Ordinal Types in Ramsey Theory and Well-Partial-Ordering Theory

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There is a big gap between the infinite Ramsey theorem $\omega \to (\omega)^n_k$ and its finite version

$$R(n; \ell_1, \dots, \ell_k) \to (\ell_1, \dots, \ell_k)^n_k.$$

The finite Ramsey theorem is much finer. In this paper we fill in the gap by defining "Ramsey numbers $R(n; \gamma_1, \dots, \gamma_k)$" for arbitrary ordinals $\gamma_1, \dots, \gamma_k$; these generalized Ramsey numbers are again ordinals, and their estimate is a quantitative strengthening of the infinite Ramsey theorem. Actually, this is just a special case of our general definition of "Ramsey numbers" which is based on an axiomatic approach. The axioms themselves imply some estimates and other facts. To obtain sharper results, however, we have to consider more concrete situations. Besides the classical one already mentioned we investigate also the Canonical Ramsey Theorem, the Erdös–Szekeres theorem on monotone sequences and the well-partial-ordering (wpo) theory. In the last case, the Ramsey numbers generalize the so-caled types of wpo sets, a concept already studied in a great detail.

The Ramsey numbers are also closely connected with independence results in finite combinatorics. This fact has been already known for the types of wpo sets. The existence of "$R(n; \omega, \dots, \omega)$" implies the Paris–Harrington modification of Ramsey theorem. As one might expect from unprovability of this theorem in PA, it holds

$$\lim_{n \to \infty} R(n; \omega, \dots, \omega) = \varepsilon_0.$$

Finally, we should remark that our approach is different from ordinal Ramsey theorems (like e.g. $\omega^2 \to (\omega^2, n)$).

1. Introduction

The well–known Finite Ramsey Theorem says that, given natural numbers $n, k, \ell_1, \ldots, \ell_k$, there is a number $R(n; \ell_1, \ldots, \ell_k)$, called the *Ramsey number*, with the following property. If r is a coloring of n–element subsets of $\{1, \ldots, N\}$ by k colors (i.e. $r : [\{1, \ldots, N\}]^n \to \{1, \ldots, k\}$) such that every set $E \subseteq \{1, \ldots, N\}$ whose all n–subsets are colored i, has at most ℓ_i elements, then $N < R(n; \ell_1, \ldots, \ell_k)$. The infinite Ramsey theorem (i.e. $\omega \to (\omega)^2_2$) gives no such number, it simply says that every infinite sequence contains an infinite homogeneous subsequence without indicating how sparse the homogeneous subsequences are. We propose a way to measure this sparsity based on a generalization of the notion of a Ramsey number. We refer to (Graham, Rothschild, Spencer 1980) or (Nešetřil 1987) for an exposition of Ramsey theory.

Our results are motivated by the well–partial–ordering theory, so let us start by recalling its rudiments. Let Q be a partially ordered set. A sequence $q_1, q_2, \ldots$ (finite or not) of elements of Q is called a *good* if there are indices i, j such that $i < j$ and $q_i \leq q_j$, and is called *bad* otherwise. The set of all bad sequences of elements of Q is denoted by $\mathrm{Bad}(Q)$. The set Q is called *well–partially–ordered (wpo)* if every infinite sequence of elements of Q is good. Let us remark that this theory is often called the well–quasi–ordering one, because it is usually sufficient to work with quasi–orderings (i.e. reflexive and transitive relations) rather than with partial orderings. But since every quasi–ordering becomes a partial–ordering after identifying all elements x, y with $x \leq y \leq x$ we found it more convenient to work with partial orderings and hence we call the existing theory the well–partial–ordering theory.

The well–partially-ordered sets have been studied for a while (see e.g. (Higman 1952), (Kruskal 1960), (Nash–Williams 1963) or (Kruskal 1972) for a survey). A recent major breakthrough was done by Robertson and Seymour (Robertson, Seymour) who proved the so–called Wagner's conjecture, an outstanding problem in the area which has been open for many decades.

The usual method in wpo theory is a minimal bad sequence argument, basically due to Nash–Williams. It is an induction–like argument, but it is highly nonconstructive. Trying to find a more constructive proofs for some wpo theorems we rediscovered the theory of types of wpo sets, initiated by de Jongh, Parikh (de Jongh, Parikh 1977) and Schmidt (Schmidt 1978), (Schmidt 1979). The constructive approach is as follows: To find an ordinal γ and a function $f : \mathrm{Bad}(Q) \to \gamma$ such that

$$f(q_1, \ldots, q_{n-1}) > f(q_1, \ldots, q_n)$$

for every $(q_1, \ldots, q_n) \in \mathrm{Bad}(Q)$. The least ordinal for which such a function exists, called the type of Q and denoted by $c(Q)$, turns out to be an interesting invariant which reflects the complexity of the wpo set and also provability and nonprovability of some wpo statements in certain logical systems.

There is another way of expressing $c(Q)$. A partially ordered set $(Q, \leq)$ is wpo if and only if every linear extension of $\leq$ is a well–ordering, hence it has an ordinal type. It is a nontrivial fact that among all these linear extensions there is a maximal one and its ordinal type is exactly $c(Q)$. This may be viewed as a minimax theorem in wpo theory (see Section 4). De Jongh, Parikh (de Jongh, Parikh 1977) and Schmidt (Schmidt 1979) have computed the types of some wpo sets.

The above facts led us to introduce the type in a more general setting which includes both the Ramsey theory and well–partial–ordering theory. This is a bit more involved. The type of a Ramsey result is not a single number, but an ordinal function of the complexity of the partition. To clarify it let us consider the simplest example. Let $\mathcal{A} \subseteq [\omega]^2$ and assume that each infinite set $X \subseteq \omega$ contains an infinite subset Y such that $[Y]^2 \subseteq \mathcal{A}$. For γ an ordinal and $g: \omega^{<\omega} \to \gamma$ let us call a set $X \subseteq \omega$ $(\mathcal{A}, g)$–bad if for any $x_1 < x_2 < \ldots < x_n \in X$ such that $[\{x_1, \ldots, x_n\}]^2 \subseteq \mathcal{A}$ we have $g(x_1, \ldots, x_{n-1}) > g(x_1, \ldots, x_n)$. Roughly, the ordinal γ and the function g measure the "killing" of homogeneous subsets of X. Note that, by our assumption about $\mathcal{A}$, there is no infinite $(\mathcal{A}, g)$–bad set. The *type* $c_{\mathcal{A}}(\gamma)$ corresponding to $\mathcal{A}$ and γ is defined to be the least ordinal δ such that for every $g: \omega^{<\omega} \to \gamma$ there is $f: \omega^{<\omega} \to \delta$ such that for every $X \subseteq \omega$ the following holds:

if X is $(\mathcal{A}, g)$–bad, then $f(x_1, \ldots, x_{n-1}) > f(x_1, \ldots, x_n)$ for any $x_1 < \ldots < x_n \in X$.

Hence if the killing of homogeneous parts of X is measured by g, then the killing of X itself is measured by f.

Such a formulation is possible not only for the Ramsey theorem, but also for the Canonical Ramsey Theorem of Erdös and Rado (Erdös, Rado 1950), for the Nash–Williams' Partition Theorem (Nash–Williams 1965) and in general for every Ramsey type theorem which has an infinitary version and where homogenity can be recognized from finite segments.

In the following section we introduce the exact definitions. The key notion is that of a *sheaf*, which corresponds to a *partition* in Ramsey theory. We consider some basic examples and prove two theorems on abstract sheaves.

As in Ramsey theory we are not interested in a single partition but in a system of partitions of the same kind. In Section 3 we introduce the corresponding concepts of R–property and strong R–property. These definitions enable us to distinguish between "uniform" and "non–uniform" estimates (with respect to systems of partitions). For a broad class of systems (the so–called standard ones) the uniform and non–uniform cases coincide (under some obvious cardinality assumptions).

Section 4 is devoted to the wpo theory considered from our point of view. Some of the theorems presented in this section were known, but the proofs are new and simpler.

In Section 5 we investigate two possible generalizations of the Erdös–Szekeres theorem on monotone sequences. Similarly as in the finite case, the Ramsey function reveals as a product of its arguments, suitably defined for ordinals.

In Section 6 we give upper and lower bounds to the Ramsey function of the k–system which corresponds to classical Ramsey theory. As in the finite case the bounds are of the form of iterated exponentiation and in fact are obtained by similar methods. The lower bound requires a somewhat tricky modification of the Stepping-Up Lemma from Ramsey theory (see Graham, Rothschild, Spencer 1980).

In Section 7 we give upper bounds for the Canonical Ramsey Theorem of Erdös and Rado (Erdös, Rado 1950).

Let us introduce some terminology. If U is an arbitrary set, then $U^{<\omega}, U^{\omega}, U^n$ and $[U]^n$ denote the set of nonempty finite sequences of elements of U, the set of infinite sequences of elements of U, the Cartesian product of n copies of U and the set of n–element subsets of U, respectively. If $a \in U^{<\omega}$, then $|a|$ is the length of a. For $a = (a_1, a_2, \ldots)$, $b = (b_1, b_2, \ldots) \in U^{<\omega} \cup U^{\omega}$ we write $a \subseteq b$ if there are $j_1 < j_2 < \ldots$ such that $(a_1, a_2, \ldots) = (b_{j_1}, b_{j_2}, \ldots)$ and $a \ll b$ if $a \neq b$ and there is n such that $a = (a_1, \ldots, a_n) = (b_1, \ldots, b_n)$. In particular $a \subseteq a$, but not $a \ll a$. If $a \subseteq b$ we say that a is a *subsequence* of b and if $a \ll b$ we say that a is a *segment* of b. For $a \in U^{<\omega}$ we put $\downarrow a := \{b \in U^{<\omega} \mid b \subseteq a\}$. If $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_m) \in U^{<\omega}$, then

$$a.b := (a_1, \ldots, a_n, b_1, \ldots, b_m) \in U^{<\omega}.$$

If $f : X \to Y$ is any function and $M \subseteq X$, then $f \restriction M$ denotes the restriction of f to M. If $f : X \to Y$ is any function and $(x_1, \ldots, x_n) \in X$, then the value of f at $(x_1, \ldots, x_n)$ will be denoted by $f(x_1, \ldots, x_n)$, to avoid cumbersome notation like $f((x_1, \ldots, x_n))$.

A *tree* is a couple $(T, \leq)$, where T is any set and $\leq$ is a partial ordering on T such that for any $t \in T$ the set $\{t' \in T \mid t' \leq t\}$ is a finite chain. A *subtree* of T is a subset S of T such that $s_1 \leq s_2 \leq s_3$ and $s_1, s_3 \in S$ imply $s_2 \in S$, together with the restriction of $\leq$ to S. A frequently used tree will be $(U^{<\omega}, \leq)$, where $a \leq b$ iff either $a = b$ or $a \ll b$. We make the convention that subsets of $U^{<\omega}$ will be regarded as trees with this ordering. If $(T, \leq)$ is a tree and t, t' are distinct elements of T, we say that t' is a *successor* of t if $t \leq t'$ and there is no t'', distinct from t and t' such that $t \leq t'' \leq t'$.

Of great interest will be trees without an infinite chain: to such a tree one can find the least ordinal $\gamma_T < |T|^+$ such that there is a function $\psi_T : T \to \gamma_T$ satisfying $\psi_T(t) > \psi_T(t')$ for any $t < t' \in T$. The ordinal γ_T is called the *type* of T and the function ψ_T is called a *character* on T. If T is a tree, the T_t denotes the subtree of all $t' \in T$ such that $t \leq t'$. Let S, T be trees. A mapping $f : S \to T$, is called a *tree homomorphism* if it is strictly increasing, i.e. if $s < s'$ implies $f(s) < f(s')$ for any $s, s' \in S$. If T contains no infinite chain and there is a tree homomorphism $f : S \to T$, then it can be seen by induction on γ_T that S contains no infinite chain and $\gamma_S \leq \gamma_T$.

The terminology about ordinals is a standard. We identify each ordinal with the set of its predecessors. If $\alpha, \beta \in On$, the class of ordinals, and

$$\alpha = \omega^{\alpha_1} + \ldots + \omega^{\alpha_n} \quad (\alpha_1 \geq \ldots \geq \alpha_n),$$

$$\beta = \omega^{\beta_1} + \ldots + \omega^{\beta_m} \quad (\beta_1 \geq \ldots \geq \beta_m)$$

are their Cantor's normal forms, then the *natural sum* of α, β is defined by

$$\alpha \# \beta := \omega^{\gamma_1} + \ldots + \omega^{\gamma_{n+m}},$$

where $\gamma_1 \geq \ldots \geq \gamma_{n+m}$ is a nonincreasing rearrangement of $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots \beta_m$. An equivalent definition is

$$\alpha \# \beta = \sup\{\alpha' \# \beta + 1, \alpha \# \beta' + 1 \mid \alpha' < \alpha,\ \beta' < \beta\}.$$

The *natural product* is defined by

$$\alpha * \beta := \#\{\omega^{\alpha_i \# \beta_j} \mid i = 1, \ldots, n,\ j = 1, \ldots, m\}.$$

If λ is an ordinal, then a set $M \subseteq \lambda$ is called *cofinal* in λ if for every $\alpha \in \lambda$ there exists $\beta \in M$ such that $\beta \geq \alpha$.

We list below some properties of # and $*$ which will be used without any further reference.

(i) $\alpha \# \beta = \beta \# \alpha$, $\alpha * \beta = \beta * \alpha$,
(ii) $\alpha \# 1 = \alpha + 1$,
(iii) $\alpha \# (\beta \# \gamma) = (\alpha \# \beta) \# \gamma$, $\alpha * (\beta * \gamma) = (\alpha * \beta) * \gamma$,
(iv) if $\alpha \leq \gamma$, $\beta \leq \delta$ and one inequality is strict, then $\alpha \# \beta < \gamma \# \delta$ and $\alpha * \beta < \gamma * \delta$,
(v) if $\beta < \omega^\alpha$ and $\gamma < \omega^\alpha$, then $\beta \# \gamma < \omega^\alpha$,
(vi) $\alpha * (\beta \# \gamma) = (\alpha * \beta) \# (\alpha * \gamma)$,
(vii) if $\beta < \omega^{\omega^\alpha}$ and $\gamma < \omega^{\omega^\alpha}$ then $\beta * \gamma < \omega^{\omega^\alpha}$.

2. Sheaves

Definition 2.1. Let U be an infinite set. A *sheaf* in U is a set $A \subseteq U^{<\omega}$ such that $a \in A$ and $b \ll a$ implies $b \in A$. A *k-sheaf* in U is a k–tuple of sheaves and it is convenient to identify sheaves and 1–sheaves. A sheaf A is said to have the *R-property* (short for Ramsey property) if every infinite sequence in U contains an infinite subsequence each finite segment of which belongs to A. A k–sheaf $(A_1, \ldots, A_k)$ is said to have the *R-property* if the sheaf $A_1 \cup \ldots \cup A_k$ has the R–property. Equivalently $(A_1, \ldots, A_k)$ has the R–property if and only if for every infinite sequence p in U there are an index $1 \leq i \leq k$ and an infinite subsequence of p each finite segment of which belongs to A_i.

Example 2.2. For $r : [U]^n \to \{1, \ldots, k\}$ and $i \in \{1, \ldots, k\}$ we define
$R_i^r := \{a \in U^{<\omega} \mid$ if $(x_1, \ldots, x_n)$ is an injective subsequence of a,
then $r(\{x_1, \ldots, x_n\}) = i\}$.
Clearly $R^r := (R_1^r, \ldots, R_k^r)$ is a k–sheaf, it will be called the *Ramsey k-sheaf* corresponding to the coloring r. By Ramsey theorem this k–sheaf has the R–property.

Definition 2.3. An ordinal–valued function f defined on a set $M \subseteq U^{<\omega}$ is called a *killing* on M if $a, b \in M$ and $a \ll b$ imply $f(a) > f(b)$.

Let $\gamma_1, \dots, \gamma_k$ be ordinals. A k–tuple of functions $g = (g_1, \dots, g_k)$ is called a $(\gamma_1, \dots, \gamma_k)$-*testing* if $g_i : U^{<\omega} \to \gamma_i$. Let $A = (A_1, \dots, A_k)$ be a k–sheaf, $\gamma_1, \dots, \gamma_k$ ordinals and $g = (g_1, \dots, g_k)$ a $(\gamma_1, \dots, \gamma_k)$–testing. A sequence $a \in U^{<\omega}$ is called (A, g)-*bad* if each g_i is a killing on $\downarrow a \cap A_i$. The tree of (A, g)-bad sequences will be denoted by $\text{Bad}(A, g)$.

Definition 2.4. Let $A = (A_1, \dots, A_k)$ be a k–sheaf, $\gamma_1, \dots, \gamma_k$ ordinals. The R–ordinal $\Phi_A(\gamma_1, \dots, \gamma_k)$ if such exists is defined as the minimal ordinal γ such that for each $(\gamma_1, \dots, \gamma_k)$–testing g there exists a function $f : U^{<\omega} \to \gamma$, called the *R-character corresponding to A and g* (or simply corresponding to g if it is clear which k–sheaf is ment), such that one of the following equivalent conditions is satisfied.

(2.4a) If a is (A, g)–bad, then f is a killing on $\downarrow a$ for every $a \in U^{<\omega}$.
(2.4b) If $b \ll a$ and a is (A, g)–bad, then $f(b) > f(a)$.
(2.4c) f is a character on the tree of (A, g)–bad sequences.

In other words $\Phi_A(\gamma_1, \dots, \gamma_k)$ exists if and only if for no $(\gamma_1, \dots, \gamma_k)$–testing g the tree $\text{Bad}(A, g)$ contains an infinite chain and equals the least upper bound of the types of $\text{Bad}(A, g)$ for all $(\gamma_1, \dots, \gamma_k)$–testings g.

Definition 2.5. The above least upper bound may be attained for some g. Such a g will be called the *universal* $(\gamma_1, \dots, \gamma_k)$-*testing*. If $\gamma_1, \dots, \gamma_k$ are finite then the supremum is always attained, namely for $g = (g_1, \dots, g_k)$ defined by $g_i(a) = (\gamma_i - |a|)^+$. We make the convention that for $\gamma_1, \dots, \gamma_k$ finite we shall understand by a universal $(\gamma_1, \dots, \gamma_k)$–testing the one defined above.

Theorem 2.6. *For a k-sheaf* $A = (A_1, \dots, A_k)$, *the following conditions are equivalent.*

(i) A has the R-property.
(ii) All the R–ordinals $\Phi_A(\gamma_1, \dots, \gamma_k)$ *exist and are* $\leq |U|^+$ *for any ordinals* $\gamma_1, \dots, \gamma_k$.
(iii) The R–ordinal $\Phi_A(|U|^+, \dots, |U|^+)$ *exists.*
(iv) All the R–ordinals $\Phi_A(\gamma_1, \dots, \gamma_k)$ *exist for* $\gamma_1, \dots, \gamma_k < |U|^+$.

Proof. (i) $\Rightarrow$ (ii): Let $\gamma_1, \dots, \gamma_k$ be given and let $g = (g_1, \dots, g_k)$ be a $(\gamma_1, \dots, \gamma_k)$–testing. By assumption, $\text{Bad}(A, g)$ contains no infinite chain. Thus $f : U^{<\omega} \to |U|^+$ defined by

$$\begin{aligned} f(a) &= \psi_{\text{Bad}(A,g)}(a) \quad \text{for } a \in \text{Bad}(A, g) \\ &= 0 \qquad\qquad\qquad \text{otherwise} \end{aligned}$$

is the desired R–character corresponding to A and g.

(ii) $\Rightarrow$ (iii): Obvious.
(iii) $\Rightarrow$ (iv): Obvious.
(iv) $\Rightarrow$ (i): Let A not have the R–property.

Then there exists an infinite sequence p in U such that the subtrees S_i of $\mathrm{Bad}(A,g)$ consisting of all $\{a \subseteq p \mid a \in A_i\}$ contain no infinite chain ($i = 1,\dots,k$). Put

$$g_i(a) = \psi_{S_i}(a) \quad \text{for } a \in S_i$$
$$= 0 \qquad \text{otherwise.}$$

Then $g = (g_1,\dots,g_k)$ is a $(\gamma_{S_1},\dots,\gamma_{S_k})$–testing such that every finite $a \subseteq p$ is (A,g)–bad. Hence if (iv) was true there would be an R–character corresponding to A and g, which would be a killing on the set of finite segments of p, i.e. if $p_1 \ll p_2 \ll \dots \ll p$, then

$$f(p_1) > f(p_2) > \dots$$

a contradiction. □

Definition 2.7. Let $A = (A_1,\dots,A_k)$ be a k–sheaf, $\gamma_1,\dots,\gamma_k$ ordinals and $a \in U^{<\omega}$. An $(A;\gamma_1,\dots,\gamma_k)$-*germ* on a is a k–tuple $g = (g_1,\dots,g_k)$ of functions $g_i : \downarrow a \to \gamma_i$ such that each g_i is a killing on $\downarrow a \cap A_i$. If $g = (g_1,\dots,g_k)$ is a $(\gamma_1,\dots,\gamma_k)$–testing, we define

$$g \restriction\downarrow a := (g_1 \restriction\downarrow a,\dots,g_k \restriction\downarrow a).$$

Thus if a is (A,g)–bad, then $g \restriction\downarrow a$ is an $(A;\gamma_1,\dots,\gamma_k)$–germ on a.

Theorem 2.8. *If a k–sheaf $A = (A_1,\dots,A_n)$ has the R–property, then $\Phi_A(\gamma_1,\dots,\gamma_k) < |U|^+$ for any ordinals $\gamma_1,\dots,\gamma_k < |U|^+$.*

Proof. Consider the tree $(S,\le)$ defined by

$S := \{(a,g) \mid a \in U^{<\omega}$ and g is an $(A;\gamma_1,\dots,\gamma_k)$–germ on $a\}$,

$(a,g) < (b,h)$ if $a \ll b$ and $g = h \restriction\downarrow a$.

We claim that S has no infinite chain. Indeed, let (a^i,g^i) be an infinite chain in S. Then there exists an infinite subsequence $s \subseteq \bigcup_{a^i}$ each finite segment of which lies in A_j, j fixed. Let $s_1 \ll s_2 \ll \dots \ll s$ and assume $s_i \subseteq a^{j(i)}$ in such a way that $j(1) < j(2) < \dots$. By the definition of $\le$ we obtain

$$\gamma_j > g_j^{j(1)} > g_j^{j(2)} > \dots$$

which is a contradiction showing that S has no infinte chain. Now $\gamma_S < |U|^+$. Let g be a $(\gamma_1,\dots,\gamma_k)$–testing, we define $f : U^{<\omega} \to \gamma_S$ by

$$f(a) = \psi_S(a, g \restriction a) \quad \text{if } a \text{ is } (A,g) - \text{bad}$$
$$= 0 \qquad \text{otherwise.}$$

Obviously, f is an R–character corresponding to A and g. □

3. Ramsey Systems

Definition 3.1. A k–system $\mathcal{M}$ in U is a set of k–sheaves. A k–system $\mathcal{M}$ is said to have the R–property, if each $A \in \mathcal{M}$ has the R-property. In that case we define the R–ordinals

$$\Phi_{\mathcal{M}}(\gamma_1, \dots, \gamma_k) := \sup\{\Phi_A(\gamma_1, \dots, \gamma_k) \mid A \in \mathcal{M}\}.$$

Definition 3.2. Let $r : [U]^n \to \{1, \dots, k\}$. The Ramsey k–sheaf $R^r = (R_1^r, \dots, R_k^r)$ was defined in 2.2. We put

$$\mathcal{R}_k^n := \{R^r \mid r : [U]^n \to \{1, \dots, k\}\}.$$

Clearly, $\mathcal{R}_k^n$ has the R–property. It will be called the *Ramsey k-system.* We shall write $\Phi_n(\gamma_1, \dots, \gamma_k)$ instead of $\Phi_{\mathcal{R}_k^n}(\gamma_1, \dots, \gamma_k)$.

Proposition 3.3. *If* $\gamma_1, \dots, \gamma_k$ *are finite, then*

$$\Phi_n(\gamma_1, \dots, \gamma_k) + 1 = R(n; \gamma_1, \dots, \gamma_k),$$

the Ramsey number.

Proof. Let us consider the universal $(\gamma_1, \dots, \gamma_k)$–testing g. Let $a_1, \dots, a_m \in U$ and let

$$r : [\{a_1, \dots, a_m\}]^n \to \{1, \dots, k\}$$

be such that there is no $E \subseteq \{a_1, \dots, a_m\}$ such that $|E| = \gamma_i$ and $[E]^n$ is colored i, and extend r to $[U]^n$ arbitrarily. Then $a = (a_1, \dots, a_m)$ is (R^r, g)–bad, hence there is a killing $f : U^{<\omega} \to \Phi_{R^r}(\gamma_1, \dots, \gamma_k)$ on $\downarrow a$, which implies

$$m \leq \Phi_{R^r}(\gamma_1, \dots, \gamma_k) \leq \Phi_n(\gamma_1, \dots, \gamma_k).$$

To show the converse inequality let a k–sheaf $R^r \in \mathcal{R}_k^n$ be given, we define $f : U^{<\omega} \to R(n; \gamma_1, \dots, \gamma_k)$ by

$$f(a) := (R(n; \gamma_1, \dots, \gamma_k) - |a| - 1)^+$$

and for $a = (a_1, \dots, a_m) \in U^{<\omega}$ we define $\overline{r} := [\{1, \dots, m\}]^n \to \{1, \dots, k\}$ by

$$\begin{aligned} \overline{r}(\{i_1, \dots, i_n\}) &= r(\{a_{i_1}, \dots, a_{i_n}) \quad \text{if } (a_{i_1}, \dots, a_{i_n}) \text{ is injective} \\ &= \text{arbitrarily} \quad \text{otherwise} \end{aligned}$$

If a is (R^r, g)–bad, then there is no $E \subseteq \{1, \dots, m\}$ such that $|E| = \gamma_i$ and $[E]^n$ is $\overline{r}$–colored i. Hence $m < R(n; \gamma_1, \dots, \gamma_k)$ and thus f is a killing on $\downarrow a$. □

Remark 3.4. The R–ordinal $\Phi_n(\omega, \dots, \omega)$ corresponds to a statement whose finite miniaturization is the Paris–Harrington principe (Paris, Harrington 1977), i.e. the statement $\forall n\ \forall k\ \forall n_1, \dots, n_k\ \exists N$ such that for any k–coloring of $[\{1, \dots, N\}]^n$ there exists $A \subseteq \{1, \dots, N\}$ and $i \in \{1, \dots, k\}$ such that $|A| \geq n_i$ and $[A]^n$ is colored i, and, moreover A is relatively large, i.e. $|A| > \min A$. Indeed, letting $U = \omega$ and $g = (g_1, \dots, g_k)$, where

$$g_i(a_1,\ldots,a_m) = \max(\gamma_i - m, \min\{a_1,\ldots,a_m\} - m, 0)$$

we see that the corresponding Ramsey character kills subsets without monochromatic relatively large subsets.

One might expect that because of unprovability of the Paris–Harrington principe from PA it would hold

$$\sup\{\Phi_n(\underbrace{\omega,\ldots,\omega}_{k \text{ times}}) \mid n,k \in \omega\} = \varepsilon_0.$$

This is in fact true, we prove it in Section 6.

In this section we stay on a rather abstract level. Of course, we have an analogy of 2.7 for Ramsey systems, but there is no general analogy of 2.8. We search for a restricted class of Ramsey systems for which such a statement would hold.

Definition 3.5. A k–system $\mathcal{M}$ is said to have the *strong R-property* if for each sequence $A^i = (A^i_1,\ldots,A^i_k)$ of elemets of $\mathcal{M}$ and for each infinite sequence p in U there exists a subsequence $s \subseteq p$ such that for each finite segment $a \ll s$ there exists an $i \geq |a|$ with $a \in A^i_1 \cup \ldots \cup A^i_k$. Note that the condition actually implies a to be in A^i_j for $j \in \{1,\ldots,k\}$ fixed and infinitely many i.

Definition 3.6. Let $\mathcal{M}$ be a k–system and $\gamma_1,\ldots,\gamma_k$ ordinals. We define the *germ tree* $T(\mathcal{M};\gamma_1,\ldots,\gamma_k) = (T,\leq)$ by

$T := \{(a,g) \restriction a \in U^{<\omega}$ and there is $A \in \mathcal{M}$ such that g is an $(A;\gamma_1,\ldots,\gamma_k)$–germ on $a\}$,

$(a,g) < (b,h)$ if $a \ll b$ and $g = h \restriction\!\downarrow a$.

If the germ tree has no infinite chain we define the *strong R-ordinal* by

$$\overline{\Phi}_{\mathcal{M}}(\gamma_1,\ldots,\gamma_k) := \gamma_{T(\mathcal{M};\gamma_1,\ldots,\gamma_k)}$$

and we write $\overline{\Phi}_n(\gamma_1,\ldots,\gamma_k)$ instead of $\overline{\Phi}_{\mathcal{R}^n_k}(\gamma_1,..,\gamma_k)$. In general the existence of $\overline{\Phi}_{\mathcal{M}}$ implies the existence of $\Phi_{\mathcal{M}}$, $\overline{\Phi}_{\mathcal{M}} \geq \Phi_{\mathcal{M}}$ and nothing more holds.

Theorem 3.7. *A k-system $\mathcal{M}$ has the strong R–property if and only if all the strong R–ordinals $\overline{\Phi}_{\mathcal{M}}(\gamma_1,\ldots,\gamma_k)$ exist for $\gamma_1,\ldots,\gamma_k < |U|^+$. In that case we have*

$$\Phi_{\mathcal{M}}(\gamma_1,\ldots,\gamma_k) \leq \overline{\Phi}_{\mathcal{M}}(\gamma_1,\ldots,\gamma_k)$$

for any ordinals $\gamma_1,\ldots,\gamma_k$ and $\overline{\Phi}_{\mathcal{M}}(\gamma_1,\ldots,\gamma_k) < |U|^+$ for $\gamma_1,\ldots,\gamma_k < |U|^+$.

Proof. The inequality $\overline{\Phi}_{\mathcal{M}}(\gamma_1,\ldots,\gamma_k) < |U|^+$ is obvious by a cardinality argument. To see $\Phi_{\mathcal{M}}(\gamma_1,\ldots,\gamma_k) \leq \overline{\Phi}_{\mathcal{M}}(\gamma_1,\ldots,\gamma_k)$, let a $(\gamma_1,\ldots,\gamma_k)$–testing g be given, and let $A \in \mathcal{M}$. We define $f: U^{<\omega} \to \overline{\Phi}_{\mathcal{M}}(\gamma_1,\ldots,\gamma_k)$ by

$$\begin{aligned} f(a) &= \psi_{T(\mathcal{M};\gamma_1,\ldots,\gamma_k)}(a, g \restriction\!\downarrow a) && \text{if } a \text{ is } (A,g)\text{–bad} \\ &= 0 && \text{otherwise.} \end{aligned}$$

It is easily seen that f is an R–character corresponding to g.

Let us pass to the proof of the equivalence.
$\Rightarrow$: We must prove that $T(\mathcal{M};\gamma_1,\dots,\gamma_k)$ has no infinite chain. Suppose that $(a^1,g^1),(a^2,g^2),\dots$ is an infinite chain in $T(\mathcal{M};\gamma_1,\dots,\gamma_k)$, let $g^i=(g_1^i,\dots,g_k^i)$ be an $(A^i,\gamma_1,\dots,\gamma_k)$–germ on a^i, let $A^i=(A_1^i,\dots,A_k^i)$. By the strong R–property there exists an infinite sequence s such that for every its finite segment $s_i \ll s$ of length i there exists $n(i)$ such that $s_i \subseteq a^{n(i)}$ and $s_i \in A_1^{n(i)} \cup \dots \cup A_k^{n(i)}$. We may assume that $s_i \in A_j^{n(i)}$ with $j \in \{1,\dots,k\}$ fixed and that $n(1)<n(2)<\dots$. Now, by the definition of $T(\mathcal{M};\gamma_1,\dots,\gamma_k)$ we have

$$\gamma_j > g_j^{n(1)}(s_1) > g_j^{n(2)}(s_2) > \dots ,$$

which is a contradiction.
$\Leftarrow$: Let $A^n=(A_1^n,\dots,A_k^n)$ be a sequence in $\mathcal{M}$ and p an infinite sequence in U each infinite subsequence s of which has a finite segment $a \ll s$ such that $a \in A_j^n$ for $j \in \{1,\dots,k\}$ implies $n<|a|$. Let S_j be the subtree of $U^{<\omega}$ consisting of all $a \subseteq p$ such that $a \in A_j^n$ for some $j \in \{1,\dots,k\}$ and some $n \geq |a|$. By the assumption, S_j contains no infinite chain. Now for $a \ll p$ let $g_j^a : \downarrow a \to \gamma_{S_j}$ be defined by

$$g_j^a = \psi_{S_j}(b) \text{ if } b \in S_j$$
$$= 0 \qquad \text{otherwise.}$$

We see easily that $g^a=(g_1^a,\dots,g_k^a)$ is an $(A^{|a|};\gamma_{S_1},\dots,\gamma_{S_k})$–germ on a. Thus, $\{(a,g^a)\}_a$, where a runs through finite segments of p, is an infinite chain in $T(\mathcal{M};\gamma_1,\dots,\gamma_k)$, showing that $\overline{\Phi}_{\mathcal{M}}(\gamma_{S_1},\dots,\gamma_{S_k})$ does not exist. □

Proposition 3.8. *The Ramsey k–system $\mathcal{R}_k^n$ has the strong R–property.*

Proof. This follows either from 3.17 or from the estimates of $\overline{\Phi}_n$ given below, but we give a direct proof. Let $(A^{r_j})_{j\in\omega}$ be a sequence of elements of $\mathcal{R}_k^n$. Let us choose a non–trivial ultrafilter $\mathcal{U}$ on ω and define $r:[U]^n \to \{1,\dots,k\}$ by

$$r(\{x_1,\dots,x_n\})=i \text{ iff } \{j \mid r_j(\{x_1,\dots,x_n\})=i\} \in \mathcal{U}.$$

By Ramsey theorem there is for each infinite sequence p in U an infinite subsequence $s \subseteq p$ such that every finite segment a of s belongs to $A_1^r \cup \dots \cup A_k^r$. By the definition of r there is $j \geq |a|$ such that $r_j(\{x_1,\dots,x_n\})=r(\{x_1,\dots,x_n\})$ for all injective subsequences $(x_1,\dots,x_n)\subseteq a$, hence a belongs to $A_1^{r_j}\cup\dots\cup A_k^{r_j}$. □

Theorem 3.9. *If $\mathcal{M}$ is finite and $\gamma_1,\dots,\gamma_k$ are also finite, then $\Phi_{\mathcal{M}}(\gamma_1,\dots,\gamma_k)$ exists if and only if $\overline{\Phi}_{\mathcal{M}}(\gamma_1,\dots,\gamma_k)$ exists and $\Phi_{\mathcal{M}}(\gamma_1,\dots,\gamma_k)=\overline{\Phi}_{\mathcal{M}}(\gamma_1,\dots,\gamma_k)$.*

Proof. Let $\mathcal{M}=\{A_1,\dots,A_m\}$, let $T=T(\mathcal{M};\gamma_1,\dots,\gamma_k)$ be the germ tree and for $i=1,\dots,m$ let T_i be the subtree T_i consisting of all $(a,g)\in T$ such that g is an $(A_i;\gamma_1,\dots,\gamma_k)$–germ on a. Then each T_i is downwards–closed in T (i.e. $(a,g)\leq(b,h)$ and $(b,h)\in T_i$ imply $(a,g)\in T_i$), hence

$$\overline{\Phi}_{\mathcal{M}}(\gamma_1,\dots,\gamma_k) = \gamma_T = \max\{\gamma_{T_i} \mid 1 \le i \le m\}$$
$$= \max\{\overline{\Phi}_{A_i}(\gamma_1,\dots,\gamma_k) \mid 1 \le i \le j\}.$$

Thus it is sufficient to prove that $\Phi_{\mathcal{M}}(\gamma_1,\dots,\gamma_k) \ge \overline{\Phi}_{\mathcal{M}}(\gamma_1,\dots,\gamma_k)$ for $|\mathcal{M}| = 1$. We proceed by transfinite induction with the following induction hypothesis. (H_α) : For any natural number k, any k–sheaf A and any finite numbers $\gamma_1,\dots,\gamma_k$ such that $\Phi_A(\gamma_1,\dots,\gamma_k) \le \alpha$ it holds $\Phi_A(\gamma_1,\dots,\gamma_k) \ge \Phi_A(\gamma_1,\dots,\gamma_k)$.

Assume that (H_β) is true for any $\beta < \alpha$ and let $k, A, \gamma_1,\dots,\gamma_k$ such that $\Phi_A(\gamma_1,\dots,\gamma_k) = \alpha$ be given. We denote by T the germ tree $T(\{A\};\gamma_1,\dots,\gamma_k)$. Let $a \in U^{<\omega}$ be a one–element sequence and let $g = (g_1,\dots,g_k)$ be an $(A;\gamma_1,\dots,\gamma_k)$–germ on a. We define a $2k$–sheaf $A^\alpha = (A^\alpha_1,\dots,A^\alpha_{2k})$ by $A^\alpha_i = A_i$, $A^\alpha_{k+i} = \{b \in U^{<\omega} \mid a.b \in A_i\}$ $(i = 1,\dots,k)$ and a tree homomorphism

$$H : T_{(a,g)} \to T(\{A^a\};\gamma_1,\dots,\gamma_k;g_1(a),\dots,g_k(a))$$

by $H(a.b,h) = (b,\overline{h})$, where $\overline{h}_i(x) = h_i(x)$ and $\overline{h}_{k+i}(x) = h_i(a.x)$ for $x \subseteq b$. Thus

$$\overline{\Phi}_{\{A\}}(\gamma_1,\dots,\gamma_k) \le \sup\{\overline{\Phi}_{\{A^a\}}(\gamma_1,\dots,\gamma_k,g_1(a),\dots,g_k(a)) + 1 \mid (a,g) \in T\}.$$

Now we are going to estimate the corresponding R–ordinals. Given finite numbers $\gamma_1,\dots,\gamma_{2k}$ such that $\gamma_i > \gamma_{i+k}$ $(i = 1,\dots,k)$, let $\overline{h} = (\overline{h}_1,\dots,\overline{h}_{2k})$ be the universal $(\gamma_1,\dots,\gamma_{2k})$–testing. We define a $(\gamma_1,\dots,\gamma_k)$–tesing $h = (h_1,\dots,h_k)$ by

$$\begin{aligned} h_i(x) &= \overline{h}_i(x) && \text{if neither } a = x \text{ nor } a \ll x \\ &= \gamma_{k+i} && \text{if } x = a \\ &= \overline{h}_{k+i}(b) && \text{if } x = a.b. \end{aligned}$$

It is easily seen that if $b \in U^{<\omega}$ is $(A^a,\overline{h})$–bad, then $a.b$ is (A,h)–bad. If f is an R–character corresponding to A and h, we may define $\overline{f} := U^{<\omega} \to f(a)$ by $\overline{f}(b) = f(a.b)$. Then $\overline{f}$ is clearly an R–character corresponding to A and $\overline{h}$, thus showing $\Phi_{\{A^a\}}(\gamma_1,\dots,\gamma_{2k}) = \gamma_{\mathrm{Bad}(A^a,\overline{h})} < \gamma_{\mathrm{Bad}(A,h)} \le \Phi_{\{A\}}(\gamma_1,\dots,\gamma_k)$. (Here the finiteness of $\gamma_1,\dots,\gamma_k$ is crucial.) Hence we may use the induction hypothesis. The above inequalities and Theorem 3.7 conclude the proof. □

Theorem 3.10. *If $\Phi_{\mathcal{M}}(\gamma_1,\dots,\gamma_k) < \omega$, then $\overline{\Phi}_{\mathcal{M}}(\gamma_1,\dots,\gamma_k) = \Phi_{\mathcal{M}}(\gamma_1,\dots,\gamma_k)$.*

Proof. If $g = (g_1,\dots,g_k)$ is an $(A;\gamma_1,\dots,\gamma_k)$–germ on $a = (a_1,\dots,a_m)$, then extending each g_i by $g_i(b) = 0$ for $b \notin \downarrow a$ we see that a is (A,g)–bad. Hence $m \le \Phi_{\mathcal{M}}(\gamma_1,\dots,\gamma_k)$ and consequently $\overline{\Phi}_{\mathcal{M}}(\gamma_1,\dots,\gamma_k) \le \Phi_{\mathcal{M}}(\gamma_1,\dots,\gamma_k)$. The converse inequality follows from 3.7. □

Remark 3.11. For no k–system $\mathcal{M}$ one can expect $\Phi_{\mathcal{M}}(\gamma_1,\dots,\gamma_k) = \overline{\Phi}_{\mathcal{M}}(\gamma_1,\dots,\gamma_k)$ for *any* ordinals $\gamma_1,\dots,\gamma_k$, since $\Phi_{\mathcal{M}}(\gamma_1,\dots,\gamma_k) \le |U|^+$ by 2.6(ii), while it is an easy exercise that $\overline{\Phi}_{\mathcal{M}}(\gamma_1,\dots,\gamma_k) \ge \min\{\gamma_1,\dots,\gamma_k\}$. On the other hand it is easy to construct k–systems such that even $\Phi_{\mathcal{M}}(1) < \overline{\Phi}_{\mathcal{M}}(1)$ or $\Phi_{\mathcal{M}}(1)$ exists and $\overline{\Phi}_{\mathcal{M}}(1)$ does not. Namely, for $a = (a_1,\dots,a_m) \in U^{<\omega}$ we define

$$A^a := \{(x_1, \ldots, x_n) \in U^{<\omega} \mid \text{ either } x_1 = \ldots = x_n \text{ or } x_i \neq a_j \text{ for all } i, j\}.$$

Clearly $\Phi_{A^a}(1) = |a|$, thus letting $\mathcal{M} = \{A^a \mid a \in U\}$ we have $\Phi_{\mathcal{M}}(1) = \omega$, while the corresponding strong R–ordinal does not exist. If $S \subseteq U^{<\omega}$ is a tree without infinite chains, then for $\mathcal{M}(S) := \{A^a \mid a \in S\}$ we have $\Phi_{\mathcal{M}(S)}(1) \leq \omega$, while $\overline{\Phi}_{\mathcal{M}(S)}(1) \geq \gamma_S$.

An example of a k–system consisting only of one k–sheaf for which the R-ordinals and strong R–ordinals differ for small ordinals is given in the next section (see Remark 4.12).

In the rest of this section we prove that for a large class of k–systems the R-ordinals and strong R–ordinals coincide for all ordinals $\gamma_1, \ldots, \gamma_k < |U|^+$.

Definition 3.12. We say that a k–system $\mathcal{M}$ is *movable*, if for any k–sheaf $A = (A_1, \ldots, A_k) \in \mathcal{M}$, any sequence $(a_1, \ldots, a_m) \in U^{<\omega}$ and any injective sequence $(b_1, \ldots, b_m) \in U^{<\omega}$ there exists a k–sheaf $B = (B_1, \ldots, B_k) \in \mathcal{M}$ such that if $(b_{i_1}, \ldots, b_{i_p}) \in B_j$ for some $1 \leq i_1 < i_2 < \ldots < i_p \leq m$ and $1 \leq j \leq k$, then $(a_{i_1}, \ldots, a_{i_p}) \in A_j$.

Let $A = (A_1, \ldots, A_k)$, $B = (B_1, \ldots, B_k)$ be two k–sheaves and let $V \subseteq U^{<\omega}$. We say that $A = B$ on V if for every sequence $a \in V$ and any $j \in \{1, \ldots, k\}$ we have $a \in A_j$ iff $a \in B_j$.

We say that a k–system $\mathcal{M}$ has the *concatenation property*, if for any family $\{A^\alpha\}_{\alpha \in A}$ of elements of $\mathcal{M}$, any family of subsets $\{V_\alpha\}_{\alpha \in A}$ of $U^{<\omega}$ such that each V_α is closed under subsequences and $A^\alpha = A^\beta$ on $V_\alpha \cap V_\beta$ for any $\alpha, \beta \in A$ there exists a k–sheaf $A \in \mathcal{M}$ such that $A^\alpha = A$ on V_α for every $\alpha \in A$.

A movable k–system which has the concatenation property will be called *standard*. For example, the Ramsey k–system is standard.

Theorem 3.13. *Let $\mathcal{M}$ be a movable k–system satisfying the strong R–property and V an infinite subset of U. For $A = (A_1, \ldots, A_k) \in \mathcal{M}$ we put $A^V = (A_1^V, \ldots, A_k^V)$, where*

$$A_i^V := A_i \cap V^{<\omega}$$

and further

$$\mathcal{M}^V := \{A^V \mid A \in \mathcal{M}\}.$$

Then $\mathcal{M}^V$ satisfies the strong R–property (in V) and

$$\overline{\Phi}_{\mathcal{M}}(\gamma_1, \ldots, \gamma_k) = \overline{\Phi}_{\mathcal{M}^V}(\gamma_1, \ldots, \gamma_k)$$

for any ordinals $\gamma_1, \ldots, \gamma_k$.

Proof. Let $a_1, a_2, \ldots$ be distinct elements of V. Let $(b, h) \in T(\mathcal{M}; \gamma_1, \ldots, \gamma_k)$, where $b = (b_1, \ldots, b_m)$ and $h = (h_1, \ldots, h_k)$; we define $H(b, h) = (a, g)$ by $a = (a_1, \ldots, a_m)$ and $g = (g_1, \ldots, g_k)$, where

$$g_i(a_{i_1}, \ldots, a_{i_p}) = h_i(b_{i_1}, \ldots, b_{i_p}).$$

Let S be the range of H. By movability, S is a subtree of $T(\mathcal{M}^V; \gamma_1, \ldots, \gamma_k)$ and H is a tree homomorphism, showing that

$$\overline{\Phi}_{\mathcal{M}}(\gamma_1,\ldots,\gamma_k) = \gamma_{T(\mathcal{M};\gamma_1,\ldots,\gamma_k)} \le \gamma_S \le \overline{\Phi}_{\mathcal{M}^V}(\gamma_1,\ldots,\gamma_k).$$

Since the converse inequality is obvious, we are done. □

Definition 3.14. Let T be a subtree of the germ–tree $T(\mathcal{M};\gamma_1,\ldots,\gamma_k)$. We say that T is *simple* if whenever $(a,g),(b,h) \in T$ are such that a and b have the same last term, then $(a,g) = (b,h)$. Similarly, $S \subseteq U^{<\omega}$ is called *simple* if whenever $a,b \in S$ have the same last term, then $a = b$.

Lemma 3.15. *If $\mathcal{M}$ is a movable k-system satisfying the strong R–property and $\gamma_1,\ldots,\gamma_k < |U|^+$ then there exists a simple subtree S of the germ tree $T(\mathcal{M};\gamma_1,\ldots,\gamma_k)$ such that $\gamma_S = \overline{\Phi}_{\mathcal{M}}(\gamma_1,\ldots,\gamma_k)$.*

Proof. Let us denote by T the germ tree $T(\mathcal{M};\gamma_1,\ldots,\gamma_k)$, let $I : T \to U$ be defined by $I((a_1,\ldots,a_m),g) = a_m$ and let $J : T \to U$ be a bijection. For a $(\gamma_1,\ldots,\gamma_k)$–testing $g = (g_1,\ldots,g_k)$ we define a $(\gamma_1,\ldots,\gamma_k)$–testing $g' = (g_1',\ldots,g_k')$ by

$$g_i'(a_1,\ldots,a_m) = g(IJ^{-1}(a_1),\ldots,IJ^{-1}(a_m))$$

and for $(a,g) = ((a_1,\ldots,a_m),g) \in T$ we define

$$\begin{aligned} a' =&(J((a_1),g \restriction\downarrow (a_1)), J((a_1,a_2),g \restriction\downarrow (a_1,a_2)),\ldots \\ &\ldots, J((a_1,\ldots,a_m),g \restriction\downarrow (a_1,\ldots,a_m))). \end{aligned}$$

Now if g is an $(A;\gamma_1,\ldots,\gamma_k)$–germ on a, there exists by movability a k–sheaf $A' \in \mathcal{M}$ such that g' is an $(A';\gamma_1,\ldots,\gamma_k)$–germ on a'. Hence the mapping $H : T \to T$ defined by $H(a,g) = (a',g')$ is a tree homomorphism. Thus denoting by S the image of T under H we see that

$$\overline{\Phi}_{\mathcal{M}}(\gamma_1,\ldots,\gamma_k) \ge \gamma_S \ge \gamma_T = \overline{\Phi}_{\mathcal{M}}(\gamma_1,\ldots,\gamma_k),$$

moreover S is clearly simple. □

Theorem 3.16. *A k–system which has the R–property and the concatenation property has the strong R–property.*

Proof. Let $\mathcal{M}$ be as above, let $\gamma_1,\ldots,\gamma_k$ be ordinals and let (a^i,g^i) be an infinite chain in $T(\mathcal{M};\gamma_1,\ldots,\gamma_k)$, $a^i = (x_1,\ldots,x_i)$, $g^i = (g_1^i,\ldots,g_k^i)$, let g^i be an $(A^i;\gamma_1,\ldots,\gamma_k)$–germ on a^i, $A^i \in \mathcal{M}$. By an easy compactness argument we may pick an increasing sequence $i_1,i_2,\ldots$ such that

$$A^{i_n} = A^{i_m} \text{ on } \downarrow a^n \text{ for any } m \ge n.$$

Now apply the concatenation property to the family of sheaves $\{A^{i_n}\}_n$ and to the family of sets $\{\downarrow a^n\}_n$. We obtain a sheaf $A \in \mathcal{M}$ with

$$A = A^{i_n} \text{ on } \downarrow a^n.$$

For $x \subseteq (x_1,\ldots,x_n)$ we put $g_j(x) = g_j^n$ $(j = 1,\ldots,k)$. Then $g = (g_1,\ldots,g_k)$ is well-defined and the sequence $(x_1,\ldots,x_n,\ldots)$ is (A,g)–bad. A contradiction. Thus $\overline{\Phi}_{\mathcal{M}}(\gamma_1,\ldots,\gamma_k)$ exists. Apply 3.7 □

Theorem 3.17. *If $\mathcal{M}$ is standard and has the R-property and $\gamma_1, \ldots, \gamma_k < |U|^+$, then*

$$\Phi_{\mathcal{M}}(\gamma_1, \ldots, \gamma_k) = \overline{\Phi}_{\mathcal{M}}(\gamma_1, \ldots, \gamma_k).$$

Proof. Let T be a simple subtree of $T(\mathcal{M}; \gamma_1, \ldots, \gamma_k)$ of type $\overline{\Phi}_{\mathcal{M}}(\gamma_1, \ldots, \gamma_k)$. We shall find a subtree S of T of the same type, a $(\gamma_1, \ldots, \gamma_k)$–testing h and a k–sheaf $A \in \mathcal{M}$ such that h is an $(A; \gamma_1, \ldots, \gamma_k)$–germ on a for every $(a, g) \in S$.

Let T_Π be the tree obtained from T by formally adding a least element Π. For $z \in T_\Pi$ we define

$$T(z) := \{z' \in T \mid z \leq z' \text{ or } z' \leq z\}$$

(so that $T(\Pi) = T$) and $V(z) := \bigcup\{\downarrow a \mid (a, g) \in T(z)\}$. Let us observe that by simplicity if $z = (a, g) \in T$ and z', z'' are its distinct successors, then $V(z') \cap V(z'') = \downarrow a$. We shall construct for every $z \in T_\Pi$ a k–sheaf A^z, a subtree $S(z)$ of $T(z)$ and a k–tuple $g^z = (g_1^z, \ldots, g_k^z)$ of functions such that

(3.17a) for every $(a, g) \in S(z)$, g^z is an $(A^z; \gamma_1, \ldots, \gamma_k)$–germ on a, and

(3.17b) $\gamma_{S(z)} = \gamma_{T(z)}$. If $z = (a, g) \in T$ is such that $\psi_T(z) = 0$, then let $S(z) = T(z)$, $g^z = g$ and let A^z be such that g is an $(A^z; \gamma_1, \ldots, \gamma_k)$–germ on a.

If $\psi_T(z)$ is a successor ordinal, then $\psi_T(z) = \psi_T(z') + 1$ for some $z \leq z'$. We put $S(z) := S(z')$, $A^z := A^{z'}$ and $g^z := g^{z'}$.

Finally if $\psi_T(z)$ is a limit ordinal, let $(z_\alpha)_{\alpha \in \Lambda}$ be successors of z in T such that $\sup\{\psi_T(z_\alpha) \mid \alpha \in \Lambda\} = \psi_T(z)$. There is a subset $\Lambda' \subseteq \Lambda$ such that

(3.17c) $\sup\{\psi_T(z_\alpha) \mid \alpha \in \Lambda'\} = \psi_T(z)$

and if $z \neq \Pi$, say $z = (a, g) \in T$, then for every $b \subseteq a$, every $\alpha, \beta \in \Lambda'$ and every $i \in \{1, \ldots, k\}$ $b \in A_i^{z_\alpha}$ iff $b \in A_i^{z_\beta}$, hence $A^{z_\alpha} = A^{z_\beta}$ on $V(z_\alpha) \cap V(z_\beta)$. By the concatenation property there exists a k–sheaf $A^z \in \mathcal{M}$ such that $A^z = A^{z_\alpha}$ on $V(z_\alpha)$ for every $\alpha \in \Lambda'$. We define $g^z = (g_1^z, \ldots, g_k^z)$ by

$$\begin{aligned} g_i^z(c) &= g_i^{z_\alpha}(c) \quad \text{if } c \in V(z_\alpha) \text{ for } \alpha \in \Lambda' \\ &= 0 \qquad\quad\;\; \text{if no such } \alpha \text{ exists.} \end{aligned}$$

Then g_i^z are well–defined and it follows that (3.17a) is satisfied for $S(z) := \bigcup\{S(z_\alpha) \mid \alpha \in \Lambda'\}$; condition (3.17b) follows from (3.17c).

Now put $S := S(\Pi)$, $h := g^\Pi$ and $A := A^\Pi$. Then $S \subseteq \mathrm{Bad}(A, h)$, hence

$$\overline{\Phi}_{\mathcal{M}}(\gamma_1, \ldots, \gamma_k) = \gamma_S \leq \gamma_{\mathrm{Bad}(A,h)} \leq \Phi_{\mathcal{M}}(\gamma_1, \ldots, \gamma_k).$$

The converse inequality follows from 3.7. □

Corollary 3.18 of the proof. *If $\mathcal{M}$ is standard and $\gamma_1, \ldots, \gamma_k < |U|^+$, then there exist $A \in \mathcal{M}$, a $(\gamma_1, \ldots, \gamma_k)$–testing g and a simple subtree S of* Bad(A, g) *such that*

$$\overline{\Phi}_{\mathcal{M}}(\gamma_1,\ldots,\gamma_k) = \Phi_{\mathcal{M}}(\gamma_1,\ldots,\gamma_k) = \Phi_A(\gamma_1,\ldots,\gamma_k) = \gamma_{\mathrm{Bad}(A,g)} = \gamma_S.$$

In particular, there exists a universal testing.

4. Well-partial-ordering

Definition 4.1. Let $(Q,\leq)$ be a partially ordered set. A sequence $q_1, q_2, \ldots$ (finite or not) of elements of Q is called *good* if there are indices i, j such that $i < j$ and $q_i \leq q_j$ and is called *bad* otherwise. The set Q is called *well-partially-ordered (wpo)* if every bad sequence is finite.

Proposition 4.2. *The following conditions on a partially ordered set $(Q,\leq)$ are equivalent.*

(i) $(Q,\leq)$ *is wpo.*

(ii) For every infinite sequence $q_1, q_2, \ldots$ of elements of Q there is an increasing sequence $i_1, i_2, \ldots$ of natural numbers such that $q_{i_1} \leq q_{i_1} \leq \ldots$.

(iii) There is neither an infinite decreasing sequence in Q nor an infinity of mutually incomparable elements of Q.

(iv) Every linear extension of $\leq$ is a well–ordering.

(v) Every nonempty subset of Q has at least one but only finitely many minimal elements.

Proof. Easy consequence of Ramsey's theorem. □

Definition 4.3. Let Q be a partially ordered set. For $q_1, q_2 \in Q$ we write $q_1 < q_2$ if $q_1 \leq q_2$, and $q_1 \neq q_2$, and $q_1 \not\leq q_2$ if not $q_1 \leq q_2$. For a cardinal κ we put $U := \kappa \times Q$ and we introduce the following sheaves in U :

$$\begin{aligned}
\mathrm{Asc}(U) &:= \{((\alpha_1,q_1),\ldots,(\alpha_m,q_m)) \in U^{<\omega} \mid q_1 \leq q_2 \leq \ldots \leq q_m\},\\
\mathrm{Bad}(U) &:= \{((\alpha_1,q_1),\ldots,(\alpha_m,q_m)) \in U^{<\omega} \mid q_i \not\leq q_j \text{ for } i<j\},\\
\mathrm{Nd}(U) &:= \{((\alpha_1,q_1),\ldots,(\alpha_m,q_m)) \in U^{<\omega} \mid q_i > q_j \text{ for no } i<j\},\\
\mathrm{Dec}(U) &:= \{((\alpha_1,q_1),\ldots,(\alpha_m,q_m)) \in U^{<\omega} \mid q_1 > q_2 < \ldots > q_m\},\\
\mathrm{Inc}(U) &:= \{((\alpha_1,q_1),\ldots,(\alpha_m,q_m)) \in U^{<\omega} \mid q_i \leq q_j \text{ for no } i \neq j\},\\
\mathrm{Comp}(U) &:= \{((\alpha_1,q_1),\ldots,(\alpha_m,q_m)) \in U^{<\omega} \mid \text{ for all } i,j \text{ either } q_i \leq q_j \text{ or }\\
&\qquad q_j \leq q_i\}.
\end{aligned}$$

For $\kappa = 1$ we identify U with Q; thus the sheaves $\mathrm{Asc}(Q), \ldots, \mathrm{Comp}(Q) \subseteq Q^{<\omega}$ are defined.

If Q is wpo, then $\mathrm{Bad}(U), \mathrm{Dec}(U)$ and $\mathrm{Inc}(U)$ are trees without infinite chains. Hence they have types, which were denoted by $\gamma_{\mathrm{Bad}(U)}, \gamma_{\mathrm{Dec}(U)}$ and $\gamma_{\mathrm{Inc}(U)}$, respectively.

Let g be the universal (1)–testing. Then a sequence $(\alpha_1, g_1), (\alpha_2, g_2), \ldots$ is $(\mathrm{Asc}(U), g)$–bad if and only if the sequence $q_1, q_2, \ldots$ is bad. (This should justify our terminology). Hence the sheaf $\mathrm{Asc}(U)$ has the R–property if and only if Q is wpo. Moreover

$$\overline{\Phi}_{\{\mathrm{Asc}(U)\}}(1) = \Phi_{\mathrm{Asc}(U)}(1) = \overline{\Phi}_{\{\mathrm{Asc}(Q)\}}(1) = \Phi_{\mathrm{Asc}(Q)}(1) = \gamma_{\mathrm{Bad}(U)} = \gamma_{\mathrm{Bad}(Q)}.$$

This ordinal, denoted by $c(Q)$, is called the *type* of the wpo set Q. Similarly, we have

$$\overline{\Phi}_{\{Nd(U)\}}(1) = \Phi_{Nd(U)}(1) = \overline{\Phi}_{\{Nd(Q)\}}(1) = \Phi_{Nd(Q)}(1) = \gamma_{\mathrm{Dec}(U)} = \gamma_{\mathrm{Dec}(Q)}.$$

This ordinal will be called the *height* of Q and will be denoted by $\mathrm{ht}(Q)$. Finally,

$$\overline{\Phi}_{\{\mathrm{Comp}(U)\}}(1) = \Phi_{\mathrm{Comp}(U)}(1) = \overline{\Phi}_{\{\mathrm{Comp}(Q)\}}(1) = \Phi_{\mathrm{Comp}(Q)}(1) = \\ = \gamma_{\mathrm{Inc}(U)} = \gamma_{\mathrm{Inc}(Q)}.$$

The last ordinal will be called the *width* of Q and will be denoted by $wd(Q)$.

For $q \in Q$ we define

$$\begin{aligned} \mathrm{ht}(q) &= 0 && \text{if } q \text{ is a minimal element of } Q \\ &= \sup\{\mathrm{ht}(q') + 1 \mid q' < q\} && \text{otherwise.} \end{aligned}$$

Clearly $\mathrm{ht}(Q) = \sup\{\mathrm{ht}(q) + 1 \mid q \in Q\}$.

Definition 4.4. For $(q_1, \ldots, q_m) \in \mathrm{Bad}(Q)$ we put

$$\begin{aligned} &Q_{(q_1,\ldots,q_m)} := \{q \in Q \mid (q_1, \ldots, q_m, q) \in \mathrm{Bad}(Q)\}, \\ &Q_{q_1} := Q_{(q_1)}, \\ &cl(q_1) := \{q \in Q \mid q_1 \leq q\} \end{aligned}$$

and we denote

$$\begin{aligned} &\lambda(Q) := \sup\{\alpha \in Qn \mid \alpha \text{ is the (ordinal) type of a linear extension of } \leq\}, \\ &\chi(Q) := \sup\{\alpha \in Qn \mid \alpha \text{ is the (ordinal) type of a chain in } Q\}. \end{aligned}$$

Let us remark that $c(Q) = \sup\{c(Q_q) + 1 \mid q \in Q\}$.

If Q_1 and Q_2 are partially ordered sets, then $Q_1 \dot{\cup} Q_2$ denotes the disjoint union of Q_1 and Q_2 whose partial ordering is the disjoint union of the partial orderings on Q_1 and Q_2, and $Q_1 \times Q_2$ denotes the Cartesian product of Q_1 and Q_2 with the partial ordering defined by

$$(q_1, q_2) \leq (q_1', q_2') \text{ iff } q_i \leq q_i' \text{ in } Q_i \ (i = 1, 2).$$

It is easy to see that if Q_1 and Q_2 are wpo, then these constructions define again wpo sets.

Lemma 4.5. *Let Q be wpo and let $(q_\alpha \mid \alpha \in \mathrm{cf}(\lambda))$ be a transfinite sequence of elements of Q. Then there exists an increasing ordinal sequence $(\alpha_\beta \mid \beta \in \mathrm{cf}(\lambda))$ such that $q_{\alpha_\beta} \leq q_{\alpha_{\beta'}}$ for $\beta \leq \beta' \in \mathrm{cf}(\lambda)$.*

Proof. Let $(q_\alpha \mid \alpha \in \mathrm{cf}(\lambda))$ be as above. For a cofinal subset M of $\mathrm{cf}(\lambda)$ we call an ordinal $\alpha \in M$ *terminal* for M if the set

$$\{\beta \in M \mid q_\alpha \not\leq q_\beta\}$$

is cofinal in λ. We claim that there is a cofinal subset of $\mathrm{cf}(\lambda)$ without a terminal element. Suppose not and put $M_0 = \mathrm{cf}(\lambda)$. If $M_0, \dots, M_n$, $\delta_0, \dots, \delta_{n-1}$ are defined, we let δ_n be the terminal element for M_n and $M_{n+1} := \{\beta \in M_n \mid q_{\delta_n} \not\leq q_\beta\}$. Then $q_{\delta_1}, q_{\delta_2}, \dots$ is a bad seuqnce in Q, a contradiction.

So let $M \subseteq \mathrm{cf}(\lambda)$ be cofinal without a terminal element. We define inductively $\alpha_0 := \min M$ and

$$\alpha_\beta := \sup\{\sup\{\delta \in M \mid q_{\alpha_{\beta'}} \not\leq q_\delta\} + 1 \mid \beta' < \beta\}.$$

□

Theorem 4.6 (de Jongh, Parikh 1977). *If Q_1 and Q_2 are wpo, then*

$$c(Q_1 \dot\cup Q_2) = c(Q_1)\#c(Q_2).$$

Proof. By induction on $c(Q_1)\#c(Q_2)$

$$\begin{aligned} c(Q_1\dot\cup Q_2) &= \sup\{c((Q_1\dot\cup Q_2)_q) + 1 \mid q \in Q_1\dot\cup Q_2\} = \\ &= \sup\{c((Q_1)_{q_1}\dot\cup Q_2) + 1, c(Q_1\dot\cup(Q_2)_{q_2}) + 1 \mid q_1 \in Q_1,\ q_2 \in Q_2\} = \\ &= \sup\{c((Q_1)_{q_1})\#c(Q_2) + 1, c(Q_1)\#c((Q_2)_{q_2}) + 1 \mid q_1 \in Q_1,\ q_2 \in Q_2\} = \\ &= c(Q_1)\#c(Q_2). \end{aligned}$$

□

Theorem 4.7. *Let $(Q, \leq)$ be wpo.*

(i) (de Jongh, Parikh 1977) $\lambda(Q)$ is attained, i.e. there always exists a maximal linear extension of $\leq$,

(ii) (The First Minimax Theorem) $c(Q) = \lambda(Q)$.

Proof. Clearly $c(Q) \geq c(\lambda(Q)) = \lambda(Q)$. For the converse inequality let $\gamma = c(Q)$ and let $(q_\alpha \mid \alpha \in \mathrm{cf}(\gamma))$ be such that $\sup c(Q_{q_\alpha}) + 1 = \gamma$. By Lemma 4.5 we may assume that $q_\alpha \leq q_\beta$ for $\alpha \leq \beta \in \mathrm{cf}(\gamma)$. We proceed by induction on γ.

Let first γ be sum–reducible, i.e. $\gamma = \xi + \eta$, where $\xi, \eta \neq 0$ and η is sum–irreducible. Choose α such that $c(Q_{q_\alpha}) \geq \xi$. Since

$$c(Q_{q_\alpha})\#c(\mathrm{cl}(q_\alpha)) \geq c(Q) \geq c(Q_{q_\alpha}) + c(\mathrm{cl}(q_\alpha)),$$

we have $\gamma > c(\mathrm{cl}(q_\alpha)) \geq \eta$. Hence by the induction hypothesis

$$\lambda(Q) \leq c(Q) = \xi + \eta \leq \lambda(Q_{q_\alpha}) + \lambda(\mathrm{cl}(q_\alpha)) \leq \lambda(Q).$$

Now let γ be sum–irreducible. Let γ_α converge to γ ($\alpha \in \mathrm{cf}(\gamma)$) and define inductively

$$\mu(\alpha) := \min\{\beta \in \mathrm{cf}(\gamma) \mid c((\bigcap\{\mathrm{cl}(q_{\mu(\alpha')}) \mid \alpha' < \alpha\})_{q_\beta}) > \gamma_\alpha\}.$$

The function μ is well-defined, since

$$(\bigcap\{\mathrm{cl}(q_{\mu(\alpha')}) \mid \alpha' < \alpha\})_{q_\beta}) \cup \bigcup\{Q_{q_{\mu(\alpha')}} \mid \alpha' < \alpha\} = Q_{q_\beta}$$

and hence

$$c((\bigcap\{\mathrm{cl}(q_{\mu(\alpha')}) \mid \alpha' < \alpha\})_{q_\beta})\#c(\bigcup\{Q_{q_{\mu(\alpha')}} \mid \alpha' < \alpha\}) \geq c(Q_{q_\beta}),$$

which converges to γ. We put

$$Q_\alpha := (\bigcap\{\mathrm{cl}(q_{\mu(\alpha')}) \mid \alpha' < \alpha\})_{q_{\mu(\alpha)}},$$

clearly $\gamma > c(Q_\alpha) > \gamma_\alpha$. By the induction hypothesis

$$\lambda(Q) \leq c(Q) \leq \sum_{\alpha\in\mathrm{cf}(\gamma)} c(Q_\alpha) = \sum_{\alpha\in\mathrm{cf}(\gamma)} \lambda(Q_\alpha) \leq \lambda(Q),$$

which proves (ii). The maximal linear extension is obtained by concatenating the maximal linear extensions on corresponding subsets of Q. □

Theorem 4.8 (de Jongh, Parikh 1977). *Let Q_1, Q_2 be wpo sets. Then*

$$c(Q_1 \times Q_2) = c(Q_1) * c(Q_2).$$

Proof. Clearly

$$c(Q_1 \times Q_2) = \lambda(Q_1 \times Q_2) \geq \lambda(c(Q_1) \times c(Q_2)) \geq c(Q_1) * c(Q_2).$$

We prove the converse inequality by induction on $c(Q_1)\#c(Q_2)$. Let first $c(Q_1) = \alpha_1 + \ldots + \alpha_n$, $c(Q_2) = \beta_1 + \ldots + \beta_m$, $\alpha_1 \geq \ldots \geq \alpha_n$, $\beta_1 \geq \ldots \beta_m$, α_i, β_i sum–irreducible, $n > 1$ or $m > 1$ (i.e. we are assuming that either $c(Q_1)$ or $c(Q_2)$ is sum–reducible). By Theorem 6.7 $Q_1 = Q_1^1 \cup \ldots \cup Q_1^n$, $Q_2 = Q_2^1 \cup \ldots \cup Q_2^m$, where $c(Q_1^i) = \alpha_i$, $c(Q_2^j) = \beta_j$ ($i = 1, \ldots n$; $j = 1, \ldots, m$). We have

$$\begin{aligned} c(Q_1 \times Q_2) &= c(\bigcup\{Q_1^i \times Q_2^j \mid 1 \leq i \leq n,\ 1 \leq j \leq m\}) \leq \#\{c(Q_1^i) * c(Q_2^j) \mid \\ &\quad 1 \leq i \leq n,\ 1 \leq j \leq m\} = \\ &= (\#_{1\leq i\leq n} c(Q_1^i)) * \left(\#_{1\leq j\leq m} c(Q_2^j)\right) = c(Q_1) * c(Q_2). \end{aligned}$$

Now let $c(Q_1)$ and $c(Q_2)$ be sum–irreducible. Then by the induction hypothesis

$$\begin{aligned} c(Q_1 \times Q_2) &= \sup\{c((Q_1 \times Q_2)_{(q_1,q_2)}) + 1 \mid q_1 \in Q_1,\ q_2 \in Q_2\} \leq \\ &\leq \sup\{c(((Q_1)_{q_1} \times Q_2) \cup (Q_1 \times (Q_2)_{q_2})) + 1 \mid q_1 \in Q_1,\ q_2 \in Q_2\} = \\ &= \sup\{c((Q_1)_{q_1} \times Q_2)\#c((Q_1 \times (Q_2)_{q_2})) \mid q_1 \in Q_1,\ q_2 \in Q_2\} = \\ &= \sup\{(c((Q_1)_{q_1}) * c(Q_2))\#(c(Q_1) * c((Q_2)_{q_2})) + 1 \mid q_1 \in Q_1,\ q_2 \in Q_2\} \leq \\ &\leq c(Q_1) * c(Q_2). \end{aligned}$$

□

Theorem 4.9. *Let Q be wpo.*

(i) *(Wolk 1967) $\chi(Q)$ is attained, i.e. there is a maximal chain in Q.*
(ii) *(The Second Minimax Theorem)* $\mathrm{ht}(Q) = \chi(Q)$.

Proof. (i) This argument is taken from (Wolk 1967). Let M_0 be the set of minimal elements of Q and define inductively M_α to be the set of minimal elements in $Q \setminus \bigcup_{\beta<\alpha} M_\beta$. By 4.2 each M_α is finite, let χ be the least ordinal such that $M_\chi = 0$. Then clearly $\chi \geq \chi(Q)$, we shall prove that there is a chain $(q_\alpha \mid \alpha \in \chi)$ such that $q_\alpha \in M_\alpha$ for any $\alpha \in \chi$, which will give (i).

If $A = \{\alpha_1 < \ldots < \alpha_n\} \subseteq \chi$ is a finite set then there is a chain $q_{\alpha_1} \leq \ldots \leq q_{\alpha_n}$ such that $q_{\alpha_i} \in M_{\alpha_i}$ $(i = 1, \ldots, n)$; we put $f_A(\alpha_i) = q_{\alpha_i}$. By Rado Selection Lemma (cf. Ore 1962) there is a function $f : \chi \to Q$ such that for every finite $A \subseteq \chi$ there is a finite B such that $A \subseteq B \subseteq \chi$ and $f_B \restriction A = f \restriction A$. Hence $(f(\alpha) \mid \alpha \in \chi)$ is the desired chain.

(ii) Clearly $\mathrm{ht}(Q) \geq \chi(Q)$. For the other inequality define $f : \mathrm{Dec}(Q) \to \chi(Q)$ for $q_1 > \ldots > q_n$ by

$$f(q_1, \ldots, q_n) = \chi(\{q \in Q \mid q_n > q\}).$$

By (i) above, f is a character, which gives $\mathrm{ht}(Q) \leq \chi(Q)$. □

Remark 4.10. Theorem 4.9 holds under weaker hypothesis than that Q be *wpo*, namely it suffices that for every infinite sequence $q_1, q_2 \ldots$ of elements of Q there are indices i, j such that $i < j$ and either $q_i \leq q_j$ or $ht(q_i) \geq ht(q_j)$. See (Křiž), (Milner, Sauer), (Pouzet 1979), (Schmidt 1981).

4.11 Theorem.

(i) $\overline{\Phi}_{\{\mathrm{Asc}(Q)\}}(\gamma) = \overline{\Phi}_{\{\mathrm{Asc}(U)\}}(\gamma) = \gamma * c(Q)$.

(ii) *If $\gamma < \kappa^+$ then* $\Phi_{\mathrm{Asc}(U)}(\gamma) = \gamma * c(Q)$.
(iii) *For $Q = \omega + 1$ we have*

$$\Phi_{\mathrm{Asc}(Q)}(\omega + 1) \leq \omega^2 + 2\omega < \omega^2 + 2\omega + 1 = \overline{\Phi}_{\{\mathrm{Asc}(Q)\}}(\omega + 1).$$

Proof.

(i) $\gamma * c(Q) \leq \overline{\Phi}_{\{\mathrm{Asc}(Q)\}}(\gamma)$: For $s = ((\alpha_1, q_1), \ldots, (\alpha_m, q_m)) \in \mathrm{Bad}(\gamma \mathrm{x} Q)$ let $a^s = (q_1, \ldots, q_m)$ and let $g^s : \downarrow a^s \to \gamma$ be defined by $g^s(x_1, \ldots, x_p) = \alpha_{i_p}$ if $(q_{i_1}, \ldots, q_{i_p})$ is the first appearance of $(x_1, \ldots, x_p) \subseteq a^s$ in a^s. Then g^s is an $(\mathrm{Asc}(Q), \gamma)$-germ on a^s. Hence $H : \mathrm{Bad}(\gamma \mathrm{x} Q) \to T(\{\mathrm{Asc}(Q)\}; \gamma)$ defined by $H(s) = (a^s, g^s)$ is a tree homomorphism, which gives

$$\gamma * c(Q) = c(\gamma \mathrm{x} Q) \leq \overline{\Phi}_{\{\mathrm{Asc}(Q)\}}(\gamma),$$

using Theorem 4.8.

$\overline{\Phi}_{\{\mathrm{Asc}(Q)\}}(\gamma) \leq \overline{\Phi}_{\{\mathrm{Asc}(U)\}}(\gamma)$: Obvious.

$\overline{\Phi}_{\{\mathrm{Asc}(U)\}}(\gamma) \leq \gamma * c(Q)$: Let $T = T(\{\mathrm{Asc}(U)\}; \gamma)$, we shall define a tree homomorphism $H : T \to \mathrm{Bad}(\gamma \times Q)$, which will give the result. So let $(a, g) \in$

We have taken the liberty to denote the last k–system of 5.2 by Ch_k, because it corresponds to a weaker version of the Chvátal's Tree–Complete Graph Ramsey Theorem. The systems $\mathcal{E}_1, \mathcal{S}_k$ and Ch_k are closely related and their R-ordinals are easily computed. Later in this section, we introduce the Erdös–Szekeres system $\mathcal{E}_n$ corresponding to n linear orderings. Generally, $\mathcal{E}_n$ is a 2^n–system. The investigation of $\Phi_{\mathcal{E}_n}$ is technically more complicated and we do not know if it was completely done even for finite values of arguments.

We obtain a lower bound for $\Phi_{\mathcal{E}_n}$ by certain ordinal product and an upper bound by a maximal product. In the finite case, of course, these bounds coincide. In the infinite case, however, the upper bound is not generally achieved. The exact form of the function $\Phi_{\mathcal{E}_n}$ seems to be rather profound.

Theorem 5.4.
(i) If $\gamma_1, \ldots, \gamma_k < |U|^+$ then $\Phi_{\mathcal{S}_k}(\gamma_1, \ldots, \gamma_k) = \Phi_{Ch_k}(\gamma_1, \ldots, \gamma_k) = \gamma_1 * \ldots * \gamma_k$.
(ii) $\overline{\Phi}_{\mathcal{S}_k}(\gamma_1, \ldots, \gamma_k) = \overline{\Phi}_{Ch_k}(\gamma_1, \ldots, \gamma_k) = \gamma_1 * \ldots * \gamma_k$ for any ordinals $\gamma_1, \ldots, \gamma_k$.

Proof. (i) Let $Q = \gamma_1 \times \ldots \times \gamma_k$ be endowed by the product partial ordering.

Proof of $\Phi_{Ch_k}(\gamma_1, \ldots, \gamma_k) \le \gamma_1 * \ldots * \gamma_k$: Let $r : U^2 \to \{1, \ldots, k\}$ and let a $(\gamma_1, \ldots, \gamma_k)$–testing $g = (g_1, \ldots, g_k)$ be given. We define a $(\gamma_1, \ldots, \gamma_k)$–testing $h = (h_1, \ldots, h_k)$ by

$$h_i(a) = \min\{g_i(b) \mid b \in \downarrow a \cap S_i^r,\ a \text{ and } b \text{ have the same last term}\}$$

for $i = 2, \ldots, k$ and by

$$\begin{aligned} h_1(a) = \min\{g_1(x) \mid x = (x_1, \ldots, x_p) \subseteq a,\ x_p = \text{ last term of } a,\\ h_i(x_1) \le h_i(x_1, x_2) \le \ldots \le h_i(x_1, ..., x_p) \text{ for } i = 2, \ldots, k\} \end{aligned}$$

for $i = 1$. If we consider h as a function $U^{<\omega} \to Q$, then $H : \mathrm{Bad}(Ch^r, g) \to \mathrm{Bad}(Q)$ defined by

$$H(a_1, \ldots, a_m) = (h(a_1), h(a_1, a_2), \ldots, h(a_1, \ldots, a_m))$$

is a tree homomorphism, showing that (use 4.8)

$$\Phi_{Ch^r}(\gamma_1, \ldots, \gamma_k) \le c(Q) = \gamma_1 * \ldots * \gamma_k.$$

$\Phi_{\mathcal{S}_k}(\gamma_1, \ldots, \gamma_k) \le \Phi_{Ch_k}(\gamma_1, \ldots, \gamma_k)$ is obvious.
$\gamma_1 * \ldots * \gamma_k \le \Phi_{Ch_k}(\gamma_1, \ldots, \gamma_k)$: We may safely assume $Q \subseteq U$. We define $r : U^2 \to \{1, \ldots, k\}$ by

$$\begin{aligned} r(a, b) &= \min(\{i \mid \alpha_i > \beta_i\} \cup \{k\}) \text{ if } a = (\alpha_1, \ldots, \alpha_k) \in Q \text{ and } b = (\beta_1, \ldots, \beta_k) \in Q\\ &= \text{arbitrarily} \quad \text{otherwise.} \end{aligned}$$

and $g_i : U^{<\omega} \to \gamma_i$ by

$$\begin{aligned} g_i(a_1, \ldots, a_m) &= \alpha_m^i \text{ if } a_j = (\alpha_j^1, \ldots, \alpha_j^k) \in Q \text{ for } j = 1, \ldots, m\\ &= 0 \quad \text{otherwise} \end{aligned}$$

It follows that the identity is a tree homomorphism $\mathrm{Bad}(Q) \to \mathrm{Bad}(S^r, g)$, hence $\gamma_1 * \ldots * \gamma_k \leq \Phi_{S^r}(\gamma_1, \ldots, \gamma_k)$.
(ii) This follows from (i), since by Theorem 3.13 we may assume that $|U|$ is as large as we wish. (The system is evidently standard.) □

Theorem 5.5.
(i) *If* $\gamma_1, \gamma_2 < |U|^+$ *then* $\Phi_{\mathcal{E}_1}(\gamma_1, \gamma_2) = \gamma_1 * \gamma_2$.
(ii) $\overline{\Phi}_{\mathcal{E}_1}(\gamma_1, \gamma_2) = \gamma_1 * \gamma_2$ *for any ordinals* γ_1, γ_2.

Proof. Similarly as in 5.4, it suffices to prove (i).
$\leq$: Given a linear ordering $\leq$ on U, we define $r : U^2 \to \{1, 2\}$ by

$$r(x, y) = 1 \text{ if} x \leq y$$
$$= 2 \text{ otherwise.}$$

Then the identity is a tree homomorphism $\mathrm{Bad}(E^{\leq}, g) \to \mathrm{Bad}(S^r, g)$ for any (γ_1, γ_2)–testing g, and the inequality follows from 5.4(i).
$\geq$: We may assume that $Q := \gamma_1 \times \gamma_2 \subseteq U$. Let $g_i(a_1, \ldots, a_m)$ be defined to be the i–th coordinate of a_m if $a_m \in Q$ and to be 0 otherwise ($i = 1, 2$). Then $g = (g_1, g_2)$ is a (γ_1, γ_2)–testing. For (α, β), $(\alpha', \beta') \in Q$ we define

$$(\alpha, \beta) \leq (\alpha', \beta') \text{ if } \beta < \beta', \text{ or } \beta = \beta' \text{ and } \alpha \geq \alpha'.$$

Then the identity is a tree homomorphism $\mathrm{Bad}(Q) \to \mathrm{Bad}(E^{\leq}, g)$ and the remaining inequality follows from 4.8. □

Definition 5.6. Let $n > 1$ be a natural number and let $\tau = (\leq_1, \ldots, \leq_n)$ be an n–tuple of linear orderings on U. We denote by Σ the set of all mappings $\{1, \ldots, n\} \to \{1, 2\}$ and for $\sigma \in \Sigma$ we define $\leq_\sigma$ by

$$x \leq_\sigma y \text{ iff } x \leq_i y \text{ for } i \in \sigma^{-1}(1) \text{ and } y \leq_i x \text{ for } i \in \sigma^{-1}(2)$$

and the sheaf

$$E_\sigma^\tau := \{(x_1, \ldots, x_m) \in U^{<\omega} \mid x_1 \leq_\sigma x_2 \leq_\sigma \ldots \leq_\sigma x_m\}.$$

This gives rise to a 2^n–sheaf

$$E^\tau := (E_\sigma^\tau \mid \sigma \in \Sigma)$$

and a 2^n–system

$$\mathcal{E}_n := \{E^\tau \mid \tau \text{ is as above}\}.$$

This system will be called the *Erdös-Szekeres system*, for it clearly generalizes the system $\mathcal{E}_1$ introduced in 5.1.

For the lower bounds to $\Phi_{\mathcal{E}_n}$ we need some more definitions.

Definition 5.7. For $\sigma, \sigma' \in \Sigma$ we define $\sigma \lhd \sigma'$ if there exists $i \in \{1, \ldots, n\}$ such that $\sigma(i) < \sigma'(i)$ and $\sigma(j) = \sigma'(j)$ for all $j = i+1, \ldots, n$. For ordinals γ_σ $(\sigma \in \Sigma)$ we define an irreflexive ordering $\lhd$ on the product $X_{\sigma \in \Sigma} \gamma_\sigma$ by

T, let $a = ((\alpha_1, q_1), \ldots, (\alpha_m, q_m))$, we put $b = (q_1, \ldots, q_m)$ and we shall define $h : \downarrow b \to \gamma$ by

$$h(x_1, \ldots, x_p) = \min g(x_{i_1}, \ldots, x_{i_s}),$$

the min taken over all $i_1 < \ldots < i_s = p$ such that $x_{i_1} \leq x_{i_2} \leq \ldots \leq x_{i_s}$. Now we define

$$H(a, g) = ((h(q_1), q_1), (h(q_1, q_2), q_2), \ldots, (h(q_1, \ldots, q_m), q_m));$$

it is easily seen that this definition gives rise to a sequence $H(a, g) \in \mathrm{Bad}(\gamma \times Q)$.

(ii) If $\gamma < \kappa^+$ then we may safely assume that $\gamma \times Q \subseteq U$, it suffices to show that $\Phi_{\mathrm{Asc}(U)}(\gamma) \geq c(\gamma \times Q)$. So let $g : U^{<\omega} \to \gamma$ be defined by

$$\begin{aligned} g(x_1, \ldots, x_m) &= \gamma_m \quad \text{if } x_i = (\gamma_i, q_i) \in \gamma \times Q \text{ and } q_1 \leq \ldots \leq q_m \\ &= 0 \quad \text{otherwise,} \end{aligned}$$

and let f be the R–character corresponding to $\mathrm{Asc}(U)$ and g. It is easily seen that if a sequence $a \in (\gamma \times Q)^{<\omega}$ is from $\mathrm{Bad}(\gamma \times Q)$, then it is $(\mathrm{Asc}(U), g)$–bad. Hence f is a character on $\mathrm{Bad}(\gamma \times Q)$ and consequently $\Phi_{\mathrm{Asc}(U)}(\gamma) \geq c(\gamma \times Q) = \gamma * c(Q)$, as desired.

(iii) Let $Q = \omega + 1$ and let $g : Q^{<\omega} \to \omega + 1$ be given. Let $g' : \omega^{<\omega} \to \omega$ be the restriction of g to $\omega^{<\omega}$, let $f' : \omega^{<\omega} \to \omega^2 + \omega$ be the R–character corresponding to $\mathrm{Asc}(\omega)$ and g'; let $g(\omega, \omega) = n \in \omega$. We define $f : Q^{<\omega} \to \omega^2 + \omega + n + 3$ by

$$f(q_1, \ldots, q_m) = f'(q_1, \ldots, q_{i_1 - 1}, q_{i_1 + 1}, \ldots, q_{i_p - 1}, q_{i_p + 1}, \ldots, q_m) + n + 2 - p,$$

where $q_{i_1}, \ldots, q_{i_p}$ are all occurences of ω within $(q_1, \ldots, q_m)$. Then f is clearly an R–character corresponding to $\mathrm{Asc}(Q)$ and g, showing that $\Phi_{\mathrm{Asc}(Q)}(\omega + 1) \leq \omega^2 + 2\omega$. □

Remark 4.12. Part (ii) of the above theorem is the essential reason for introducing the cardinal κ. For the other results the value of κ is irrelevant. Let us remark that 4.11(iii) can be used to construct a sheaf A for which a universal $(\omega + 1)$–testing exists, but $\Phi_{\{A\}}(\omega + 1) < \overline{\Phi}_{\{A\}}(\omega + 1)$. This shows that the assumptions in 3.9 cannot be weakend.

Theorem 4.13 (The Height-Width Theorem). *We have*

$$c(Q) \leq \mathrm{ht}(Q) * \mathrm{wd}(Q)$$

for any wpo set Q.

Proof. Let g be a character on $\mathrm{Inc}(Q)$. We define, for $(q_1, \ldots, q_n) \in \mathrm{Bad}(Q)$

$$\begin{aligned} h(q_1, \ldots, q_n) = \min\{g(q_{i_1}, \ldots, q_{i_m}) \mid i_1 < i_2 < \ldots < i_m = n,\ \mathrm{h}(q_{i_1}) \leq \ldots \\ \leq \mathrm{h}(q_{i_m})\} \end{aligned}$$

and

$$f(q_1,\dots,q_n) = ((\mathrm{ht}(q_1),\mathrm{h}(q_1)),(\mathrm{ht}(q_2),\mathrm{h}(q_1,q_2)),\dots$$
$$\dots(\mathrm{ht}(q_n),h(q_1,\dots,q_n))).$$

It is easily seen that h is well–defined and that $f(g_1,\dots,g_n) \in \mathrm{Bad}(\mathrm{ht}(Q)\times \mathrm{wd}(Q))$. Hence $f : \mathrm{Bad}(Q) \to \mathrm{Bad}(\mathrm{ht}(Q) \times \mathrm{wd}(Q))$ is a tree homomorphism, which gives $c(Q) \leq c(\mathrm{ht}(Q) \times \mathrm{wd}(Q)) = \mathrm{ht}(Q) * \mathrm{wd}(Q)$ by 4.8 □

4.14 Remark. The above theorem generalizes the result that a partially ordered set with at least $rs+1$ elements either contains a chain of length $r+1$ or an antichain of $s+1$ elements.

On the contrary to $c(Q)$ and $\mathrm{ht}(Q)$ we did not find any reasonable characterization of $\mathrm{wd}(Q)$. Of course, for $\mathrm{wd}(Q) < \omega$ the Dilworth's decomposition theorem (Dilworth 1950) gives one, but the width behaves much worse, when it is infinite. For example, there is a wpo set Q of width $\omega+1$, which cannot be decomposed into two sets, one of width ω and one of width 1.

5. Erdös-Szekeres Theorem

Definition 5.1. Let $\leq$ be a linear ordering on U. We define a 2–sheaf $E^{\leq} = (E_1^{\leq}, E_2^{\leq})$ by

$$E_1^{\leq} = \{(x_1,\dots,x_m) \mid x_1 \leq \dots \leq x_m\},$$
$$E_2^{\leq} = \{(x_1,\dots,x_m) \mid x_1 \geq \dots \geq x_m\}.$$

This gives rise to Ramsey 2–system

$$\mathcal{E}_1 := \{E^{\leq} \mid \leq \text{ is a linear ordering on } U\},$$

which will be called the *Erdös-Szekeres system.*

Definition 5.2. Let $r : U^2 \to \{1,\dots,k\}$ be given. We define a k–sheaf $S^r = (S_1^r,\dots,S_k^r)$ by

$$S_i^r := \{(x_1,\dots,x_m) \in U^{<\omega} \mid r(x_j,x_{j+1}) = i \text{ for } j = 1,\dots,m-1\}.$$

This defines a Ramsey system $\mathcal{S}_k := \{S^r \mid r : U^2 \to \{1,\dots k\}\}$, which will be called the *generalized Erdös-Szekeres system.*

A minor modification of this system yields a more general system, which, however, has the same R-ordinals. We define

$$R_1^r := \{(x_1,\dots,x_m) \in U^{<\omega} \mid r(x_i,x_j) = 1 \text{ for all } 1 \leq i,\ j \leq m\}.$$

(Note that R_1^r was for $r : [U]^2 \to \{1,\dots,k\}$ already defined in 2.2; and this just extends the definition to $r : U^2 \to \{1,\dots,k\}$.) We put $Ch^r := (R_1^r, S_2^r,\dots,S_k^r)$ and

$$Ch_k := \{\mathrm{Ch}^r \mid r : U^2 \to \{1,\dots,k\}\}.$$

Remark 5.3. It is an easy exercise that for ℓ_1,ℓ_2 finite the Erdös–Szekeres theorem is equivalent to the statement $\Phi_{\mathcal{E}_1}(\ell_1,\ell_2) = \ell_1.\ell_2$.

$$(\alpha_\sigma)_{\sigma\in\Sigma} \lhd (\beta_\sigma)_{\sigma\in\Sigma} \text{ if there exists a } \sigma \in \Sigma \text{ such that } \alpha_\sigma < \beta_\sigma \text{ and}$$
$$\alpha_{\sigma'} = \beta_{\sigma'} \text{ for any } \sigma' \lhd \sigma.$$

Let us denote by Q the partially ordered set $(X_{\sigma\in\Sigma}\gamma_\sigma, \lhd)$. The set Q is in fact well–ordered, we denote its type by $\prod_{\sigma\in\Sigma}\gamma_\sigma$. Let us remark that $\prod_{\sigma\in\Sigma}\gamma_\sigma$ is the usual ordinal product of the ordinals γ_σ in the order given by $\lhd$ on Σ.

Lemma 5.8. *Let n be a natural number and let Q be as in 5.7. Then there exists an n–tuple $\tau = (\leq_1, \ldots, \leq_n)$ of linear orderings on Q such that for any $\sigma \in \Sigma$ and for any $x, y \in Q$, $x \rhd y$ and $x \leq_\sigma y$ implies $x_\sigma > y_\sigma$, where x_σ and y_σ are the σ–th coordinates of x and y, respectively.*

Proof. We proceed by induction on n. For $n = 1$ we identify Σ with $\{1, 2\}$ and define $\leq_1$ as $\leq$ in 5.5(i). Now assume the lemma to be proved for $n - 1$. Let us prove it for n. For $i = 1, 2$ we put

$$\Sigma^i := \{\sigma \in \Sigma \mid \sigma(n) = i\}$$

and

$$Q^i := (X_{\sigma\in\Sigma^i}\gamma_\sigma, \lhd).$$

Let $p^i : Q \to Q^i$ be the projections. By the induction hypothesis, there are linear orderings $\leq_1^i, \ldots, \leq_{n-1}^i$ on Q^i $(i = 1, 2)$ with the desired property. We define for $j = 1, \ldots, n-1$

$$x \leq_j y \quad \text{if } p^1(x) \leq_j^1 p^1(y), \text{ and } p^1(x) \geq_j^1 p^1(y) \text{ implies } p^2(x) \leq_j^2 p^2(y)$$

and

$$x \leq_n y \quad \text{if either } p^1(x) \lhd p^1(y), \text{or } p^1(x) = p^1(y) \text{ and } x \rhd y, \text{or } x = y.$$

Now let $x \rhd y$ and $x \leq_\sigma y$ for some $\sigma \in \Sigma$. We distinguish two cases.
Case 1: $\sigma \in \Sigma^2$. Then $x \geq_n y$ and it follows that $p^1(x) \rhd p^1(y)$, since otherwise we would have a contradiction to $x \rhd y$. It follows that for $j = 1, \ldots, n-1$ we have

$$p^1(x) \leq_j^1 p^1(y) \text{ if } \sigma(j) = 1, \text{ and}$$
$$p^1(x) \geq_j^1 p^1(y) \text{ if } \sigma(j) = 2.$$

Thus we are done by the induction hypothesis.
Case 2: $\sigma \in \Sigma^1$. Then $x \leq_n y$. Since p^1 is $\lhd$–nondecreasing (!), we have $p^1(x) \rhd p^1(y)$ or $p^1(x) = p^1(y)$. The former case cannot occur and hence the latter one occurs. But then $p^2(x) \rhd p^2(y)$ and for $j = 1, \ldots, n-1$

$$p^2(x) \leq_j^2 p^2(y) \text{ if } \sigma(j) = 1, \text{ and}$$
$$p^2(x) \geq_j^2 p^2(y) \text{ if } \sigma(j) = 2.$$

We may again use the induction hypothesis. □

Theorem 5.9. *Let $n > 1$ and let γ_σ $(\sigma \in \Sigma)$ be ordinals. Then*
(i) If all γ_σ are $< |U|^+$, then $\prod_{\sigma\in\Sigma}\gamma_\sigma \leq \Phi_{\mathcal{E}_n}((\gamma_\sigma)_{\sigma\in\Sigma}) \leq *_{\sigma\in\Sigma}\gamma_\sigma$
(ii) $\prod_{\sigma\in\Sigma}\gamma_\sigma \leq \overline{\Phi}_{\mathcal{E}_n}((\gamma_\sigma)_{\sigma\in\Sigma}) \leq *_{\sigma\in\Sigma}\gamma_\sigma$ for any ordinals γ_σ.

Proof. Again, it is suficient to prove (i). We shall skip the expression "$\sigma \in \Sigma$" whenever it will be possible.
$\prod\gamma_\sigma \leq \Phi_{\mathcal{E}_n}((\gamma_\sigma)_\sigma)$: Let Q and $\tau := (\leq_1, \ldots, \leq_n)$ be as in Lemma 5.8. We may assume that $Q \subseteq U$ and we extend the orderings $\leq_i$ to U arbitrarily. Let $g_\sigma : Q \to \gamma_\sigma$ be the projection to the σ–th coordinate. Extend g_σ by 0 outside Q. Then $g = (g_\sigma)_\sigma$ is a $(\gamma_\sigma)_\sigma$–testing and it follows from Lemma 5.8 that the identity is a tree homomorphism $\mathrm{Bad}(Q) \to \gamma_{\mathrm{Bad}(E^\tau, g)}$, which proves the inequality.
$\Phi_{\mathcal{E}_n}((\gamma_\sigma)_\sigma) \leq *_{\sigma\in\Sigma}\gamma_\sigma$: Let $\tau = (\leq_1, \ldots, \leq_n)$ and let a $(\gamma_\sigma)_\sigma$–testing be given. Let $\Sigma = \{\sigma_1, \ldots, \sigma_{2^n}\}$. We define $r : U^2 \to \{1, \ldots, 2^n\}$ by

$$r(x, y) = \min\{i \mid x \leq_{\sigma_i} y\}.$$

Then the identity is a tree homomorphism $\mathrm{Bad}(E^\tau, g) \to \mathrm{Bad}(S^\tau, g)$ and the inequality follows from 5.4. □

Definition 5.10. We are going to show that, for $n \geq 2$, the upper bound from 5.9 is not attained. In the rest of this section we put $n = 2$ so that $|\Sigma| = 4$.

Suppose that $\leq_1, \leq_2$ are linear orderings on U. Then for $a, b \in U$, $a \neq b$, there is a unique $\sigma(a, b) \in \Sigma$ such that $a \leq_{\sigma(a,b)} b$. Let us call σ, σ' *opposite* if $\sigma(i) + \sigma'(i) = 3$ for every $i = 1, 2$.

Lemma 5.11. *Let $a, b, c \in U$ be distinct and let $\sigma(a, b)$, $\sigma(b, c)$, $\sigma(a, c)$ be distinct. Then $\sigma(a, b)$, $\sigma(a, c)$ are not opposite.*

Proof. Suppose the contrary. Because of symetry and possible reversing of the orderings we may assume that $\sigma(a, b)$ is equal to 1 identically. Then we have $a \leq_1 b$, $a \leq_2 b$, $a \geq_1 c$, $a \geq_2 c$. Thus $c \leq_1 b$, $c \leq_2 b$ and hence $\sigma(a, b) = \sigma(a, c)$, a contradicition. □

Definition 5.12. Let $Q = X_{\sigma\in\Sigma}(\omega + 1)$ be equipped with the product partial ordering. For $q = (q_\sigma)\sigma \in \Sigma \in Q$ we define the *pattern* of q by

$$\pi(q) := \{\sigma \in \Sigma \mid q_\sigma < \omega\} \subseteq \Sigma.$$

A sequence $(q_1, \ldots, q_m) \in Q^{<\omega}$ is called *jolly* if
(5.12a) for $i \neq j$, $\pi(q_i) \neq \pi(q_j)$,
(5.12b) for $1 \leq i < j \leq n$, $2 \leq |\pi(q_i)| \leq |\pi(q_j)| \leq 3$, and
(5.12c) for $1 \leq i < j \leq n$, if q_i^σ, q_j^σ are the σth coordinates of q_i, q_j, respectively, and if $q_i^\sigma < \omega$, then $q_i^\sigma \leq q_j^\sigma$.

Thus the length of a jolly sequence is at most 10 and every jollly sequence is bad. We put

$$T := \{a \in \mathrm{Bad}(Q) \mid a \text{ contains no jolly subsequence of length } 10\}.$$

Lemma 5.13. $\gamma_T \leq \omega^4 + 4\omega^3 + 5\omega^2 + 4\omega + 1$.

Proof. The idea is straightforward: since T contains no jolly subsequence of maximal possible length, the essential part of some subset of type $\geq \omega^2$ is not included in T. To make it precise we need some more definitions.

We say that $(q_\sigma)\sigma \in \Sigma \in Q$ is *controlled* by $(q'_\sigma)\sigma \in \Sigma \in Q$ if $\omega > q'_\sigma > q_\sigma$ for some $\sigma \in \Sigma$. For $\pi \subseteq \Sigma$ and $q \in Q$ we define

$$Q_\pi := \{q \in Q \mid \pi(q) = \pi\},$$
$$Q_2^q := \{q' \in Q \mid q' \text{ is controlled by } q \text{ and } |\pi(q')| = 2\},$$
$$Q_3^q := \{q' \in Q \mid q' \text{ is controlled by } q \text{ and } 2 \leq |\pi(q')| \leq 3\}.$$

It is easily seen that $c(Q_\pi) = \omega^{|\pi|}$, $c(Q_2^q) < \omega^2$, $c(Q_3^q) < \omega^3$; let

$$g_\pi : Q_\pi \to \omega^{|\pi|},$$
$$g_2^q : Q_2^q \to c(Q_2^q),$$
$$g_3^q : Q_e^q \to c(Q_3^q)$$

be characters. In this proof we shall use a convention that g_π of the empty sequence is $\omega^{|\pi|}$ and similarly for g_2^q and g_3^q. If $s \in Q^{<\omega}$ and $\pi \subseteq \Sigma$, we denote by $s \restriction Q_\pi$ the (possibly empty) subsequence of s consisting of all those terms of s which belong to Q_π. It is worth noting that

$$f : \mathrm{Bad}(Q) \to \omega^4 + 4\omega^3 + 6\omega^2 + 4\omega + 1$$

defined by

$$f(s) := \#_{\pi \subseteq \Sigma} g_\pi(s \restriction Q_\pi)$$

is a character on $\mathrm{Bad}(Q)$.

For some $s \in Q^{<\omega}$ and $q \in Q$ we define $J(s) \in Q^{<\omega}$, $\Pi(s) \subseteq [\Sigma]^2 \cup [\Sigma]^3$, $s^0 \in Q^{<\omega}$ and $s^q \in Q^{<\omega}$ by induction on the length of s. The intended meaning is the following. $J(s)$ will be the "first jolly subsequence", $\Pi(s)$ will be the set of patterns of terms of $J(s)$, s^0 will be the subsequence of elements controlled by no term of $J(s)$ and s^q will be the subsequence of all elements controlled by q.

For $s =$ empty sequence we initialize all these objects to be empty sequences or empty sets. Now let $s = x.(q)$, where x is possibly empty. We distinguish several cases; each one is ment to assume negation of preceding ones.

(i) If $\Pi(s)$ is undefined, or $|\pi(q)| = 3$ and $[\Sigma]^2 \backslash \Pi(x) \neq 0$, then let $J(s)$, $\Pi(s)$, s^0, s^q be undefined.

(ii) If $|\pi(q)| \neq 2$ or $|\pi(q)| \neq 3$ or $\pi(q) \in \Pi(x)$, then let $J(s) := J(x)$, $\Pi(s) := \Pi(x)$, $s^0 := x^0.(q)$, $s^q := x^q$ for any $q \in Q$.

(iii) If q is controlled by no term of $J(x)$, then let $J(s) := J(x).(q)$, $\Pi(s) := \Pi(x) \cup \{\pi(q)\}$, $s^0 := x^0.(q)$, $s^q := x^q$ for any $q \in Q$.

(iv) Let q' be a term of $J(x)$ such that q is controlled by q', we put $J(s) := J(x)$, $\Pi(s) := \Pi(x)$, $s^0 := x^0$, $s^{q'} := x^{q'}.(q)$, $s^q := x^q$ for any $q \in Q\backslash\{q'\}$.

Let $M(s)$ be the set of terms of $J(s)$. We claim the following.

(5.13a) $J(s)$ is a jolly sequence.

(5.13b) If $[\Sigma]^2 \backslash \Pi(s) \neq 0$ then $[\Sigma]^3 \cap \Pi(s) = 0$.

(5.13c) If $s \in T$ and $[\Sigma]^2 \backslash \Pi(s) = 0$ then $[\Sigma]^3 \backslash \Pi(s) \neq 0$.

(5.13d) If s^q is a nonempty sequence, then $s^q \in (Q_3^q)^{<\omega}$ and if moreover $[\Sigma]^2 \backslash \Pi(s) \neq 0$, then $s^q \in (Q_2^q)^{<\omega}$.

(5.13e) If q is a term of s and $\Pi(s)$ is defined, then q is a term either of s^0 or of $s^{q'}$ for some $q' \in M(s)$.

Condition (5.13c) follows from (5.13a) and the definition of T, the other conditions follow from the construction.

Now we define

$$f : T \to \omega^4 + 4\omega^3 + 5\omega^2 + 4\omega + 1$$

by

$$\begin{aligned} f(s) &:= \#_{\pi \neq \kappa}^{\pi \subseteq \Sigma} g_\pi(s^0 \restriction Q_\pi) \# \mathop{\#}_{q \in M(s)} g_2^q(s^q) \quad \text{if } [\Sigma]^2 \backslash \Pi(s) \neq \emptyset, \text{ where } x \in [\Sigma]^2 \backslash \Pi(s) \\ &:= \mathop{\#}_{\pi \subseteq \Sigma} q_\pi(s \restriction Q_\pi) \quad \text{if } \Pi(s) \text{ is undefined} \\ &:= \#_{\pi \neq \kappa}^{\pi \subseteq \Sigma} g_\pi(s^0 \restriction Q_\pi) \# \mathop{\#}_{q \in M(s)} g_3^q(s^q) \quad \text{if} [\Sigma]^2 \backslash \Pi(s) = \emptyset, \text{ where } \kappa \in [\Sigma]^3 \backslash \Pi(s) \end{aligned}$$

It is easily seen that f is a character on T. □

Theorem 5.14. *We have*

$$\begin{aligned} \overline{\Phi}_{\mathcal{E}_2}(\omega+1, \omega+1, \omega+1, \omega+1) &\leq \omega^4 + 4\omega^3 + 5\omega^2 + 4\omega + 1 \\ &< (\omega+1) * (\omega+1) * (\omega+1) * (\omega+1). \end{aligned}$$

Proof. By standardness, it is sufficient to prove the inequality for $\Phi_{\mathcal{E}_2}$. So let $\tau = (\leq_1, \leq_2)$ be a pair of linear orderings on U and let $g = (g_\sigma)_{\sigma \in \Sigma}$ be an $(\omega+1, \omega+1, \omega+1, \omega+1)$–testing. We claim that

$$\gamma_{\mathrm{Bad}(E^\tau, g)} \leq \omega^4 + 4\omega^3 + 5\omega^2 + 4\omega + 1.$$

This will be done by defining a tree homomorphism $H : \mathrm{Bad}(E^\tau, g) \to T$. We define $h = (h_\sigma)_{\sigma \in \Sigma}$ by

$$h_\sigma(a) := \min\{g_\sigma(b) \mid b \in \downarrow a \cap E_\sigma^\tau,\ a \text{ and } b \text{ have the same last term}\}$$

and put, for $a = (a_1, \ldots, a_m) \in \mathrm{Bad}(E^\tau, g)$

$$H(a) = (h(a_1), h(a_1, a_2), \ldots, h(a_1, \ldots, a_m)),$$

where h is considered as a function $\mathrm{Bad}(E^\tau, g) \to Q$. It is easily seen that if $\sigma = \sigma(a_i, a_j)$, then

(5.14a) $h_\sigma(a_1,\ldots,a_i) < h_\sigma(a_1,\ldots,a_j)$,
and if $a_i = a_j$, then

(5.14b) $h_\sigma(a_1,\ldots,a_i) > h_\sigma(a_1,\ldots,a_j)$ for all $\sigma \in \Sigma$.

Hence H is a tree homomorphism $\mathrm{Bad}(E^\tau, g) \to \mathrm{Bad}(Q)$ and thus our aim is to show that its range is in fact contained in T. To this end suppose the contrary, namely that

$$b^1 = (b^1_\sigma)_{\sigma\in\Sigma} := h(a_1,\ldots,a_{i_1 0}),$$

$$\vdots \quad \vdots \quad \vdots$$

$$b^{10} = (b^{10}_\sigma)_{\sigma\in\Sigma} := h(a_1,\ldots,a_{i_{10}}),$$

is a jolly sequence for some $1 \le i_1 < \ldots < i_{10} \le m$. By (5.14b) we have $a_{i_j} \neq a_{i_p}$ for $1 \le j < p \le m$ and thus $\sigma_{j,p} := \sigma(a_{i_j}, a_{i_p})$ is well–defined. Now for $j = 7, 8, 9, 10$ let $\pi(b_j) = \{\sigma_j\}$ ($\pi(b_j)$ consists of one element by (5.12b)). There are p, ℓ such that $7 < p < \ell \le 10$ and σ_p, σ_ℓ are opposite. Now let $j \in \{1,\ldots,6\}$ be such that $\pi(b_j) = \Sigma\backslash\{\sigma_p, \sigma_\ell\}$. By (5.14a)

$$b^7_{\sigma_{7,p}} > b^p_{\sigma_{7,p}}.$$

From this and (5.12c) we deduce that $\sigma_{j,p} = \sigma_\ell$, $\sigma_{j,\ell} = \sigma_p$, $\sigma_{7,p} = \sigma_{7,\ell} = \sigma_7$, $\sigma_{j,7} \in \{\sigma_p, \sigma_\ell\}$ (see fig. 1, where $\star$ denotes a value less then ω).

$$\sigma_\ell \ \sigma_p \ \sigma_7$$

$$b^j = (\omega, \omega, \star, \star)$$
$$b^7 = (\star, \star, \omega, \star)$$
$$b^p = (\star, \omega, \star, \star)$$
$$b^\ell = (\omega, \star, \star, \star)$$

Fig. 1

By (5.12a), σ_7, σ_p, σ_ℓ are distinct. Now if $\sigma_{j,7} = \sigma_p$, then $\sigma_{j,7}$ is opposite to $\sigma_{j,p}$, while $\sigma_{j,7}$, $\sigma_{7,p}$, $\sigma_{j,p}$ are distinct. If, on the other hand, $\sigma_{j,7} = \sigma_\ell$, then $\sigma_{j,7}$ is opposite to $\sigma_{j,\ell}$, while $\sigma_{j,7}$, $\sigma_{7,\ell}, \sigma_{j,\ell}$ are distinct. In both cases we obtain a contradiction to Lemma 5.11. □

6. Ramsey Systems

In this section we give bounds for $\Phi_n(\gamma_1,\ldots,\gamma_k)$ and $\overline{\Phi}_n(\gamma_1,\ldots,\gamma_k)$. Recall that, according to 3.17, $\Phi_n(\gamma_1,\ldots,\gamma_k) = \overline{\Phi}_n(\gamma_1,\ldots,\gamma_k)$ for $\gamma_1,\ldots,\gamma_k < |U|^+$.

Theorem 6.1.
(i) $\Phi_n(\gamma_1,\ldots,\gamma_k)=0$ *if some* $\gamma_i=0$.
(ii) *If* $\gamma_1,\ldots,\gamma_k>0$ *then* $\overline{\Phi}_1(\gamma_1,\ldots,\gamma_k)=\gamma_1\#\ldots\#\gamma_k$.
(iii) *If* $0<\gamma_1,\ldots,\gamma_k<|U|^+$ *then* $\Phi_1(\gamma_1,\ldots,\gamma_k)=\gamma_1\#\ldots\#\gamma_k$.
(iv) *For* $n>1$ *and* $\gamma_1,\ldots,\gamma_k>0$

$$\Phi_n(\gamma_1,\ldots,\gamma_k)\le$$
$$\le \sup_{\gamma_i'<\gamma_i}[\Phi_{n-1}(\Phi_n(\gamma_1',\ldots,\gamma_k),\ldots,\Phi_n(\gamma_1,\ldots,\gamma_i',\ldots,\gamma_k),\ldots,\Phi_n(\gamma_1,\ldots,\gamma_k'))+1].$$

Proof.
(i) Obvious.
(ii) $\le$: For $(a,g)\in T(\mathcal{R}^1_k;\gamma_1,\ldots,\gamma_k)$ define

$$f(a,g)=\max_r[\#_{i=1}^k \min\{g_i(b_1,\ldots,b_m)\mid(b_1,\ldots,b_m)\subseteq a, r(b_1)=\ldots=$$
$$=r(b_m)=i\}],$$

the max taken over all colorings $[U]^1\to\{1,\ldots,k\}$ such that g is an $(R^r;\gamma_1,\ldots,\gamma_k)$–germ on a. It is easily seen that f is a character on $T(\mathcal{R}^1_k;\gamma_1,\ldots,\gamma_k)$.
$\ge$: Let $\gamma_1,\ldots,\gamma_k$ be given, let us choose distinct elements $x_1,\ldots,x_k\in U$ and define

$$r(x)=i \qquad \text{if } x=x_i$$
$$= \text{arbitrarily otherwise.}$$

Let Q be the set consisting of all pairs (i,α), where $i\in\{1,\ldots,k\}$ and $\alpha\in\gamma_i$, partially ordered by the rule $(i,\alpha)\le(j,\beta)$ if $i=j$ and $\alpha\le\beta$. Then $c(Q)=\gamma_1\#\ldots\#\gamma_k$ by 4.6.

For $s=((i_1,\alpha_1),\ldots,(i_m,\alpha_m))\in \mathrm{Bad}(Q)$ let $a^s=(a_1,\ldots,a_m)=(x_{i_1},\ldots,x_{i_m})$ and define $g_i^s:\downarrow a^s\to\gamma_i$ by

$$g_i^s(a_{j_1},\ldots,a_{j_p})=\alpha_{j_p} \text{ if } a_{j_1},\ldots a_{j_p} \text{ are the first } p$$
$$\text{occurences of } x_i \text{ within } a^s$$
$$=0 \quad \text{otherwise.}$$

Then $g^s=(g_1^s,\ldots,g_k^s)$ is an $(R^r;\gamma_1,\ldots,\gamma_k)$–germ on a^s. Hence if we define a tree homomorphism $H:\mathrm{Bad}(Q)\to T(\{R^r\};\gamma_1,\ldots,\gamma_k)$ by $H(s)=(a^s,g^s)$, we see that $\gamma_1\#\ldots\#\gamma_k=\gamma_{\mathrm{Bad}(Q)}\le\gamma_{T(\{R^r\};\gamma_1,\ldots,\gamma_k)}\le\overline{\Phi}_1(\gamma_1,\ldots,\gamma_k)$.
(iii) This follows from (ii) above and Theorem 3.17, but we give a direct proof. It suffices to show that if $\gamma_1,\ldots,\gamma_k$ are nonzero and $<|U|^+$, then $\gamma_1,\ldots,\gamma_k\le\Phi_1(\gamma_1,\ldots,\gamma_k)$. Let Q be as in (ii), we may safely assume that $Q\subseteq U$. We define a $(\gamma_1,\ldots,\gamma_k)$–testing $g=(g_1,\ldots,g_k)$ by

$$g_i(a_1,\ldots,a_m)=\alpha_m \text{ if } a_j=(i,\alpha_j)\in Q \text{ for } j=1,\ldots,m$$
$$=0 \quad \text{otherwise}$$

and a coloring $r:[U]^1\to\{1,\ldots,k\}$ by

$$r(a) = i \text{ if } a = (i, \alpha) \in Q$$
$$= 1 \text{ otherwise.}$$

Let $f : U^{<\omega} \to \Phi_1(\gamma_1, \dots, \gamma_k)$ be the R–character corresponding to R^r and g. Since, as easily seen, every sequence from Bad(Q) is (R^r, g)–bad, it follows that f is a character on Bad(Q), hence $\Phi_1(\gamma_1, \dots, \gamma_k) \geq \gamma_1 \# \dots \# \gamma_k$, as desired.
(iv) For $r : [U]^n \to \{1, \dots, k\}$ and $x \in U$ let $r' : [U]^{n-1} \to \{1, \dots, k\}$ be defined by

$$r'(\{x_1, \dots, x_{n-1}\}) = r(\{x_1, \dots, x_{n-1}, x\}) \text{ if } x \neq x_i \text{ for } i = 1, \dots, n-1$$
$$= \text{arbitrarily} \qquad \text{otherwise.}$$

Let g be a $(\gamma_1, \dots, \gamma_k)$–testing and let T be the tree of (R^r, g)–bad sequences. We are going to estimate the type of $T_{(x)}$. Let $\gamma_i' = g_i(x)$, we define $(\gamma_1, \dots, \gamma_i', \dots, \gamma_k)$–testings $g^i = (g_1^i, \dots, g_k^i)$ by

$$g_j^i(a) = g_j((x).a) \text{ if } j \neq i \text{ or } g_i((x).a) < \gamma_i'$$
$$= 0 \qquad \text{otherwise.}$$

Let h^i be the R–character corresponding to R^r and g^i. Then $h = (h^1, \dots, h^k)$ is a $(\Phi_n(\gamma_1', \dots, \gamma_k), \dots, \Phi_n(\gamma_1, \dots, \gamma_k'))$–testing, let f be the R–character corresponding to this testing and the k–sheaf R^{r}. We claim that f is a killing on every sequence $a \in U^{<\omega}$ such that $(x).a$ is (R^r, g)–bad. Indeed, each g_j^i is a killing on $\downarrow a \cap R_1^{r'} \cap R_1^r$, hence h^i is a killing on $\downarrow a \cap R_i^{r'}$ and consequently f is a killing on a. Thus

$$\gamma_{T_{(x)}} \leq \Phi_{n-1}(\Phi_n(\gamma_1', \dots, \gamma_k), \dots, \Phi_n(\gamma_1, \dots, \gamma_k'))$$

and (iv) follows. □

Corollary 6.2. $\Phi_2(\gamma_1, \dots, \gamma_k) \leq (k+1)^{\gamma_1 \# \dots \# \gamma_k}$.

Remark 6.3. For $n = 2$, Theorem 6.1 gives the same estimate as the one known is the finite case. The estimate contained in Corollary 6.2 is slightly weaker because of the difficulties with limit ordinals. On the other hand, for $n > 2$, the bound from 6.1(iv) is of little interest; in the finite case, for instance, it is not even primitively recursive in n. To obtain sharper estimates for $\Phi_n(\gamma_1, \dots, \gamma_k)$ $(n > 2)$ one has to use different methods. It is convenient to use the strong R–ordinals here.

Definition 6.4. For $n \geq 3$ and colorings $r_i : [U]^i \to \{1, \dots j_i\}$ $(i = 1, \dots, n)$ put

$$B^{r_1, \dots, r_n} := \{a \in U^{<\omega} \mid r_i(\{x_1, \dots, x_i\}) = r(\{x_1, \dots, x_{i-1}, x_i'\}) \text{ for any}$$
$$i = 1, \dots, n \text{ and any two injective subsequences}$$
$$(x_1, \dots, x_i), (x_1, \dots, x_{i-1}, x_i').$$

We define a 1–system $\mathcal{B}_{j_1,\dots,j_n}$ by

$$\mathcal{B}_{j_1,\dots,j_n} := \{B^{r_1,\dots,r_n} \mid r_i : [U]^i \to \{1,\dots,j_i\}\}$$

and put

$$\tilde{\Phi}_{j_1,\dots,j_n}(\gamma) := \Phi_{\mathcal{B}_{j_1,\dots,j_n}}(\gamma),$$
$$\check{\Phi}^n_k(\gamma) = \tilde{\Phi}_{1,\dots,1,k}(\gamma) \quad (n-1 \text{ occurences of } 1).$$

We denote by $\mathbf{1}$ the constant mapping $U^{<\omega} \to \{1\}$.

Theorem 6.5.
(i) $\tilde{\Phi}_{j_1,\dots,j_n}(0) = 0$, *and for* $\gamma > 0$
(ii) $\tilde{\Phi}_{j_1,j_2,\dots,j_n}(\gamma) \leq j_1 * \tilde{\Phi}_{1,j_2,\dots,j_n}(\gamma)$,
(iii) $\tilde{\Phi}_{1,j_2,\dots,j_n}(\gamma) \leq \sup\{\tilde{\Phi}_{1,j_2,j_2\cdot j_3,\dots,j_{n-1}\cdot j_n,j_n}(\gamma') + 1 \mid \gamma' < \gamma\}$.

Proof.
(i) Obvious.
(ii) Let the colorings $r_i : [U]^i \to \{1,\dots,j_i\}$ $(i = 1,\dots,n)$, an ordinal $\gamma > 0$ and a (γ)–testing g be given. Let $h : U^{<\omega} \to \tilde{\Phi}_{1,j_2,\dots,j_n}(\gamma)$ be the character corresponding to $B^{\mathbf{1},r_2,\dots,r_n}$ and g, and let $f : U^{<\omega} \to j_1 * \tilde{\Phi}_{1,j_2,\dots,j_n}(\gamma)$ be the R–character corresponding to R^{r_1} and $(h,\dots,h)$ (j_1 times), which exists by 6.1(ii). Now if a is $(B^{r_1,\dots,r_n},g)$–bad, then h is a killing on $\downarrow a \cap R^{r_1}_j$ for every $j = 1,\dots,j_1$, hence f is a killing on $\downarrow a$, which proves (ii).
(iii) Let the colorings $r_i : [U]^i \to \{1,\dots,j_i\}$ $(i = 2,\dots,n)$, $x \in U$, an ordinal $\gamma > 0$ and a (γ)–testing g be given. We put $j_1 = 1$ for definiteness. Let T be the tree of $(B^{\mathbf{1},r_2,\dots,r_n},g)$–bad sequences. We are going to estimate the type of $T_{(x)}$. We define colorings $\tilde{r}_i : [U]^i \to \{1,\dots,j_i.j_{i+1}\}$ $(i = 1,\dots,n-1)$, $\tilde{r}_n : [U]^n \to \{1,\dots,j_n\}$ by
$\tilde{r}_n := r_n$,

$$\begin{aligned}\tilde{r}_i(\{a_1,\dots,a_i\}) &= j_{i+1}.(r_i(\{a_1,\dots,a_i\}) - 1) + r_{i+1}(\{x,a_1,\dots,a_i\}) && \text{if } x \notin \{a_1,\dots,a_i\}\\ &= j_{i+1}.(r_i(\{a_1,\dots,a_i\}) - 1, 1 && \text{otherwise}\end{aligned}$$

for $i = 1,\dots,n-1$ and a $(g(x))$–testing h by

$$\begin{aligned}h(b) &= g((x).b) \text{ if } g((x).b) < g(x)\\ &= 0 \qquad \text{otherwise.}\end{aligned}$$

Now if $b \in B^{\tilde{r}_1,\dots,\tilde{r}_n}$, then $(x).b \in B^{\mathbf{1},r_2,\dots,r_n}$. Thus if $(x).a$ is $(B^{\mathbf{1},r_2,\dots,r_n},g)$–bad, then a is $(B^{\tilde{r}_1,\dots,\tilde{r}_n},h)$–bad. Hence

$$\gamma_{T_{(x)}} \leq \Phi_{B^{\tilde{r}_1,\dots,\tilde{r}_n}}(g(x))$$

and (iii) follows. □

Corollary 6.6. *If* γ *is finite, then* $\check{\Phi}^n_k(\gamma)$ *is finite. For any* γ *we have* $\check{\Phi}^n_k(\gamma) \leq \omega^\gamma$.

Theorem 6.7. *For $n \geq 3$ we have*

$$\Phi_n(\gamma_1,\ldots,\gamma_k) \leq \check{\Phi}_k^n(\Phi_{n-1}(\gamma_1,\ldots,\gamma_k)).$$

Proof. Let $r : [U]^n \to \{1,\ldots,k\}$, let $\gamma_1,\ldots,\gamma_k$ be ordinals and let $g = (g_1,\ldots,g_k)$ be a $(\gamma_1,\ldots,\gamma_k)$–testing. We define $h : U^{<\omega} \to \overline{\Phi}_{n-1}(\gamma_1,\ldots,\gamma_k)$ by

$$\begin{aligned} h(a) &= \Psi_{T(\mathcal{R}_k^{n-1};\gamma_1,\ldots,\gamma_k)}(a, g \restriction\downarrow a) && \text{if } g \restriction\downarrow a \text{ is an}(R^{\tilde{r}};\gamma_1,\ldots,\gamma_k)\text{–germ on } a \\ & && \text{for some } \tilde{r} : [U]^{n-1} \to \{1,\ldots,k\} \\ &= 0 && \text{otherwise.} \end{aligned}$$

We claim that if a is (R^r, g)–bad, then a is $(B^{1,\ldots,1,r}, h)$–bad. Indeed, let $b \in\downarrow a \cap B^{1,\ldots,1,r}$, then for an injective sequence $(x_1,\ldots,x_n) \subseteq b$ we may define $\tilde{r}(\{x_1,\ldots,x_{n-1}\}) = r(\{x_1,\ldots,x_n\})$, since the right hand side does not depend on x_n. Thus $g \restriction\downarrow b$ is an $(R^{\tilde{r}};\gamma_1,\ldots,\gamma_k)$–germ on b and hence h is a killing on$\downarrow b$. Thus, a is $(B^{1,\ldots,1,r}, h)$–bad. We conclude that the identity is a tree homomorphism $\text{Bad}(R^r, g) \to \text{Bad}(B^{1,\ldots,1,r}, h)$. Hence

$$\Phi_{R^r}(\gamma_1,\ldots,\gamma_k) \leq \Phi_{B^{1,\ldots,1,r}}(\overline{\Phi}_{n-1}(\gamma_1,\ldots,\gamma_k)),$$

which gives the theorem by 3.17. □

Remark 6.8. Note that Theorem 3.17 is used essentially in the proof of 6.7.

Corollary 6.9. *For $n \geq 2$ we have*

$$\Phi_n(\gamma_1,\ldots,\gamma_k) \leq \omega^{\omega^{\cdot^{\cdot^{\cdot^{\omega^{(k+1)^{\gamma_1\#\ldots\#\gamma_k}}}}}}} \Big\}(n-2) \text{ times}.$$

Now we are going to obtain lower bounds for $\Phi_n(\gamma_1,\ldots,\gamma_k)$.

Definition 6.10. We need to consider another set $\overline{U}$ and for every tree $S \subseteq U^{<\omega}$ its dual tree $\overline{S} \subseteq \overline{U}^{<\omega}$. The set $\overline{U}$ is defined as the set of all functions $\varepsilon : U^{<\omega} \to \{0,1\}$ with the property that there exists an $a \in U^{<\omega}$ such that $\varepsilon(b) = 0$ for every $b \in U^{<\omega}$ which is not a segment of a. The letter ε (with or without dashes or suffixes) is reserved to designate elements of $\overline{U}$ or $\overline{U}^{<\omega}$.

For $\varepsilon, \varepsilon' \in \overline{U}$ we define $\varepsilon \lhd \varepsilon'$ if there exists an $a \in U^{<\omega}$ such that $\varepsilon(a) < \varepsilon'(a)$ and for every $b \in U^{<\omega}$, $\varepsilon(b) \neq \varepsilon'(b)$ implies $a = b$ or $a \ll b$. In this case we define $D(\varepsilon, \varepsilon') := a$; this determines $D(\varepsilon, \varepsilon')$ for $\varepsilon \lhd \varepsilon'$ uniquely. The relation $\lhd$ is easily seen to be an ordering. Observe the following properties of $\lhd$ and D :

(6.10a) If $\varepsilon \lhd \varepsilon' \lhd \varepsilon''$, then $D(\varepsilon, \varepsilon') \neq D(\varepsilon', \varepsilon'')$.

(6.10b) If $\varepsilon_1 \lhd \varepsilon_2 \lhd \ldots \lhd \varepsilon_n$, then $D(\varepsilon_1, \varepsilon_n) = \ll -\min_{1<i\leq n} D(\varepsilon_{i-1}, \varepsilon_i)$. [For a partial ordering $<, < -\min M$ means the $<$ –least element of M, if such exists.]

If $S \subseteq U^{<\omega}$ is a tree, we define the dual tree $\overline{S} \subseteq \overline{U}^{<\omega}$ as the set of all sequences $(\varepsilon_1,\ldots,\varepsilon_m) \in \overline{U}^{<\omega}$ such that

(6.10c) if $\varepsilon_i(a) = 1$ then $a \in S$ for every $a \in U^{<\omega}$ and $i = 1, \ldots, m$,
(6.10d) $\varepsilon_1 \lhd \varepsilon_2 \lhd \ldots \lhd \varepsilon_m$.

Lemma 6.11. *Let $S \subseteq U^{<\omega}$ contain no infinite chain and let $\overline{S} \subseteq \overline{U}^{<\omega}$ be the dual tree. Then $\overline{S}$ contains no infinite chain and we have*

$$\gamma_{\overline{S}} \geq 2^{\gamma_S}.$$

Proof. Suppose that

$$(\varepsilon_1),\ (\varepsilon_1, \varepsilon_2), \ldots, (\varepsilon_1, \ldots, \varepsilon_n), \ldots$$

is an infinite chain in $\overline{S}$. We put

$$D_i := \lim_{n \to \infty} D(\varepsilon_i, \varepsilon_n).$$

(By (6.10b) the right hand sequence is eventually constant.) By (6.10b), too, the sequence $D_1, D_2, \ldots$ is $\ll$–nondecreasing and by (6.10a) is not eventually constant. Thus it yields an infinite chain in S.

This proves the former statement. To prove the latter one we proceed by induction on γ_S. Let S be fixed and suppose that the lemma holds for every tree $S' \subseteq U^{<\omega}$ such that $\gamma_{S'} < \gamma_S$. Let $\gamma < \gamma_S$ be given. We denote by T the tree of all $(\varepsilon_1, \ldots, \varepsilon_m) \in \overline{S}$ such that $\varepsilon_1(x) = \ldots = \varepsilon_m(x) = 0$ for every $x \in U^{<\omega}$ such that $|x| = 1$. If $x \in S$ then $\overline{(S_x)}$ (i.e. the dual tree to S_x) is contained in T; hence, by the induction hypothesis
(6.11a) $\gamma_T \leq \gamma_{\overline{(S_x)}} \geq \sup_{x \in S} 2^{\gamma_{S_x}} \geq 2^{\gamma}$.
We claim that
(6.11b) $\gamma_{\overline{S}_t} \geq 2^{\gamma}$ for every $t \in T$. To prove (6.11b) let $x \in S$ be such that $|x| = 1$; we define $H : \overline{(S_x)} \to \overline{S}_t$ by

$$H(\varepsilon_1, \ldots, \varepsilon_m) = t.(\varepsilon'_1, \ldots, \varepsilon'_m);$$

where

$$\begin{aligned} \varepsilon'_i &= \varepsilon_1(a) \text{ if } a \neq x \\ &= 1 \qquad \text{if } a = x. \end{aligned}$$

Then H is a tree homomorphism showing that

$$\gamma_{\overline{S}_t} \geq \sup_x \gamma_{\overline{(S_x)}} \geq \sup_x 2^{\gamma_{S_x}} \geq 2^{\gamma},$$

which proves (6.11b).

Now (6.11a) and (6.11b) imply

$$\gamma_{\overline{S}} \geq 2^{\gamma} + 2^{\gamma} = 2^{\gamma+1}$$

which proves the lemma, since $\gamma < \gamma_S$ was arbitrary. □

6.12 Stepping-Up Lemma. *For $n \geq 3$ and ordinals $\gamma_1, \ldots, \gamma_k$ we have*

$$\Phi_{n+1}(\gamma_1', \gamma_2', \ldots, \gamma_k') \geq 2^{\Phi_n(\gamma_1, \ldots, \gamma_k)},$$

where

$$\begin{aligned} \gamma_i' &= \min(\gamma_1, \omega) + \sup\{\gamma' + n \mid \gamma' < \gamma_1\} && \text{for } i = 1 \\ &= \gamma_2 + \min(\gamma_2 + n - 1, \omega) && \text{for } i = 2 \\ &= \sup\{\gamma' + n \mid \gamma' < \gamma_i\} && \text{for } i = 3, \ldots, k \end{aligned}$$

Proof. We may assume that $\gamma_1, \ldots, \gamma_k < |U|^+$, for if $\gamma_i \geq |U|^+$ for some i and all γ_i are nozero, then $\Phi_{n+1}(\gamma_1', \ldots, \gamma_k') = \Phi_n(\gamma_1, \ldots, \gamma_k) = |U|^+ = 2^{|U|^+}$ by 2.6, 6.1(iii) and obvious monotonicity of the R–ordinals. By 3.18 there exists a $(\gamma_1, \ldots, \gamma_k)$–testing $g = (g_1, \ldots, g_k)$, a coloring $r : [U]^n \to \{1, \ldots, k\}$ and a simple substree S of $\mathrm{Bad}(R^r, g)$ of type $\Phi_n(\gamma_1, \ldots, \gamma_k)$. We shall define a coloring $\overline{r} : [\overline{U}]^{n+1} \to \{1, \ldots, k\}$ and a $(\gamma_1', \ldots, \gamma_k')$–testing $h = (h_1, \ldots, h_k)$ such that $\overline{S} \subseteq \mathrm{Bad}(R^{\overline{r}}, h)$. Then the lemma will follow from 6.11.

For $\delta_1, \delta_2 \in U$ we define $\delta_1 \lhd \delta_2$ if $(\delta_1, \delta_2) \subseteq a$ for some $a \in S$. This is an ordering by simplicity of S and the definition of a subtree. For $\varepsilon, \varepsilon' \in \overline{S}$ let $\delta(\varepsilon, \varepsilon')$ be the last term of $D(\varepsilon, \varepsilon')$. By (6.10a), (6.10b) and simplicity of S we have

(6.12a) if $\varepsilon, \varepsilon', \varepsilon'' \in \overline{S}$ and $\varepsilon \lhd \varepsilon' \lhd \varepsilon''$, then $\delta(\varepsilon, \varepsilon') \neq \delta(\varepsilon', \varepsilon'')$,

(6.12b) if $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m \in \overline{S}$ and $\varepsilon_1 \lhd \varepsilon_2 \lhd \ldots \lhd \varepsilon_m$, then $\delta(\varepsilon_1, \varepsilon_m) = \lhd - \min_{1 < i \leq n} \delta(\varepsilon_{i-1}, \varepsilon_i)$.

Let $E = \{\varepsilon_1, \ldots, \varepsilon_{n+1}\} \in [\overline{U}]^{n+1}$. If E is not linearly ordered by $\lhd$, then define $\overline{r}(E)$ arbitrarily. In the opposite case assume that $\varepsilon_1 \lhd \varepsilon_2 \lhd \ldots \lhd \varepsilon_{n+1}$ and put $\delta_i = \delta(\varepsilon_i, \varepsilon_{i+1})$. Now define

$$\begin{aligned} \overline{r}(E) &= r(\{\delta_1, \ldots, \delta_n\}) && \text{if } \delta_1 \lhd \ldots \lhd \delta_n \text{ or } \delta_n \lhd \ldots \lhd \delta_1, \\ &= 1 && \text{if } \delta_1 \lhd \delta_2 \rhd \delta_3, \\ &= 2 && \text{if } \delta_1 \rhd \delta_2 \lhd \delta_3, \\ &= \text{arbitrarily} && \text{otherwise.} \end{aligned}$$

Claim 6.13. *Let $\varepsilon_1 \lhd \varepsilon_2 \lhd \ldots \lhd \varepsilon_m$ be elements of $\overline{U}$, $m \geq n$ and assume that $(\varepsilon_1, \ldots, \varepsilon_m) \in R_i^{\overline{r}}$. Put, for $j = 1, \ldots, m-1$, $\delta_j = (\varepsilon_j, \varepsilon_{j+1})$. Then there exists a p such that $1 \leq p \leq m - n + 1$ and*

(6.13a) $(\delta_1, \ldots, \delta_p) \in R_i^r$ and $(\delta_p, \ldots, \delta_{m-n+1}) \in R_i^r$, and one of the following possibilities occurs.

(6.13b) $\delta_1 \lhd \delta_2 \ldots \lhd \delta_{p-1} \lhd \delta_p \rhd \delta_{p+1} \rhd \ldots \rhd \delta_{m-n+1}$ or

(6.13c) $\delta_1 \rhd \delta_2 \rhd \ldots \rhd \delta_{p-1} \rhd \delta_p \lhd \delta_{p+1} \lhd \ldots \lhd \delta_{m-n+1}$.

Moreover, if $1 < k < m - n + 1$, then either $i = 1$ and (6.13b) holds, or $i = 2$ and (6.13c) holds.

Proof. For $2 \leq j \leq m - n + 1$, let us call j *local max* if $\delta_{j-1} \lhd \delta_j \rhd \delta_{j+1}$ and *local min* if $\delta_{j-1} \lhd \delta_j \rhd \delta_{j+1}$. If $i \neq 1$ then there can be no local max j, since otherwise $(\varepsilon_{j-1}, \varepsilon_j, \varepsilon_{j+1}, \ldots, \varepsilon_{n+1}) \in R_1^{\overline{r}}$. Similarly there can be no local min if $i \neq 2$. Since between any two local max's there must be a local min and vice versa, all of the claim except (6.13a) follows.

To prove (6.13a) let $(\delta_{i_1}, \dots, \delta_{i_n}) \subseteq (\delta_1, \dots, \delta_{m-n+1})$ and let, say, $\delta_{i_1} \lhd \delta_{i_2} \lhd \dots \lhd \delta_{i_n}$. The result follows by stepping–up to the set

$$E = \{\varepsilon_{i_1}, \dots, \varepsilon_{i_n}, \varepsilon_{i_n+1}\}.$$

For $1 \leq j < n$

$$\delta(\varepsilon_{i_j}, \varepsilon_{i_{j+1}}) = \lhd - \min\{\delta_\ell \mid i_j \leq \ell < i_{j+1}\} = \delta_{i_j}$$

by monotonicity and

$$\delta(\varepsilon_{i_n}, \varepsilon_{i_n+1}) = \delta_{i_n}.$$

The δ_{i_j} are monotonic so E is colored "by the δ", and $r(\{\delta_{i_1}, \dots, \delta_{i_n}\}) = \overline{r}(E) = i$. If $\delta_{i_1} \rhd \dots \rhd \delta_{i_n}$ the same argument works with

$$E = \{\varepsilon_{i_1}, \varepsilon_{i_1+1}, \dots, \varepsilon_{i_n+1}\}.$$

□

6.14 Proof of 6.12 continued. For $\delta_m \in U$ we put $\|\delta_m\| := m$ if $(\delta_1, \dots, \delta_m) \in S$ for some $\delta_1, \dots, \delta_{m-1} \in U$. Note that $\|.\|$ is unique, if defined. It remains to define h. So let $(\varepsilon_1, \dots, \varepsilon_m) \subseteq \varepsilon \in \overline{S}$, let $(\varepsilon_1, \dots, \varepsilon_m) \in R_i^{\overline{r}}$ and let p be as in 6.13. We define $\delta_j := \delta(\varepsilon_j, \varepsilon_{j+1})$ for $j = 1, \dots, m-1$ and observe the following.

(6.14a) If $\delta_j \rhd \delta_k$ then $\|\delta_j\| < \|\delta_k\|$.

(6.14b) If $\delta_{i_1} \lhd \dots \lhd \delta_{i_s}$ then $g_i(\delta_{i_1}, \dots, \delta_{i_{s-1}}) < g_i(\delta_{i_1}, \dots, \delta_{i_s})$.

Now we define for $i = 1$

$$\begin{aligned} h_1(\varepsilon_1, \dots, \varepsilon_m) &= \min(\gamma_1, \omega) + g_1(\delta_1) + n - m && \text{if } m \geq n \\ &= \min(\gamma_1, \omega) + g_1(\delta_1, \dots, \delta_{m-n+1}) && \text{if } m \geq n \ p = m - n + 1 \\ &= \|\delta_{m-1}\| && \text{if } m \geq n,\ p < m - n + 1,\ \gamma_1 \geq \omega \\ &= \gamma_1 - m && \text{if } m \geq n,\ p < m - n + 1,\ \gamma_1 < \omega \end{aligned}$$

for $i = 2$

$$\begin{aligned} h_2(\varepsilon_1, \dots, \varepsilon_m) &= \gamma_2 + \|\delta_1\| + n - m && \text{if } m \leq n,\ \gamma_2 \geq \omega \\ &= 2\gamma_2 + n - m - 1) && \text{if } m \leq n \text{ or } p = m - n + 1, \text{ and } \gamma_2 < \omega \\ &= \gamma_2 + \|\delta_{m-n+1}\| && \text{if } m \geq n,\ p = m - n + 1,\ \gamma_2 \geq \omega \\ &= g_2(\delta_p, \dots, \delta_{m-n+1}) && \text{if } m \geq n,\ p < m - n + 1 \end{aligned}$$

and for $i = 3, \dots, k$

$$\begin{aligned} h_i(\varepsilon_1, \dots, \varepsilon_m) &= \max(\|\delta_1\|, g_i(\delta_1)) + n - m && \text{if } m \leq n,\ \gamma_i \geq \omega \\ &= g_i(\delta_1, \dots, \delta_{m+n-1}) && \text{if } \delta_1 \lhd \dots \lhd \delta_{m+n-1},\ \gamma_i \geq \omega \\ &= \|\delta_{m-n+1}\| && \text{if } \delta_1 \rhd \dots \rhd \delta_{m+n-1},\ \gamma_i \geq \omega \\ &= \gamma_i + n - m + 1 && \text{if } \gamma_i < \omega \end{aligned}$$

□

Corollary 6.15. *For p, ℓ finite we have*

$$\Phi_n(2p+n-1, 2\ell+n-1, \omega, \ldots, \omega) \geq 2^{\Phi_n(p,\ell,\omega,\ldots,\omega)}.$$

Using the trivial lower bound $\Phi_3(\omega, \ldots, \omega) \geq \omega^k$ (k repetitions of ω), which follows from 5.4 we obtain

Theorem 6.16. *For $k \geq 3$ we have*

$$\left.\omega^{\omega^{\cdot^{\cdot^{\cdot^{\omega^k}}}}}\right\}{\scriptstyle (n-1)\text{ times}} \geq \Phi_n\underbrace{(\omega, \ldots, \omega)}_{k\text{ times}} \geq \left.\omega^{\omega^{\cdot^{\cdot^{\cdot^{\omega^k}}}}}\right\}{\scriptstyle (n-2)\text{ times}}.$$

7. Canonical Ramsey Theorem

In this section we assume that U is linearly ordered by an ordering $<$, n will be a fixed number and and κ a subset of $\{1, ..., n\}$.

Definition 7.1. *Let $r : [U]^n \to [U]^n$ and $\kappa \subseteq \{1, ..., n\}$. We say that a sequence $a \in U^{<\omega}$ is (r, κ)-canonical if for any $2n$ terms $x_1 < ... < x_n$, $y_1 < ... < y_n$ of a we have $r(\{x_1, ..., x_n\}) = r(\{y_1, ..., y_n\})$ iff $x_i = y_i$ for every $i \in \kappa$. Now we define a 2^n-sheaf $C^r = (C^r_\kappa)_{\kappa \subseteq \{1,...,n\}}$ by*

$$C^r_\kappa := \{a \in U^{<\omega} | a \text{ is } (r, \kappa)\text{-canonical}\}$$

and put

$$\mathcal{C}_n := \{C^r | r : [U]^n \to [U]^n\}.$$

It is easily seen that $\mathcal{C}^n$ is a standard 2^n-system, it has the R-property by the canonical Ramsey theorem of Erdös and Rado.

Definition 7.2. *Let $b \in U^{<\omega}$, let $x = \{x_1 < ... < x_n\}$, $y = \{y_1 < ... < y_n\}$, $u = \{u_1, ..., u_n\}$ and $v = \{v_1, ..., v_n\}$ be sets consisting of terms of b. We put $x : y = u : v$ if we have*

$$x_i \leq y_i \text{ iff } u_i \leq v_i, \text{ and } x_i \geq y_i \text{ iff } u_i \geq v_i.$$

Let $r : [U]^n \to [U]^n$. We say that $b \in U^{<\omega}$ is r-invariant if for any sets x, y, u, v consisting of terms of b such that $x : y = u : v$ and $r(x) = r(y)$ we have $r(u) = r(v)$.

Lemma 7.3. *Let $r : [U]^n \to [U]^n$ and $b \in U^{<\omega}$. If b is r-invariant and contains at least $2n+1$ distinct elements, then it is (r, κ)-canonical for some $\kappa \subseteq \{1, ..., n\}$.*

Proof. This is a standard argument. See e.g. Rado's paper in this volume. □

Theorem 7.4. *Let k be the number of equivalence relations on the set $[\{1,...,2n\}]^n$. Then*

$$\Phi_{\mathcal{C}^n}((\gamma_\kappa)_{\kappa\subseteq\{1,...,n\}}) \leq \Phi_{2n}((\gamma_\kappa + 2n - 1)_{\kappa\subseteq\{1,...,n\}}, 3n, ..., 3n),$$

where the argument $3n$ is repeated $(k - 2^n)$ times.

Proof. Since the system $\mathcal{C}^n$ is standard, we may use Lemma 3.15 and choose a simple subtree T of the germ tree $T(\mathcal{C}_n; (\gamma_\kappa)_\kappa)$ such that $\gamma_T = \Phi_{\mathcal{C}^n}((\gamma_\kappa)_\kappa)$. Now if $(a,g) \in T$, then a is an injective sequence.

Let $\mathcal{E}$ designate the set of all equivalence relations on $[\{1,...,2n\}]^n$. For $\kappa \subseteq \{1,...,n\}$ let $E_\kappa \in \mathcal{E}$ be defined by

$$\{\alpha_1 < ... < \alpha_n\}E_\kappa\{\beta_1 < ... < \beta_n\} \text{ iff } \alpha_i = \beta_i \text{ for every } i \in \kappa.$$

Let $(a,g) \in T$ and let $g = (g_\kappa)_\kappa$ be a $(C^r, (\gamma_\kappa)_\kappa)$-germ on a, where $C^r = (C^r_\kappa)_\kappa$ and $r : [U]^n \to [U]^n$. We define

$$\overline{r} : [U]^{2n} \to \mathcal{E}$$

so that $\overline{r}(\{z_1,...,z_{2n}\}) = E$ if and only if it holds

$$\{\alpha_1 < ... < \alpha_n\}E\{\beta_1 < ... < \beta_n\} \text{ iff } r(\{z_{\alpha_1},...,z_{\alpha_n}\}) = r(\{z_{\beta_1},...,z_{\beta_n}\}),$$

where $z_1 < ... < z_{2n}$ in the ordering on U. Finally, we define $h = (h_E)_{E\in\mathcal{E}}$ by the rule

$$\begin{aligned} h_E(b) &= \gamma_\kappa + 2n - 1 - |b| && \text{if } E = E_\kappa \text{ for some } \kappa \text{ and } |b| < 2n \\ &= g_\kappa(b) && \text{if } E = E_\kappa \text{ for some } \kappa \text{ and } |b| \geq 2n \\ &= \max(3n - |b|, 0) && \text{otherwise .} \end{aligned}$$

We are going to show that $H : T \to T(\mathcal{R}^{2n}_k; (\gamma_\kappa + 2n - 1)_\kappa, 3n, ..., 3n)$ defined by $H(a,g) = (a,h)$ is a tree homomorphism, which will give the theorem. To this end we must show that h is an $(R^{\overline{r}}; (\gamma_\kappa + 2n - 1)_\kappa, 3n, ..., 3n)$-germ on a. To see this it is sufficient to show that

(7.4a) if $b \in R^{\overline{r}}_{E_\kappa} n \downarrow a$ and $|b| \geq 2n$ then $b \in C^r_\kappa$, and

(7.4b) if $b \in R^{\overline{r}}_{E^n} \downarrow a$ and $|b| \geq 3n + 1$, then $E = E_\kappa$ for some κ.

To prove (7.4a) let $b \in R^{\overline{r}}_{E_\kappa} n \downarrow a$ and $|b| \geq 2n$. Let $x_1 < ... < x_n, y_1 < ... < y_n$ be terms of b. Let $z_1 < ... < z_{2n}$ be terms of b such that $x_i = z_{\alpha_i}$, $y_i = z_{\beta_i}$, $\alpha_1 < ... < \alpha_n, \beta_1 < ... < \beta_n$. Now we have $r(\{x_1,...,x_n\}) = r(\{y_1,...,y_n\}) \Leftrightarrow r(\{z_{\alpha_1},...,z_{\alpha_n}\}) = r(\{z_{\beta_1},...,z_{\beta_n}\}) \Leftrightarrow \{\alpha_1,...,\alpha_n\}E_\kappa\{\beta_1,...,\beta_n\} \Leftrightarrow \alpha_i = \beta_i$ for every $i \in \kappa \Leftrightarrow x_i = y_i$ for every $i \in \kappa$, which shows that $b \in C^r_\kappa$.

To prove (7.4b) let $b \in R^{\overline{r}}_E n \downarrow a$ and $|b| \geq 3n + 1$. Let $c = (c_0 < ... < c_{2n})$ be a sequence consisting of $2n + 1$ terms of b such that the remaining n terms of b are all $< c_0$ (recall that b is injective since a is). We claim that c is r-invariant. So let $x = \{x_1 < ... < x_n\}, y = \{y_1 < ... < y_n\}, u = \{u_1 < ... < u_n\}, v = \{v_1 < ... < v_n\}$ be subsets of $\{c_0,...,c_{2n}\}$, let $x : y = u : v$ and

$r(x) = r(y)$. It follows from $x : y = u : v$ that we can find $4n$ terms $z_1 < \dots < z_{2n}, w_1 < \dots < w_{2n}$ of b (here we need that $|b| \geq 3n+1$) and sets $\{\alpha_1 < \dots < \alpha_n\} \subseteq \{1, \dots, 2n\}$, $\{\beta_1 < \dots < \beta_n\} \subseteq \{1, \dots, 2n\}$ such that $x_i = z_{\alpha_i}, y_i = z_{\beta_i}, u_i = w_{\alpha_i}$ and $v_i = w_{\beta_i}$. Since $\overline{r}(\{z_1, \dots, z_{2n}\}) = \overline{r}(\{w_1, \dots, w_{2n}\}) = E$ we have $r(\{x_1, \dots, x_n\}) = r(\{y_1, \dots, y_n\}) \Rightarrow r(\{z_{\alpha_1}, \dots, z_{\alpha_n}\}) = r(\{z_{\beta_1}, \dots, z_{\beta_n}\}) \Rightarrow \{\alpha_1, \dots, \alpha_n\} E \{\beta_1, \dots, \beta_n\} \Rightarrow r(\{w_{\alpha_1}, \dots, w_{\alpha_n}\}) = r(\{w_{\beta_1}, \dots, w_{\beta_n}\}) \Rightarrow r(\{u_1, \dots, u_n\}) = r(\{v_1, \dots, v_n\})$, which proves that c is r-invariant.

Now by Lemma 7.3 there exists a $\kappa \subseteq \{1, \dots, n\}$ such that c is (r, κ)-canonical. Hence $E = E_\kappa$, which proves (7.4b) and thus completes the proof of the theorem.

□

Corollary 7.5. *We have for $n \geq 1$*

$$\Phi_{C^n}((\gamma_\kappa)_{\kappa \subseteq \{1, \dots, n\}}) \leq \left.\omega^{\omega^{\cdot^{\cdot^{\omega^{\#\{\gamma_\kappa | \kappa \subseteq \{1, \dots, n\}\}}}}}\#\omega}\right\}(2n-1) times.$$

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