j=1,2$, no circuit C of H_j' has $|V(C)\cap X|\leqslant 1$. The only subsets of cardinality $\leqslant 3$ of the vertex set of a hexad that have at least two vertices in every circuit are the sides of the hexad. Consequently, X is a side of both H_1' and H_2', and so both H_1' and H_2' are unions of three of $T_1, ..., T_k$. We may assume that $H_1'=T_1\cup T_2\cup T_3$. Now H_1', H_2' is a bad pair by (5.1) and so $H_1'\neq H_2'$. Thus we may assume that $T_4\subseteq H_2'$ and $V(T_1\cap H_2')=X$. Then $(A_1\cup A_2, B_1\cap B_2)$ is a 3-separation of G dividing H_1' and H_2', a contradiction.

This proves that there exists i with $1\leqslant i\leqslant k$ such that $A_i\cap H_1$ is not a forest, and hence $B_i\cap H_1$ is a forest. For definiteness let us assume that $i=1$, and so $B_1\cap H_1$ is a forest. Since (A_1, B_1) does not divide H_1 and H_2, it follows that $B_1 \cap H_2$ is a forest. Since $A_1 \backslash V(A_1 \cap B_1)$ is connected, it follows from (4.4) that $C(H_j) \subseteq A_1 \cup T_2$ ($j=1, 2$), contrary to (5.2). Hence $k \leqslant 2$, as required. ∎

(5.5) *If (A, B) is a $(\leqslant 3)$-separation of G and $V(A), V(B) \neq V(G)$, then $A\backslash V(A \cap B)$, $B\backslash V(A \cap B)$ are both connected. Moreover, if H_1, H_2 is a bad pair and $H_1 \cap A$ is a forest then A can be drawn in a disc with $V(A \cap B)$ on the boundary.*

Proof. Since $A\backslash V(A \cap B), B\backslash V(A \cap B)$ are both non-empty and are expressible as unions of components of $G\backslash V(A \cap B)$, it follows that both are connected, by (5.4). Let H_1, H_2 be a bad pair with $H_1 \cap A$ a forest. Suppose that there is a triad $T \subseteq A$ with feet $V(A \cap B)$ and with $V(T) \neq V(A)$. Since $H_1 \cap A$ and $H_2 \cap A$ are forests and $B\backslash V(A \cap B)$ is connected, it follows from (4.4) that $C(H_1), C(H_2) \subseteq T \cup B$. But $V(T) \neq V(A)$, contrary to (5.2). Hence there is no such T, and so by (4.5), A can be drawn in a disc with $V(A \cap B)$ on the boundary. ∎

(5.6) *If H_1, H_2 is a bad pair then $C(H_1) \cup C(H_2) = G$.*

Proof. As in the proof of (5.2), it suffices to show that $H_1 \cup H_2 = G$. Suppose not; then by (5.2) there is an edge $e \notin E(H_1 \cup H_2)$. Now H_1, H_2 is not a bad pair in $G\backslash e$ by the minimality of G, but H_1 and H_2 do not 3-communicate in $G\backslash e$, and so there is a $(\leqslant 3)$-separation (A, B) of $G\backslash e$ dividing H_1 and H_2. Let $H_1 \cap B, H_2 \cap A$ be forests. Since $(A+e, B)$ is not a $(\leqslant 3)$-separation of G dividing H_1, H_2 it follows that not both ends of e are in $V(A)$. Similarly, not both ends of e are in $V(B)$, and $|V(A \cap B)| = 3$. Let e have ends $a \in V(A) - V(B)$ and $b \in V(B) - V(A)$. Let $V(A \cap B) = X = \{x_1, x_2, x_3\}$.

(1) *There is a triad $T_1 \subseteq A$ with feet X and with $a \notin V(T_1)$.*

For suppose not. Then for each component C of $A\backslash(X \cup \{a\})$ there exists $x \in X$ with no neighbour in C. Hence there exists mutually edge-disjoint subgraphs A_1, A_2, A_3 of A with union A, and with $a \in V(A_i)$ and $V(A_i) \cap X = X - \{x_i\}$ $(1 \leqslant i \leqslant 3)$. Since $H_2 \cap A$ is a forest, so is $H_2 \cap A_i$. Choose $B_i \subseteq G$ such that (A_i, B_i) is a separation of G and $V(A_i \cap B_i) = \{a\} \cup (X - \{x_i\})$. By (5.5) applied to (A_i, B_i), A_i can be drawn in a disc with $\{a\} \cup (X - \{x_i\})$ on the boundary $(1 \leqslant i \leqslant 3)$. Since $A = A_1 \cup A_2 \cup A_3$, A can be drawn in a disc with X on the boundary. But since $H_1 \cap B$ is a forest it follows that $A \cup (H_1 \cap B)$ is planar, and hence so is H_1, a contradiction. This proves (1).

Now by (5.4), $G\backslash X$ has at most two connected components, and so $G\backslash X\backslash e$ has at most three. Hence one of $A\backslash X, B\backslash X$ includes only one of them, and so one of $A\backslash X, B\backslash X$ is connected. Let T_1 be as in (1); then by (4.4) applied to $G\backslash e, H_2, T_1$ we deduce that there is a hexad H_4 of G with $a \notin V(H_4)$ which 3-communicates with H_2. Similarly there is a hexad H_3 with $b \notin V(H_3)$ which 3-communicates with H_1. By (5.1), H_3, H_4 is a bad pair in G. By (2.4) applied to G, H_1, H_2, H_3, H_4, no $(\leqslant 3)$-separation of G/e divides H_3 and H_4. From the minimality of G, H_3 and H_4 3-communicate in $G\backslash e$ and, hence, in G by (3.2), a contradiction. The result follows. ∎

(5.7) *If H_1, H_2 is a bad pair and (A, B) is a $(\leqslant 3)$-separation of G, and $H_1 \cap B$ is a forest, then either $V(A) = V(G)$, or $|V(B) - V(A)| = 1$ and no two vertices in $V(A \cap B)$ are adjacent.*

Proof. Let $X = V(A \cap B)$, and suppose that $V(A) \neq V(G)$. Let A' be obtained from A by deleting all edges of A with both ends in X, and let B' be the subgraph of G induced on $V(B)$. Then (A', B') is a 3-separation of G. Since $H_1 \cap B$ is a forest, it follows from (2.1) that $V(A) = V(A')$ contains at least five nodes of H_1, and so $H_1 \cap B'$ is a forest, by (2.1) again. Let $T \subseteq B'$ be a triad with feet X. By (5.5) $B'\backslash X$ is connected, and so by (4.4), there is a hexad H_1' communicating with H_1, with $H_1' \cap B' \subseteq T$. Hence, $C(H_1) \cap B' \subseteq T$ and, similarly, $C(H_2) \cap B' \subseteq T$, since $H_2 \cap B$ is a forest. By (5.6), $T = B'$, and so $|V(B) - V(A)| = 1$ by (5.3), and the claim follows. ∎

(5.8) *If H_1, H_2 is a bad pair and v is a node of (H_1), and some arc of H_1 incident with v has an internal vertex, then $v \in V(H_2)$.*

Proof. Suppose not; then by (5.6), every edge of G incident with v is in $E(C(H_1))$, and hence in $E(H_1)$. Since v has valency $\geqslant 3$ in H_1 and hence in G, it follows that v has exactly three neighbours in G. Let $e \in E(G)$ have ends u, v say, so that u is a 2-valent vertex of H_1.

Now e is H_1- and H_2-contractible. By (3.2) and the minimality of G, there is a $(\leqslant 3)$-separation of G/e dividing H_1/e and H_2/e. Since no $(\leqslant 3)$-separation of G divides H_1 and H_2, it follows that there are subgraphs A, B of G with union G, $E(A \cap B) = \{e\}$ and $|V(\cap B)| = 4$, so that $(A/e, B/e)$ is a 3-separation of G/e with $(A/e) \cap (H_2/e)$ and $(B/e) \cap (H_1/e)$ forests. Consequently, $A \cap H_2$ and $B \cap H_1$ are forests. Let $V(A \cap B) = \{u, v, x, y\}$.

(1) *There is a connected component F of $B\backslash V(A \cap B)$, such that u, v, x, y all have neighbours in F, and $V(F \cap H_2) \neq \varnothing$.*

For let $F_1, \ldots, F_k$ be the connected components of $B\backslash V(A \cap B)$ with non-empty intersection with H_2. We suppose, for a contradiction, that for $1 \leqslant i \leqslant k$, at most three of u, v, x, y have neighbours in F_i. By (5.7), applied to a $(\leqslant 3)$-separation (C, D) say with $C\backslash V(C \cap D) = F_i$, we deduce that $|V(F_i)| = 1$, $F_i = \{v_i\}$, say, and no two neighbours of v_i are adjacent. Hence v_i is not adjacent to both u and v, and so its set of neighbours is either $\{u, x, y\}$ or $\{v, x, y\}$. By (2.1), since $(H_2/e) \cap (A/e)$ is a forest, at least two of $v_1, \ldots, v_k$ are nodes of H_2, say v_1, v_2. Since $v \notin V(H_2)$ and v_1, v_2 are

3-valent in H_2, it follows that v_1, v_2 both have neighbour set $\{u, x, y\}$. But then $G\backslash\{u, x, y\}$ has $\geqslant 3$ connected components, contrary to (5.4). This proves (1).

Let F be as in (1), and let $T \subseteq B$ be a triad with feet u, x, y and $V(T) \subseteq V(F) \cup \{u, x, y\}$. Then T/e is a triad of G/e, with feet w, x, y, where w is the vertex of G/e formed by identifying u and v. Now $T/e \subseteq B/e$, and the component of $(G/e)\backslash\{w, x, y\}$ containing $V(T) - \{w, x, y\}$ has non-empty intersection with H_1/e, for this component has vertex set $V(F)$ and F contains a neighbour of v. From (4.4), there is a hexad H'_3 of G/e 3-communicating with H_1/e in G/e, such that $H'_3 \cap (B/e) \subseteq T/e$. Let H_3 be the source of H'_3. Then H_3 3-communicates with H_1 by (3.2), and $E(H_3 \cap B) \subseteq E(T) \cup \{e\}$. Since every edge of G incident with v belongs to $E(C(H_1))$ and, hence, to $E(H_3)$, and no edge of T is incident with v, it follows that e is the only edge of B incident with v, contrary to (1). ∎

(5.9) *If H_1, H_2 is a bad pair then $V(H_2) = V(G)$.*

Proof. Suppose not, and let $v \in V(G) - V(H_2)$. Since v has valency $\geqslant 3$ in G and every edge incident with it belongs to $E(H_1)$ by (5.6), it follows that v is a node of H_1 and is 3-valent in G. Let the neighbours of v in G be u_1, u_2, u_3. Let H be an arbitrary hexad which 3-communicates with H_1.

(1) *v, u_1, u_2, u_3 are nodes of H.*

For certainly v is a node of H since all the edges incident with v belong to $E(C(H_1)) \subseteq E(H)$ by (5.6). By (5.8), no arc of H incident with v has an internal vertex, since H, H_2 is a bad pair and $v \notin V(H_2)$. This proves (1).

Let $v = u_6$ and let the other two nodes of H be u_4, u_5. Let L_{ij} be the arc of H with ends u_i, u_j for $1 \leqslant i \leqslant 3$ and $4 \leqslant j \leqslant 6$.

(2) *Every H-path from $V(L_{14} \cup L_{24} \cup L_{34})$ to $V(L_{15} \cup L_{25} \cup L_{35})$ with no end in $\{u_1, u_2, u_3\}$ has ends u_4 and u_5.*

For the otherwise, let L be an H-path with one end, x, in $V(L_{14})$ say, with $x \neq u_1, u_4$, and the other end, y, in $V(L_{15} \cup L_{25} \cup L_{35})$ with $y \neq u_1, u_2, u_3$. If $y \in V(L_{15}) - \{u_5\}$, let H' be the hexad obtained from H by deleting the interior of the subpath of L_{14} between x and u_1, and adding L. If $y \in V(L_{i5})$, where $i = 2$ or 3, let H' be obtained from H by deleting the interior of L_{15} and adding L. Then H' 3-communicates with H and hence with H_1, and yet u_1 is not a node of it, contrary to (1).

(3) *There is an H-path with ends u_4, u_5*

For otherwise, by (2), $G\backslash\{u_1, u_2, u_3\}$ has $\geqslant 3$ connected components, contrary to (5.4).

(4) *For $1 \leqslant i \leqslant 3$ and $j = 4, 5$, L_{ij} has no internal vertices.*

For suppose that L_{14} say has an internal vertex. By (5.3) there is an H-path L with one end, x, an internal vertex of L_{14}, and the other end, y, in $V(H) - V(L_{14})$. By (2) and the fact that u_6 is 3-valent in G, $y \in V(L_{24} \cup L_{34})$, and we may assume from the symmetry that $y \in V(L_{24})$. Let H' be the hexad obtained from H by deleting the interior of the subpath of L_{24} between y and u_4, and adding L. Then H' has nodes $u_1, u_2, u_3, x, u_5, u_6$, and it 3-communicates with H and hence with H_1. By (3) applied to H', there is an H'-path between x and u_5. This therefore does not pass through u_4, since $u_4 \in V(H')$ and $u_4 \neq x, u_5$, and thus it includes an H-path from $V(L_{14} \cup L_{24} \cup L_{34})$ to u_5, contradicting (2). This proves (4).

Now $|V(G)| > 6$, since $|V(H_2)| \geqslant 6$ and $u_6 \neq V(H_2)$. But by (4), $|V(H)| = 6$, and so $V(G) \neq V(H)$. Let C be a connected component of $G\backslash V(H)$ and let $X \subseteq V(H)$ be the set of all $u \in V(H)$ with a neighbour in C. Now $u_6 \notin X$, since all the neighbours of u_6 are in $V(H)$. By (5.3), $|X| \geqslant 3$, and so we may assume that $u_1 \in X$. If $u_4 \in X$, let L be an H-path from u_1 to u_4 with $V(L) \subseteq \{u_1, u_4\} \cup V(C)$. The hexad obtained from H by deleting the edge joining u_1 and u_4, and adding L, has an arc with an internal vertex contrary to (4). Thus $u_4 \notin X$ and, similarly, $u_5 \notin X$. Hence $X = \{u_1, u_2, u_3\}$, and so $\{u_1, u_2, u_3\}$ has $\geqslant 3$ connected components, contrary to (5.4). The result follows. ∎

(5.10) *$V(H) = V(G)$ for every hexad H in G, and every hexad in G has an arc with an internal vertex.*

Proof. Let H_1, H_2 be a bad pair. Then H does not 3-communicate with both H_1 and H_2, and so we may assume that it does not 3-communicate with H_1. But no ($\leqslant 3$)-separation of G divides H_1 and H, by (2.1) and (5.7), and so H_1, H is a bad pair. From (5.9), $V(H) = V(G)$. Now $|V(G)| \geqslant 7$, since if $|V(G)| = 6$ then every two hexads are 3-adjacent. Hence $|V(H)| \geqslant 7$, and the result follows. ∎

Actually it is possible to complete the proof of (1.3) at this stage by listing explicitly all graphs G satisfying (5.7) and (5.10) and checking that (1.3) holds for them. (A similar result was obtained by Wagner [5], who found all the graphs G such that every subdivision of K_5 or $K_{3,3}$ in G used all vertices of G.) One way to do so is by applying a result of Robertson [3], on the structure of the graphs with no V_8 minor, for the graphs with a V_8 minor satisfying (5.7) and (5.10) can also be found explicitly. But this involves a good deal of case analysis, and we shall give another way to complete the proof of (1.3) without invoking Robertson's theorem.

If H is a hexad in G, we denote by $N(H)$ the set of all vertices $v \in V(G)$ such that v is a node of every hexad H' in G that 3-communicates with H.

(5.11) *If H_1, H_2 is a bad pair and $v \in N(H_1)$ then no arc of H_1 incident with v has an internal vertex.*

Proof. Let H_1 have nodes $u_1, \ldots, u_6$ and arcs L_{ij} $(1 \leqslant i \leqslant 3, 4 \leqslant j \leqslant 6)$ as usual, where $v = u_6$, and suppose that L_{16} has an internal vertex; and let x be such an internal vertex, adjacent in L_{16} to u_1. Now $V(H_1) = V(G)$ by (5.9), and so by (5.3), there is an edge e of G with ends x and a vertex $y \in V(H_1) - V(L_{16})$. If $y \in V(L_{24} \cup L_{25}) - \{u_4, u_5\}$, let H be the hexad obtained from H_1 by deleting the interior of L_{26} and adding e; then H 3-communicates with H_1 and yet $u_6 = v$ is not a node of H, a contradiction since $v \in N(H_1)$. If $y \in V(L_{26})$, we obtain a contradiction similarly by deleting the interior of the subpath of L_{26} between y and u_6 and adding e. Thus, if $y \in V(L_{24} \cup L_{25} \cup L_{26})$ then $y \in \{u_4, u_5\}$ and, similarly, if $y \in V(L_{34} \cup L_{35} \cup L_{36})$ then $y \in \{u_4, u_5\}$. Hence $y \in V(L_{14} \cup L_{15})$, and from the symmetry we may assume that $y \in V(L_{14})$. There is a hexad, obtained from H_1 by deleting the interior of the subpath of L_{14} between u_1 and y and adding e. By (5.10), this hexad has vertex set $V(G)$, and so y is adjacent to u_1 in H_1. Since two neighbours of x in G are adjacent, it follows from (5.7) that x has valency $\geqslant 4$ in G. Let e' be an edge of G with ends x, y', where $y' \neq y$ and $e' \notin E(H_1)$. By the same argument as above for e, y, it follows that $y' \in V(L_{14} \cup L_{15})$ and y' is adjacent to u_1. Since $y \neq y'$ it follows that $y' \in V(L_{15})$. But then there is a hexad obtained from H_1 by deleting u_1 and adding e and e', contrary to (5.10). Hence L_{16} has no internal vertex, and the result follows. ▮

(5.12) *If H_1, H_2 is a bad pair then* $|N(H_1)| \leqslant 3$.

Proof. Let H_1 have nodes $u_1, \ldots, u_6$ and arcs L_{ij} $(1 \leqslant i \leqslant 3, 4 \leqslant j \leqslant 6)$ as usual. By (5.10) we may assume that L_{36} has an internal vertex. By (5.11), $u_3, u_6 \notin N(H_1)$. By (5.3), there is an edge e of G with one end an internal vertex of L_{36} and the other, y, in $V(H_1 - V(L_{36})$. From the symmetry we may assume that $y \in V(L_{14} \cup L_{16})$ and $y \neq u_4$. Let H be the hexad obtained from H by deleting the interior of the subpath of L_{16} between y and u_6 if $y \in V(L_{16})$, or deleting the interior of L_{16} if $y \in V(L_{14})$, and adding e. Then H 3-communicates with H_1, u_2 is a node of H, and u_6 is an internal vertex of an arc of H incident with u_2. By (5.11), $u_2 \notin N(H_1)$, and so $N(H_1) \subseteq \{u_1, u_4, u_5\}$. The result follows. ▮

(5.13) *If $v_1, v_2, v_3 \in V(G)$ are mutually adjacent and H_1, H_2 is a bad pair then one of $N(H_1), N(H_2)$ equals $\{v_1, v_2, v_3\}$.*

Proof. Let e_1, e_2, e_3 be edges with ends v_2v_3, v_3v_1, v_1v_2. By (5.6) we may assume that $e_2, e_3 \in E(C(H_1))$. Let H be a hexad which 3-communicates with H_1. Then $e_2, e_3 \in E(H)$. If v_1 is 2-valent in H then v_1, v_2, v_3 belong to the same arc of H, and then there is a hexad H' with $V(H') = V(H) - \{v_1\}$, contrary to (5.10). Thus v_1 is a node of H, and so $v_1 \in N(H_1)$. By (5.11), no arc of H incident with v_1 has an internal vertex, since H, H_2 is a bad pair and $v_1 \in N(H_1) = N(H)$. Hence v_2, v_3 are nodes of H, and so $v_2, v_3 \in N(H_1)$. By (5.12), $|N(H_1)| \leqslant 3$, and the result follows. ▮

(5.14) *G is triangle-free.*

Proof. By (5.3) G has no circuit with $\leqslant 2$ edges. Suppose that it has one with three edges, and let v_1, v_2, v_3 be mutually adjacent vertices of G. By (5.13) we may assume that $N(H_1) = \{v_1, v_2, v_3\}$. Let H_1 have nodes $u_1, \ldots, u_6$ and arcs L_{ij} $(1 \leqslant i \leqslant 3, 4 \leqslant j \leqslant 6)$ as usual. By (5.10), we may assume that L_{36} has an internal vertex. Hence by (5.11), $u_3, u_6 \notin N(H_1)$. Since $N(H_1) \subseteq \{u_1, \ldots, u_6\}$, we may assume that

$$\{v_1, v_2, v_3\} = N(H_1) = \{u_1, u_2, u_4\}.$$

By (5.11), $L_{14}, L_{15}, L_{16}, L_{24}, L_{25}, L_{26}, L_{34}$ have no internal vertices. Hence u_1, u_2, u_5 are mutually adjacent in G, and yet $u_5 \notin N(H_1)$, and so by (5.13), $N(H_2) = \{u_1, u_2, u_5\}$. But similarly $N(H_2) = \{u_1, u_2, u_6\}$, a contradiction. Hence G is triangle-free. ▮

(5.15) *Some vertex G has valency* 3.

Proof. Let G have $n+1$ vertices, and let v be a vertex of minimum valency d say. Then $|E(G)| \geqslant \frac{1}{2}(n+1)d$, and so $G \backslash v$ has at least $\frac{1}{2}(n-1)d$ edges. But $G \backslash v$ is triangle-free by (5.14) and has no hexad and has $n \geqslant 3$ vertices, and so $|E(G \backslash v)| \leqslant 2n - 4$ by (2.5). Hence

$$\tfrac{1}{2}(n-1)d < 2(n-1)$$

and so $d < 4$. The result follows. ▮

(5.16) *If $v \in V(G)$ is 3-valent and H_1, H_2 is a bad pair then $v \notin N(H_1)$.*

Proof. Suppose that $v \in N(H_1)$. Let H_1 have nodes $u_1, \ldots, u_6$, where $u_6 = v$, and arcs L_{ij} $(1 \leqslant i \leqslant 3, 4 \leqslant j \leqslant 6)$ as usual. By (5.11), none of L_{16}, L_{26}, L_{36} has an internal vertex. By (5.10) we may assume that L_{14} has an internal vertex. Let x be the vertex of L_{14} adjacent to u_4. By (5.3) there is an edge e of G with ends x, y, where $e \notin E(H_1)$. By (5.10), $y \notin V(L_{14})$. If $y \in V(L_{15} \cup L_{25} \cup L_{35}) - \{u_1, u_2, u_3\}$, then by deleting the interior of L_{15} (or the interior of the subpath of L_{15} between u_1 and y, if $y \in V(L_{15})$) and adding e, we obtain a hexad H 3-communicating with H_1 such that u_1 is an internal vertex of an arc of H incident with u_6, contrary to (5.11), since H, H_2 is a bad pair and $v \in N(H_1) = N(H)$. Consequently, $y \notin V(L_{15} \cup L_{25} \cup L_{35}) - \{u_1, u_2, u_3\}$. Since every edge of G incident with u_6 belongs to H_1 and L_{16}, L_{26}, L_{36} have no internal vertices, it follows that $y \in V(L_{24} \cup L_{34})$, and from the symmetry we may assume that $y \in V(L_{24})$. Now x, y, and u_4 are not mutually adjacent, by (5.14), and so the subpath of L_{24} between y and u_4 has an internal vertex. By deleting the interior of this subpath and adding e, we obtain a hexad contradicting (5.10). The result follows. ∎

Now we complete the proof of (1.3) by proving a contradiction. It will follow that the global assumption of this section, that G exists, is false, and so (1.3) is true as required.

(5.17) *A contradiction.*

Proof. By (5.15), G has a 3-valent vertex v. By (5.16) and (5.1), we may assume that v is not a node of H_1 or H_2, and so by (5.10), v is 2-valent in H_1 and in H_2. Hence there is an edge e of $H_1 \cap H_2$ with ends v, u say. Now e is H_1- and H_2-contractible, and by (3.2), H_1/e and H_2/e do not 3-communicate in G/e. From the minimality of G, there is a ($\leqslant 3$)-separation of G/e dividing H_1/e and H_2/e. But G/e is 3-connected, by (5.7). By (4.7) there is a hexad H in G/e with $V(H) \neq V(G/e)$. Let H' be the source of H; then H' is a hexad in G with $V(H') \neq V(G)$, contrary to (5.10). ∎

6. Kuratowski Subgraphs

Now we derive (1.2) from (1.3). First, let us prove the easy "only if" half of (1.2).

(6.1) *If H_1, H_2 are Kuratowski subgraphs of G that communicate, then no ($\leqslant 3$)-separation of G divides H_1, H_2.*

Proof. We may assume (as in the proof of (2.3)) that H_1, H_2 are 1- or 2-adjacent; and by (2.2) we may assume that not both H_1, H_2 are hexads. Thus, let H_2 be a pentad, and let H_1 be either a hexad or a pentad, such that H_1, H_2 are 1-adjacent. Let $J = H_1 \cup H_2$, and let L be an arc of J so that H_2 is obtained from J by deleting the interior of L. Suppose that (A, B) is a ($\leqslant 3$)-separation of G such that $E(A)$ meets $\leqslant 3$ arcs of H_2. It follows that every node of H_2 is in $V(B)$. If H_1 is a pentad then since H_1, H_2 are 1-adjacent pentads, they have the same nodes (as is easily seen) and hence every node of H_1 is in $V(B)$; and so (A, B) does not divide H_1, H_2 as required. We may therefore assume that H_1 is a hexad. Since $V(B)$ contains every node of H_2, it follows that every node of J in $V(A) - V(B)$ is an end of L. We suppose that $H_1 \cap B$ is a forest. Then $V(A)$ contains $\geqslant 5$ nodes of H_1, by (2.1)(i), and hence $V(A)$ contains $\geqslant 5$ nodes of J. Consequently, $|V(A \cap B)| = 3$, every vertex in $V(A \cap B)$ is a node of H_1, and both ends of L are in $V(A) - V(B)$, and are nodes of H_1, and L is an arc of H_1. But there is no arc of H_1 joining the node of H_1 in $V(B) - V(A)$ with either end of L, contradicting that H is a hexad. This proves that $H_1 \cap B$ is not a forest, and so (A, B) does not divide H_1, H_2, as required. ∎

Now we turn to the "if" half of (1.2). We begin with the following.

(6.2) *If H, H' are hexads in G that 3-communicate then they communicate.*

Proof. It suffices to show that if H, H' are 3-adjacent then they communicate. Let $J = H_1 \cup H_2$, isomorphic to a subdivision of the graph J_3 of Fig. 1. Let u_1, u_2 be the two 5-valent nodes of J, and u_3, u_4, u_5, u_6 the other four nodes, in order on the circuit of $J \backslash \{u_1, u_2\}$ through them. Let H_1 be obtained from J by deleting the interiors of the arcs with ends $u_1u_3, u_2u_4, u_1u_5, u_2u_6$, and let H_2 be obtained similarly with u_1, u_2 exchanged. Let H_3 be the pentad obtained from J by deleting the interiors of the arcs with ends u_1u_3, u_2u_3. Then H_i, H_3 are 1-adjacent ($i = 1, 2$) and so H_1, H_2 communicate, as required. ∎

(6.3) *Let H be a pentad in G that communicates with no hexad. Let H have nodes $u_1, ..., u_5$. Then there are 10 subgraphs A_{ij} ($1 \leqslant i < j \leqslant 5$) of G such that*

(i) *for $1 \leqslant i < j \leqslant 5$, A_{ij} contains the arc of H with ends u_i, u_j*

(ii) *for $1 \leqslant i < j \leqslant 5$ and $1 \leqslant i' < j' \leqslant 5$, if $(i, j) \neq (i', j')$ then $E(A_{ij} \cap A_{i'j'}) = \emptyset$ and $V(A_{ij} \cap A_{i'j'}) = \{v_i, v_j\} \cap \{v_{i'}, v_{j'}\}$*

(iii) *$G = \bigcup(A_{ij}\colon 1 \leqslant i < j \leqslant 5)$.*

Proof. Let L_{ij} be the arc of H with ends u_i, u_j ($1 \leqslant i < j \leqslant 5$).

(1) *For $1 \leqslant i < j \leqslant 5$, no H-path in G has one end an internal vertex of L_{ij} and the other end not in $V(L_{ij})$.*

For if P is such an H-path, then $H \cup P$ includes a hexad, which is therefore 1-adjacent to H, contrary to hypothesis.

(2) *Let C be a component of $G \backslash V(H)$; then there exist i, j with $1 \leqslant i < j \leqslant 5$ such that every vertex of H with a neighbour in $V(C)$ belongs to L_{ij}.*

This follows from (1) if some vertex of H with a neighbour in $V(C)$ is not a node of H. Thus, it suffices to show that at most two nodes of H have neighbours in $V(C)$. Suppose that u_1, u_2, u_3 (say) all have neighbours in $V(C)$. Let H' be the pentad obtained from H by replacing L_{12} by an H-path L'_{12} with ends u_1, u_2, with $V(L'_{12}) \neq \{u_1, u_2\}$, and with $V(L'_{12}) - \{u_1, u_2\} \subseteq V(C)$. Then H is 1-adjacent to H'; but H' is 1-adjacent to a hexad (obtained by adding an H'-path from the interior of L'_{12} to u_3 and deleting the interiors of L_{13}, L_{23}, and L_{45}), and so H communicates with a hexad, contrary to the hypothesis. This proves (2).

From (1) and (2), the result follows. ∎

Proof of (1.2). The "only if" part of (1.2) is proved in (6.1), and it remains to prove the "if" part. Thus, let H_1, H_2 be Kuratowski subgraphs of G, such that no ($\leqslant 3$)- separation of G divides H_1, H_2. Suppose first that for $i = 1, 2$, H_i communicates with a hexad H_i' ($i = 1, 2$). Then by (6.1) (as in the proof of (5.1)), no ($\leqslant 3$)-separation divides H_1', H_2'. But then H_1', H_2' 3-communicate by (1.3) and, hence, H_1', H_2' communicate by (6.2); consequently H_1, H_2 communicate, as required. We may therefore assume that H_1 say communicates with no hexad and, in particular, H_1 is a pentad. Let H_1 have nodes $u_1, ..., u_5$. By (6.3), there are subgraphs A_{ij} ($1 \leqslant i < j \leqslant 5$) as in (6.3). If some $v \in V(A_{12}) - \{u_1, u_2\}$ is a node of H_2, then

$$(A_{12}, \bigcup (A_{ij}\colon 3 \leqslant j \leqslant 5 \text{ and } 1 \leqslant i < j))$$

is a 2-separation of G dividing H_1 and H_2, contrary to hypothesis. Thus every node of H_2 is in $\{u_1, ..., u_5\}$. Consequently, H_2 is a pentad with node set $\{u_1, ..., u_5\}$. It suffices, therefore, to show that H_1 communicates with every pentad H with node set $\{u_1, ..., u_5\}$, and we shall prove this by induction on $|E(H \cup H_1)|$. If $H = H_1$ the result is trivial, and we assume not. For $1 \leqslant i < j \leqslant 5$, let the arc of H_1 with ends u_i, u_j be L_{ij}, and let the arc of H with ends u_i, u_j be P_{ij}. We see that $H_1 \cap A_{ij} = L_{ij}$ and $H \cap A_{ij} = P_{ij}$ for all i, j. Since $H \neq H_1$, there is an edge of H_1 not in H, belonging to L_{12}, say; and consequently there is an H-path $L \subseteq L_{12}$, with ends u, v, say. Since $L \subseteq L_{12} \subseteq A_{12}$ and $u, v \in V(H \cap A_{12}) = V(P_{12})$, there is a path $P \subseteq P_{12}$ with ends u, v. Let H' be the pentad obtained from H by deleting the interior of P and adding L. Then H and H' are 1-adjacent. But $E(P) \nsubseteq E(H)$, since $P \neq L$ (because L is an H-path) and L is the only path of $H \cap A_{ij}$ with ends u, v. Thus $|E(H_1 \cup H')| < |E(H_1 \cup H)|$, and so from the inductive hypothesis, H_1 communicates with H'. Since H, H' are 1-adjacent, it follows that H_1 communicates with H. This completes the inductive proof, and the result follows. ∎

A *cutset* of G is a separation (A, B) of G such that there is a component C of $A \backslash V(A \cap B)$ and a component D of $B \backslash V(A \cap B)$ with the property that every vertex in $V(A \cap B)$ has a neighbour in both $V(C)$ and $V(D)$. A separation (A, B) is *non-planar* if it is a cutset, it has order $\leqslant 3$ and neither A nor B can be drawn in a disc with $V(A \cap B)$ drawn on the boundary. G is *Kuratowski connected* if it has no non-planar separation.

(6.4) *A graph is Kuratowski connected if and only if there is no* ($\leqslant 3$)*-separation dividing two Kuratowski subgraphs.*

Proof. Suppose first that (A, B) is a non-planar separation. We claim it divides two Kuratowski subgraphs. For since (A, B) is a cutset, there is a subgraph T of B such that T is a triad with feet $V(A \cap B)$ if $|V(A \cap B)| = 3$, T is a path with both ends in $V(A \cap B)$ if $|V(A \cap B)| = 2$, and T is null otherwise. Since A cannot be drawn in a disc with $V(A \cap B)$ on the boundary, it follows that $A \cup T$ is non-planar. By Kuratowski's theorem, there is a Kuratowski subgraph H of $A \cup T$. Since $H \cap B \subseteq T$ it follows that $E(B)$ meets $\leqslant 3$ arcs of H. Similarly, there is a Kuratowski subgraph H' of G such that $E(A)$ meets $\leqslant 3$ arcs of H'. Hence (A, B) divides H and H', as required.

For the converse, suppose that there is ($\leqslant 3$)-separation (A, B) of G dividing two Kuratowski subgraphs and choose such (A, B) with $|V(A \cap B)|$ minimum. Suppose that no component of $A \backslash V(A \cap B)$ contains a neighbour of every member of $V(A \cap B)$. Then there are subgraphs $A_1, ..., A_k$ of G where $V(A \cap B) = \{v_1, ..., v_k\}$, such that

(i) $V(A_i) \cap V(A \cap B) = V(A \cap B) - \{v_i\}$ ($1 \leqslant i \leqslant k$),

(ii) $A_1 \cup ... \cup A_k = A$, and

(iii) for $1 \leqslant i < j \leqslant k$, $E(A_i \cap A_j) = \varnothing$, and $V(A_i \cap A_j) = V(A \cap B) - \{v_i, v_j\}$.

There is a Kuratowski subgraph H such that $E(B)$ meets $\leqslant 3$ arcs of H; hence, there is a node v of H in $V(A) - V(B)$. Let $v \in V(A_1) - V(A \cap B)$ say. The separation $(A_1, A_2 \cup ... \cup A_k \cup B)$ has order $k - 1$ and divides two Kuratowski subgraphs (since $k - 1 \leqslant 2$ and a node of H is in $V(A_1) - V(B)$)), contrary to the minimality of $|V(A \cap B)|$. Thus there is a component of $A \backslash V(A \cap B)$ containing a neighbour of every member of $V(A \cap B)$, and similarly for $B \backslash V(A \cap B)$. Hence (A, B) is a cutset.

Let H be a Kuratowski subgraph of G such that $E(B)$ meets $\leqslant 3$ arcs of H. Hence $B \cap H$ can be drawn in a disc with the vertices of $V(A \cap B \cap H)$ on the boundary. Since H is non-planar, it follows that $A \cap H$ cannot be drawn in a disc with the vertices of $V(A \cap B \cap H)$ on the boundary, and therefore A cannot be drawn in a disc with the vertices of $V(A \cap B)$ on the boundary. By a similar argument for B, we deduce that (A, B) is non-planar, as required. ∎

We deduce the following from (1.2) and (6.3).

(6.5) *A graph is Kuratowski connected if and only if every two Kuratowski subgraphs communicate.*

In view of the fact that the graph J_2 of Fig. 1 is not Kuratowski connected, one might conjecture a strengthening of (6.5): that if G is Kuratowski connected, than any two Kuratowski subgraphs of G are linked by a sequence of Kuratowski subgraphs, each 1-adjacent to the next.