On the Orientable Genus of Graphs Embedded in the Klein Bottle

Neil Robertson
DEPARTMENT OF MATHEMATICS
OHIO STATE UNIVERSITY
COLUMBUS, OHIO 43210

Robin Thomas
SCHOOL OF MATHEMATICS
GEORGIA INSTITUTE OF TECHNOLOGY
ATLANTA, GEORGIA 30332

ABSTRACT

Let G be a graph embedded in the Klein bottle with "representativity" at least four. We give a formula for the orientable genus of G, which also implies a polynomially bounded algorithm. The formula is in terms of the number of times certain closed curves on the Klein bottle intersect the graph. In particular, it shows that a cut-and-paste technique for re-embedding graphs is the best possible.

1. INTRODUCTION

Let ψ be an embedding of a graph G in a surface Σ. We say that ψ is an embedding with *representativity* ρ if ρ is maximum such that every nonnull-homotopic closed curve intersects the image of G at least ρ times. The following is a result of [2].

1.1. Let G be a graph embedded in the projective plane with representativity $\rho \geq 3$. Then the orientable genus of G is $\lfloor \rho/2 \rfloor$.

In this paper we prove an analogue of 1.1 for the Klein bottle, which also yields a polynomially bounded algorithm. Testing genus is NP-complete for general graphs [5], and therefore it would be interesting to characterize all

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lower ideals for which testing genus is polynomial-time solvable. Our result can be viewed as a step towards this problem.

If X is a topological space, a *closed curve* (in X) is a continuous mapping $f: S \to X$, where S is the unit circle in the complex plane $\mathbb{C}$. A closed curve is called *simple* if it is $1-1$. The Klein bottle, denoted by N_2, is the space obtained by identifying sides a, b and c, d of a square with sides a, b, c, d as shown in Figure 1. Then the identified sides a, b form a closed curve α_1 and the identified sides c, d form a closed curve α_2; we may assume that $\alpha_1(1) = \alpha_2(1)$. Given two closed curves α, β with $\alpha(1) = \beta(1)$, we define a closed curve $\alpha\beta$ by

$$\alpha\beta(e^{it}) = \begin{cases} \alpha(e^{2it}), & 0 \le t < \pi; \\ \beta(e^{2it}), & \pi \le t < 2\pi. \end{cases}$$

Let $x_0 = \alpha_1(1) = \alpha_2(1)$. We recall that the fundamental group $\pi(N_2, x_0)$ has a presentation in terms of generators $\{\alpha_1, \alpha_2\}$ and one relation $\alpha_1^2\alpha_2^2$. For $\gamma \in \pi(N_2, x_0)$ we say that a closed curve f in N_2 is a γ-curve, if f is (freely) homotopic to γ or γ^{-1}. (Warning: sometimes a γ_1-curve can be homotopic to a γ_2-curve for $\gamma_1, \gamma_2 \in \pi(N_2, x_0)$ such that $\gamma_1 \ne \gamma_2$ and $\gamma_1 \ne \gamma_2^{-1}$). Thus an α_1^2-curve (same as α_2^2-curve) separates the Klein bottle into two Möbius bands.

By a *surface* we mean a compact connected 2-manifold with (possibly null) boundary. Let $(f_1, \ldots, f_n)$ be a family of closed curves in a surface Σ. We put $M(f_1, \ldots, f_n) = \{\{(i, z), (i', z')\}: (i, z) \ne (i', z'), i, i' \in \{1, \ldots, n\}, z, z' \in S \text{ and } f_i(z) = f_{i'}(z')\}$. The family $(f_1, \ldots, f_n)$ is said to be *finitary* if $M(f_1, \ldots, f_n)$ is finite. For a finitary family $(f_1, \ldots, f_n)$ of closed curves we define $\mathrm{cr}(f_1, \ldots, f_n)$, the *crossing number,* as the cardinality of the set

$$\{\{(i, z), (i', z')\} \in M(f_1, \ldots, f_n): f_i, f_{i'} \text{ cross at } (z, z')\},$$

where closed curves f, h are said to cross at (z, z') if $f(z) = h(z')$ and there exists a homeomorphism of a neighborhood of $f(z) = h(z')$ onto the unit disk in $\mathbb{C}$ such that the image of f follows the real axis and the image of h follows the imaginary axis. The family $(f_1, \ldots, f_n)$ is said to be *cross-free* if $\mathrm{cr}(f_1, \ldots, f_n) = 0$.

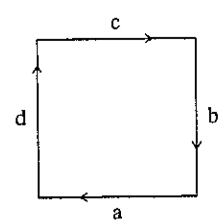

FIGURE 1

A *1-blockage* is a finitary 1-element family (f) consisting of an $\alpha_1 \cdot \alpha_2$-curve f in N_2. A *2-blockage* is a finitary (ordered) pair (f_1, f_2) of two closed curves in N_2 such that f_i is an α_i-curve $(i = 1, 2)$. A *blockage* is a 1-blockage or a 2-blockage.

Let G be a graph embedded in a surface Σ. The G-*degree* (or simply *degree* when no confusion is likely) of a closed curve f in Σ, denoted by $\deg f$, is the cardinality of the set $\{z \in S: f(z) \in G\}$. Now let $\Sigma = N_2$. The *order* of a 1-blockage (f) is $\lceil \deg f/2 \rceil$, the order of a 2-blockage (f_1, f_2) is $\lfloor \deg f_1/2 \rfloor + \lfloor \deg f_2/2 \rfloor$. We can state our main result now.

1.2. Let G be a graph embedded in N_2 in such a way that every α_i-curve has degree at least four $(i = 1, 2)$, and let $g \ge 4$ be an integer. Then the following conditions are equivalent.

(i) G has orientable genus $\ge g$.
(ii) Every cross-free blockage has order $\ge g$.
(iii) Every blockage has order $\ge g$.

Since (iii) above is easy to test in polynomial time (hint: consider a lifting of G in the universal covering surface), we have the following algorithmic result.

1.3. There exists a polynomial algorithm to solve the following problem.
Instance: A graph G, embedded in N_2 as in (1.2), and an integer $g \ge 0$.
Question: Is the orientable genus of G at least g?

For if $g = 0$, 1, 2, or 3, we can use the algorithm of [3]. If $g \ge 4$ we apply (1.2).

The assumption about the degree of every α_i-curve in (1.2) is necessary, because in the embedding of a graph G shown in Figure 2 every blockage has order ≥ 4 and yet Figure 3 shows the embedding of G in S_3, the orientable surface without boundary of genus 3. In fact, it follows from our proof of 1.2 that even if the embedding of G into N_2 does not satisfy the assumption about the degree of every α_i-curve, then the orientable genus of G can only be either g or $g - 1$. We have found a polynomial-time algorithm to decide whether it is g or $g - 1$ (notice that the existence of such an algorithm does not seem to follow from [3]), but the algorithm is quite complicated and does not seem to contain any essentially new ideas, and so is not described here.

We conjecture the following generalization of 1.2. Let N_k be the nonorientable surface with cross-cap number k, that is, the surface obtained by identifying k pairs of consecutive sides of a $2k$-gon. A blockage (in N_k) is a finitary n-tuple $B = (f_1, \ldots, f_n)$ of closed curves such that there exist closed curves $f_1', f_2', \ldots, f_n'$ with the property that f_i is homotopic to f_i'

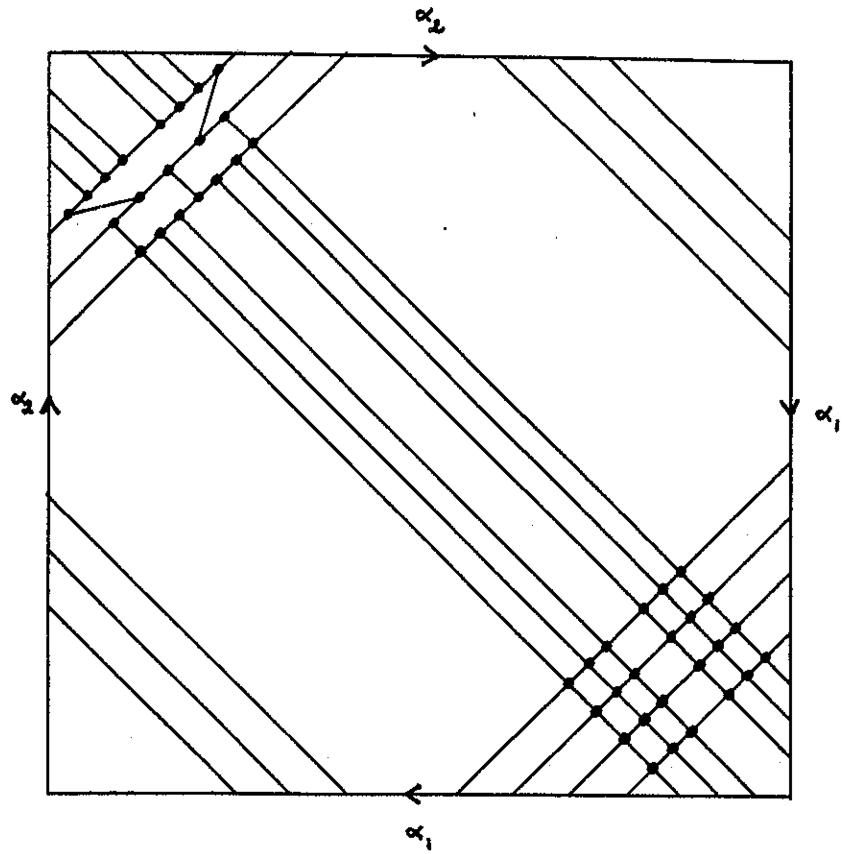

FIGURE 2

$(i = 1, \ldots, n)$, $M(f_1', \ldots, f_n') = \varnothing$ and the surface Σ obtained by cutting open along all of $f_1', f_2', \ldots, f_n'$ is orientable. We define res(B), the *residuum* of B, to be the orientable genus of the surface Σ. We define the order of f_i $(i = 1, \ldots, n)$ by

$$\operatorname{ord}(f_i) = \begin{cases} \left\lfloor \dfrac{\deg f_i}{2} \right\rfloor, & \text{if } f_i \text{ is orientation reversing,} \\ \left\lceil \dfrac{\deg f_i}{2} \right\rceil, & \text{if } f_i \text{ is orientation preserving,} \end{cases}$$

where a curve g is said to be *orientation reversing* (*orientation preserving*, respectively) if passing once around g changes (does not change, respectively) the orientation. Finally, we define the order of B as

$$\operatorname{res}(B) + \sum_{i=1}^{n} \operatorname{ord}(f_i).$$

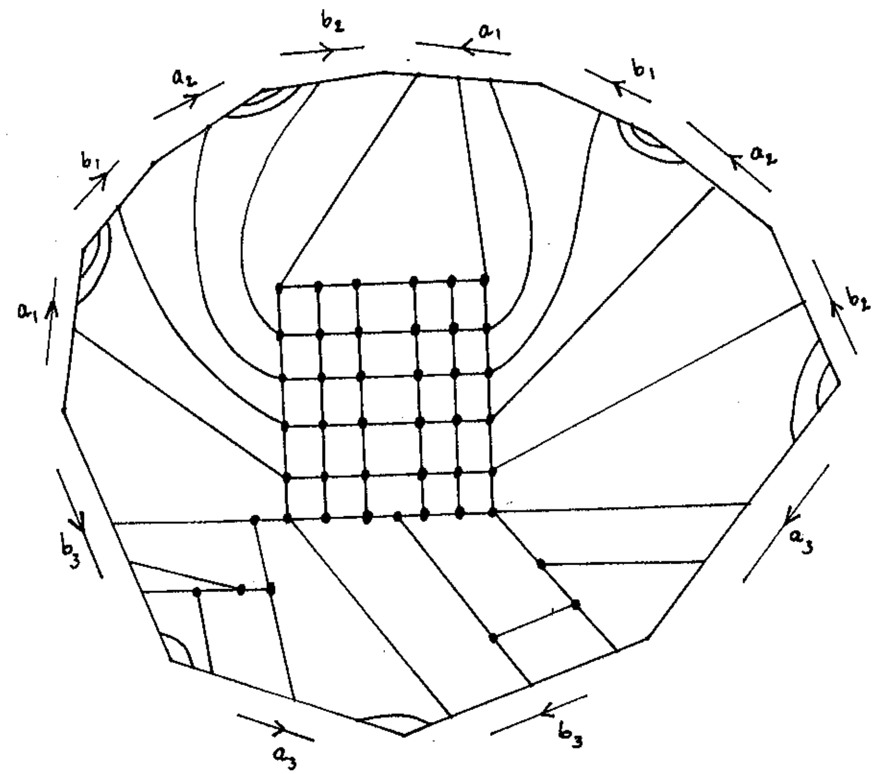

FIGURE 3

1.4. Conjecture. Let $k \geq 2$ be a fixed integer and let G be a graph embedded in N_k with sufficient representativity, and let $g \geq 2k$ be an integer. Then the following conditions are equivalent:

(i) The orientable genus of G is at least g.
(ii) Every cross-free blockage has order at least g.
(iii) Every blockage has order at least g.

We end this section with some more terminology. A *graph* is a pair $G = (U, V)$, where U is a Hausdorff topological space and $V \subseteq U$ is a finite set such that

(i) $U - V$ is the disjoint union of finitely many open subsets, called *edges*,
(ii) for each edge e, its boundary is a subset of V consisting of one or two points. If it consists of two points, then the pair $(\bar{e}, e)$ is homeomorphic to the pair $([0,1], (0,1))$; if the boundary consists of one point, then the pair $(\bar{e}, e)$ is homeomorphic to the pair $(S, S - \{1\})$,

(iii) a subset $A \subseteq U$ is closed (open) if and only if $A \cap \bar{e}$ is closed (open) for all edges e.

[Here $\bar{e}$ denotes the closure of the set e.] We write $U(G) = U$, $V(G) = V$, and we denote the set of edges by $E(G)$. The rest of our graph terminology is standard, we remark that paths and circuits have no "repeated" vertices and no "repeated" edges. We say that a mapping ψ is an *embedding* of a graph G into a topological space X if ψ: $U(G) \to X$ is a homeomorphism onto its image. We say that a graph G is *embedded* in a topological space X if the identity is an embedding of G into X. Obviously, if ψ is an embedding of G into X, then the image graph $(\psi(U(G)), \psi(V(G)))$ is isomorphic to G and is embedded in X. Therefore we shall usually assume that the graph in question is actually embedded in X.

We say that closed curves f, g in X are *homotopic* if there exists a continuous function F: $S \times [0,1] \to X$ such that $F(z,0) = f(z)$ and $F(z,1) = g(z)$ for every $z \in S$. A closed curve f is *null-homotopic* if it is homotopic to some constant curve, and is *nonnull-homotopic* otherwise.

If f: $X \to Y$ is a mapping then $f(X)$ denotes the set $\{f(x): x \in X\}$. Let G be a graph embedded in N_2 and let f be a simple γ-curve for some $\gamma \in \pi(N_2, x_0)$ such that $f(S)$ is a circuit in G. We say that $f(S)$ is a *γ-circuit* in G.

2. DISJOINT PATHS IN THE KLEIN BOTTLE

In this section we prove three results that we shall need for the proof of 1.2 in Section 3. We start with a well-known and easy lemma.

2.1. Let f_1, f_2 be two nonnull-homotopic closed curves in N_1, the projective plane. Then $f_1(S) \cap f_2(S) \neq \varnothing$.

A *pseudocylinder* is a surface obtained from a sphere by deleting interiors of two regions R_1, R_2 homeomorphic to closed discs with $R_1 \cap R_2$ finite. We refer to the boundaries of R_1 and R_2 as the boundary components.

2.2. Let $k \geq 1$ be an integer and let G be a graph embedded in a pseudocylinder Σ. Then either there exists a simple nonnull-homotopic closed curve f in Σ of degree $<k$ separating the two boundary components, or else there are k disjoint paths in G, each joining the two boundary components of Σ.

Proof. This follows easily from Menger's theorem. ∎

The following is the first result of this section.

2.3. Let $k \geq 0$ be an integer and let G be a graph embedded in N_2 in such a way that every cross-free 1-blockage has order $\geq k$. Then there

exist disjoint circuits $C_1, \ldots, C_k$ in G such that C_i separates C_{i-1} and C_{i+1} $(i = 2, \ldots, k-1)$ and one of the following conditions is satisfied:

(a) C_1 is an α_1-circuit and C_i is an α_1^2-circuit $(i = 2, \ldots, k)$;
(b) C_1 is an α_2-circuit and C_i is an α_1^2-circuit $(i = 2, \ldots, k)$;
(c) C_i is an α_1^2-circuit $(i = 1, \ldots, k)$.

Proof. Let (h) be a cross-free 1-blockage with $n = |\{z \in S: h(z) \in U(G)\}|$ minimum. Then $n \geq 2k - 1$. Let $v_1, \ldots, v_n$ (in this order) be the intersections of h with $U(G)$. We cut open along h and thus obtain an embedding of a graph G' in a pseudocylinder Σ with, say, vertices $x_1, \ldots, x_n$ (in this order) on one boundary component, and say vertices $y_n, \ldots, y_1$ (in this order) on the other boundary component, so that G is obtained from G' by identifying x_i and y_i into v_i $(i = 1, \ldots, n)$. There is no simple closed curve h' in Σ separating the two boundary components of G'-degree $< n$, because after pasting the two boundary components together, (h') becomes a 1-blockage in N_2. Hence by 2.2 there are n disjoint paths in G, each joining x_i and $y_{j(i)}$ $(i = 1, \ldots, n)$. It follows that $j(j(i)) = i$, and therefore, after pasting the two boundary components together, the paths $P_1, \ldots, P_n$ become the desired k disjoint circuits. ∎

The proof of the following lemma is left to the reader.

2.4. Let G be a graph embedded in N_1, the projective plane, with representativity ≥ 1. Then there exists a nonnull-homotopic simple closed curve h in N_1 such that $h(S)$ is a circuit in G.

Let X be a topological space. A curve (in X) is a continuous mapping f: $I \to X$, where I is the closed unit interval $[0,1]$.

Let f be a null-homotopic simple closed curve in N_1. Then f bounds a disk with interior U. Let $\Sigma = N_1 - U$. Then Σ is a *Möbius band* and f is called a *boundary curve* of Σ. A curve h: $I \to \Sigma$ is called a *chord* (in Σ) if $h(0)$ and $h(1)$ belong to the boundary of Σ. A chord h: $I \to \Sigma$ is called *essential* if the closed curve g is nonnull-homotopic in N_1, where

$$g(e^{it}) = \begin{cases} h((t - t_0)/(t_1 - t_0)), & \text{if } t_0 \leq t < t_1, \\ f(e^{it}), & \text{if } t_1 \leq t < t_0 + 2\pi, \end{cases}$$

and t_0, t_1 are such that $f(e^{it_0}) = h(0)$, $f(e^{it_1}) = h(1)$, $|t_0 - t_1| \leq 2\pi$ and $t_0 < t_1$.

The following lemma follows easily from 2.1.

2.5. Let Σ be the Möbius band and let h_1, h_2 be two essential chords with $h_1(I) \cap h_2(I) = \varnothing$. Then $h_1(0)$, $h_2(0)$, $h_1(1)$, and $h_2(1)$ occur on the boundary of Σ in this (or reverse) order.

2.6. Let G be a graph embedded in the Möbius band Σ with distinct vertices u, v on the boundary of Σ. Then either there exists a closed curve of degree ≤ 1, homotopic to the boundary curve or else there exists an essential chord h in Σ such that $h(0) = u, h(1) = v$, and $h(I)$ is a path in G.

Proof. Let G be as stated and assume that every closed curve homotopic to the boundary curve has degree ≥ 2. Let G' be the graph obtained from G by identifying the vertices u and v; the embedding of G induces an embedding of G' in N_1. By 2.4 there exists a nonnull-homotopic simple closed curve f in N_1 such that $f(S)$ is a circuit in G', let C be the circuit or path generated by $f(S)$ in G. It follows from the assumption that there exist two disjoint paths in G from $\{u, v\}$ to C, combining these paths with a suitable segment of C yields the desired chord. ∎

The following is the second result.

2.7. Let G be a graph embedded in N_2 in such a way that every simple closed α_1^2-curve has degree at least 2. Then there exists an $\alpha_1\alpha_2$-circuit in G.

Proof. Suppose, if possible, that the theorem fails and let G be a counterexample with $|V(G)|$ minimum. Then there exists a simple α_1^2-curve h with degree 2, let u, v be the intersections of h with $U(G)$. We may assume that $u, v \in V(G)$. When we cut open along h we obtain two Möbius bands Σ_1, Σ_2. By 2.6 there exists an essential chord h_i in Σ_i with $h_i(0) = u$, $h_i(1) = v$, and such that $h_i(I)$ is a path in G $(i = 1, 2)$. It is easily seen that $h_1(I) \cup h_2(I)$ is as desired. ∎

For an orientable surface, the last result of this section would follow from a theorem of Schrijver [4]. However, his theorem is still open for nonorientable surfaces. We say that curves $f, g\colon I \to N_2$ are *equivalent* if there exists a continuous mapping $F\colon I \times I \to N_2$ such that $F(t, 0) = f(t)$, $F(t, 1) = g(t)$, $F(0, t) = f(0)$ and $F(1, t) = f(1)$ for all $t \in I$.

2.8. Let f_i be a simple α_i-curve in N_2 $(i = 1, 2)$, assume that f_1, f_2 cross each other, and that (f_1, f_2) is finitary. Then there exist t_1, t_2, t_1', t_2' with $|t_1 - t_1'| < 2\pi$, $|t_2 - t_2'| < 2\pi$, $t_1 < t_1', t_2 < t_2'$ such that f_1, f_2 cross at (e^{it_1}, e^{it_2}) and at $(e^{it_1'}, e^{it_2'})$ and such that if we define, for $t \in [0, 1]$

$$g_1(t) = f_1(e^{it_1(1-t)+it_1't}),$$
$$g_2(t) = f_1(e^{it_1(i-t)+i(t_1'-2\pi)t}),$$
$$h_1(t) = f_2(e^{it_2(1-t)+it_2't}),$$
$$h_2(t) = f_2(e^{it_2(1-t)+i(t_2'-2\pi)t}),$$

then there exist $i, j \in \{1, 2\}$ such that g_i is equivalent to h_j.

Proof. Let $u_1, \ldots, u_n \in N_2$ be the points where f_1 and f_2 cross. We cut open along f_1 and obtain a Möbius band Σ, with say, vertices $x_1, \ldots, x_n$, $y_1, \ldots, y_n$ (in this order) on the boundary, where x_i, y_i are obtained by splitting u_i. The segments of f_2 become disjoint chords in Σ. By 2.5 at least one of these chords is not essential (otherwise each chord would join x_i and y_i, which is impossible, since f_2 is a closed curve). The theorem now follows from the definition of an essential chord. ∎

3. MAIN RESULT

We shall prove 1.2 by proving (i) ⇒ (ii) ⇒ (iii) ⇒ (i). We denote by S_g the orientable surface without boundary of genus g.

3.1. Let G be a graph embedded in N_2. If there exists a cross-free blockage of order $\leq g$, then the orientable genus of G is at most g.

Proof. We prove the result by an elementary technique called "surgery." Let B be a cross-free blockage of order $\leq g$. We first assume that B is a 1-blockage. Let $B = (f)$, and let $v_1, \ldots, v_k$ $(k \leq 2g)$ be the intersections (in this order) of f with $U(G)$. We cut open along f and split each v_i into two vertices x_i and y_i, to obtain another graph G' embedded into a pseudocylinder Σ with the vertices $x_1, \ldots, x_k$ (in this order) on one boundary component, and the vertices $y_1, \ldots, y_k$ (in this order) on the other boundary component. It is easily seen that we can add $\leq g$ handles to Σ, to obtain an orientable surface S_g of genus g, and identify vertices x_i and y_i $(i = 1, \ldots, k)$ along the added handles. Thus we can produce an embedding of G into S_g.

Now if B is a 2-blockage, say $B = (f_1, f_2)$, we proceed similarly. Let $v_1, \ldots, v_k$ be the intersections of f_1 with $U(G)$, and let $v_1', \ldots, v_{k'}'$ be the intersections of f_2 with $U(G)$. We first cut open along f_1 and split each v_i into x_i and y_i, and then cut open along f_2 and split each v_i' into x_i' and y_i'. We obtain a graph G' embedded into a pseudocylinder with vertices $x_1, \ldots, x_k, y_1, \ldots, y_k$ (in this order) on one boundary component, and with vertices $x_1', \ldots, x_{k'}', y_1', \ldots, y_{k'}'$ (in this order) on the other boundary component. Again, by adding at most $\lfloor k/2 \rfloor + \lfloor k'/2 \rfloor \leq g$ handles, we can identify x_i and y_i $(i = 1, \ldots, k)$, and x_i' and y_i' $(i = 1, \ldots, k')$ to obtain an embedding of G into S_g. ∎

3.2. Let G be a graph embedded in N_2 as in 1.2 and let B be a blockage of order $g \geq 0$. Then there exists a cross-free blockage of order $\leq g$.

Proof. Let us choose a blockage B of order $\leq g$ with $\text{cr}(B)$ minimum, and subject to that with $|M(B)|$ minimum. We suppose that $\text{cr}(B) > 0$ and seek a contradiction. Let $B = (f_1, \ldots, f_n)$ (so that $n = 1$ or 2) and let H be the pair $(f_1(S), \{f_1(z)\colon \{(1, z), (1, z')\} \in M(f_1)$ for some $z' \in S\})$. Then H is a graph unless f_1 is simple.

(1) If f_1 is not simple then every α_1-curve has H-degree ≥ 1, and if $n = 1$ then every α_2-curve has H-degree ≥ 1.

For this follows from 2.1.

(2) If f_1 is not simple then there exists a simple α_1^2-curve h in N_2 with H-degree at most 1.

For otherwise there exists by 2.7 a simple $\alpha_1\alpha_2$-curve f in N_2 with $f(S) \subseteq U(H)$. Then (f) is a cross-free blockage of order $\leq g$ (this is obvious if $n = 1$, and if $n = 2$ then it follows from the fact that f_2 has G-degree ≥ 4), a contradiction.

(3) $n = 2$ and f_1, f_2 are simple.

For suppose that either $n = 1$ (in which case f_1 is not simple because $\mathrm{cr}(B) > 0$), or that f_1 is not simple. The curve h from (2) separates N_2 into two Möbius bands Σ_1, Σ_2. We may assume that the notation is chosen so that Σ_i contains an α_i-curve. By (1) and 2.4 there exists a simple α_1-curve f_1' with $f_1'(S) \subseteq f_1(S) \cap \Sigma_1$. If $n = 1$ then similarly there exists a simple α_2-curve f_2' with $f_2'(S) \subseteq f_1(S) \cap \Sigma_2$, but then (f_1', f_2') is a cross-free blockage of order $\leq g$, a contradiction. If $n = 2$ then $B' = (f_1', f_2)$ is a blockage of order $\leq g$ with $\mathrm{cr}(B') \leq \mathrm{cr}(B)$ and $|M(B')| < |M(B)|$, a contradiction. Hence $n = 2$ and f_1 is simple. We deduce from the symmetry that f_2 is also simple. This proves (3).

We deduce that f_1 and f_2 cross each other and so we may apply 2.8. We may assume without loss of generality that (using the notation of 2.8) $t_1 = t_2, t_1' = t_2'$, and that g_1 and h_1 are equivalent. We may also assume that

$$|\{t \in (0,1)\colon g_1(t) \in U(G)\}| \leq |\{t \in (0,1)\colon h_1(t) \in U(G)\}|.$$

Let

$$f_2'(e^{it}) = \begin{cases} f_1(e^{it}), & \text{if } t_1 \leq t < t_1'; \\ f_2(e^{it}), & \text{if } t_1' \leq t < t_1 + 2\pi. \end{cases}$$

It follows that f_2' is homotopic to f_2 and that its G-degree is at most the G-degree of f_2. We can shift f_2' slightly into a closed curve f_2'' of the same G-degree and homotopic to f_2 such that (f_1, f_2'') is finitary and $\mathrm{cr}(f_1, f_2'') < \mathrm{cr}(f_1, f_2)$. This is a contradiction, because (f_1, f_2'') is a blockage of order $\leq g$. ∎

We require the following result of Battle, Harary, Kodama, and Youngs [1].

3.3. Let the graph G be a disjoint union of graphs G_1 and G_2. Then the orientable genus of G is $g_1 + g_2$, where g_i is the orientable genus of G_i $(i = 1, 2)$.

3.4. Let G_1, G_2 be two disjoint connected graphs of genera $g_1, g_2 \geq 1$, respectively, embedded in $S_{g_1+g_2}$. Then there exists a simple closed curve h in $S_{g_1+g_2}$ separating the surface into two surfaces Σ_1, Σ_2 with common boundary in such a way that Σ_1 contains G_1 and Σ_2 contains G_2 (and therefore the orientable genus of Σ_i is g_i).

Proof. There exist simple closed curves $h_1, \ldots, h_n$ in $S_{g_1+g_2}$ such that $\bigcup_{i=1}^n h_i(S)$ separates the two graphs. Let us choose such a system with n minimum. It follows that none of the curves is null-homotopic and by 3.3 each separates the surface. Hence $n = 1$ and the partition is as desired, again by 3.3. ∎

3.5. Let G be a graph embedded in N_2 as in 1.2 and assume that every blockage has order $\geq g$, where $g \geq 4$ is some integer. Then G has orientable genus $\geq g$.

Proof. Suppose for a contradiction that G is a graph that satisfies the hypothesis but not the conclusion of the theorem with $|V(G)|$ minimum. We shall assume that (a) of 2.3 occurs, for the case (b) is symmetric and (c) is handled similarly and is in fact easier. So let $C_1, \ldots, C_g$ be as in 2.3(a). Let $\rho_1 \geq 4$ and $\rho_2 \geq 4$ be such that $\lfloor \rho_1/2 \rfloor + \lfloor \rho_2/2 \rfloor = g$ and such that every α_i-curve in N_2 has degree at least ρ_i. Let us choose a simple α_1^2-curve f with the following properties:

(i) $\mathrm{Im}\, f$ separates $C_{\lfloor \rho_1/2 \rfloor}$ and $C_{\lfloor \rho_1/2 \rfloor + 1}$.
(ii) $\mathrm{Im}\, f \cap V(G) = \varnothing$.
(iii) $|\mathrm{Im}\, f \cap U(G)|$ is minimum subject to (i) and (ii).

Let E be the set of edges intersected by $\mathrm{Im}\, f$, and let G_0 be the graph obtained from G by deleting all edges from E. The curve f separates N_2 into two Möbius bands Σ_1, Σ_2 such that Σ_1 contains C_1, and it separates G_0 into two disjoint subgraphs G_1 and G_2. Let G_1 be the one that contains C_1. Let $\Sigma_i = \Sigma_i^* - U_i$, where Σ_i^* is homeomorphic to N_1 and U_i is an open disk in Σ_i^*. Then the embedding of G into N_2 induces an embedding of G_i into Σ_i^* $(i = 1, 2)$.

(1) The orientable genus of G_1 is at least $\lfloor \rho_1/2 \rfloor - 1$.

For we consider the embedding of G_1 in Σ_1^*. Let h be a nonnull-homotopic closed curve in Σ_1^*, then h can be regarded as an α_1-curve in N_2; and if h intersects an edge from E, or if it intersects G_2, then it meets C_1 at least once, and each of $C_2, \ldots, C_{\lfloor \rho_1/2 \rfloor}$ at least twice. Since h has G-degree $\geq \rho_1$ it follows that it has G_1-degree $\geqslant 2\lfloor \rho_1/2 \rfloor - 1$. Hence G_1 is embedded in Σ_1^* with representativity $\geq 2\lfloor \rho_1/2 \rfloor - 1$ and thus its orientable genus is $\geq \lfloor \rho_1/2 \rfloor - 1$ by 1.1.

(2) The orientable genus of G_2 is at least $\lfloor \rho_2/2 \rfloor$.

For this follows similarly as (1).

(3) The graphs G_1 and G_2 are connected.

For let $i \in \{1,2\}$. We consider G_i as being embedded in Σ_i^*. By the argument of (1) and 2.4, G_i contains a nonnull-homotopic circuit; let G_i' be the component of G_i containing this circuit. By 2.1 every other component B of G_i contains no nonnull-homotopic circuit and is therefore contained in a disk. It follows from the choice of f that B is a component of G, which is impossible by the minimality of $|V(G)|$.

Let G' be the graph obtained from G by contracting G_2 to a vertex v_0.

(4) The orientable genus of G' is at least $\lfloor \rho_1/2 \rfloor$.

For the embedding of G_1 in Σ_1^* has the property that all endpoints of edges of E that are in $V(G_1)$ lie on the boundary of the face containing U_1. Therefore we can construct an embedding ψ' of G' into Σ_1^* by choosing $\psi'(v_0) \in U_1$ and $\psi'(e)$ for $e \in E(G') - E(G_1)$ appropriately. We claim that the representativity of this embedding is at least $2\lfloor \rho_1/2 \rfloor$. For suppose, if possible, that h is a nonnull-homotopic simple closed curve in Σ_1^* of G'-degree $< 2\lfloor \rho_1/2 \rfloor \le \rho_1$. Then h intersects neither v_0, nor any edge in E, for otherwise it would have to intersect each C_i ($i = 2, \ldots, \lfloor \rho_1/2 \rfloor$) at least twice and C_1 at least once, which would be $\ge 2\lfloor \rho_1/2 \rfloor$ intersections altogether. But now it follows that h can be regarded as an α_1-curve in N_2 of G-degree $< \rho_1$, a contradiction. Hence no such h exists and (4) follows from 1.1.

Now suppose for a contradiction that there exists an embedding ψ of G into S_{g-1}. By (1), (2), (3), and 3.4 there exists a simple closed curve h in S_{g-1} separating S_{g-1} into two surfaces Σ_1' and Σ_2' such that Σ_1' has genus $\lfloor \rho_1/2 \rfloor - 1$ and contains the image of G_1. Hence G_1 has an embedding ψ_1 into Σ_1' with the property that all end points of edges of E that belong to $V(G_1)$ lie on the boundary of the face incident with $h(S)$. Similarly, as in the proof of (4) we deduce that G' has an embedding into $S_{\lfloor \rho_1/2 \rfloor - 1}$, contrary to (4). ∎

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