LARGE NON-PLANAR GRAPHS AND AN APPLICATION TO CROSSING-CRITICAL GRAPHS

Guoli Ding[1]
Department of Mathematics
Louisiana State University
Baton Rouge, Louisiana 70803, USA

Bogdan Oporowski
Department of Mathematics
Louisiana State University
Baton Rouge, Louisiana 70803, USA

Robin Thomas[2]
School of Mathematics
Georgia Institute of Technology
Atlanta, Georgia 30332, USA

and

Dirk Vertigan[3]
Department of Mathematics
Louisiana State University
Baton Rouge, Louisiana 70803, USA

ABSTRACT

We prove that, for every positive integer k, there is an integer N such that every 4-connected non-planar graph with at least N vertices has a minor isomorphic to $K_{4,k}$, the graph obtained from a cycle of length $2k+1$ by adding an edge joining every pair of vertices at distance exactly k, or the graph obtained from a cycle of length k by adding two vertices adjacent to each other and to every vertex on the cycle. We also prove a version of this for subdivisions rather than minors, and relax the connectivity to allow 3-cuts with one side planar and of bounded size. We deduce that for every integer k there are only finitely many 3-connected crossing-critical graphs with no subdivision isomorphic to the graph obtained from a cycle of length $2k$ by joining all pairs of diagonally opposite vertices.

5 November 1998, Revised 21 December 2009.

---

[1] Partially supported by NSF under Grant No. DMS-0556091.

[2] Partially supported by NSF under Grants No. DMS-9623031 and DMS-0701077, and by NSA under Grant No. MDA904-98-1-0517.

[3] Partially supported by NSA under Grant No. MDA904-97-I-0042.

1. INTRODUCTION

In this paper *graphs* are finite and may have loops or multiple edges. A graph is a *subdivision* of another if the first can be obtained from the second by replacing each edge by a non-zero length path with the same ends. Our first theorem follows the pattern of the following results. The first two are easy.

(1.1) *For every positive integer k, there is an integer N such that every connected graph with at least N vertices has either a path on k vertices, or a vertex with at least k distinct neighbors.*

(1.2) *For every positive integer k, there is an integer N such that every 2-connected graph with at least N vertices has either a cycle of length at least k, or a subgraph isomorphic to a subdivision of $K_{2,k}$.*

These two results were generalized to 3- and 4-connected graphs in [4]. To state the theorems we need to define a few families of graphs. Let $k \geq 3$ be an integer. The *k-spoke wheel*, denoted by W_k, has vertices $v_0, v_1, \ldots, v_k$, where $v_1, v_2, \ldots, v_k$ form a cycle, and v_0 is adjacent to all of $v_1, v_2, \ldots, v_k$. The *$2k$-spoke alternating double wheel*, denoted by A_k, has vertices $v_0, v_0', v_1, v_2, \ldots, v_{2k}$, where $v_1, v_2, \ldots, v_{2k}$ form a cycle in this order, v_0 is adjacent to $v_1, v_3, \ldots, v_{2k-1}$, and v_0' is adjacent to $v_2, v_4, \ldots, v_{2k}$. The vertices v_0 and v_0' will be called the *hubs* of A_k. The *k-rung ladder*, denoted by L_k, has vertices $v_1, v_2, \ldots, v_k, u_1, u_2, \ldots, u_k$, where $v_1, v_2, \ldots, v_k$ and $u_1, u_2, \ldots, u_k$ form paths in the order listed, and v_i is adjacent to u_i for $i = 1, 2, \ldots, k$. The graph W_k' is obtained from L_k by adding an edge between v_1 and v_k, and contracting the edges joining u_1 to v_1 and u_k to v_k. The graph O_k, called the *k-rung circular ladder*, is obtained from L_k by adding edges between v_1 and v_k and between u_1 and u_k; and the *k-rung Möbius ladder*, denoted by M_k, is obtained from L_k by adding edges between v_1 and u_k and between u_1 and v_k. The graph $K_{4,k}'$ is obtained from $K_{4,k}$ by splitting each of the k vertices of degree four in the same way. More precisely, it has vertices $x, y, x', y', v_1, v_2, \ldots, v_k, v_1', v_2', \ldots, v_k'$, where v_i is adjacent to v_i', x, and y, and v_i' is adjacent to v_i, x', and y' for $i = 1, 2, \ldots, k$. We remark that W_k, W_k', and $K_{3,k}$ are 3-connected. The following is proved in [4].

(1.3) *For every integer $k \geq 3$, there is an integer N such that every 3-connected graph with at least N vertices has a subgraph isomorphic to a subdivision of one of W_k, W'_k, and $K_{3,k}$.*

For the second result we need a couple more definitions. A *separation* of a graph is a pair (A, B) of subsets of $V(G)$ such that $A \cup B = V(G)$, and there is no edge between $A - B$ and $B - A$. It is *nontrivial* if $A - B \neq \emptyset \neq B - A$. The *order* of (A, B) is $|A \cap B|$. A graph G is said to be *almost 4-connected* if it is 3-connected and, for every separation (A, B) of G of order three, one of $A - B, B - A$ contains at most one vertex. (We remark that this is called "internally 4-connected" in [4], but that term unsually has a different meaning.) Clearly every 4-connected graph is almost 4-connected, and if $k \geq 4$, then $A_k, O_k, M_k, K_{4,k}$, and $K'_{4,k}$ are almost 4-connected. The following is the second result from [4].

(1.4) *For every integer $k \geq 4$, there is an integer N such that every almost 4-connected graph with at least N vertices contains a subgraph isomorphic to a subdivision of one of A_k, O_k, M_k, $K_{4,k}$, and $K'_{4,k}$.*

Our first objective is to prove a version of (1.4) for non-planar graphs, as follows. We define B_k to be the graph obtained from A_k by adding an edge joining its hubs.

(1.5) *For every integer $k \geq 4$, there is an integer N such that every almost 4-connected non-planar graph with at least N vertices has a subgraph isomorphic to a subdivision of one of B_k, M_k, $K_{4,k}$, and $K'_{4,k}$.*

A graph is a *minor* of another if the first can be obtained from a subgraph of the second by contracting edges. For the minor containment (1.5) has the following corollary, which was stated for 4-connected graphs in the abstract.

(1.6) *For every integer $k \geq 4$, there is an integer N such that every almost 4-connected non-planar graph with at least N vertices has a minor isomorphic to $K_{4,k}$, or the graph obtained from a cycle of length $2k+1$ by adding an edge joining every pair of vertices at distance exactly k, or the graph obtained from a cycle of length k by adding two vertices adjacent to each other and to every vertex on the cycle.*

Proof. This follows immediately from (1.5), because $K_{4,k}$ is a minor of $K'_{4,k}$; the second outcome graph is a minor of M_{2k+1}; and the third outcome graph is a minor of B_{2k}. □

In fact, in (3.4) we prove a stronger result than (1.5). We relax the connectivity requirement on G to allow separations of order three as long as one side of the separation is planar and has bounded size.

We apply the stronger form of (1.5) to deduce a theorem about crossing-critical graphs. Traditionally, a graph G is called *crossing-critical* if it cannot be drawn in the plane with at most one crossing, but $G\backslash e$ can be so drawn for every edge $e \in E(G)$. (We use $\backslash$ for deletion.) But then every graph obtained from a crossing-critical graph by subdividing and edge is again crossing-critical, and there is another simple operation that can be used to generate arbitrarily large crossing-critical graphs. To avoid these easily understood constructions we define a graph G to be *X-minimal* if

(i) G has crossing number at least two,

(ii) $G\backslash e$ has crossing number at most one for every edge $e \in E(G)$,

(iii) G has no vertices of degree two, and

(iv) G does not have a vertex of degree four incident with two pairs of parallel edges.

If v is a vertex of degree two in a graph G, and G' is obtained from G by contracting one of the edges incident with v, then G satisfies (i) if and only G' satisfies (i), and the same holds for condition (ii). Similarly, if $u \in V(G)$ has degree four and is incident with two pairs of parallel edges, and if G'' is obtained from $G\backslash u$ by adding a pair of parallel edges joining the two neighbors of u, then the same conclusion holds for G and G''. Thus the notion of X-minimality provides a reasonable concept of being "minimal with crossing number at least two". Our second result then states the following.

(1.7) *For every integer k there exists an integer N such that every X-minimal graph on at least N vertices has a subgraph isomorphic to a subdivision of M_k.*

This is of interest, because of a belief by many experts on crossing numbers that X-minimal graphs with an M_7 subdivision can be completely described. There are infinitely many of them, but they all seem to fall within a well-described infinite family. The sequel to [1] promises to prove that. Another proof of (1.7) appears in [1].

To prove (1.7) we need the following lemma, which may be of independent interest.

(1.8) *Let G be an X-minimal graph on at least 17 vertices. Then for every separation (A, B) of G of order at most three, one of $G|A$, $G|B$ has at most 129 vertices and can be drawn in a disk with $A \cap B$ drawn on the boundary of the disk.*

The bound of 129 is far from best possible, and we make no attempt to optimize it.

The paper is organized as follows. In Section 2 we state two lemmas from other papers that will be used later. In Section 3 we prove (1.5), and in Section 4 we prove a lemma about planar graphs that we use in the final Section 5, where we first prove (1.8) and then (1.7).

The ideas of our paper were initially developed in November 1998 and written in manuscript form [2]. In October 2009 the authors of [1] kindly informed us of their work, and that prompted us to revise [2], resulting in the present article. The authors of [1] graciously withdrew their paper to allow for simultaneous submission with the present manuscript.

2. PLANAR SUBGRAPHS OF NON-PLANAR GRAPHS

We formalize the concept of a subdivision as follows. Let G, H be graphs. A mapping η with domain $V(G) \cup E(G)$ is called a *homeomorphic embedding* of G into H if for every two vertices v, v' and every two edges e, e' of G

(i) $\eta(v)$ is a vertex of H, and if v, v' are distinct then $\eta(v), \eta(v')$ are distinct,

(ii) if e has ends v, v', then $\eta(e)$ is a path of H with ends $\eta(v), \eta(v')$, and otherwise disjoint from $\eta(V(G))$, and

(iii) if e, e' are distinct, then $\eta(e)$ and $\eta(e')$ are edge-disjoint, and if they have a vertex in common, then this vertex is an end of both.

We shall denote the fact that η is a homeomorphic embedding of G into H by writing $\eta : G \hookrightarrow H$. If K is a subgraph of G we denote by $\eta(K)$ the subgraph of H consisting of all vertices $\eta(v)$, where $v \in V(K)$, and all vertices and edges that belong to $\eta(e)$ for some

$e \in E(K)$. It is easy to see that H has a subgraph isomorphic to a subdivision of G if and only if there is a homeomorphic embedding $G \hookrightarrow H$. The reader is advised to notice that $V(\eta(K))$ and $\eta(V(K))$ mean different sets. The first is the vertex-set of the graph $\eta(K)$, whereas the second is the image of the vertex-set of K under the mapping η. An *η-path* in H is a path in H with both ends in $\eta(G)$ and otherwise disjoint from it.

A cycle C in a graph G is called *peripheral* if it is induced and $G\backslash V(C)$ is connected. Let $\eta : G \hookrightarrow H$, let C be a peripheral cycle in G, and let P_1 and P_2 be two disjoint η-paths with ends u_1, v_1 and u_2, v_2, respectively, such that u_1, u_2, v_1, v_2 belong to $V(\eta(C))$ and occur on $\eta(C)$ in the order listed. In those circumstances we say that the pair P_1, P_2 is an *η-cross*. We also say that it is an *η-cross in C*. We say that u_1, v_1, u_2, v_2 are the *feet* of the cross. We say that the cross is *free* if

(F1) for $i = 1, 2$ there is no $e \in E(G)$ such that P_i has both ends in $V(\eta(e))$, and

(F2) whenever $e_1, e_2 \in E(G)$ are such that all the feet of the cross belong to $V(\eta(e_1)) \cup V(\eta(e_2))$, then $e_1 \neq e_2$ and e_1 and e_2 have no end in common.

The following is shown in [6].

(2.1) *Let G be an almost 4-connected planar graph on at least seven vertices, let H be a non-planar graph, and let $\eta : G \hookrightarrow H$ be a homeomorphic embedding. Then there exists a homeomorphic embedding $\eta' : G \hookrightarrow H$ such that $\eta(v) = \eta'(v)$ for every vertex $v \in V(G)$ of degree at least four and one of the following conditions holds:*

(i) there exists an η'-path in H such that both of its ends belong to $V(\eta'(C))$ for no peripheral cycle C in G,

(ii) there exists a free η'-cross, or

(iii) there exists a separation (X, Y) of H of order at most three such that $|\eta'(V(G)) \cap X - Y| \leq 1$ and $H|X$ does not a have a drawing in a disk with $X \cap Y$ drawn on the boundary of the disk.

If η is a homeomorphic embedding of G into H, an *η-bridge* is a connected subgraph B of H with $E(B) \cap E(\eta(G)) = \emptyset$, such that either

(i) $|E(B)| = 1, E(B) = \{e\}$ say, and both ends of e are in $V(\eta(G))$, or

(ii) for some component C of $H \backslash V(\eta(G)), E(B)$ consists of all edges of H with at least one end in $V(C)$.

It follows that every edge of H not in $\eta(G)$ belongs to a unique η-bridge. We say that a vertex v of H is an *attachment* of an η-bridge B if $v \in V(\eta(G)) \cap V(B)$.

Let η be a homeomorphic embedding of G into H. We say that an η-bridge B is *unstable* if there exists an edge $e \in E(G)$ such that $V(B) \cap V(\eta(G)) \subseteq V(\eta(e))$, and otherwise we say that it is *stable*. The following result is probably due to Tutte. A proof may be found in [3, Lemma 6.2.1] or [6] or elsewhere.

(2.2) *Let G be a graph, let H be a simple 3-connected graph, and let $\eta : G \hookrightarrow H$ be a homeomorphic embedding. Then there exists a homeomorphic embedding $\eta' : G \hookrightarrow H$ such that every η'-bridge is stable and $\eta(v) = \eta'(v)$ for every vertex $v \in V(G)$ of degree at least three.*

3. LARGE NON-PLANAR GRAPHS

We need the following minor strengthening of (1.4).

(3.1) *For every two integers $k, t \geq 4$ there is an integer N such that every 3-connected graph with at least N vertices either contains a subgraph isomorphic to a subdivision of one of A_k, O_k, M_k, $K_{4,k}$, and $K'_{4,k}$, or it has a separation (A, B) of order at most three such that $|A| \geq t$ and $|B| \geq t$.*

Proof. For $t = 5$ this is (1.4). For $t > 5$ the result follows by making obvious modifications to the proof of (1.4) in [4]. □

(3.2) *Let $k \geq 4$ be an integer, let H be a non-planar graph, and let $\eta : A_{2k+1} \hookrightarrow H$ be a homeomorphic embedding. Then one of the following holds.*

(i) There exist a homeomorphic embedding $\eta' : A_k \hookrightarrow H$ and an η'-path P in H such that η' maps the hubs of A_k to the same pair of vertices η maps the hubs of A_{2k+1} to, and the ends of P are the images of the hubs of A_k under η'.

(ii) There exist a homeomorphic embedding $\eta' : A_{2k+1} \to H$ and a separation (A, B) of H of order at most three such that $|\eta'(V(A_{2k+1})) \cap A - B| \leq 1$ and $H|A$ cannot be drawn in a disk with $A \cap B$ drawn in the boundary of the disk.

Proof. By (2.1) we may assume (by replacing η by a different homeomorphic embedding that maps the hubs of A_{2k+1} to the same pair of vertices of H as η) that η satisfies (i), (ii), or (iii) of (2.1). If it satisfies (iii), then the result holds, and so we may assume that η satisfies (2.1)(i) or (2.1)(ii).

Assume first that η satisfies (2.1)(i), and let P be the corresponding η-path. Let $v_0, v_0', v_1, v_2, \ldots, v_{4k+2}$ be as in the definition of A_{2k+1}. If P has one end in $V(\eta(v_0v_i)) - \{\eta(v_i)\}$ and the other in $V(\eta(v_0'v_j)) - \{\eta(v_j)\}$ for some i and j, then $A_{2k+1} \backslash \{v_0v_i, v_0'v_j\}$ has a subgraph A that is isomorphic to a subdivision of A_{2k-1}. Let η' be the restriction of η to A and let P' be the $\eta(v_0)\eta(v_0')$-path in the union of P, $\eta(v_0v_i)$, and $\eta(v_0'v_j)$. Then η' and P' satisfy (i).

Thus we may assume by symmetry that both ends of P are in $V(\eta(A_{2k+1} \backslash v_0)) - \{\eta(v_0')\}$. In fact, we may further assume by symmetry that both ends of P are in $V(\eta(A_{2k+1} \backslash \{v_0, v_1, v_2, \ldots, v_{2k}\})) - \{\eta(v_0')\}$. Since $P \cup \eta(A_{2k+1})$ is non-planar, there exist $i, j \in \{2k+1, 2k+2, \ldots, 4k+2\}$ with $|i - j| = 1$ such that P is vertex-disjoint from $\eta(Q)$, where Q is the path with vertex-set $\{v_0, v_i, v_j, v_0'\}$. Let $\eta'(x) = \eta(x)$ for all vertices and edges x of $A_{2k+1} \backslash \{v_{2k+1}, v_{2k+2}, \ldots, v_{4k+2}\}$. We define $\eta'(v_1v_{2k})$ to be the path in H with ends $\eta(v_1)$ and $\eta(v_{2k})$ consisting of P and two subpaths of $\eta(G) \backslash \{\eta(v_0), \eta(v_0'), \eta(v_2), \eta(v_i)\}$. Then $\eta' : A_k \hookrightarrow H$ and $P' = \eta(Q)$ satisfy (i).

The argument is similar when η satisfies (2.1)(ii). □

(3.3) *Let $k \geq 1$ be an integer, and let H be a non-planar graph such that there exists a homeomorphic embedding $\eta : O_{4k} \hookrightarrow H$. Then either H has a subgraph isomorphic to a subdivision of M_k, or there exist a homeomorphic embedding $\eta' : O_{4k} \hookrightarrow H$ and a separation (A, B) of H of order at most three such that $|\eta'(V(O_{4k})) \cap A - B| \leq 1$ and $H|A$ cannot be drawn in a disk with $A \cap B$ drawn in the boundary of the disk.*

Proof. The proof is similar to that of (3.2). We omit the details. □

Let us recall that B_k is the graph obtained from A_k by adding an edge joining its hubs. A graph G is *t-shallow* if for every separation (A, B) of order at most three, one of $G|A$, $G|B$ has fewer than t vertices and can be drawn in a disk with $A \cap B$ drawn on the boundary of the disk. The following is the main result of this section. It implies (1.5), because every almost 4-connected graph is 5-shallow.

(3.4) *For every two integers $k, t \geq 4$ there is an integer N such that every 3-connected t-shallow non-planar graph with at least N vertices contains a subgraph isomorphic to a subdivision of one of B_k, M_k, $K_{4,k}$, and $K'_{4,k}$.*

Proof. Let k, t be given. By replacing k by a larger integer we may assume that $8k \geq t+1$. Let N be the integer that satisfies (3.1) with k replaced by $4k$. We claim that N satisfies the conclusion of (3.4). To see this let G be a 3-connected t-shallow non-planar graph on at least N vertices. By (3.1) G has a subgraph isomorphic to a subdivision of one of A_{4k}, O_{4k}, M_{4k}, $K_{4,4k}$, and $K'_{4,4k}$. If G has a subgraph isomorphic to a subdivision of M_{4k}, $K_{4,4k}$, or $K'_{4,4k}$, then the result holds.

Assume now that there exists a homeomorphic embedding $\eta : A_{4k} \hookrightarrow G$. By (3.2) either G has a subgraph isomorphic to a subdivision of B_k, or there exists a separation (A, B) as in (3.2)(ii). In the former case the theorem holds, and so we may assume the latter. Since G is t-shallow we see that $|B| < t$, but all but possibly one vertex of $\eta(V(A_{4k}))$ belong to B, contrary to $8k \geq t + 1$. The argument when there exists a homeomorphic embedding $\eta : O_{4k} \hookrightarrow G$ is similar, using (3.3) instead. □

4. A LEMMA ABOUT PLANAR GRAPHS

Let G be a *plane* graph; that is, a graph drawn in the plane with no crossings. Then every cycle C bounds a disk in the plane, and we define $\text{ins}(C)$ to be the set of edges of G drawn in the open disk bounded by C. (By definition, an edge of a drawing does not include its ends.) The following will be a hypothesis common to several lemmas, and so we give it a name in order to avoid repetition.

(4.1) Hypothesis. Let G be a loopless plane graph drawn in the closed unit disk Δ, let x_1, x_2, x_3 be distinct vertices of G, and let them be the only vertices of G drawn in the boundary of Δ. Assume that there is no separation (A, B) of order at most two with $x_1, x_2, x_3 \in A$ and $B - A \neq \emptyset$.

The last assumption of (4.1) will be referred to as the *internal* 3*-connectivity of* G.

Assume (4.1), let C be a cycle in G with $\{x_1, x_2, x_3\} \not\subseteq V(C)$ and $\text{ins}(C) \neq \emptyset$. We say that C is *robust* if there exists an edge $f \in \text{ins}(C)$ such that for every $e \in E(C)$ the graph $G \backslash \{x_1, x_2, x_3\} \backslash e \backslash f$ has a component containing a neighbor of each of x_1, x_2, x_3. Let Z be the set of all vertices $v \in V(C)$ such that either $v \in \{x_1, x_2, x_3\}$ or v is incident with an edge not in $E(C) \cup \text{ins}(C)$. We say that C is *flexible* if $|Z| \leq 3$ and at least two vertices in $Z - \{x_1, x_2, , x_3\}$ are incident with exactly one edge not in $E(C) \cup \text{ins}(C)$. Our objective in this section is to prove that if G has sufficiently many vertices and satisfies Hypothesis (4.1), then it has a robust cycle or a flexible cycle.

(4.2) *Assume* (4.1)*. Then every cycle of* $G \backslash \{x_1, x_2, x_3\}$ *that does not bound a face is robust.*

Proof. Let C be a cycle of $G \backslash \{x_1, x_2, x_3\}$ that does not bound a face, and let $f \in \text{ins}(C)$. By the internal 3-connectivity of G there exist three internally disjoint paths from $\{x_1, x_2, x_3\}$ to $V(C)$, and hence $G \backslash e \backslash f$ has a component containing neighbors of all of x_1, x_2, x_3 for all $e \in E(C)$. Thus C is robust, as desired. □

Let us recall that a *block* is a graph with no cutvertices, and a *block of a graph* is a maximal subgraph that is a block. The *block graph* of a graph G is the graph whose vertices are all the blocks of G and all the cut vertices of G, with the obvious incidences. An *end-block* of a graph G is a block that has degree one in the block graph of G.

(4.3) *Assume* (4.1)*, and that* G *has no robust cycle. Then every two distinct cycles of* $G \backslash \{x_1, x_2, x_3\}$ *are edge-disjoint. Consequently, every block of* $G \backslash \{x_1, x_2, x_3\}$ *is a cycle or a complete graph on at most two vertices.*

Proof. This follows from (4.2), because otherwise some cycle of $G\backslash\{x_1, x_2, x_3\}$ is not facial. $\square$

(4.4) *Assume* (4.1)*, and assume that* G *has at least* 16 *vertices and no robust cycle. Let* B_1, B_2 *be two distinct end-blocks of* $G\backslash\{x_1, x_2, x_3\}$*. For* $i = 1, 2$ *let* v_i *be the unique cut vertex of* $G\backslash\{x_1, x_2, x_3\}$ *that belongs to* B_i*, and let* $N_i \subseteq \{x_1, x_2, x_3\}$ *be the set of vertices of* $\{x_1, x_2, x_3\}$ *that have a neighbor in* $B_i\backslash v_i$*. Then* $|N_1| = |N_2| = 2$ *and* $|N_1 \cap N_2| = 1$*.*

Proof. We first notice that N_1 and N_2 have at least two elements by the internal 3-connectivity of G. Thus it suffices to show that $|N_1 \cap N_2| \leq 1$. Let us assume for a contradiction that $x_1, x_2 \in N_1 \cap N_2$. The fact that G is drawn in a disk with x_1, x_2, x_3 on the boundary of the disk implies that either x_3 has no neighbor outside $B_1\backslash v_1$, or it has no neighbor outside $B_2\backslash v_2$, and hence from the symmetry we may assume the latter. But x_3 has at least one neighbor in $B_2\backslash v_2$ by the internal 3-connectivity of G. Since G has at least 16 vertices, it follows from (4.3) that $G\backslash\{x_1, x_2, x_3\}$ has at least seven vertices with at most two neighbors. Each of those vertices has a neighbor in $\{x_1, x_2, x_3\}$, and hence there is an index $i \in \{1, 2, 3\}$ such that x_i has at least three neighbors in $G\backslash\{x_1, x_2, x_3\}$. Furthermore, if B_2 has a unique edge, then i and the three neighbors of x_i can be chosen to be not in $B_2\backslash v_2$. Thus there is a cycle C of G containing x_i but no other x_j such that $\text{ins}(C)$ includes an edge f incident with x_i; and if B_2 has a unique edge, then C does not use that edge. Since x_1, x_2 and x_3 all have a neighbor in $B_2\backslash v_2$, it follows that C is robust, a contradiction. $\square$

(4.5) *Assume* (4.1)*, and assume that* G *has at least* 16 *vertices and no robust cycle. Then the block graph of* $G\backslash\{x_1, x_2, x_3\}$ *is a path.*

Proof. Suppose for a contradiction that the block graph of $G\backslash\{x_1, x_2, x_3\}$ is not a path. Then $G\backslash\{x_1, x_2, x_3\}$ has at least three end-blocks, say B_1, B_2, and B_3. For $i = 1, 2, 3$ let N_i be as in (4.4). By (4.4) we may assume that the blocks B_1, B_2, B_3 are numbered in such a way that $N_1 = \{x_2, x_3\}$, $N_2 = \{x_1, x_3\}$, and $N_3 = \{x_1, x_2\}$. Let C be a cycle containing an edge joining x_i to a vertex of N_j for all distinct integers $i, j \in \{1, 2, 3\}$, such that all other edges of C belong to $B_1 \cup B_2 \cup B_3$. Let T be a connected subgraph of $G\backslash\{x_1, x_2, x_3\}$ such that $V(T \cap C) = \{u_1, u_2, u_3\}$, where $u_i \in V(B_i)$. Then $x_1, u_3, x_2, u_1, x_3, u_2$ appear on C in the order listed. Since G has at least 16 vertices there exist an edge $f \in E(G)-E(T)-E(C)$ and index $i \in \{1, 2, 3\}$ such that $f \in \text{ins}(C')$, where C' is the unique cycle in $(C \cup T)\backslash u_i$. It follows that C' is robust, a contradiction. □

(4.6) *Assume* (4.1)*, and assume that G has at least* 130 *vertices. Then G has a robust cycle or a flexible cycle.*

Proof. Assume for a contradiction that G has neither a robust cycle nor a flexible cycle. Let $B := G\backslash\{x_1, x_2, x_3\}$, let $a_1b_1, a_2b_2, \ldots, a_tb_t$ be all the cut edges of B, and let $D_0, D_1, \ldots, D_t$ be all the components of $B\backslash\{a_1b_1, a_2b_2, \ldots, a_tb_t\}$. By (4.5) the numbering can be chosen so that $a_j \in V(D_{j-1})$ and $b_j \in V(D_j)$ for all $j = 1, 2, \ldots, t$. By (4.4) we may assume that x_1 and x_3 have a neighbor in D_0, and that x_2 and x_3 have a neighbor in D_t.

(1) *For $i \in \{1, 2, 3\}$ and $j \in \{0, 1, \ldots, t\}$ there are at most two edges with one end x_i and the other end in D_j.*

To prove (1) suppose for a contradiction that there are three edges with one end x_i and the other end in D_j. Then there exists a cycle C using two of those edges such that the third edge, say f, belongs to $\text{ins}(C)$ and $C\backslash x_i$ is a subgraph of D_j. If $0 < j < t$, then there exists a path P in $D_j\backslash E(C)$ with ends b_j and a_{j+1}. By considering the edge f and path P (when $0 < j < t$) we deduce that C is robust, a contradiction. This proves (1).

(2) *For $j = 0, 1, \ldots, t$ the graph D_j has at most* 18 *vertices.*

To prove (2) we first notice that the block graph of D_j is a path by (4.5). Since D_j is 2-edge-connected, each block of D_j is a cycle by (4.3). By the internal 3-connectivity of G no two consecutive blocks of D_j are both a cycle of length two, unless their shared vertex is adjacent to at least one of x_1, x_2, x_3. Since every vertex of D_j except possibly b_j (if $j > 0$) and a_{j+1} (if $j < t$) has at least three distinct neighbors by the internal 3-connectivity of G, the claim follows from (1). This proves (2).

(3) *There is at most one index* $j \in \{1, 2, \ldots, t-1\}$ *such that the graph* D_j *includes a neighbor of* x_1.

To prove (3) we suppose for a contradiction that there exist two such indices j, j' with $0 < j' < j < t$. Since x_1 has also a neighbor in B_0, there exists a cycle C through the vertex x_1 with $V(C) \subseteq V(D_0 \cup D_1 \cup \cdots \cup D_j) \cup \{x_1\}$ and such that some edge f incident with x_1 belongs to $\text{ins}(C)$. Since x_2 and x_3 have a neighbor in D_t, and D_j is 2-edge-connected, it follows that C is robust, a contradiction. This proves (3).

From the symmetry between x_1 and x_2 we deduce

(4) *There is at most one index* $j \in \{1, 2, \ldots, t-1\}$ *such that the graph* D_j *includes a neighbor of* x_2.

We are now ready to complete the proof of the lemma. Since G has at least 130 vertices, it follows from (2) that $t \geq 8$, and hence by (3) and (4) there exists an integer $j \in \{1, 2, \ldots, t-2\}$ such that both D_j and D_{j+1} include no neighbor of x_1 or x_2. Thus each of them includes a neighbor of x_3 by the internal 3-connectivity of G, and hence there exists a cycle C with $V(C) \subseteq V(D_j \cup D_{j+1}) \cup \{x_3\}$, $x_3, b_j, a_{j+2} \in V(C)$, and such that $a_j b_j$ is the only edge of G incident with b_j that does not belong to $E(C) \cup \text{ins}(C)$, and $a_{j+2} b_{j+2}$ is the only such edge incident with a_{j+2}. By considering the set $Z = \{a_{j+2}, b_j, x_3\}$ we deduce that C is flexible, as desired. □

We also need the following mild strengthening of (4.6). If C is a subgraph of a graph G, then by a *C-bridge* we mean an η-bridge, where $\eta : C \hookrightarrow G$ is the homeomorphic embedding that maps every vertex and edge of C onto itself.

(4.7) *Assume* (4.1), *and let* C *be a robust or flexible cycle in* G *with* $\text{ins}(C)$ *maximal. Then for every* C*-bridge* B *of* G *either* $E(B) \subseteq \text{ins}(C)$, *or at least one of* x_1, x_2, x_3 *belongs to* $V(B) - V(C)$.

Proof. Assume first that C is robust, let $f \in \text{ins}(C)$ be as in the definition of robust, and suppose for a contradiction that B is a C-bridge that satisfies neither conclusion of the lemma. By the internal 3-connectivity of G the bridge B includes a path P of

$G\backslash\{x_1, x_2, x_3\}$ with both ends on C, and otherwise disjoint from it. The graph $C \cup P$ includes a cycle $C' \neq C$ with ins(C) properly contained in ins(C'). Since every edge of P belongs to a cycle of $G\backslash f$ it follows that C' is robust, contrary to the maximality of C.

The argument when C is flexible is similar. In that case the set Z from the definition of flexible is the same for C and C'. □

5. LARGE GRAPHS WITH CROSSING NUMBER AT LEAST TWO

Recall that a graph G is *X-minimal* if

(i) G has crossing number at least two,

(ii) $G\backslash e$ has crossing number at most one for every edge $e \in E(G)$,

(iii) G has no vertices of degree two, and

(iv) G does not have a vertex of degree four incident with two pairs of parallel edges.

(5.1) *Every X-minimal graph on at least* 17 *vertices is* 3*-connected.*

Proof. Let G be an X-minimal graph on at least 17 vertices, and suppose for a contradiction that it is not 3-connected. Thus it has a nontrivial separation (A, B) of order at most two. We may assume that (A, B) has the minimum order among all nontrivial separations of G.

Assume first that the order of (A, B) is at most one. Both $G|A$ and $G|B$ have crossing number at most one by the X-minimality of G. They are both non-planar, for otherwise G itself would have crossing number at most one. Thus both $G|A$ and $G|B$ have subgraphs isomorphic to subdivisions of K_5 or $K_{3,3}$ by Kuratowski's theorem. Now the X-minimality of G implies that $G|A$ and $G|B$ have at most seven vertices, contrary to the fact that G has at least 17 vertices.

We may therefore assume that G is 2-connected and that the order of (A, B) is two. Let $A \cap B = \{u, v\}$. Let G_1 be the graph obtained from $G|A$ as follows. If $G|B$ has two edge-disjoint paths with ends u and v, then G_1 is obtained from $G|A$ by adding two edges with ends u and v; otherwise G_1 is obtained from $G|A$ by adding one edge with ends u and v. We define G_2 analogously (with the roles of A and B interchanged).

(1) *The graphs G_1 and G_2 have crossing number at most one.*

To prove (1) it suffices to argue for G_1. Assume first that $G|B$ does not have two edge-disjoint paths with ends u and v. Since $G|B$ has a path with ends u and v by the 2-connectivity of G, we deduce that a subdivision of G_1 is isomorphic to a subgraph of G, and that the containment is proper. Thus G_1 has crossing number at most one by the X-minimality of G. We may therefore assume that $G|B$ has two edge-disjoint paths P_1 and P_2 with ends u and v. Then by choosing the paths with $P_1 \cup P_2$ minimum it can be arranged that both P_1 and P_2 pass through the vertices of $V(P_1) \cap V(P_2)$ in the same order. The graph $(G|A) \cup P_1 \cup P_2$ is a proper subgraph of G by the X-minimality of G, and hence has crossing number at most one. It follows that G_1 has crossing number at most one. This proves (1).

(2) *The graphs G_1 and G_2 are non-planar.*

To prove (2) it again suffices to argue for G_1. Suppose for a contradiction that G_1 is planar. By (1) there exists a planar drawing of G_2 with at most one crossing. If none of the edges of $E(G_2)-E(G|B)$ is involved in the crossing, then this drawing and a planar drawing of G_1 can be combined to produce a planar drawing of G with at most one crossing. Thus we may assume that an edge of $E(G_2) - E(G|B)$ is crossed by another. Therefore we may assume that $E(G_2) - E(G|B)$ consists of a unique edge, say e, and hence, by construction, G_1 does not have two edge-disjoint paths with ends u and v. By Menger's theorem G_1 has an edge f such that $G_1 \backslash f$ has no path between u and v. Using the drawings of G_1 and G_2 it is now possible to obtain a drawing of G, where e and f are the only two edges that cross, contrary to the fact that G has crossing number at least two. This proves (2).

From (2) and Kuratowski's theorem it follows that for $i = 1, 2$ the graph G_i has a subgraph H_i isomorphic to a subdivision of K_5 or $K_{3,3}$. But $H_1 \cup H_2$ has crossing number at least two, and hence the X-minimality of G implies that both G_1 and G_2 have at most eight vertices, contrary to the fact that G has at least 17 vertices. This proves that G is 3-connected. $\square$

(5.2) *Let G be a graph, let C be a cycle in G, and let $B_0, B_1, \ldots, B_k$ be the C-bridges of G such that the graph $C \cup B_1 \cup B_2 \cup \ldots \cup B_k$ has a planar drawing with no crossings in which C bounds a face. Let H denote the graph $C \cup B_0$, and let $f \in E(B_1)$. Assume further that either $G\backslash e\backslash f$ is non-planar for every $e \in E(C)$, or that the C-bridge B_0 has exactly three attachments, two of which have degree three in H. If $G\backslash f$ has crossing number at most one, then so does G.*

Proof. Let Γ be a drawing of $G\backslash f$ with at most one crossing. Our first objective is to modify Γ to produce a drawing of H with at most one crossing such that no edge of C is crossed by another edge. If no edge of C is crossed by another edge in the drawing Γ, then its restriction to H is as desired. Thus we may assume that an edge $e \in E(C)$ is crossed by another edge e' in Γ. It follows that $G\backslash e\backslash f$ is planar, and hence, by hypothesis, the C-bridge B_0 has exactly three attachments, say v_1, v_2, v_3, such that v_1 and v_2 have degree three in H. If $e' \notin E(B_0)$, then it is easy to convert Γ to a desired drawing of H. Thus we may assume that $e' \in E(B_0)$. It follows that $B_0\backslash e'$ has two components, say J_1 and J_2, such that J_1 is drawn in the closed disk bounded by C and J_2 is drawn in the closure of the other face of C. Using the fact that v_1 and v_2 have degree three in H it is now easy to draw J_2 in the closed disk bounded by C so as to obtain a desired drawing of H. This proves our claim that H has a drawing with at most one crossing such that no edge of C is crossed by another edge in that drawing. Thus C bounds a face. By hypothesis it is possible to draw $B_1 \cup B_2 \cup \ldots \cup B_k$ in that face, showing that G has crossing number at most one, as desired. □

Let G be a graph, let u, u_1, u_2, u_3 be distinct vertices of G, and let Q_1, Q_2, Q_3 be three paths in G such that Q_i has ends u and u_i and such that Q_1, Q_2, Q_3 are disjoint except for u. We say that $Q_1 \cup Q_2 \cup Q_3$ is a *triad* in G, and that the vertices u_1, u_2, u_3 are its *feet*. Let G be a graph, and let P_1, P_2, P_3 be three pairwise disjoint paths in G, where P_i has ends u_i and v_i. Let T_1 and T_2 be two triads with feet v_1, v_2, v_3 such that the graphs $P_1 \cup P_2 \cup P_3$, T_1, T_2 are pairwise disjoint, except for v_1, v_2, v_3. In those circumstances we say that $P_1 \cup P_2 \cup P_3 \cup T_1 \cup T_2$ is a *tripod*, and that the vertices u_1, u_2, u_3 are its *feet*. We need the following result of [5].

(5.3) *Let G be a graph, and let u_1, u_2, u_3 be three vertices of G such that there is no separation (A, B) of G of order at most two with $u_1, u_2, u_3 \in A$ and $B - A \neq \emptyset$. If G has no planar drawing with the vertices u_1, u_2, u_3 incident with the unbounded face, then G has a tripod with feet u_1, u_2, u_3.*

(5.4) *Let G be an X-minimal graph on at least 17 vertices, and let (A, B) be a separation in G of order three. Then one of $G|A$, $G|B$ has a planar drawing with the vertices $A \cap B$ drawn on the boundary of the unbounded face.*

Proof. Suppose for a contradiction that the conclusion does not hold. By (5.1) the graph G is 3-connected. By (5.3) $G|A$ has a tripod T_1 with feet $A \cap B$, and $G|B$ has a tripod T_2 with feet $A \cap B$. The graph $T_1 \cup T_2$ has crossing number at least two, as is easily seen. Thus $G = T_1 \cup T_2$ by the X-minimality of G. Moreover, the X-minimality of G implies that G has at most 10 vertices, a contradiction. □

We are now ready to prove (1.8), which we restate.

(5.5) *Every X-minimal graph on at least 17 vertices is 130-shallow.*

Proof. Let G be an X-minimal graph on at least 17 vertices, and let (A, B) be a separation in G of order at most three with $A - B \neq \emptyset \neq B - A$. By (5.1) the separation (A, B) has order exactly three. By (5.4) we may assume that $G|B$ is drawn in a disk with the vertices of $A \cap B$ drawn in the boundary of the disk. It follows that $G|B$ satisfies (4.1), where $A \cap B = \{x_1, x_2, x_3\}$. We may assume for a contradiction that $|B| \geq 130$. By (4.6) applied to the graph $G|B$ we deduce that $G|B$ has a cycle C that is robust or flexible. By (4.7) we may choose C so that exactly one C-bridge B_0 of G satisfies $E(B_0) \not\subseteq \mathrm{ins}(C)$. If C is robust, then let f be as in the definition of robust; otherwise let $f \in \mathrm{ins}(C)$ be arbitrary. If C is flexible, then the bridge B_0 has exactly three attachments, and two of them have degree three in $C \cup B_0$. Now let C be robust, and let $e \in E(C)$. We claim that $G \backslash e \backslash f$ is not planar. To prove this we first notice that $G|A$ cannot be drawn in a disk with $A \cap B$ drawn in the boundary of the disk, because $G|B$ can be so drawn and G is not planar. By (5.3) the graph $G|A$ has a tripod T with feet $A \cap B$. Since C is robust the graph $(G|B)\backslash e\backslash f$ has a connected subgraph R that includes $A \cap B$. It follows that $T \cup R$ is a subdivision of $K_{3,3}$, which proves our claim that $G\backslash e\backslash f$ is not planar. The graph $G\backslash f$ has crossing number at most one by the X-minimality of G, and hence by (5.2) the graph G has crossing number at most one, a contradiction. ☐

(5.6) *Let G be the graph obtained from A_4 by subdividing the edges v_1v_2 and v_5v_6, and joining the new vertices by an edge. Then G has crossing number at least two.*

Proof. This follows from the fact that the new edge is the only edge $e \in E(G)$ such that $G\backslash e$ is planar. ☐

(5.7) *No X-minimal graph has a subgraph isomorphic to a subdivision of B_{65}.*

Proof. Let H be an X-minimal graph, and suppose for a contradiction that it has a subgraph isomorphic to a subdivision of B_{65}. Let $\eta : B_{65} \hookrightarrow H$ be a homeomorphic embedding, and let η_0 be the restriction of η to A_{65}. Let e_0 be the edge of B_{65} joining the two hubs. Let J be the union of $\eta_0(A_{65})$ and all η_0-bridges except the one that includes $\eta(e_0)$. We claim that J is planar. To prove this claim suppose for a contradiction that it is not. From (3.2) applied to A_{65}, J, and η_0 we deduce that (i) or (ii) of (3.2) holds. If (i) holds, then we conclude that the graph obtained from B_{32} by adding an edge parallel to e_0 is isomorphic to a subdivision of H. That is a contradiction, because said graph is not X-minimal, as is easily seen. Thus we may assume that H has a separation (A, B) as in (3.2)(ii). But $|B| \geq |V(B_{65})| - 1 \geq 130$, and $H|A$ does not have a planar drawing with the vertices in $A \cap B$ incident with the infinite face, contrary to (5.5). This proves our claim that J is planar. Thus we may regard J as a graph drawn in the sphere.

Let the vertices of A_{65} be numbered as in the definition of A_{65}. Assume first that $\eta(e_0)$ has only one edge. Let C_0 be a cycle in J with $v_0 \notin V(C_0)$ such that the open disk bounded by C_0 that includes v_0 is as small as possible. Let C_0' be defined analogously, with v_0' replacing v_0. The cycles C_0, C_0' are edge-disjoint, for otherwise H has crossing number at most one. But now it follows that the graph obtained from H by deleting an edge of $\eta(v_0v_1)$ has crossing number at least two, contrary to the X-minimality of H. This completes the case when $\eta(e_0)$ has only one edge.

We may therefore assume that $\eta(e_0)$ has at least one internal vertex. Let us say that an η-bridge of H is *solid* if either it has at least two edges, or it has a unique edge and that edge is not parallel to an edge of $\eta(B_{65})$. By (2.2) we may assume that every solid η-bridge is stable. Let us say that a vertex $v \in V(\eta_0(A_{65})) - \{\eta_0(v_0), \eta_0(v_0')\}$ is *exposed* if there exists an η-path between an internal vertex of $\eta(e_0)$ and v. It follows from (5.1) that there exists at least one exposed vertex. For an integer $i \in \{1, 3, \ldots, 129\}$ let C_i denote the cycle of A_{65} with vertex-set $\{v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}, v_0\}$ (index arithmetic modulo 130), and let F_i be the set of edges of A_{65} with at least one end in $V(C_i)$. From (5.6) we deduce that there exists an integer i such that $\eta(e)$ includes an exposed vertex for no $e \in F_i$. Let $J_0, J_1, \ldots, J_k$ be all the $\eta(C_i)$-bridges of H, where J_0 includes v_0'. Then J_0 includes $\eta(e_0)$, and hence $J_1, J_2, \ldots, J_k$ are also $\eta_0(C_i)$-bridges of J. Since every solid η-bridge is stable, it follows that $J_1, J_2, \ldots, J_k$, when regarded as $\eta_0(C_i)$-bridges of J, are drawn in the closed disk Δ bounded by $\eta_0(C_i)$ that does not include v_0'; hence $\eta_0(C_i) \cup J_1 \cup J_2 \cup \cdots \cup J_k$ has a planar drawing with no crossings in which $\eta_0(C_i)$ bounds a face. Since in the planar drawing of J the path $\eta(v_0v_{i+2})$ is drawn in Δ we deduce that $k \geq 1$. Thus we may select $f \in E(J_1)$. Since there exists an exposed vertex, but none in $\eta(e)$ for any $e \in F_i$, it follows that $H\backslash e\backslash f$ is non-planar for every edge $e \in E(C_i)$. The graph $H\backslash f$ has crossing number at most one by the X-minimality of G, contrary to (5.2). □

We are finally ready to prove (1.7), which we restate.

(5.8) *For every integer k there exists an integer N such that every X-minimal graph on at least N vertices has a subgraph isomorpic to a subdivision of M_k.*

Proof. We may assume that $k \geq 65$. Let N be such that (3.4) holds for k and $t := 130$, and let G be an X-minimal graph on at least N vertices. By (5.5) the graph G is 130-shallow. By (3.4) it has a subgraph isomorphic to a subdivision of one of B_k, M_k, $K_{4,k}$, and $K_{4,k}'$. But G clearly has no subgraph isomorphic to a subdivision of $K_{4,k}$ or $K_{4,k}'$, and it has no subgraph isomorphic to a subdivision of B_k by (5.7), because $k \geq 65$. Thus G has a subgraph isomorphic to a subdivision of M_k, as desired. $\square$

References

1. D. Bokal, R. B. Richter and G. Salazar, Characterization of 2-crossing-critical graphs I: Low connectivity or no V_{2n} minor, manuscript, October 17, 2009.
2. G. Ding, B. Oporowski, R. Thomas and D. Vertigan, Large 4-connected nonplanar graphs, manuscript, August 1999.
3. B. Mohar and C. Thomassen, Graphs on surfaces, Johns Hopkins University Press, Baltimore, MD, 2001.
4. B. Oporowski, J. Oxley and R. Thomas, Typical subgraphs of 3- and 4-connected graphs, *J. Combin. Theory Ser. B* **57** (1993), 239–257.
5. N. Robertson and P. D. Seymour, Graph Minors IX. Disjoint crossed paths, *J. Combin. Theory Ser. B* **49** (1990), 40–77.
6. N. Robertson, P. D. Seymour and R. Thomas, Non-planar extensions of planar graphs, manuscript.