MONOCHROMATIC VS. MULTICOLORED PATHS

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ABSTRACT

Let l and k be positive integers, and let $X = \{0, 1, \ldots, l^{k-1}\}$. Is it true that for every coloring $\delta : X \times X \to \{0, 1, \ldots\}$ there either exist elements $x_0 < x_1 < \ldots < x_l$ of X with $\delta(x_0, x_1) = \delta(x_1, x_2) = \ldots = \delta(x_{l-1}, x_l)$, or else there exist elements $y_0 < y_1 < \ldots < y_k$ of X with $\delta(y_{i-1}, y_i) \neq \delta(y_{j-1}, y_j)$ for all $1 \leq i < j \leq k$? We prove here that this is the case if either $l \leq 2$, or $k \leq 4$, or $l \geq (3k)^{2k}$. The general question remains open.

1. INTRODUCTION

Erdös and Szekeres proved in [2] that every sequence of l^2+1 real numbers contains a monotone subsequence of length $l+1$. Further extensions of this simple and elegant statement have been extensively studied and some of these extensions form combinatorial folklore today. We start with one such statement. Throughout this paper, X will denote a linearly ordered set with linear ordering $\leq$.

(1.1) *Let $k, l \geq 2$ be integers, and let $\delta : X \times X \to \{1, 2, \ldots, k-1\}$. If $|X| \geq l^{k-1}+1$ then there exist elements $x_0 < x_1 < \ldots < x_l$ such that $\delta(x_0, x_1) = \delta(x_1, x_2) = \ldots = \delta(x_{l-1}, x_l)$. Moreover, the number $l^{k-1}+1$ is best possible in the sense that there exists a linearly ordered set X with $|X| = l^{k-1}$ for which the above fails.*

Proof. We begin with the first half of the statement. Suppose that the required elements $x_0 < x_1 < \ldots < x_l$ do not exist. For $x \in X$ let $\Phi(x) = (l_1, l_2, \ldots, l_{k-1})$, where l_i is maximum such that there are elements $x_0 < x_1 < \ldots < x_{l_i} = x$ of X with $\delta(x_0, x_1) = \delta(x_1, x_2) = \ldots = \delta(x_{l_i-1}, x_{l_i}) = i$. Then $0 \leq l_i \leq l-1$ for $i = 1, 2, \ldots, k-1$. Moreover $\Phi(x) \neq \Phi(x')$ for $x < x'$, because $\Phi(x)$ and $\Phi(x')$ differ in the $\delta(x, x')$-th coordinate. Thus $|X| \leq l^{k-1}$, a contradiction.

To prove the second half let X consist of all vectors $(l_1, l_2, \ldots, l_{k-1})$ with $0 \leq l_i \leq l-1$ for $i = 1, 2, \ldots, k-1$, and let $\leq$ be a linear ordering on X with the property that if $l_i \leq l'_i$ for all $i = 1, 2, \ldots, k-1$, then $(l_1, l_2, \ldots, l_{k-1}) \leq (l'_1, l'_2, \ldots, l'_{k-1})$. Now let $x < x'$ be elements of X, say $x = (l_1, l_2, \ldots, l_{k-1})$ and $x' = (l'_1, l'_2, \ldots, l'_{k-1})$. Then there is an index i such that $1 \leq i \leq k-1$ and $l_i < l'_i$. We choose one such index i and define $\delta(x, x') = i$. We deduce that if $x_0 < x_1 < \ldots < x_p$ are elements of X with $\delta(x_0, x_1) = \delta(x_1, x_2) = \ldots = \delta(x_{p-1}, x_p)$, then $p < l$, as desired. □

In this paper we shall be concerned with the following attempt to generalize (1.1). Let Δ be an arbitrary infinite set. The elements of Δ will be referred to as *colors*. Let l, k be positive integers, and let $f(l, k)$ be the least integer with the property that if $|X| \geq f(l, k)+1$, then for every mapping $\delta : X \times X \to \Delta$ either there exist elements $x_0 < x_1 < \ldots < x_l$ of X with $\delta(x_0, x_1) = \delta(x_1, x_2) = \ldots = \delta(x_{l-1}, x_l)$, or else there exist elements $y_0 < y_1 < \ldots < y_k$ of X

with $\delta(y_{i-1}, y_i) \neq \delta(y_{j-1}, y_j)$ for all $1 \leq i < j \leq k$. That such an integer indeed exists may be deduced from the Canonical Ramsey Theorem of Erdös and Rado [1]. The bound obtained this way, however, is not a good one. We conjecture the following.

(1.2) *Conjecture. For all positive integers l and k, $f(l,k) = l^{k-1}$.*

It follows from the second part of (1.1) that $f(l,k) \geq l^{k-1}$. Thus the number l^{k-1} in (1.2) is best possible. In this paper we prove the following.

(1.3) *Let l, k be positive integers.*

(i) If $l \leq 2$, then $f(l,k) = l^{k-1}$.

(ii) If $k \leq 4$, then $f(l,k) = l^{k-1}$.

(iii) If $l \geq (3k)^{2k}$, then $f(l,k) = l^{k-1}$.

Parts (i) and (ii) are proved in Section 2, part (iii) in Section 3. An immediate consequence of (iii) is the following.

(1.4) *For every positive integer k there exists an integer c_k such that for every positive integer l, $f(l,k) \leq l^{k-1} + c_k$.*

Another generalization of (1.1) was obtained by Tuza [5], who proved that if $|X| \geq l^{k-1} + 1$, then the conclusion of (1.1) holds for every mapping $\delta : X \times X \to \Delta$ with the property that for every $x \in X$, the set $\{\, \delta(y,x\} \,|\, y \in X, y < x \,\} \cup \{\, \delta(x,y) \,|\, y \in X, y > x \,\}$ has at most $k-1$ elements.

Let us establish some terminology. By a *path* we mean a sequence $x_0, x_1, \ldots, x_p$ of elements of X such that $x_0 < x_1 < \ldots < x_p$. The number p is the *length* of the path. (Thus we really mean "monotone path", but we skip the word monotone, because we do not consider paths in graphs.) The element x_0 is called the *tail* of the path, whereas the element x_p is called its *head*. Let $\delta : X \times X \to \Delta$. We say that a path $x_0, x_1, \ldots, x_p$ is a *flash* if $\delta(x_0, x_1) = \delta(x_1, x_2) = \ldots = \delta(x_{p-1}, x_p)$, and we say that it is a *rainbow* if the colors $\delta(x_0, x_1), \delta(x_1, x_2), \ldots, \delta(x_{p-1}, x_p)$ are all distinct. A flash of length p is called a p-flash (in X), whereas a rainbow of length p is called a *p-rainbow* (in X). With this terminology (1.2) can be reformulated by saying that if $|X| \geq l^{k-1} + 1$, then there exists either an l-flash or a k-rainbow. Let $\alpha \in \Delta$. We say that a path $x_0, x_1, \ldots, x_p$ *uses* α if $\delta(x_{i-1}, x_i) = \alpha$ for some $1 \le i \le p$.

2. SMALL l OR SMALL k

In this section we prove (i) and (ii) of (1.3). We begin with (i). Since it clearly holds for $l = 1$, it is enough to prove it for $l = 2$. To do so we prove the following stronger result.

(2.1) *Let k be a positive integer, let $|X| \ge 2^{k-1} + 1$, and let $\delta : X \times X \to \Delta$ be such that there is no 2-flash. Then there exists a k-rainbow with head* $\max(X)$.

Proof. We proceed by induction on k. The statement is obviously satisfied for $k = 1$. Let $k > 1$ be given, and assume that (2.1) holds for $k - 1$. Let $|X| \ge 2^{k-1} + 1$, and let $\delta : X \times X \to \Delta$ be such that there is no 2-flash. Let $a = \max(X)$, and let $b = \max(X - \{a\})$. Let $\alpha = \delta(b, a)$, and let A be the set of all $x \in X$ such $\delta(x, y) = \alpha$ for some $y \in X$ with $x < y$. Then $b \in A$ and $a \notin A$. Assume first that $|A| \ge 2^{k-2} + 1$. Then by the induction hypothesis applied to the set A there exists a $(k-1)$-rainbow $x_0, x_1, \ldots, x_{k-1}$ with $x_0, \ldots, x_{k-1} \in A$ and $x_{k-1} = b$. We claim that $\delta(x_{i-1}, x_i) \ne \alpha$ for every i with $1 \le i \le k-1$. Indeed, suppose that $\delta(x_{i-1}, x_i) = \alpha$ for some i with $1 \le i \le k-1$, and let $y \in X$ be such that $x_i < y$ and $\delta(x_i, y) = \alpha$. Such a y exists because $x_i \in A$. Then x_{i-1}, x_i, y is a 2-flash, contrary to our assumption. This proves our claim that $\delta(x_{i-1}, x_i) \ne \alpha$ for every i with $1 \le i \le k-1$. It follows that $x_0, x_1, \ldots, x_{k-1}, a$ is the desired k-rainbow.

Now assume that $|A| \le 2^{k-2}$. By the induction hypothesis applied to the set $(X - A - \{a\}) \cup \{b\}$ there exists a $(k-1)$-rainbow $x_0, x_1, \ldots, x_{k-1}$ with $x_0, \ldots, x_{k-2} \notin A$ and $x_{k-1} = b$. It follows that $\delta(x_{i-1}, x_i) \ne \alpha$ for all i with $1 \le i \le k-1$, and hence $x_0, x_1, \ldots, x_{k-1}, a$ is as desired. □

We now turn to the proof of (ii). We first establish that it is enough to prove it for $k = 4$.

(2.2) *Let $l \ge 1$ and $k \ge 2$ be integers. If $f(l, k) = l^{k-1}$ then $f(l, k-1) = l^{k-2}$.*

Proof. We know from Section 1 that $f(l, k-1) \ge l^{k-2}$. Assume that $f(l, k-1) > l^{k-2}$. Thus there exist a linearly ordered set X with $|X| \ge l^{k-2} + 1$ and a coloring $\delta : X \times X \to \Delta$ such that there is neither an l-flash, nor a $(k-1)$-rainbow. Let $Y = X \times \{0, 1, \dots, l-1\}$ with linear ordering defined by saying that $(x, p) \leq (x', p')$ if either $x < x'$, or $x = x'$ and $p \leq p'$. Let $\alpha \in \Delta$ be such that $\alpha \neq \delta(x, y)$ for all $x, y \in X$. We define $\gamma : Y \times Y \to \Delta$ as follows:

$$\gamma((x,p),(x',p')) = \begin{cases} \delta(x,x') & \text{if } x < x' \\ \alpha & \text{otherwise.} \end{cases}$$

It follows that in Y there exists neither an l-flash nor a k-rainbow. Thus $f(l, k) > l^{k-1}$, as desired.

□

Now we are ready to finish the proof of (ii) of (1.3). For $x \in X$ we put $\delta^+(x) = \{\delta(x, y) | y \in X, y > x\}$.

(2.3) *For every positive integer l, $f(l, 4) = l^3$.*

Proof. We proceed by induction on l. The statement of (2.3) is obviously satisfied if $l = 1$. Now let $l > 1$ and assume that (2.3) holds for $l - 1$. Let $\delta : X \times X \to \Delta$ be such that there is neither an l-flash, nor a 4-rainbow. We will show that $|X| \leq l^3$. To this end we define subsets L, M, R of X as follows:

$$R = \{x \in X | \text{ there are no } y, z, w \in X \text{ with } x < y < z \text{ and } x < w$$
$$\text{such that } \delta(x, y) \neq \delta(y, z) = \delta(x, w)\},$$

$$L = \{x \in X | \text{ there is no } (l-1)\text{-flash with head } x\},$$

$$M = X - (R \cup L).$$

We claim that

(1) $|L| \leq (l-1)^3$.

For this follows from the induction hypothesis applied to the set L.

Now we claim that

(2) $|R| \leq 3l - 2$.

For let $a = \max(X)$. Then certainly $a \in R$. We are going to construct a mapping $\phi : R - \{a\} \to \{0, 1, 2\}$, and distinct colors $\alpha_0, \alpha_1, \alpha_2 \in \Delta$ such that for every $i \in \{0, 1, 2\}$ the following two conditions are satisfied.

(i) If $x \in R - \{a\}$ is such that $\phi(x) = i$, then there exists $y \in R$ with $y > x$ such that $\delta(x, y) = \alpha_i$, and

(ii) if $x_1, x_2, \ldots, x_p \in R - \{a\}$ are all the elements of $R - \{a\}$ with $\phi(x_j) = i$ $(j = 1, 2, \ldots, p)$, and $x_1 < x_2 < \ldots < x_p$, then $\delta(x_{j-1}, x_j) = \alpha_i$ for all j with $2 \leq j \leq p$.

For the construction we shall proceed in terms of an algorithm. At the beginning of the algorithm we set $r_0 = \max(R - \{a\})$, $\phi(r_0) = 0$ and $\alpha_0 = \delta(r_0, a)$; the remaining variables are undefined. Now assume that $x \in R - \{a\}$ is such that $\phi(x)$ is undefined and $\phi(y)$ has already been defined for all $y \in R - \{a\}$ with $y > x$ in such a way that (i) and (ii) are satisfied. For $i = 0, 1, 2$, if $\phi^{-1}(i) \neq \emptyset$ we denote by y_i the smallest element of $\phi^{-1}(i)$, otherwise y_i is undefined. Note that y_0 is well-defined, because $r_0 \in \phi^{-1}(0)$. To define $\phi(x)$ we distinguish three cases.

Case 1. There exists an $i \in \{0, 1, 2\}$ such that $\phi^{-1}(i) \neq \emptyset$ and $\delta(x, y_i) = \alpha_i$. In this case we choose the smallest such i and define $\phi(x) = i$.

For the remaining two cases we may therefore assume that

(∗) for every $i \in \{0, 1, 2\}$, if $\phi^{-1}(i) \neq \emptyset$ then $\delta(x, y_i) \neq \alpha_i$.

Case 2. $\phi^{-1}(1) = \emptyset$. We let $r_1 = x, \phi(x) = 1$ and $\alpha_1 = \delta(r_1, r_0)$. We claim that $\alpha_1 \neq \alpha_0$. Indeed, if $\alpha_1 = \alpha_0$ let $y > y_0$ be such that $\delta(y_0, y) = \alpha_0$. Since $\delta(x, y_0) \neq \alpha_0$ by (∗), we deduce that the elements y_0, y, r_0 contradict the fact that $x \in R$. This proves our claim that $\alpha_1 \neq \alpha_0$.

Case 3. $\phi^{-1}(1) \neq \emptyset$ and $\phi^{-1}(2) = \emptyset$. Then y_1 is well-defined. We let $r_2 = x$, $\phi(x) = 2$ and $\alpha_2 = \delta(r_2, r_1)$. We claim that $\alpha_2 \notin \{\alpha_0, \alpha_1\}$. Indeed, if $\alpha_2 = \alpha_i$ $(i = 0$ or $1)$, let $y > y_i$ be such that $\delta(y_i, y) = \alpha_i$. Since $\delta(x, y_i) \neq \alpha_i$ by (∗), we deduce that the elements y_i, y, r_1 contradict the fact that $x \in R$.

To complete our construction we must show that the three cases above are exhaustive, that is, that under the assumption (∗) one of $\phi^{-1}(1)$, $\phi^{-1}(2)$ is empty. For suppose that both $\phi^{-1}(1)$ and $\phi^{-1}(2)$ are nonempty. Then y_0, y_1, y_2 are well-defined. Since $\delta(r_2, r_1) = \alpha_2, \delta(r_1, r_0) = \alpha_1, \delta(r_0, a) = \alpha_0$ are all distinct, we deduce from the fact that no 4-rainbow exists that $\delta(x, r_2) \in \{\alpha_0, \alpha_1, \alpha_2\}$. Let $\delta(x, r_2) = \alpha_i$, where $i \in \{0, 1, 2\}$, and let $y > y_i$ be such that $\delta(y_i, y) = \alpha_i$. Since

$\delta(x, y_i) \neq \alpha_i$ by $(*)$, we deduce that the elements of y_i, y, r_2 contradict the fact that $x \in R$. This completes the construction.

Now we are ready to finish the proof of (2). Let $i \in \{0, 1, 2\}$ and let $x_1, \ldots, x_p \in R - \{a\}$ be all the elements of $R - \{a\}$ with $\phi(x_j) = i$ $(j = 1, 2, \ldots, p)$ and let $x_1 < x_2 < \cdots < x_p$. Then, by (ii), $\delta(x_{j-1}, x_j) = \alpha_i$ for all j with $2 \leq j \leq p$, and, by (i), there exists $x_{p+1} \in R$ with $x_{p+1} > x_p$ such that $\delta(x_p, x_{p+1}) = \alpha_i$. Hence $p \leq l - 1$, because no l-flash exists. Since ϕ takes on values $0, 1, 2$ we deduce that $|R - \{a\}| \leq 3(l - 1)$. Thus $|R| \leq 3l - 2$, which proves (2).

To complete the proof of (2.3) we must show that $|M| \leq 3(l-1)^2$. This will be done in several steps. We say that a set $M_0 \subseteq M$ is *nice* if, writing $x_0 = \max(M_0)$,

(a) there exist elements $y_0, z_0, w_0 \in X$ and distinct colors $\alpha, \beta \in \Delta$ such that $x_0 < y_0 < z_0, x_0 < w_0, \delta(x_0, y_0) = \alpha$ and $\delta(x_0, w_0) = \delta(y_0, z_0) = \beta$, and

(b) either

  (b1) every $(l - 1)$-flash with head in M_0 uses neither α nor β, or

  (b2) there exists $\gamma \in \Delta - \{\alpha, \beta\}$ such that for every $x \in M_0$ there is an $(l - 1)$-flash with head x which uses γ.

In the next three claims M_0 is a nice set with $x_0, y_0, z_0, w_0, \alpha, \beta$ as in the above definition.

(3) *For every $x \in M_0$ there exists an $(l - 1)$-flash with head x which uses neither α nor β.*

For this is clear when (b2) holds. Assume then that (b1) holds. Since $x \notin L$ there is an $(l - 1)$-flash with head x, which is as desired by (b1). This proves (3).

(4) *For every $x \in M_0$ there exist elements $y, z \in X$ with $x < y < z$ such that either $\delta(x, y) = \alpha$ and $\delta(y, z) = \beta$, or $\delta(x, y) = \beta$ and $\delta(y, z) = \alpha$.*

For this holds for $x = x_0$ by (a). So let $x \in M_0 - \{x_0\}$. From (3) and the fact that no l-flash exists it follows that there exists an $x' \in X$ with $x' < x$ such that $\delta(x', x)$ is distinct from α, β and $\delta(x, x_0)$. We deduce that $\delta(x, x_0) \in \{\alpha, \beta\}$, for otherwise x', x, x_0, y_0, z_0 is a 4-rainbow. If $\delta(x, x_0) = \alpha$ we put $y = x_0$ and $z = w_0$; if $\delta(x, x_0) = \beta$ we put $y = x_0$ and $z = y_0$. Then (4) is clearly satisfied.

(5) *Let $x, y \in M_0$ with $x < y$. Then $\delta(x,y) \in \{\alpha, \beta\}$.*

For by (3) and the fact that no l-flash exists there exists $x' \in X$ with $x' < x$ such that $\delta(x', x)$ is distinct from $\delta(x,y), \alpha, \beta$. By (4) there exist $y', y'' \in X$ with $y < y' < y''$ such that either $\delta(y, y') = \alpha$ and $\delta(y', y'') = \beta$, or $\delta(y, y') = \beta$ and $\delta(y', y'') = \alpha$. In either case $\delta(x,y) \in \{\alpha, \beta\}$ for otherwise x', x, y, y', y'' is a 4-rainbow.

(6) *Every nice set contains at most $(l-1)^2$ elements.*

To prove (6) we claim that M_0 contains no $(l-1)$-flash. If (b1) holds then this follows from (5). Suppose now for a contradiction that (b2) holds and that $x \in M_0$ is the head of an $(l-1)$-flash F contained in M_0. By (5), F uses α or β. From the symmetry we may assume that it uses α. Let x' be the next-to-last element of F; then $x' < x$ and $\delta(x', x) = \alpha$. By (b2) there exists $x'' \in X$ with $x'' < x'$ and $\delta(x'', x') = \gamma$. Since $x \in M$ there are $y, z, w \in X$ such that $x < y < z$, $x < w$, and $\delta(x,y) \neq \delta(x,w) = \delta(y,z)$. Since there is an $(l-1)$-flash with head x which uses α, and one which uses γ, we deduce that $\delta(x,y), \delta(x,w) \notin \{\alpha, \beta\}$. Then x'', x', x, y, z is a 4-rainbow, a contradiction. This proves our claim that M_0 contains no $(l-1)$-flash. Claim (6) now follows from (5) and (1.1).

Now we are ready to show that $|M| \leq 3(l-1)^2$. Let $x_0 = \max(M)$, and let $w_0, y_0, z_0 \in X$ be such that $x_0 < y_0 < z_0$, $x_0 < w_0$ and $\delta(x_0, y_0) \neq \delta(x_0, w_0) = \delta(y_0, z_0)$. Let $\alpha = \delta(x_0, y_0)$ and let $\beta = \delta(x_0, w_0) = \delta(y_0, z_0)$. We define

$$M_1 = \{x \in M | \text{ there exists an } (l-1)-\text{flash with head } x \text{ which uses } \alpha\},$$

$$M_2 = \{x \in M | \text{ there exists an } (l-1)-\text{flash with head } x \text{ which uses } \beta\},$$

$$M_3 = \{x \in M | \text{ every } (l-1)-\text{flash with head } x \text{ uses neither } \alpha \text{ nor } \beta\}.$$

Clearly $M_1 \cup M_2 \cup M_3 = M$.

(7) *M_3 is nice.*

For we claim that $x_0 \in M_3$. Indeed, every $(l-1)$-flash with head x_0 uses neither α nor β, because $\alpha, \beta \in \delta^+(x_0)$ and no l-flash exists. Thus $x_0 = \max(M_3)$, and M_3 satisfies (a) and (b1) with x_0, y_0, w_0, α and β. This proves (7).

(8) *M_1 and M_2 are nice.*

From the symmetry it is enough to argue for M_1. Let $x'_0 = \max(M_1)$. Since $x'_0 \in M$ there exist elements $y'_0, z'_0, w'_0 \in X$ such that $x'_0 < y'_0 < z'_0$, $x'_0 < w'_0$, and $\delta(x'_0, y'_0) \neq \delta(x'_0, w'_0) = \delta(y'_0, z'_0)$. Let $\alpha' = \delta(x'_0, y'_0)$ and $\beta' = \delta(x'_0, w'_0) = \delta(y'_0, z'_0)$. As there is an $(l-1)$-flash with head x_0 using α, and $\alpha', \beta' \in \delta^+(x_0)$, and no l-flash exists, we deduce that $\alpha' \neq \alpha \neq \beta'$. Thus M_1 satisfies (a) and (b2) with $y'_0, z'_0, w'_0, \alpha'$, β', and $\gamma = \alpha$. This proves (8).

Now we have

$$|X| \leq |L| + |M| + |R| \leq |L| + |M_1| + |M_2| + |M_3| + |R| \leq (l-1)^3 + 3(l-1)^2 + 3l - 2 = l^3$$

by (1), (6), (7), (8) and (2), as desired. □

3. LARGE l

We start with the following generalization of (1.1) which follows from a theorem of Gallai [3]. The proof can also be obtained by a slight variation of the proof of (1.1).

(3.1) *Let $\delta : X \times X \to \Delta$, let $\alpha \in \Delta$, and let n, m be integers. If $|X| > n \cdot m$ then either there exist elements $x_0 < x_1 < \ldots < x_n$ of X such that $\delta(x_{i-1}, x_i) = \alpha$ for every integer i with $1 \leq i \leq n$, or else there exist elements $y_0 < y_1 < \ldots < y_m$ of X such that $\delta(y_i, y_j) \neq \alpha$ for all integers i, j with $0 \leq i < j \leq m$.*

We need the following weak form of (1.2).

(3.2) *Let l, k be positive integers, let $|X| \geq l^{k-1} + 1$, let $\delta : X \times X \to \Delta$ be such that there is no l-flash, and let $A \subseteq \Delta$ with $|A| \leq k-1$ be such that every 2-flash uses an element of A. Then there exists a k-rainbow.*

Proof. We may assume that $|A| = k-1$. Let $A = \{\alpha_1, \alpha_2 \ldots, \alpha_{k-1}\}$. For $x \in X$ let $\phi_i(x)$ be maximum such that there exists a $\phi_i(x)$-flash with head x which uses α_i. Let $X_1 \subseteq X$ be such that $|X_1| \geq \frac{2}{l} l^{k-1} + 1$ and $|\{\phi_1(x) | x \in X_1\}| \leq 2$. Let $X_2 \subseteq X_1$ be such that $|X_2| \geq \left(\frac{2}{l}\right)^2 l^{k-1} + 1$ and $|\{\phi_2(x) | x \in X_2\}| \leq 2$. We continue in the same way and finally choose $X_{k-1} \subseteq X_{k-2}$ such that $|X_{k-1}| \geq 2^{k-1} + 1$ and $|\{\phi_{k-1}(x) | x \in X_{k-1}\}| \leq 2$. We claim that X_{k-1} contains no 2-flash. For suppose that $x, y, z \in X_{k-1}$ are such that $x < y < z$ and $\delta(x, y) = \delta(y, z)$. Then

$\delta(x,y) = \delta(y,z) \in A$, say $\delta(x,y) = \delta(y,z) = \alpha_i$. But $\phi_i(x) < \phi_i(y) < \phi_i(z)$, contrary to the choice of X_i. This proves our claim that X_{k-1} contains no 2-flash. Thus there exists a k-rainbow in X_{k-1} (and hence in X) by (2.1). $\square$

Let $Y \subseteq X$. We say that an element $x \in Y$ is *left k-special relative to Y* if for every $\alpha \in \Delta$ there exists a $(k-2)$-rainbow in Y with head x which does not use α. We say that an element $x \in X$ is *left k-special* if it is left k-special relative to X.

(3.3) *Let $k, l, r \geq 2$ be integers. If there is neither an l-flash nor an r-rainbow in X, and*

$$|X| > rl + (rl)^2 + \ldots + (rl)^{k-2},$$

then there exists a left k-special element in X.

Proof. Let $l \geq 2$ be fixed. We proceed by induction on k. If $k = 2$ every element of X is left k-special, and so the statement clearly holds. Now let $k \geq 3$ be an integer, and assume that (3.3) holds for $k-1$. Assume that there is no left k-special element in X. Then for every element $x \in X$ there exists a color $\alpha(x) \in \Delta$ such that every $(k-2)$-rainbow in X with head x uses $\alpha(x)$. For $\alpha \in \Delta$ let Y_α be the set of all elements $y \in X$ for which there exists an element $x \in X$ with $x > y$, $\alpha(x) = \alpha$ and $\delta(y,x) \neq \alpha$. We claim that

(1) $\qquad |Y_\alpha| \leq l(rl + (rl)^2 + \ldots + (rl)^{k-3})$.

To prove (1) we apply (3.1). Indeed, there is no l-flash in Y_α, because there is none in X. Moreover, let Z be a subset of Y_α with the property that $\delta(z,z') \neq \alpha$ for all $z, z' \in Z$ with $z < z'$. We claim that no $z \in Z$ is left $(k-1)$-special relative to Z. For suppose that $z \in Z$ is left $(k-1)$-special relative to Z. Since $z \in Y_\alpha$ there exists an element $x \in X$ with $x > z$, $\alpha(x) = \alpha$ and $\delta(z,x) \neq \alpha$. Let $x_0, x_1, \ldots, x_{k-3}$ be a $(k-3)$-rainbow in Z with head z which does not use $\delta(z,x)$. Then $x_0, x_1, \ldots, x_{k-3}, x$ is a $(k-2)$-rainbow in X with head x which does not use α, contrary to $\alpha(x) = \alpha$. This contradiction proves our claim that no $z \in Z$ is left $(k-1)$-special relative to Z. Hence

$$|Z| \leq rl + (rl)^2 + \ldots + (rl)^{k-3}$$

by (3.3) applied to Z and $k-1$. Now (1) follows from (3.1).

For $\alpha \in \Delta$ let Z_α be the set of all $z \in X$ such that either $z \in Y_\alpha$ or $\alpha(z) = \alpha$. We claim that

(2) $\quad |Z_\alpha| \leq l(1 + rl + (rl)^2 + \ldots + (rl)^{k-3}).$

For if $z, z' \in Z_\alpha - Y_\alpha$ with $z < z'$ then $\delta(z, z') = \alpha$. Thus $|Z_\alpha - Y_\alpha| \leq l$, because no l-flash exists, and (2) follows from (1).

Now we finish the proof of (3.3). Clearly $X \neq \emptyset$. Let $x_1 = \max(X)$ and let $\alpha_1 = \alpha(x_1)$. If $X = Z_{\alpha_1}$ we stop, otherwise let $x_2 = \max(X - Z_{\alpha_1})$ and let $\alpha_2 = \alpha(x_2)$. We remark that $\alpha_1 \neq \alpha_2$. If $X = Z_{\alpha_1} \cup Z_{\alpha_2}$ we stop, otherwise we let $x_3 = \max(X - (Z_{\alpha_1} \cup Z_{\alpha_2}))$ and let $\alpha_3 = \alpha(x_3)$. Then $\alpha_3 \notin \{\alpha_1, \alpha_2\}$. We continue this process until $X = Z_{\alpha_1} \cup Z_{\alpha_2} \cup \ldots \cup Z_{\alpha_i}$ for some $i \leq r$, or until we construct $x_1, x_2, \ldots, x_{r+1}$.

The latter is impossible, because $x_{r+1}, x_r, \ldots, x_2, x_1$ would be an r-rainbow in X (since $x_{j+1} \notin Y_{\alpha_j}$ and $\alpha(x_j) = \alpha_j$ we deduce that $\delta(x_{j+1}, x_j) = \alpha_j$), which we are assuming does not exist. Hence $X = Z_{\alpha_1} \cup Z_{\alpha_2} \cup \ldots \cup Z_{\alpha_i}$ for some $i \leq r$ and the theorem follows from (2). □

We deduce

(3.4) *Let $k \geq 2$ and $l \geq 2$ be integers. If there is neither an l-flash nor a k-rainbow, and $|X| > (2k^{k-2} - 2)l^{k-2}$, then there exists a left k-special element in X.*

Proof. This follows from (3.3) by setting $r = k$. □

In this section we prove the remainder of (1.3), which we now restate.

(3.5) *Let l, k be positive integers with $l \geq (3k)^{2k}$. Then $f(l, k) = l^{k-1}$.*

The proof of (3.5) is complex and will be divided into several steps, which will occupy the remainder of the paper. We proceed by induction on k. We know from (2.2) and (2.3) that (3.5) holds when $k \leq 4$. So let $k \geq 5$, let $l \geq (3k)^{2k}$ and assume that (3.5) holds for l and $k-1$. Let $\delta : X \times X \to \Delta$ be such that there is no l-flash nor a k-rainbow. We must show that $|X| \leq l^{k-1}$.

Let $x \in X$ be left k-special. We denote by $A(x)$ a minimal subset of Δ with the property that for every $\alpha \in \Delta$ there exists a $(k-2)$-rainbow with head x using only colors from $A(x) - \{\alpha\}$.

(3.6) $|A(x)| \le (k-1)(k-2)$ *for every left k-special element $x \in X$.*

Proof. Let $\alpha \in \Delta$. There exists a $(k-2)$-rainbow with head x which uses only colors from $A(x) - \{\alpha\}$; say it uses $\alpha_1, \alpha_2, \ldots, \alpha_{k-2} \in A(x) - \{\alpha\}$. For every integer i with $1 \le i \le k-2$ there exists a $(k-2)$-rainbow with head x which uses only colors from a set $A_i \subseteq A(x) - \{\alpha_i\}$ such that $|A_i| = k-2$. By the minimality of $A(x)$, $A(x) \subseteq \{\alpha_1, \alpha_2, \ldots, \alpha_{k-2}\} \cup \bigcup_{i=1}^{k-2} A_i$. But the latter set contains at most $(k-1)(k-2)$ elements. □

(3.7) *Let $x \in X$ be left k-special, and let $y, z \in X$ be such that $x \le y < z$ and $\delta(y,z) \notin A(x)$. Then $|\{\, v \in X \,|\, v > x \,\}| \le 2k^{k-2}l^{k-2}$.*

Proof. Let $x, y, z \in X$ be as stated.

(1) *Let $u \in X$ be such that $x < u < \max(X)$. If $\delta(x,u) \notin A(x)$ or $\delta^+(u) - A(x) \ne \emptyset$ then $\delta^+(u) = \{\delta(x,u)\}$.*

For let $\alpha = \delta(x,u)$. If $\delta^+(u) - A(x) \ne \emptyset$ we choose $u' \in X$ with $u' > u$ such that $\delta(u,u') \notin A(x)$, otherwise we choose $u' \in X$ with $u' > u$ arbitrarily. Let $\alpha' = \delta(u,u')$; then $\{\alpha, \alpha'\} \not\subseteq A(x)$. Now $\alpha = \alpha'$, for otherwise there exists a $(k-2)$-rainbow $x_0, x_1, \ldots, x_{k-2}$ with head x using none of α, α', in which case $x_0, x_1, \ldots, x_{k-2}, u, u'$ is a k-rainbow, a contradiction which proves (1).

(2) *There exists $u \in X$ such that $x < u < \max(X)$ and $\delta(x,u) \notin A(x)$.*

Indeed, if $x = y$ we put $u = z$. If $x < y$ we put $u = y$ and deduce from (1) that $\delta(x,u) = \delta(y,z) \notin A(x)$. This proves (2).

Let $u \in X$ be minimum with $x < u < \max(X)$ and $\delta(x,u) \notin A(x)$. Let $\alpha = \delta(x,u)$.

(3) $|\{v \in X | x < v \le u\}| \le (2k^{k-2} - 2)l^{k-2}$.

For let $Y = \{v \in X | x < v \le u\}$. We claim that no element of Y is left k-special relative to Y. We first notice that if $v, v' \in Y$ with $v < v'$, then $\delta(v,v') \in A(x)$, for otherwise $\delta(x,v) = \delta(v,v') \notin A(x)$ by (1), contrary to the choice of u. Now suppose for a contradiction that $v \in Y$ is left k-special relative to Y. Then there exists a $(k-2)$-rainbow $x_0, x_1, \ldots, x_{k-2}$ in Y with head v not using $\delta(v,u)$ (and using only colors from $A(x)$ by the fact which we noticed above). Let

$u' \in X$ be such that $u' > u$. Then $\delta(u, u') = \alpha$ by (1). Thus $x_0, x_1, \ldots, x_{k-2}, u, u'$ is a k-rainbow, a contradiction which proves our claim that no element of Y is left k-special relative to Y. Now (3) follows from (3.4) applied to Y.

(4) $\quad |\{v \in X | v > u, \alpha \in \delta^+(v)\}| < l$

For let $v_0 = u < v_1 < \ldots < v_p$ be such that $\alpha \in \delta^+(v_i)$ for all i with $0 \leq i \leq p$. Then $\delta(v_{i-1}, v_i) = \alpha$ for all i with $1 \leq i \leq p$ by (1), and so $p < l$, as desired.

(5) $\quad |\{v \in X | v > u, \alpha \notin \delta^+(v)\}| \leq l^{k-2}$

For the above set contains no $(k-1)$-rainbow. Indeed, if $v_1, v_2, \ldots, v_k$ was one, then $u, v_1, \ldots, v_k$ would be a k-rainbow (because $\delta(u, v_1) = \alpha$ and $\delta(v_{i-1}, v_i) \neq \alpha$ for all i with $2 \leq i \leq k$ by (1)), a contradiction. Hence (5) follows from the induction hypothesis that (3.5) holds for $k-1$.

From (3), (4), and (5) we deduce the statement of (3.7). $\square$

Let L be the set of all left k-special elements $x \in X$ with $|A(y)| \geq k$ for every left k-special element $y \in X$ with $y \leq x$.

(3.8) $\quad |L| \leq (e(k-1))^{2(k-1)} l^{k-2}$.

Proof. Let $a = \min(L)$. Suppose for a contradiction that $|L| > (e(k-1))^{2(k-1)} l^{k-2}$.

(1) $\quad |\delta^+(x)| \leq k-1$ *for every* $x \in L - \{a\}$.

For suppose that $|\delta^+(x_{k-1})| \geq k$ for some $x_{k-1} \in L - \{a\}$, let $x_0 < x_1 < \ldots < x_{k-2} = a$ be a $(k-2)$-rainbow with head a not using $\delta(a, x_{k-1})$, and let $x_k > x_{k-1}$ be such that $\delta(x_{k-1}, x_k)$ is different from all of $\delta(x_{i-1}, x_i)$ $(i = 1, ..., k-1)$. Then $x_0, x_1, \ldots, x_k$ is a k-rainbow, a contradiction.

(2) $\quad \delta^+(x) \subseteq A(a)$ *for every* $x \in L - \{a\}$.

For otherwise $|\{y \in X | y > a\}| \leq 2k^{k-2} l^{k-2}$ by (3.7), and $|\{y \in X | y \leq a\}| \leq 2k^{k-2} l^{k-2}$ by (3.4) because no $y < a$ is left k-special. Thus $|L| \leq (e(k-1))^{2(k-1)} l^{k-2}$, a contradiction.

Since $|A(a)| \leq (k-1)(k-2)$ by (3.6), and

$$\binom{(k-1)(k-2)}{0} + \binom{(k-1)(k-2)}{1} + \cdots + \binom{(k-1)(k-2)}{k-1} \leq (e(k-1))^{k-1},$$

there exists a set $L' \subseteq L$ with $|L'| > (e(k-1))^{k-1} l^{k-2}$ such that $\delta^+(x) = \delta^+(x')$ for all $x, x' \in L'$. For $x \in L'$ let $B_x = \{ \delta(x,y) \mid y \in L', y > x \}$. Similarly there exists a set $L'' \subseteq L$ with $|L''| > l^{k-2}$ such that $B_x = B_{x'}$ for all $x, x' \in L''$. Let B denote this set.

(3) $|B| \leq k-2$.

For suppose that $|B| \geq k-1$. Then $|B| = k-1$ by (1). Let $x = \min(L'')$. Since $|A(x)| \geq k$ there exists a $(k-2)$-rainbow $x_0 < x_1 < \ldots < x_{k-2} = x$ with head x which uses a color not in B. Let $x_{k-1} \in L'$ be such that $x_{k-1} > x_{k-2}$ and $x_0, x_1, \ldots, x_{k-2}$ does not use $\delta(x_{k-2}, x_{k-1})$. Let $x_k \in X$ be such that $x_k > x_{k-1}$ and $x_0, x_1, \ldots, x_{k-2}, x_{k-1}$ does not use $\delta(x_{k-1}, x_k)$. Then $x_0, x_1, \ldots, x_k$ is a k-rainbow, a contradiction.

From (3) and (1.1) applied to L'' we deduce that $|L''| \leq l^{k-2}$, a contradiction. □

We say that an element $x \in X$ is *right k-special* if for every $\alpha \in \Delta$ there exists a $(k-2)$-rainbow in X with tail x which does not use α. If $x \in X$ is right k-special we denote by $B(x)$ a minimal subset of Δ with the property that for every $\alpha \in \Delta$ there exists a $(k-2)$-rainbow with tail x using only colors from $B(x) - \{\alpha\}$. By symmetry we deduce "mirror image" versions of (3.4), (3.6), (3.7) and (3.8). Now we are ready to prove (3.5).

Proof of (3.5). Since $l \geq (3k)^{2k}$ we deduce that

(1) $2k^{k-2} l^{k-2} + (e(k-1))^{2(k-1)} l^{k-2} < \frac{1}{2} l^{k-1}$.

We may assume that $|X| \geq l^{k-1}$, for otherwise we are done. Then by (3.4), (3.8) and (1) there exists a left k-special element $x \in X$ with $|A(x)| \leq k-1$, and from the symmetry there exists a right k-special element $y \in X$ with $|B(y)| \leq k-1$. Let a be the minimum left k-special element of X with $|A(a)| \leq k-1$ and let b be the maximum right k-special element with $|B(b)| \leq k-1$. We deduce from (3.4), (3.8), their mirror image versions and (1) that $a < b$. From (3.7) and its mirror image version we deduce that $A(a) = B(b)$, and that if $\delta(x, x') \notin A(a) = B(b)$ for some $x, x' \in X$ with $x < x'$, then $x \leq a$ and $x' \geq b$. Thus we may apply (3.2) with $A = A(a) = B(b)$ to deduce that $|X| \leq l^{k-1}$, as desired. □

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