Observe that (Ω, d) is a complete metric space homeomorphic to the Cantor set, the metric topology being the topology of coordinatewise convergence in Ω. We are now ready to give the promised proof.

THEOREM (Steinhaus) [PS], [S]. *A regular matrix summability method cannot sum all sequences of* 0*'s and* 1*'s.*

Proof. Let $A = (a_{n,k})$ be regular. By (3), $\Sigma_{k=1}^{\infty}|a_{j,k}|$ is convergent for each $j \in N$, hence $f_j(x) = \Sigma_{k=1}^{\infty} a_{j,k} x_k$ is defined for all $x = (x_k) \in \Omega$, and f_j is a continuous complex-valued function on Ω because $|f_j(x) - f_j(y)| \leqslant \Sigma_{k=i+1}^{\infty}|a_{j,k}| < \varepsilon$ whenever $d(x, y) < 2^{-i}$ and i is sufficiently large.

Now, suppose A sums all sequences of 0's and 1's, or in other words, $f(x) = \lim_j f_j(x)$ is defined for all $x \in \Omega$. Notice that any open ball about any point in Ω contains two points z and w where z is eventually all 1's and w is eventually all 0's. Since A is regular, we have that $|f(z) - f(w)| = 1$. Hence, f is discontinuous at every point of Ω.

f, however, is the pointwise limit of a sequence of continuous functions, and consequently must be continuous at at least one point of Ω. This is a contradiction. ■

References

[PS] R. E. Powell and S. M. Shah, Summability Theory and Its Applications, Van Nostrand Reinhold, London, 1972.

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A COMBINATORIAL CONSTRUCTION OF A NONMEASURABLE SET

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We give a slightly unusual construction of a Lebesgue nonmeasurable set on the real line using a fundamental graph-theoretical assertion (Proposition 1).

First recall some graph-theoretical notions. A graph is a pair (V, E), where V, the set of vertices, is an arbitrary set (possibly infinite) and E, the set of edges, is contained in $\binom{V}{2}$ (= the collection of all two-element subsets of V). A path in a graph (V, E) is a sequence $v_0, \dots, v_n$ of distinct elements of V such that $\{v_i, v_{i+1}\} \in E$ for $i = 0, 1, \dots, n-1$. The number of edges, n, is the length of the path. A cycle is a closed path, that is, $v_0 = v_n$. A graph (V, E) is called bipartite if there are disjoint sets V_1 and V_2 such that $V_1 \cup V_2 = V$ and $\binom{V_i}{2}$ are disjoint from E, that is, neither V_1 nor V_2 contains both endpoints of an edge.

The following two propositions are well known, but for the reader's convenience their proofs are given here.

PROPOSITION 1. *Any graph (V, E), which has no cycles of odd length, is bipartite.*

Proof. It suffices to prove the proposition for connected graphs only. Choose any $x \in V$ and define

$$V_1 = \{ y \in V\text{: any path connecting } x \text{ and } y \text{ is of odd length}\},$$

$$V_2 = \{ y \in V\text{: any path connecting } x \text{ and } y \text{ is of even length}\}.$$

Since there are no odd cycles in (V, E), any two paths connecting x and y are of the same parity. Thus V_1 and V_2 yield the desired partition of V.

PROPOSITION 2. *Let M be a Lebesgue measurable set of real numbers with positive measure. Then there exists a $\delta > 0$ such that $M \cap (x + M) \neq \varnothing$ for any $x \in \mathbb{R}$ with $|x| < \delta$.*

Proof. Find a closed set F and an open set G with $F \subseteq M$ and $F \subseteq G$ such that $3\lambda G < 4\lambda F$ (where λ is the Lebesgue measure). Since G is a countable union of disjoint open intervals, there is one among them, say I, such that $3\lambda I < 4\lambda(F \cap I)$. Let $\delta = \lambda I/2$ and suppose that $|x| < \delta$. Then $I \cup (x + I)$ is an interval of length less than $\frac{3}{2}\lambda I$ which contains both $F \cap I$ and $x + (F \cap I)$. The last two sets cannot be disjoint, since otherwise

$$\frac{3}{2}\lambda I = \frac{3}{4}\lambda I + \frac{3}{4}\lambda I < \lambda((F \cap I) \cup (x + (F \cap I))) \leqslant \lambda(I \cup (x + I)) < \frac{3}{2}\lambda I,$$

which is a contradiction. Hence,

$$\varnothing \neq (F \cap I) \cap (x + (F \cap I)) \subseteq M \cap (x + M),$$

which completes the proof.

We now proceed to the construction. Define a graph $(\mathbb{R}, E)$ as follows: $\{x, y\} \in E$ when $|x - y| = 3^k$ for some integer k. We claim that $(\mathbb{R}, E)$ contains no cycles of odd length. Indeed, suppose that $x_0, \dots, x_n$ is a cycle of length n in $(\mathbb{R}, E)$. Then for $i = 1, 2, \dots, n$, $x_i = x_{i-1} + \xi_i 3^{k_i}$, where $\xi_i = \pm 1$ and each k_i is an integer. Then

$$\xi_1 \cdot 3^{k_1} + \cdots + \xi_n \cdot 3^{k_n} = 0.$$

For sufficiently large N,

$$\xi_1 \cdot 3^{k_1 + N} + \cdots + \xi_n \cdot 3^{k_n + N}$$

is a sum of odd integers, so in order for the sum to be 0, there must be an even number of them. This shows that the cycle is even. By Proposition 1 there are disjoint sets A and B with $\mathbb{R} = A \cup B$ such that neither A nor B contains both endpoints of an edge. Since x and $x + 3^k$ are joined by an edge, we have $A + 3^k \subseteq B$ and $B + 3^k \subseteq A$ for any integer k. Now by Proposition 2 the sets A and B are nonmeasurable.

REMARKS. (i) In fact we see that $A + 3^k = B$ and $A - 3^k = B$ for any integer k. This follows from the inclusions

$$A = A + 3^k - 3^k \subseteq B - 3^k \subseteq A,$$
$$B = B + 3^k - 3^k \subseteq A - 3^k \subseteq B.$$

(ii) The same idea has recently been used by Nešetřil and Rődl [1] to obtain a short proof of the existence of non-Ramsey sets.

EXERCISE. It is known that any construction of a nonmeasurable set must involve the Axiom of Choice. At which step of our construction is the Axiom of Choice necessary?

Reference

1. J. Nešetřil and V. Rődl, Two Remarks on Ramsey Theorem, to appear in Discrete Math.

WHEN IS AN ORDINARY DIFFERENTIAL EQUATION SEPARABLE?

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Elementary differential equations texts generally present the method of separation of variables by stating that if one can write $f(x, y) = g(x)/h(y)$, then the equation $y' = f(x, y)$ can be solved by integrating each side of $h(y)\,dy = g(x)\,dx$. They don't raise the question of whether $f(x, y)$ factors into a product of a function of x times a function of y. Most of the exercises give $f(x, y)$ in factored form, and many students cannot solve $y' = 1 + x + y^2 + xy^2$ unless they are told it is separable.

Most texts mention that a separable equation gives rise to the exact differential $g(x)\,dx - h(y)\,dy$, while treating separation of variables and exact differentials as distinct methods. The