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Graphs without K_4 and Well-Quasi-Ordering

ROBIN THOMAS

Department of Mathematical Analysis,
Charles University,
Sokolovská 83, Prague, Czechoslovakia

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It is proved that given an infinite sequence $G_1, G_2, G_3, \ldots,$ of series–parallel graphs there are indices $i<j$ such that G_j contains an induced subgraph contractable onto G_i. An example is given showing that for planar graphs the preceding theorem fails. © 1985 Academic Press, Inc.

0. INTRODUCTION

By a Kuratowski type theorem for a property P of graphs we mean any assertion of the form:

G does not have P iff $H \leqslant G$ for some H in L, where L is a finite list of graphs and $\leqslant$ is a quasi-ordering (i.e., a reflexive and transitive relation). Such a theorem exists, e.g., for the properties "to be planar" (the classical Kuratowski's theorem) or "to be embeddable in the projective plane" (see [1, 2]), where the lists are explicitly known. We are interested not in constructing such lists, but in their existence. The connection with the well-quasi-ordering theory may be stated as follows: If the property P is $\leqslant$-closed (i.e., G has P and $H \leqslant G$ implies H has P) and the class of all graphs is wqo by $\leqslant$, then there is a Kuratowski type theorem for P. In this context, the following conjecture due to Wagner is important:

Conjecture. The class of all graphs is wqo by the relation $\preccurlyeq$ ($G \preccurlyeq H$ if H contains a subgraph contractable onto G).

Partial results are due to Kruskal [4], Mader [5], and Robertson and Seymour [7]. Very recently the existence of a Kuratowski type theorem for higher surfaces has been proved by Robertson and Seymour.

There are properties, which are not $\preccurlyeq$-closed, for example, the property $P =$ "to be a string graph" (see [8]). On the other hand this property P is $\trianglelefteq$-closed, where $\trianglelefteq$ is a strengthening of $\preccurlyeq$ (see Sect. 2), and that leads to a natural question whether the class of all graphs is wqo by $\trianglelefteq$. A negative

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swer to this is given in Section 3, and an affirmative result concerning $\preceq$ n be found in Section 2. Section 1 is devoted to definitions only.

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1. Definitions.

A graph is a pair (V, E), where V is a finite set and E is a subset of the lection of all 2-element subsets of V. $\mathscr{K}_4$ denotes the class of connected phs which contain no subdivision of K_4.

Let G, H be arbitrary graphs. A contraction of H onto G is a mapping $(H) \to^{\text{onto}} V(G)$ such that

(i) For every $v \in V(G)$ the graph induced by $f^{-1}(v)$ in H is con-ted.

(ii) For every $u \neq v \in V(G)$ the following equivalence holds: $v\} \in E(G)$ if and only if there are $y, z \in V(H)$, $y \in f^{-1}(u)$, $z \in f^{-1}(v)$ such t $\{y, z\} \in E(H)$.

Let G be a graph and $v_1, ..., v_n$ distinct vertices of G. The pair $(G, ..., v_n))$ is called an n-rooted graph and will be denoted simply $_1, ..., v_n)$. The graph G itself is considered to be a 0-rooted graph. If $_1, ..., v_n)$ is an n-rooted graph and Q an arbitrary set we define a Q-elled n-rooted graph to be an $(n+2)$-triple $(g, G, v_1, ..., v_n)$, where $(G) \to Q$. For $|Q| = 1$, the mapping g brings no further structure on G, the n-rooted graph $G(v_1, ..., v_n)$ itself will be considered to be Q-labelled , e.g., $Q = \{0\}$).

f $\mathscr{G}$ is a class of n-rooted graphs and Q an arbitrary set, then $\mathscr{G}(Q)$ otes the collection of all Q-labelled graphs $(g, G, v_1, ..., v_n)$ such that $_1, ..., v_n) \in \mathscr{G}$.

Put also

$= \{G(u, v)$: G is a block from $\mathscr{K}_4$ and $\{u, v\} \in E(G)\}$

$= \{G(u, v)$: $\{u, v\} \notin E(G)$, H is 2-connected and $H \in \mathscr{K}_4$, where H is obtained from G by joining the edge $\{u, v\}\}$

$= \mathscr{B}^+ \cup \mathscr{B}^-$.

2. Affirmative result

We shall briefly recall some facts concerning wqo theory. For a nice lanation of the method used the reader is referred to [6]. Let Q be a set which a quasi-ordering (i.e., reflexive and transitive relation) $\leqslant$ is defined. Such sets are said to be quasi-ordered (qo). By a Q-sequence we mean any mapping $f: X \to Q$, where X is an infinite subset of ω. The letters X, Y (with or without dashes or suffixes) will always denote infinite subsets of ω. A Q-sequence $f: X \to Q$ is called good, if there are $i < j \in X$ such that $f(i) \leqslant f(j)$ and is called bad otherwise. A qo set Q is called well-quasi-ordered (wqo) if every Q-sequence is good. If Q, Q' are quasi-ordered via quasi-orderings $\leqslant$, $\leqslant'$, then we define a quasi-ordering $\leqslant \times \leqslant'$ on $Q \times Q'$ as follows:

$$[q_1, q_1'] \leqslant \times \leqslant' [q_2, q_2'] \qquad \text{if } q_1 \leqslant q_2 \text{ and } q_1' \leqslant' q_2'.$$

Proposition 2.1 (Higman [3]). *If Q, Q' are wqo, then $Q \times Q'$ is wqo.*

By $Q^{<\omega}$ we mean the set of finite sequences of elements of Q. $Q^{<\omega}$ will be quasi-ordered by the rule that $(a_1, ..., a_n) \leqslant_s (b_1, ..., b_m)$ if there is a strictly increasing mapping $f: \{1, ..., n\} \to \{1, ..., m\}$ such that $a_i \leqslant b_{f(i)}$ for any $i = 1, ..., n$.

Proposition 2.2 (Higman [3]). *If Q is wqo, then $Q^{<\omega}$ is wqo.*

Suppose that a quasi-ordering $\leqslant$ on a set Q is given and suppose that $\gamma = (g, G, v_1, ..., v_n)$, $\eta = (h, H, w_1, ..., w_n)$ are Q-labelled n-rooted graphs. Define $\gamma \preceq \eta$ if H contains an induced subgraph H' and there is a contraction f of H' onto G such that

For any $i = 1, ..., n$, $w_i \in V(H')$ and $f(w_i) = v_i$

For any $v \in V(G)$, there is $w \in f^{-1}(v)$ such that $g(v) \leqslant h(w)$.

We will say that γ is smaller than η (and write $\gamma \prec \eta$) if $\gamma \preceq \eta$ and $|E(G)| < |E(H)|$. Note that according to our agreements $\preceq$ and $\prec$ are defined on the class of n-rooted graphs as well as on the class of graphs itself. In what follows we shall be concerned with the wqo property of the quasi-ordering $\preceq$.

Let $\mathscr{G}$ be an arbitrary class of graphs and Q a qo set, and let $f: X \to \mathscr{G}(Q)$, $f': X' \to \mathscr{G}(Q)$ be two $\mathscr{G}(Q)$-sequences. We define $f \leqslant * f'$ if $X \subseteq X'$ and $f(i) \preceq f'(i)$ for every $i \in X$ and similarly $f < * f'$ if $X \subseteq X'$ and $f(i) \prec f'(i)$ for every $i \in X$. A $\mathscr{G}(Q)$-sequence $f: X \to \mathscr{G}(Q)$ is called minimal bad if it is bad and there is no bad $\mathscr{G}(Q)$-sequence $f' < * f$.

Lemma 2.3. *If $f: X \to \mathscr{G}(Q)$ is a bad $\mathscr{G}(Q)$-sequence, then there is a minimal bad $\mathscr{G}(Q)$-sequence $f_0 \leqslant * f$.*

Proof. Let $X = \{i_1 < i_2 < \cdots\}$. Choose $f_0(i_1)$ such that it is a first term of a bad $\mathscr{G}(Q)$-sequence which is $\leqslant * f$ and there is no smaller element of $\mathscr{G}(Q)$ with this property. Then choose $f_0(i_2)$ such that $f_0(i_1)$, $f_0(i_2)$ (in that order) are first two terms of a bad $\mathscr{G}(Q)$-sequence which is $\leqslant * f$ and there is no smaller element of $\mathscr{G}(Q)$ with this property. Continuing this process

get a bad $f_0: X \to \mathscr{G}(Q)$. We claim it is the desired one. For if there is a
d $f': X' \to \mathscr{G}(Q)$, $f' < *f_0$, we may define $e: Y \to \mathscr{G}(Q)$ by

$$Y = \{i \in X: i < \min X'\} \cup X'$$

$$\begin{aligned} e(i) &= f_0(i) \qquad i < \min X', i \in X, \\ &= f'(i) \qquad i \in X'. \end{aligned}$$

w e contradicts the choice of f_0. ∎

LEMMA 2.4. *If $\mathscr{B}(Q)$ is not wqo, then there is a minimal bad $\mathscr{B}(Q)$-uence $f_0: X_0 \to \mathscr{B}(Q)$ such that* $\mathrm{Im}\, f_0 \subseteq \mathscr{B}^-(Q)$.

Proof. By Lemma 2.3 there is a minimal bad $f: X \to \mathscr{B}(Q)$, we denote $) = (g_i, G_i, u_i, v_i)$. We may assume that either $\mathrm{Im}\, f \subseteq \mathscr{B}^-(Q)$ or Im $\mathscr{B}^+(Q)$. In the second case we define $f^-: X \to \mathscr{B}^-(Q)$ by $f^-(i) =$ $, G_i^-, u_i, v_i)$, where G_i^- is obtained from G_i by removing the edge $, v_i\}$. f^- is bad but it may fail to be minimal bad. By Lemma 2.3 there is minimal bad $f_1: X_1 \to \mathscr{B}(Q)$ such that $f_1 \leqslant *f^-$. We may again assume t either $\mathrm{Im}\, f_1 \subseteq \mathscr{B}^-(Q)$ or $\mathrm{Im}\, f_1 \subseteq \mathscr{B}^+(Q)$. In the first case we are done d in the second one $f_1 < *f$, which contradicts the minimality of f. ∎

LEMMA 2.5. *If $G(u, v) \in \mathscr{B}^-$ then either*

(i) *there exist $G_1(u_1, v_1)$, $G_2(u_2, v_2) \in \mathscr{B}$, vertex-disjoint and smaller n $G(u, v)$, so that $G(u, v)$ is obtained by identifying $u_1 = u_2 = u$, $= v_2 = v$, or* (ii) *there exist $G_1(u, w_1)$, $G_2(w_2, v) \in \mathscr{B}$, vertex-disjoint and aller than $G(u, v)$, so that $G(u, v)$ is obtained by identifying $w_1 = w_2$.*

Proof. Is well-known and we shall just sketch it. Let $G(u, v) \in \mathscr{B}^-$ be en. Then either there are two disjoint paths joining u and v or (by nger's theorem) there is a cutpoint between u and v. The first case gives while the second one leads to (ii). ∎

LEMMA 2.6. *If Q is wqo, then $\mathscr{B}(Q)$ is wqo by* $\trianglelefteq$.

Proof. Suppose that the lemma fails for some Q which is wqo. Then by mma 2.4 there is a minimal bad $\mathscr{B}(Q)$-sequence $f: X \to \mathscr{B}(Q)$ such that $f \subseteq \mathscr{B}^-(Q)$. Denote $f(i) = (g_i, G_i, u_i, v_i)$. We may assume that either

(i) holds for $G_i(u_i, v_i)$ for any $i \in X$

(ii) holds for $G_i(u_i, v_i)$ for any $i \in X$.

fine the graphs $G_i^1(u_i^1, v_i^1)$, $G_i^2(u_i^2, v_i^2)$ to be those from Lemma 2.5 and

define g_i^1, g_i^2 to be the restrictions of g_i to $V(G_i^1)$, $V(G_i^2)$, respectively. Then $f_1, f_2: X \to \mathscr{B}(Q)$ defined by

$$f_1(i) = (g_i^1, G_i^1, u_i^1, v_i^1), \qquad f_2(i) = (g_i^2, G_i^2, u_i^2, v_i^2)$$

are smaller than f. The sets $\mathrm{Im}\, f_1$, $\mathrm{Im}\, f_2$ are wqo by minimality of f, and by Proposition 2.1 there are $i < j \in X$ such that $f_1(i) \trianglelefteq f_1(j)$, $f_2(i) \trianglelefteq f_2(j)$. This yields

$$f(i) \trianglelefteq f(j)$$

which contradicts the badness of f and proves the lemma. ∎

THEOREM 2.7. *Let Q be wqo and let $\mathscr{F}$ be a class of 1-rooted blocks satisfying*

$$\textit{If } R \textit{ is wqo then } \mathscr{F}(R) \textit{ is wqo by } \trianglelefteq. \qquad (*)$$

Denote

$\mathscr{G} = \{G(v): G$ *is connected and* $B(b) \in \mathscr{F}$ *for every block B of G and every* $b \in V(B)\}$.

Then $\mathscr{G}(Q)$ is wqo by $\trianglelefteq$.

Proof. Suppose that Q and $\mathscr{F}$ satisfy the assumptions, but $\mathscr{G}(Q)$ is not wqo. Let $f: \omega \to \mathscr{G}(Q)$ be a minimal bad $\mathscr{G}(Q)$-sequence, we denote $f(i) = (g_i, G_i, v_i)$. Clearly each G_i contains at least two vertices and thus we may choose a block B_i in each G_i such that $v_i \in V(B_i)$. Denote $W_i = \{w: \{w\} = V(B_i) \cap V(B)$ for some block B in G distinct from $B_i\}$ and let $w_i^1, \ldots, w_i^{s_i}$ be elements of W_i. Consider the graph obtained from G_i by deleting edges from B_i and denote by B_i^j that component of this graph which contains w_i^j. Define h_i^j to be the restriction of g_i to $V(B_i^j)$. Put $\mathscr{H}_i = \{(h_i^j, H_i^j, w_i^j): j = 1, \ldots, s_i\}$, $\mathscr{H} = \bigcup_{i \in \omega} \mathscr{H}_i \cup \{\varnothing\}$. We define a quasi-ordering on $\mathscr{H}$, the least one containing the restriction of $\trianglelefteq$ to $\bigcup_{i \in \omega} \mathscr{H}_i$. It follows easily from minimality of f that $\mathscr{H}$ is wqo. Define $b_i: V(B_i) \to \mathscr{H} \times Q$ by the rule

$$\begin{aligned} x &\mapsto ((h_i^j, H_i^j, w_i^j), g_i(x)) \qquad \text{for } x = w_i^j \\ &\mapsto (\varnothing, g_i(x)) \qquad \text{otherwise.} \end{aligned}$$

By Proposition 2.1, $\mathscr{H} \times Q$ is wqo and hence $\mathscr{F}(\mathscr{H} \times Q)$ is wqo. Thus $e: \omega \to \mathscr{F}(\mathscr{H} \times Q)$ defined by

$$e(i) = (b_i, B_i, v_i)$$

s good, namely there are $i<j$ such that

$$e(i) \preceq e(j).$$

Then it is easily verified that

$$f(i) \preceq f(j)$$

which contradicts the badness of f. ∎

THEOREM 2.8. *If Q is wqo, then $\mathscr{K}_4(Q)$ is wqo.*

Proof. Denote

$$\mathscr{A} = \{G(u) : G \text{ is a block from } \mathscr{K}_4\}.$$

Let R be an arbitrary wqo set. Then $\mathscr{B}(R)$ is wqo by Lemma 2.6 and it follows that $\mathscr{A}(R)$ is wqo. Hence $\mathscr{A}$ satisfies (*) and we may use Theorem 2.7. ∎

Using Proposition 2.2 one can pass to non-connected graphs. Thus we have

COROLLARY 2.9. *If Q is wqo, then the class of all Q-labelled graphs which contain no subdivision of K_4 is wqo by $\preceq$.*

3. NEGATIVE RESULT

The following example shows that the class of planar graphs is not wqo by $\preceq$.

EXAMPLE 3.1. Define $G_n = (V_n, E_n)$, where

$$V_n = \{0, ..., 2n, 2n+1\}$$

$$E_n = \{\{1,2\}, \{2,3\}, ..., \{2n-1, 2n\}, \{2n, 1\}\} \cup \{\{0, 2k\} \mid k = 1, ..., n\}$$
$$\cup \{\{2n+1, 2k+1\} \mid k = 0, ..., n-1\}.$$

Denote by C_n the graph obtained from G_n by removing vertices 0 and $2n+1$. We claim that the sequence

$$G_5, G_6, G_7, ...,$$

is bad (with respect to $\preceq$).

Suppose not, thus there are $4<k<n$ such that $G_k \preceq G_n$. Then there is an induced subgraph G'_n of G_n and a contraction f of G'_n onto G_k. Denote by H_n the graph induced by the set $f^{-1}(V(C_k))$ in G'_n. Every connected subgraph of G'_n containing neither 0 nor $2n+1$ has at most four neighbours, but 0 and $2k+1$ (vertices of G_k) are of degree at least five, and hence $0, 2n+1 \in G'_n$ and $f(0), f(2n+1) \in \{0, 2k+1\}$. This yields $H_n \subseteq C_n$, and moreover, since C_k is a cycle, H_n must contain a cycle and it follows that $H_n = C_n$. (This shows that $G'_n = G_n$.) Thus there are i, j adjacent in C_n such that $f(i) = f(j)$. Now $f(i)$ is of degree four but that is impossible since any vertex from C_k is of degree three.

4. CONCLUDING REMARKS

1. Theorem 2.8 can be extended to infinite graphs. This is a more technical statement which will appear elsewhere.

2. We remark that the above results concerning the quasi-ordering $\preceq$ do not cover all possible cases. For instance, any of the graphs from Example 3.1 can be contracted onto K_5^- (the complete graph K_5 minus one edge). It is natural to ask which classes of graphs are wqo by $\preceq$. Particularly we propose the following problem:

Is the class of all graphs that cannot be contracted onto K_5^- wqo by $\preceq$? (Compare [5].)

The referee points out that concerning this conjecture the following (unpublished) theorem of Seymour may be relevant:

The only 3-connected graphs not contractable onto K_5^- are

(i) wheels
(ii) $K_{3,3}$ and the prism
(iii) graphs with $\leqslant 4$ vertices.

REFERENCES

1. D. ARCHDEACON, A Kuratowski theorem for the projective plane, Thesis, Ohio State University, 1980.
2. H. GLOVER, P. HUNEKE, AND C. S. WANG, 103 graphs that are irreducible for the projective plane, *J. Combin. Theory Ser. B* **27** (1979), 332–370.
3. G. HIGMAN, Ordering by divisibility in abstract algebras, *Proc. London Math. Soc.* (3) **2** (1952), 326–336.
4. J. B. KRUSKAL, Well-quasi-ordering, the tree theorem, and Vázsony's conjecture, *Trans. Amer. Math. Soc.* **95** (1960), 210–225.

5. W. Mader, Wohlquasigeordnete Klassen endlicher Graphen, *J. Combin. Theory Ser. B* **12**, No. 2 (1972), 105–122.
6. C. St. J. A. Nash-Williams, On well-quasi-ordering finite trees, *Proc. Cambridge Phil. Soc.* **59** (1963), 833–835.
7. N. Robertson and P. D. Seymour, Graph width and well-quasi-ordering, preprint.
8. F. W. Sinden, Topology of thin film RC circuits, *Bell System Tech. J.* **45** (1966), 1639–1662.

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