Odd $K_{3,3}$ subdivisions in bipartite graphs[1]

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Abstract

We prove that every internally 4-connected non-planar bipartite graph has an odd $K_{3,3}$ subdivision; that is, a subgraph obtained from $K_{3,3}$ by replacing its edges by internally disjoint odd paths with the same ends. The proof gives rise to a polynomial-time algorithm to find such a subdivision. (A bipartite graph G is *internally* 4*-connected* if it is 3-connected, has at least five vertices, and there is no partition (A, B, C) of $V(G)$ such that $|A|, |B| \geq 2$, $|C| = 3$ and G has no edge with one end in A and the other in B.)

1 Introduction

All graphs in this paper are finite and simple. Kuratowski's theorem [5] gives a characterization of planar graphs as those graphs that have no subgraph isomorphic to a subdivision of K_5 or $K_{3,3}$. Both K_5 and $K_{3,3}$ are necessary in the statement, but one can argue that $K_{3,3}$ is the more important of the two. It is easy to reduce planarity testing to 3-connected graphs, and for 3-connected graphs subdivisions of K_5 are not needed, in the following sense.

Theorem 1. *A* 3*-connected graph is not planar if and only if either it is isomorphic to* K_5 *or it has a subgraph isomorphic to a subdivision of* $K_{3,3}$*.*

Theorem 1 is well-known and can easily be derived from Kuratowski's theorem. Since we will be concerned with subdivisions of $K_{3,3}$, we make the following definition.

Definition 2. Let G be a graph and H a subgraph of G isomorphic to a subdivision of $K_{3,3}$. Let $v_1, v_2, \ldots, v_6$ be the degree three vertices of H and for $i = 1, 2, 3$ and $j = 4, 5, 6$ let P_{ij} be the paths in H between v_i and v_j. We then refer to H as a *hex* or a *hex of* G, the vertices v_i as the *feet* of H, and the paths P_{ij} as the *segments* of H. A segment is *odd* if it has an odd number of edges, and *even* otherwise. A hex H is *odd* if every segment of H is odd.

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While working on Pfaffian orientations (described later) we were led to the following variation of Theorem 1. If G is 3-connected, bipartite and non-planar, must it have an odd hex? Unfortunately, that is not true, as the graph depicted in Figure 1 shows, but it is true if we increase the connectivity slightly. We say that a bipartite graph G is *internally* 4-*connected* if it is 3-connected, has at least five vertices, and there is no partition (A, B, C) of $V(G)$ such that $|A|, |B| \geq 2$, $|C| = 3$ and G has no edge with one end in A and the other in B. The following is our main result.

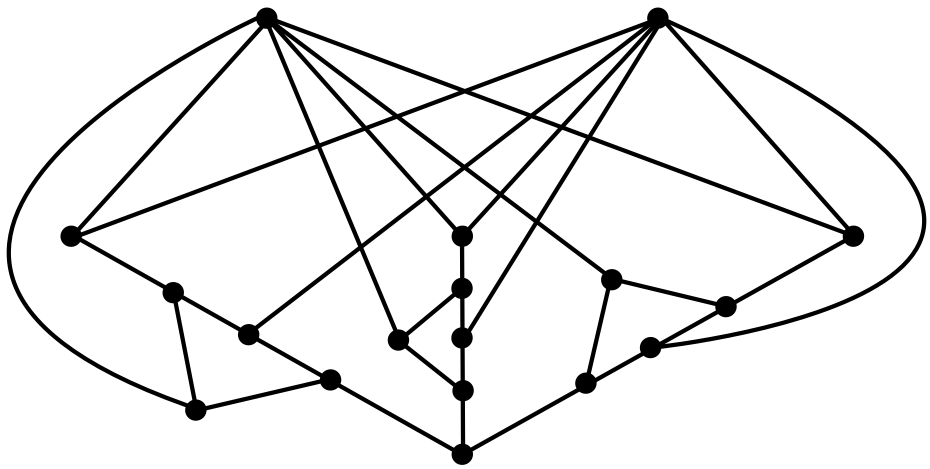

Figure 1: A graph showing that Theorem 3 does not extend to 3-connected graphs

Theorem 3. *Every internally* 4*-connected bipartite non-planar graph has an odd hex.*

Let us now explain the notion of a Pfaffian orientation, and how it led us to the above theorem. Let H be a subgraph of a graph G. We say that H is a *central subgraph* if $G \backslash V(H)$ has a perfect matching. (We use $\backslash$ for deletion and $-$ for set-theoretic difference.) An orientation D of a graph G is called *Pfaffian* if every even central cycle has an odd number of edges directed in either direction of the cycle. A graph is called *Pfaffian* if it admits a Pfaffian orientation. Pfaffian orientations have been introduced by Kasteleyn [2, 3, 4], who demonstrated that one can enumerate perfect matchings in a Pfaffian graph in polynomial time. That is significant, because counting the number of perfect matchings is #P-complete [10] in general graphs. It is not known whether there is a polynomial-time algorithm to test if a graph is Pfaffian, but we shall see below that there is one for bipartite graphs. The latter is noteworthy, because it implies polynomial-time algorithms for other problems of interest, such as Pólya's permanent problem, the even directed cycle problem, the sign non-singular matrix problem, and others. A survey of Pfaffian orientations may be found in [9]. The following is a result of Little [7].

Theorem 4. *A bipartite graph is Pfaffian if and only if it does not have an odd hex as a central subgraph.*

While Little's theorem is elegant, it is not clear how to use it to decide in polynomial time whether a bipartite graph is Pfaffian. A polynomial-time decision algorithm was obtained in [8] using a different method—it is based on a structure theorem proved in [8] and independently in [6]. However, both proofs of the structure theorem are fairly long.

We were wondering whether a simpler-to-justify algorithm may be obtained using Theorem 4, as follows. First a definition. A bipartite graph G is a *brace* if G is connected, has at least five vertices and every matching of size at most two is a subset of a perfect matching. It is easy to see that the decision problem whether a bipartite graph is Pfaffian can be reduced to braces, and that every brace is internally 4-connected. So let G be a brace. By Theorem 4 we want to test whether G has an odd hex as a central subgraph. To that end we may assume that G has a hex, for otherwise G is Pfaffian. In fact, we can find a hex in linear time using one of the linear-time planarity algorithms. The next step is to decide whether G has an odd hex. It follows from [8, Theorem (1.5)] that if a brace has a hex, then it has an odd hex, but it occurred to us that this should be true more generally than for braces, and that is how we were led to Theorem 3. The next step in our program is, given an odd hex in a brace G, decide whether there is an odd hex that is a central subgraph. We were able to do that, and will report on it elsewhere.

The paper is organized as follows. In the next section we prove two lemmas, and in Section 3 we prove Theorem 3. We start with an arbitrary hex, and gradually increase the number of odd segments in it.

2 Lemmas

In this section we prove two lemmas that we will need for the proof of Theorem 3.

Lemma 5. *Let G be an internally 4-connected bipartite graph with bipartition (A, B). Let $a, v \in A$ and $b, c \in B$ with paths $P_1 = v \dots a, P_2 = v \dots b, P_3 = v \dots c$ vertex disjoint except for v. Let $X \subseteq V(G)$ with $|X| \geq 2$ be disjoint from $V(P_1) \cup V(P_2) \cup V(P_3)$. Then at least one of the following holds:*

(1) *There exist $v' \in A, u \in B, x \in X$ and paths $P'_1 = v' \dots a, P'_2 = v' \dots b, P'_3 = v' \dots c, P'_4 = u \dots x$ such that $u \in V(P'_1)$ and all of the P'_i are vertex disjoint and are disjoint from X except that $v' \in V(P'_1) \cap V(P'_2) \cap V(P'_3), u \in V(P'_1) \cap V(P'_4)$ and $x \in X \cap V(P'_4)$.*

(2) *There exists $v', s \in A, u, t \in B, x \in X$ and paths $P'_1 = v' \dots a, P'_2 = v' \dots t \dots s \dots b, P'_3 = v' \dots c, P'_4 = u \dots s, P'_5 = t \dots x$ such that $u \in V(P'_1)$ and all of the P_i are vertex disjoint and are disjoint from X except that $v' \in V(P'_1) \cap V(P'_2) \cap V(P'_3), u \in V(P'_1) \cap V(P'_4), s \in V(P'_2) \cap V(P'_4), t \in V(P'_2) \cap V(P'_5), x \in V(P'_5) \cap X$.*

Definition 6. We will refer to the paths P'_1, P'_2, and P'_3 as the *replacement paths* and the paths P'_4 and, when appropriate, P'_5 as the *new paths*. In the forthcoming arguments we will apply either the induction hypothesis or Lemma 5 to various carefully selected paths R_1, R_2, R_3 to obtain replacement paths R'_1, R'_2, R'_3 and new paths R'_4 and, when appropriate,

R_5'. However, we will be able to assume that $R_1' = R_1, R_2' = R_2$ and $R_3' = R_3$ which will simplify our notation. We will refer to this assumption as assuming that the *replacement paths do not change.*

Proof of Lemma 5. Let the paths P_1, P_2, P_3 be fixed. By an *augmenting sequence* we mean a sequence of paths $Q_1, \ldots, Q_k$, where the ends of Q_i are v_{2i-1} and v_{2i}, $v_{2k} \in X, v_1 \in V(P_1) - \{a, v\}$, each other v_i is in $P_j \backslash v$ for some $j \in \{1, 2, 3\}$, and all the Q_i are vertex disjoint from one another and disjoint from the P_i and X except for their ends. Further, for $j > 1$ odd, v_j and v_{j-1} are distinct and both lie on the same P_i and v_j lies between v and v_{j-1} on P_i. If $v_i, v_j \in V(P_l)$ and $i < j-1$, then v, v_i, v_j appear in P_l in the order listed (possibly $v_i = v_j$). We refer to each Q_i as an *augmentation*. The *length* of an augmenting sequence is the number of augmentations it has. We define the *index* of the augmenting sequence $Q_1, Q_2, \ldots, Q_k$ to be the smallest integer i such that either i is odd and $v_i \in A$, or i is even and $v_i \in B$, or $i = 2k+1$.

We proceed by induction on the size of $V(G) - X$. Since G is internally 4-connected it follows by the standard "augmenting path" argument from network flow theory or from Lemmas 3.3.2 and 3.3.3 in [1] that there exists an augmenting sequence.

Choose the vertex v, paths P_1, P_2, P_3, and an augmenting sequence $S = (Q_1, \ldots, Q_k)$ such that the length of S is as small as possible, and, subject to that, the index of S is as large as possible. Let $v_1, v_2, \ldots, v_{2k}$ be the ends of the paths Q_i, numbered as above. Then it follows that for $j = 2, 4, \ldots, 2k-2$ the vertex v_j lies on a different path P_i than v_{j-1}. Note that this lemma is equivalent to showing that the length of S is at most 2 and that the index of S is at least twice the length of S.

Suppose first that the length of S is 1. Then we may assume that the index of S is 1, so $v_1 \in A$. Let $X' = X \cup V(v_1P_1a \cup Q_1 \backslash v_1)$. Then apply the induction hypothesis to the paths vP_1v_1, P_2, P_3 and set X'. We may assume that the replacement paths do not change. Suppose we have outcome (1). Thus there exists a path P_4 with ends $u \in B \cap V(vP_1v_1)$ and $x \in X'$, disjoint from $V(P_1 \cup P_2 \cup P_3) \cup X'$, except for its ends. If $x \in X$, then this is exactly outcome (1) in the original situation. If $x \in V(Q_1)$, then take $P_4' = P_4 \cup xQ_1v_2$ to find outcome (1) in the original situation. If $x \in V(v_1P_1a) - \{v_1\}$, then take $P_1' = vP_1u \cup P_4' \cup xP_1a$, $P_4' = uP_1v_1 \cup Q_1$ to again have outcome (1). So we must have outcome (2), and so there exist paths P_4, P_5 as stated in (2). Again, if $x \in X$, this is exactly outcome (2), and if $x \in V(Q_1)$, then taking $P_5' = P_5 \cup xQ_1v_2$ again gives outcome (2). So we may assume $x \in V(v_1P_1a)$. We then take $v' = v$, $P_1' = vP_2tP_5xP_1a$, $P_2' = vP_1uP_4sP_2b$, $P_3' = vP_3c, P_4' = tP_2s, P_5' = uP_1v_1Q_1v_2$, which is an instance of outcome (2).

So we may assume that the length of S is at least 2. Suppose that the index of S is 1, so $v_1 \in A$. Without loss of generality, we may assume that $v_2 \in V(P_2)$. Let $u \in B$ lie on P_1' between v_1 and a. Since $\{a, v_1\}$ is not a 2-separation in G, we can apply Menger's Theorem to find three paths from u, one to a, one to v_1 and one to $V(X) \cup V(Q_1) \cup V(Q_2) \cup V(vP_1v_1) \cup V(P_2) \cup V(P_3)$ labeled R_1, R_2, R_3 respectively. We replace v_1P_1a by $R_1 \cup R_2$ and simply refer to R_3 as R. Let the ends of R be u and r. If $r \in X$, then we have an augmenting sequence of length 1 contrary to the choice of S. If $r \in V(Q_i)$, then we have found an augmenting sequence of at most the same length as S, but with index at least 2. If $r \in V(P_1)$, then we take $P_1' = vP_1rRuP_1a$ and $Q_1' = uP_1v_1Q_1v_2$, which gives an augmenting sequence of the same length but with higher index. If r is on P_2 between v_3 and b with $r \neq v_3$, then taking $Q_1' = R$ is immediately an augmenting sequence with the same length and higher index. If r is on P_2' between v_3 and v', then let $v' = v_1$, $P_1' = v_1P_1a$, $P_2' = v_1Q_1v_2P_2b$, $P_3' = v_1P_1vP_3c$, and $Q_1 = uRrP_2v_3Q_2v_4$ which gives an augmenting sequence of shorter length. Similarly, if r is on P_3', let $v' = v_1$, $P_1' = v_1P_1a$, $P_2' = v_1Q_1v_2P_2b$, $P_3' = v_1P_1vP_3c$, and $Q_1' = R$, $Q_2' = vP_2v_3Q_2v_4$ which is an augmenting sequence of the same length but higher index.

So we may assume that the index of S is at least 2. Suppose the index is exactly 2, so $v_2 \in B$. Then apply the induction hypothesis to the paths Q_1, v_1P_1a and v_1P_1v and set $X' := X \cup V(P_2) \cup V(P_3) \cup V(Q_2) \cup V(Q_3) \cup \cdots \cup V(Q_k) - \{v, v_2\}$. We assume, since we may, that the replacement paths do not change. Suppose we have outcome (1). This gives a vertex $u \in A$ on Q_1 and x in X' with a path P_4 between them. If $x \in X$, then take $u' = v_1$, $P_4' = v_1Q_1uP_4x$ to get outcome (1). If $x \in Q_i$ for $i > i$, then take $Q_1' = v_1Q_1uP_4xQ_i$ which gives the shorter augmenting sequence $Q_1', Q_{i+1}, Q_{i+2}, \ldots, Q_k$. If x is on P_2 between v_2 and b, then take $P_2' = vP_1v_2Q_1uP_4xP_2b$, $v_2' = u$, and $Q_1' = v_1Q_1u$ which gives an augmenting sequence of higher index. If x is on P_2 between v and v_2, then we take $P_2' = vP_2xP_4uQ_1v_2P_2b$, $v_3' = x$, and $Q_2' = xP_2v_3Q_2v_4$ if $v_3 \in v_2P_2x$ and $Q_2' = Q_2$ if $v_3 \in xP_2v$ which gives an augmenting sequence with higher index. Finally, if x is on P_3, then take $v' = u, P_1' = uQ_1v_1P_1a, P_2' = uQ_1v_2P_2b, P_3' = uP_4xP_3c$, and $Q_1' = v_1P_1vP_2v_3Q_2v_4$ which gives a shorter augmenting sequence.

So instead we have outcome (2) and we use the notation for u, s, t, x, P_4, P_5 as listed in the outcome. It follows that $P_1 = vP_1sP_1tP_1v_1P_1a$. If $x \in X$, then taking $P_1' = vP_1sP_4uQ_1v_1P_1a, P_2' = P_2, P_3' = P_3, P_4' = sP_1tP_5x$ gives outcome (1). If $x \in V(P_2')$ between v_2 and b, we take $v' = t, P_1' = tP_1a, P_2' = tP_5xP_2b, P_3' = tP_1vP_3c, Q_1' = v_1Q_1v_2P_2v_3Q_2v_4$ and then $Q_1', Q_3, Q_4, ..., Q_k$ is an augmenting sequence of length $k-1$, a contradiction. If x is on P_2' between v and v_2, take $v' = u, P_1' = uQ_1v_1P_1a, P_2' = uQ_1v_2P_2b, P_3' = uP_4sP_1vP_3c, Q_1' = v_1P_1tP_5xP_2v_3Q_2v_4$ which gives a shorter augmenting sequence. If $x \in V(Q_i)$ with $i > 1$, then taking $Q_1' = tP_5xQ_iv_{2i}$ gives the augmenting sequence $Q_1', Q_{i+1}, Q_{i+2}, ..., Q_k$ which contradicts the choice of S. Finally, if x is on P_3', take $v' = t, v_2' = u, v_3' = s, P_1' = tP_1a, P_2' = tP_1sP_4uQ_1v_2P_2b, P_3' = tP_5xP_3c, Q_1' = v_1Q_1u$, and $Q_2' = sP_1vP_2v_3Q_2$ to get an augmenting sequence of the same length and higher index.

So we may assume the index of S is at least 3. Suppose the index is exactly 3, so $v_3 \in A$. Note that v_2 and v' are completely symmetric with respect to this augmenting sequence (up to v_3). We apply induction to the paths $v_2P_2v_3, Q_1, v_2P_2b$ and set $X' := X \cup V(P_1 \cup vP_2v_3 \cup P_3 \cup Q_2 \cup Q_3 \cup \cdots \cup Q_k) - \{v_1, v_3\}$. Since we may, we assume the replacement paths do not change. Suppose first we find outcome (2). We use the notation in the outcome for u, s, t, x. If $x \in X$, then taking $P_4' = v_1Q_1tP_5x$ with $u' = v_1$ gives outcome (1). If x is on $Q_i, i > 1$, we take $Q_1' = v_1Q_1tP_5xQ_i$ which gives a shorter augmenting sequence. If x is on P_2 between v and v_3, P_3, or on P_1 between v_1 and v, let i be such that P_i contains x, then we can take $v' = v_2, P_1' = v_2P_2uP_4sQ_1v_1P_1a, P_2' = v_2P_2b, P_3' = v_2Q_1tP_5xP_ivP_3c$, $v_1' = u, Q_1' = uP_2v_3Q_2v_4$ to find a shorter augmenting sequence. Finally, if x is on P_1 between v_1 and a, we take $v' = v_2$, $P_1' = v_2Q_1tP_5xP_1a, P_2' = v_2P_2b, P_3' = v_2P_2uP_4sQ_1v_1P_1vP_3c, v_1' =$

$t, v_2' = s, v_3' = u, Q_1' = tQ_1s, Q_2' = uP_2v_3Q_2v_4$ which gives an augmenting sequence with the same length and higher index.

So instead we consider outcome (1) with the notation for u, x, P_4 as in the outcome. If $x \in X$, we can take $u' = v_1, s' = v_2, t' = u, P_4' = Q_1, P_5' = P_4$ to find outcome (2). If x is on a $Q_i, i > 1$, then we take $v_3' = u, Q_2' = uP_4xQ_i$ which gives an augmenting sequence of at most the same length but higher index. If x is on P_2 between v' and v_3, then take $P_2' = vP_2xP_4uP_2b$, $v_3' = u, Q_2' = uP_2v_3Q_2v_4$ to find an augmenting sequence of the same length with higher index. If x is on P_3, we take $v' = v_2, P_1' = v_2Q_1v_1P_1a, P_2' = v_2P_2b, P_3' = v_2P_2uP_4xP_3c, Q_1' = v_1P_2vP_2v_3Q_2v_4$ which gives a shorter augmenting sequence. If x is on P_1 between v_1 and a, then take $v' = v_2, P_1' = v_2P_2uP_4xP_1a, P_2' = v_2P_2b, P_3' = v_2Q_1v_1P_1vP_3c, v_1' = u, Q_1' = uP_2v_3Q_2v_4$ which gives a shorter augmenting sequence. Finally, suppose x is on P_1 between v and v_1 and $x \in A$. Then consider $P_1' = v_2Q_1v_1P_1a, P_2' = v_2P_2b, P_3' = v_2P_2uP_4xP_1vP_3c, Q_1' = v_1P_1x, Q_2' = uP_2v_3Q_2$ which gives an augmenting sequence of the same length and lower index.

So we must have $x \in B$ on P_1 between v and v_1. We apply induction to the paths vP_2v_3, vP_1x, P_3 and set $X' := X \cup V(xP_1a \cup v_3P_2b \cup P_4 \cup Q_1 \cup Q_2 \cup Q_3 \cup \cdots \cup Q_k) - \{x, v_3\}$. This gives us $u_2 \in A$ between v and v_3 with a path P_5 to x_2. Suppose we have outcome (1). Then if x_2 is not on Q_1, P_4, or xP_1v_1, then by symmetry we can apply the analysis of the previous paragraph. If x_2 is on P_4, then we replace P_4 with $uP_4x_2P_5u_2$ and have one of the outcomes above. If x_2 is on Q_1, then take $v' = v, P_1' = P_1, P_2' = vP_2u_2P_5x_2Q_1v_2P_2b, P_3' = P_3, v_1' = x, Q_1' = xP_4uP_2v_3Q_2v_4$ to find a shorter augmenting sequence. Finally, if x_2 is on xP_1v_1, then take $v' = v_2, P_1' = v_2Q_1v_1P_1a, P_2' = v_2P_2b, P_3' = v_2P_2uP_4xP_1vP_3c, Q_1' = v_1P_1x_2P_5u_2P_2v_3Q_2$ which is a shorter augmenting sequence. So we must have outcome (2) of the lemma which gives vertices u_2, s, t, x_2 and paths P_5 and P_6. If x_2 is not on either P_4 or v_1P_1x, then we can apply the analysis from the previous two paragraphs. Suppose $x_2 \in V(P_4)$. Then take $v' = v, P_1' = vP_2u_2P_5sP_1a, P_2' = vP_1tP_6x_2P_4uP_2b, P_3' = P_3, Q_1' = u_2P_2v_3Q_2$ to get a shorter augmenting sequence. Finally, suppose $x_2 \in V(v_1P_1x)$. Then take $v' = s, P_1' = sP_1tP_6x_2P_1a, P_2' = sP_1xP_4uP_2b, P_3' = sP_5u_2P_2vP_3c, Q_1' = Q, Q_2' = uP_2v_3Q_2$ which is an augmenting sequence of the same length and lower index.

So we may finally assume that the length of S is at least 3 with index at least 4. Note that v_4 must be on P_3' (still assuming that v_2 was on P_2'), since otherwise we get a shorter augmenting sequence. But then take $v' = v_2, P_1' = v_2Q_1v_1P_1a, P_2' = v_2P_2b, P_3' = v_2P_2v_3Q_2v_4P_3c, Q_1' = v_1P_1vP_3v_5Q_3v_6$, which gives a shorter augmenting sequence. □

Lemma 5 will suffice for most of our arguments. However, on one occasion we will need the following strengthening.

Lemma 7. *Let G be an internally 4-connected bipartite graph with bipartition (A, B). Let $a, v \in A$ and $b, c \in B$ with paths $P_1 = v...a, P_2 = v...b, P_3 = v...c$ vertex disjoint except for v. Let $X \subseteq V(G)$ be disjoint from $V(P_1) \cup V(P_2) \cup V(P_3)$. Then at least one of the following holds:*

(A) *There exist vertices $v' \in A, u \in B, x \in X \cap A$ and paths $P_1' = v'...a, P_2' = v'...b, P_3' =$*

$v'...c, P'_4 = u...x$ such that $u \in V(P'_1)$ and all of the P'_i are vertex disjoint and are disjoint from X except as specified,

(B) *There exist vertices $v', s \in A, u, t \in B, x \in X \cap B$ and paths $P'_1 = v'...a, P'_2 = v'...b, P'_3 = v'...c, P'_4 = u...s...x, P'_5 = s...t$ such that $u \in V(P'_1)$, $t \in V(X \cup P'_2 \cup P'_3)$, and all of the P'_i are vertex disjoint and are disjoint from X except as specified and except that t may lie on P'_2 or P'_3,*

(C) *There exist vertices $v', s \in A, u, t \in B, x \in X$ and paths $P'_1 = v'...a$, $P'_2 = v'...t...s...b$, $P'_3 = v'...c$, $P'_4 = u...s$, $P'_5 = t...x$ such that $u \in V(P'_1)$ and all of the P'_i are vertex disjoint and are disjoint from X except as specified,*

(D) *There exist vertices $v', s, w \in A, u, t \in B, x, y \in X \cap B$ and paths $P'_1 = v'...u...w...t...a$, $P'_2 = v'...b, P'_3 = v'...c, P'_4 = u...s...x, P'_5 = s...t, P'_6 = w...y$ such that all of the P'_i are vertex disjoint and are disjoint from X except as specified and except that x may equal y.*

Proof. We proceed by induction on the size of $|V(G)| - |X|$. Apply Lemma 5. We may assume that we are in the first outcome of that lemma since the second outcome is outcome (C) of this lemma. We may assume that the replacement paths do not change. Thus there exists a path P_4 with ends $u \in B \cap V(P_1)$ and $x \in X$, vertex-disjoint from $P_1 \cup P_2 \cup P_3$, except for u. We may assume that $x \in B$, for otherwise (A) holds.

We apply the induction hypothesis to the paths P_4, uP_1a and uP_1v, and set $X' = X \cup V(P_2 \cup P_3 \setminus \{x, v\})$. Note that $|X'| > |X|$ since b and c are distinct, not in X, and in $P_2 \cup P_3$ and v was not in X originally. We consider each of the four outcomes separately. We may assume that the replacement paths do not change.

If outcome (A) holds, then we obtain outcome (B) of the lemma. Next, let us assume that the induction hypothesis yields outcome (B). Thus there exist vertices $s \in A \cap V(P_4)$, $t \in B$, $y \in A \cap (V(P_2 \cup P_3) \cup X)$ and $w \in A \cap (V(P_1 \cup P_2 \cup P_3) \cup X)$, and paths P_5 from s to y and P_6 from t to w such that the paths P_5 and P_6 are disjoint and disjoint from $V(P_1 \cup P_2 \cup P_3) \cup X$, except as stated. If $y \in X$, then we have the outcome (A) by taking $P'_1 = P_1, P'_2 = P_2, P'_3 = P_3, P'_4 = uP_4sP_5y$ and $v' = v, u = u, x = y$. So without loss of generality, we may assume $y \in V(P_2)$. We are now interested in where w lies. If w lies on P_2, then we may assume that it lies between y and b since y and w are then symmetric. In that case, take $P'_1 = P_1, P'_2 = vP_2yP_5tP_6wP_2b, P'_3 = P_3, P'_4 = P_4, P'_5 = sP_5t, x = x, t = t, s = s, u = u, v' = v$ which is exactly outcome (B). If w lies on P_1 between v and u, take $v' = w, s = s, t = t, u = u, x = x, P_1 = wP_1a, P_2 = wP_6tP_5yP_2b, P_3 = wP_1vP_3c, P_4 = uP_4x, P_5 = sP_5t$ which is again outcome (B). If w lies on P_3, , take $v' = w, s = s, t = t, u = u, x = x, P_1 = wP_3vP_1a, P_2 = wP_6tP_5yP_2b, P_3 = wP_3c, P_4 = uP_4x, P_5 = sP_5t$, which is again outcome (B). So we have that w lies on P_1 between u and a (or is a).

Let $r \in B$ lie between v and y on P_2. By Menger's theorem and by replacing vP_2y if necessary we may assume that there exists a path P_7 from r to a vertex z not on vP_2y that is disjoint from $P_1 \cup \cdots \cup P_6$, except for its ends. If $z \in X$, then keeping $v = v', P'_1 = P_1, P'_2 = P_2, P'_3 = P_3$ and taking $u = u, t = r, s = y, x = z, P'_4 = uP_4sP_5y, P'_5 = P_7$, this is outcome (C). If $z \in V(P_1)$ between v and u (note that this is symmetric with $z \in V(P_3)$), then take $v' = y, u = t, s = s, t = r, x = x, P'_1 = yP_5tP_6wP_1a, P'_2 = yP_2b, P'_3 = yP_2vP_3c, P'_4 = tP_5sP_4x, P'_5 = sP_4uP_1zP_7r$ to find outcome (B). If z is on P_1 between u and w, take $v' = y, u = t, s = s, t = u, x = x, P'_1 = yP_5tP_6wP_1a, P'_2 = yP_2b, P'_3 = yP_3rP_7zP_1vP_3c, P'_4 = tP_5sP_4x, P'_5 = sP_4u$, which is outcome (B). If z is on P_1 between w and a, take $v' = y, u = r, t = u, s = v, x = x, P'_1 = yP_2rP_7zP_1a, P'_2 = yP_2b, P'_3 = yP_5tP_6wP_1vP_3c, P'_4 = rP_2a, P'_5 = uP_4x$, which gives outcome (C). If z is on P_2 between y and b, then take $v = v', P'_1 = P_1, P'_2 = vP_2rP_7zP_2b, P'_3 = P_3, u = u, s = s, x = x, t = r, P'_4 = P_4, P'_5 = sP_5yP_2r$ to again find outcome (B). If z is on P_4 between u and s, take $v' = y, u = t, s = s, t = r, x = x, P'_1 = yP_5tP_6wP_1a, P'_2 = yP_2b, P'_3 = yP_2vP_3c, P'_4 = tP_5sP_4x, P'_5 = sP_4zP_7r$ which is outcome (B). If z is on P_4 between s and x, take $v' = v, u = u, s = y, t = r, P'_1 = P_1, P'_2 = P_2, P'_3 = P_3, P'_4 = uP_4sP_5y, P'_5 = rP_7zP_4x$ to get outcome (C). If z is on P_5 or P_6, then take $v = v', P'_1 = P_1, P'_2 = P_2, P'_3 = P_3, u = u, s = s, x = x, t = r, P'_4 = P_4$. If z is on P_5, then take $P_5 = sP_5zP_7r$ to find outcome (B) and if z is on P_6, take $P_5 = sP_5tP_6zP_7r$ to find outcome (B). This completes the case when induction yields outcome (B).

Next we assume that induction yields outcome (C). Thus there exist vertices $s \in A \cap V(P_4)$, $t \in B \cup V(P_1)$, $w \in A \cap V(P_1)$ and $y \in X'$, and paths P_5, P_6 such that v, u, w, t, a occur on P_1 in the order listed, P_5 has ends s and t, P_6 has ends w and y, and P_5, P_6 are disjoint and disjoint from P_1, P_2, P_3, P_4, except for their ends. Note that if $y \in X \cup B$, this is exactly outcome (D), including the notation. Suppose first that $y \in X \cup A$. Then take $v' = v, u = u, x = y, P'_1 = vP_1uP_4sP_5tP_1a, P'_2 = P_2, P'_3 = P_3, P'_4 = uP_1wP_6y$ to find outcome (A). So we may assume that $y \in V(P_2)$. Then take $v' = w, u = t, s = s, t = u, x = x, P'_1 = wP_1a, P'_2 = wP_6yP_2b, P'_3 = wP_1vP_3c, P'_4 = tP_5sP_4x, P'_5 = sP_4u$, which is outcome (C). Since $y \in V(P_3)$ is symmetric with this case, that completes this outcome.

Finally, we assume that induction yields outcome (D). Thus there exist vertices $s, t \in A \cap V(P_4)$, $r \in B \cap V(P_4)$, $w \in B$ and $y, z \in A \cap X'$, and paths P_5, P_6, P_7 such that u, s, r, t, x occur on P_4 in the order listed, P_5 has ends s and y and includes w, P_6 has ends w and t, P_7 has ends r and z, and the paths P_5, P_6, P_7 are pairwise disjoint and disjoint from P_1, P_2, P_3, P_4, except for their ends. Note that y and z are completely symmetric as are r and w. Suppose that $y \in X$. Then take $v' = v, u = u, x = y, P'_1 = P_1, P'_2 = P_2, P'_3 = P_3, P'_4 = uP_4sP_5y$ to get outcome (A). So we may assume $y \in V(P_2)$. Suppose z lies on P_2 between v and y (by symmetry, if z lies on P_2, this assumption is without loss of generality). Then take $v' = v, u = u, t = r, x = x, s = s, P'_1 = P_1, P'_2 = vP_2zP_7rP_4sP_5yP_2b, P'_3 = P_3, P'_4 = uP_4s, P'_5 = rP_4x$ which is outcome (C). So z lies on P_3. Then take $v' = y, u = u, s = s, t = r, x = x, P'_1 = yP_2vP_1a, P'_2 = yP_2b, P'_3 = yP_5sP_4rP_7zP_3c, P'_4 = uP_4s, P'_5 = rP_4x$ which is again outcome (C). This completes this outcome and the proof. □

3 Proof of Theorem 3

Let H be a subgraph of a graph G. By an *H-path* in G we mean a path in G with at least one edge, both ends in $V(H)$ and no other vertex or edge in H.

Let H be a hex in a graph G, let P be the union of a set of H-paths in G, and let Q

be a subgraph of H. We denote by $H + P - Q$ the graph obtained from $H \cup P$ by deleting all edges of Q and then deleting all resulting isolated vertices. A typical application will be when P and Q are paths, but we will need more complicated choices.

A hex in a graph G is *optimal* if no hex of G has strictly more odd segments. We proceed by a series of lemmas, each improving a lower bound on the number of odd segments in an optimal hex.

Lemma 8. *Let G be a 3-connected bipartite graph. Then every optimal hex of G has at least four odd segments.*

Proof. Let (A, B) be a bipartition of G and let H be an optimal hex in G with feet and segments numbered as in the definition of a hex. We may assume for a contradiction that H has at most three odd segments. It follows that at least five feet of H belong to the same set A or B, and so we may assume that $v_1, v_2, \ldots, v_5$ all belong to A. Thus P_{14} has an internal vertex u that belongs to B. Since G is 3-connected we may assume, by replacing P_{14} if necessary, that there exists an H-path Q with one end u and the other end, say w, in $V(H) - V(P_{14})$. By symmetry, we may assume that w belongs to $P_{15}, P_{16}, P_{24}, P_{25}$, or P_{26}. Let R be defined as $v_1P_{15}w$, $v_1P_{16}w$, $v_4P_{24}w$, P_{24} or P_{24}, respectively. Then $H + Q - R$ is a hex with strictly more odd segments than H, contrary to the optimality of H. □

Lemma 9. *Let G be an internally 4-connected bipartite graph. Then every optimal hex of G has at least five odd segments.*

Proof. Let (A, B) be a bipartition of G and let H be an optimal hex in G with feet and segments numbered as in the definition of a hex. By Lemma 8 we may assume for a contradiction that H has exactly four odd segments. It follows that two feet of H in $\{v_1, v_2, v_3\}$ and two feet of H in $\{v_4, v_5, v_6\}$ belong to the same set A or B, and so we may assume that v_1, v_2, v_4, v_5 all belong to A. Thus P_{14} has an internal vertex u that belongs to B. Since G is 3-connected we may assume, by replacing P_{14} if necessary, that there exists an H-path Q with one end u and the other end, say w, in $V(H) - V(P_{14})$. By symmetry, we may assume that w belongs to $P_{15}, P_{16}, P_{25}P_{26}$, or P_{36}. If w belongs to P_{15}, P_{16}, P_{36}, $A \cap V(P_{25} \cup P_{26})$, or $B \cap V(P_{26})$, let R be defined as $v_1P_{15}w, v_1P_{15}w$, P_{24}, P_{24}, or P_{16}, respectively. Then $H+Q-R$ is a hex with strictly more odd segments than H, contrary to the optimality of H.

So we may assume that $w \in B \cap V(P_{25})$. Note that this is the last case and is not symmetric with anything else, so it suffices to reduce to any of the previous cases. We now apply Lemma 5 to the paths $v_1P_{14}u, uP_{14}v_4, Q$ and $X := V(H) - (V(P_{14}) - \{w\})$ with A and B swapped. We may assume that the replacement paths do not change. If outcome (1) of the lemma holds, then there exist vertices $y \in A \cap V(Q)$, $z \in X$ and an $H \cup Q$-path R from y to z. If $z \in B \cap V(P_{25})$, we may assume it belongs to $v_5P_{25}w$. Then we can replace P_{25} by $v_2P_{25}wQyRzP_{25}v_5$, Q by vQy, and apply the case above where $w \in A \cap V(P_{25})$. If $z \notin B$ or z is not on P_{25}, then we can replace Q with $uQyRz$ and apply one of the previous cases.

So we may assume that the second outcome of the lemma holds. Thus there exist vertices $a \in A \cap V(Q)$, $b \in B \cap V(P_{14})$, $c \in A \cap V(P_{14})$, and $d \in X$ and disjoint $H \cup Q$-paths R between a and b and S between c and d. We may assume that b belongs to $v_1P_{14}u$. Then we can replace

P_{14} by $v_1P_{14}bRaQuP_{14}v_4$ and Q by $uP_{14}cRd$ which puts us in one of the previous cases unless $d \in B \cap V(P_{25})$. So we may assume $d \in B \cap V(P_{25})$, and that it belongs to $wP_{25}v_5$. Let H' be the hex obtained from $H \cup S \cup R \cup Q$ by deleting $V(P_{15} \cup v_5P_{25}d \cup P_{34} \cup P_{35} \cup P_{36}) - \{v_1, d, v_4, v_6\}$. Then H' has nine odd segments, contrary to the optimality of H. □

Lemma 10. *Let G be an internally 4-connected bipartite graph. Then every optimal hex of G has at least six odd segments.*

Proof. Let (A, B) be a bipartition of G and let H be an optimal hex in G with feet and segments numbered as in the definition of a hex. By Lemma 9 we may assume H has exactly five odd segments and that $v_1, v_2, v_4 \in A$ and $v_3, v_5, v_6 \in B$. We now apply Lemma 5 to the paths P_{14}, P_{15}, P_{16} and set $X := V(H) - V(P_{14} \cup P_{15} \cup P_{16})$. We may assume that the replacement paths do not change.

Suppose first that outcome (2) of the lemma holds. Thus we may assume that there exist vertices $u \in B \cap V(P_{14})$, $w \in A \cap V(P_{15})$, $y \in B \cap V(v_1P_{15}w)$ and $z \in X$, and disjoint H-paths Q from u to w and R from y to z. Then by symmetry we may assume that z belongs to one of $P_{24}, P_{25}, P_{34}, P_{35}$. Let T be, respectively, $v_2P_{24}z \cup P_{25} \cup P_{26}$, $v_5P_{25}z \cup P_{26}$, $P_{25} \cup v_3P_{34}z$, $P_{36} \cup v_5P_{35}z$. Then the hexes $H + (Q \cup R) - T$ have at least six odd segments which is a contradiction.

So we may assume that outcome (1) of the lemma holds. Thus there exists a vertex $u \in B \cap V(P_{14})$ and an H-path Q from it to $w \in X$. Then by symmetry we may assume that w belongs to one of P_{24}, P_{25}, P_{34}, and P_{35}. For w on P_{24}, P_{34}, P_{35} or $A \cap V(P_{25})$, let R be, respectively, $v_4P_{24}w, P_{36}, v_4P_{34}w, P_{34}$. Then $H + Q - R$ is a hex with more odd segments than H which contradicts the optimality of H.

So we may assume $w \in B \cap V(P_{25})$. We now apply Lemma 5 to the paths $v_1P_{14}u, uP_{14}v_4, Q$ and set $X := V(H) - V(P_{14}) - \{w\}$. Suppose that outcome (1) of the lemma holds. Thus there exist vertices $y \in A \cap V(Q)$ and $z \in X$ and an $H \cup Q$-path R from y to z. If $z \in B \cap V(P_{25})$, we may assume it belongs to $v_5P_{25}w$. Then we can replace P_{25} by $v_2P_{25}wQyRzP_{25}v_5$, Q by uQy, and apply the case above where $w \in A \cap V(P_{25})$. If $z \in B \cap V(P_{26})$, then the hex $H + (Q \cup R) - (v_5P_{25}w \cup v_2P_{26}z \cup P_{35})$ has nine odd segments which contradicts the optimality of H. If $z \notin B$ or z is not on P_{25} or P_{26}, then we can replace Q with $uQyRz$ and apply one of the previous cases.

So we may assume that the second outcome of the lemma holds. Thus there exist vertices $a \in A \cap V(Q)$, $b \in B \cap V(v_1P_{14}u)$, $c \in A \cap V(P_{14})$ and $d \in X$, and disjoint $H \cup Q$-paths R between a and b and S between c and d. Then we can replace P_{14} by $v_1P_{14}bRaQuP_{14}v_4$ and Q by $uP_{14}cRd$ which puts us in one of the previous cases, unless $d \in B$ and d is on P_{25} or P_{26}. Let $F = S \cup Q \cup R$. If d is on $wP_{25}v_5$, let $J = P_{35} \cup P_{26} \cup v_5P_{25}d \cup P_{15}$; if d is on $wP_{25}v_2$, let $J = P_{15} \cup P_{24} \cup P_{26} \cup v_2P_{25}d$; and if d is on P_{26}, let $J = P_{24} \cup P_{35} \cup v_6P_{26}d$. Then the hexes $H + F - J$ have at least six odd segments, which contradicts the optimality of H. □

Lemma 11. *Let G be an internally 4-connected bipartite graph. Then in every optimal hex of G every segment is odd.*

Proof. Let (A, B) be a bipartition of G and let H be an optimal hex in G with feet and segments numbered as in the definition of a hex. By Lemma 10 we may assume H has exactly six odd segments and that that $v_1, v_2, v_3, v_4 \in A$ and $v_5, v_6 \in B$. We now apply Lemma 5 to the paths P_{14}, P_{15}, P_{16} and set $X := V(H) - V(P_{14} \cup P_{15} \cup P_{16})$.

Suppose first that outcome (2) of the lemma holds. Thus there exist vertices $u \in B \cap V(P_{14})$, $w \in A \cap V(P_{15})$, $y \in B$ and $z \in X$, and disjoint H-paths Q from u to w and R from y to z. By symmetry we may assume that z belongs to $P_{24} \backslash v_2, P_{25}, or P_{26}$. Let J be, respectively, $P_{35} \cup zP_{24}v_2$, $P_{34} \cup zP_{25}v_5$, or $P_{34} \cup zP_{26}v_6$. Then the hexes $H + (Q \cup R) - J$ each have nine odd segments, which contradicts the optimality of H.

So we may assume that outcome (1) one of the lemma holds. Thus there exist $u \in B \cap V(P_{14})$ and an H-path Q from it to $w \in X$. Then we may assume that w belongs to P_{24} or P_{25}. If $w \in V(P_{24})$, let $R = v_4P_{24}w$. If $w \in A \cap V(P_{25})$, let $R = P_{24}$. Then $H + Q - R$ is a hex with nine odd segments, a contradiction. Thus it remains to handle the case when $w \in B \cap V(P_{25} \cup P_{26} \cup P_{35} \cup P_{36})$.

We now forget w, Q, u and instead apply Lemma 7 to the paths P_{14}, P_{15}, P_{16} and set $X := V(H) - V(P_{14} \cup P_{15} \cup P_{16})$. Outcomes (A) and (C) give results already ruled out by the case analysis from applying Lemma 5, so we may assume that outcomes (B) or (D) hold.

Suppose that outcome (B) holds. Thus there exist vertices $u \in B \cap V(P_{14})$, $w \in B \cap X$, $t \in A$ and $s \in B$, and a H-path $Q = u...t...w$ and a $H \cup Q$-path $R = t...s$. By the previous analysis and symmetry we may assume that $w \in V(P_{25})$, so we are interested in where s lies. The above case analysis handles the cases where s is on P_{24} or P_{34}, so we need only worry about the case where s belongs to P_{15}, $wP_{25}v_5$, P_{35}, P_{16}, P_{26}, P_{36}. Then, respectively, let J be defined as $P_{24} \cup wP_{25}v_5$, $P_{24} \cup wP_{25}s$, $P_{24} \cup wP_{25}v_5$, $P_{34} \cup P_{35} \cup P_{36}$, $P_{34} \cup P_{35} \cup P_{36}$, $P_{34} \cup P_{35} \cup v_3P_{36}s$. Then $H + (Q \cup R) - J$ is a hex with nine odd segments, which contradicts the optimality of H.

So we must have outcome (D). Thus there exist vertices $u, w \in V(P_{14}) \cap B$, $r \in V(P_{14}) \cap A$, $s \in A$ and $x, y \in X \cap B$, such that v_1, w, r, u, v_4 occur on P_{14} in the order listed, and there exists a H-path $Q = u...s...x$ and disjoint $H \cup Q$-paths $R = w...s$ and $S = r...y$. Without loss of generality, we may assume that $x \in V(P_{25})$. By symmetry and taking advantage of the previous cases, we may assume that y belongs to $xP_{25}v_5, P_{26}, P_{35}$, or P_{36}. Let J be $P_{15} \cup P_{34} \cup P_{35} \cup P_{36} \cup v_5P_{25}y$, $P_{24} \cup xP_{25}v_5 \cup P_{36}$, $P_{15} \cup v_3P_{35}y \cup P_{34} \cup P_{36}$, or $P_{24} \cup xP_{25}v_5 \cup v_3P_{36}y$, respectively. Then $H + (Q \cup R \cup S) - J$ are each hexes with nine odd segments, a contradiction. □

Proof of Theorem 3. Let G be an internally 4-connected non-planar bipartite graph. By Theorem 1 the graph G has a hex, and hence it has an optimal hex H. By Lemma 11 every segment of H is odd, as desired. □

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