# PERMANENTS, PFAFFIAN ORIENTATIONS, AND EVEN DIRECTED CIRCUITS

(Extended Abstract)

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## ABSTRACT

We give a polynomial-time algorithm for the following problem of Pólya. Given an $n \times n$ 0-1 matrix, either find a matrix obtained from it by changing some of the 1's to $-1$'s in such a way that the determinant of the new matrix equals the permanent of the old one, or determine that no such matrix exists. This is equivalent to finding Pfaffian orientations of bipartite graphs and to the even circuit problem for directed graphs. The algorithm is based on a structural characterization of bipartite graphs that admit a Pfaffian orientation.

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## 1. INTRODUCTION

Computing the permanent of a matrix seems to be of a different computational complexity from computing the determinant. While the determinant can be computed using Gaussian elimination, no efficient algorithm for computing the permanent is known, and, in fact, none is believed to exist. More precisely, Valiant [14] has shown that computing the permanent is #P-complete even when restricted to 0-1 matrices.

It is therefore reasonable to ask if perhaps computing the permanent can be somehow reduced to computing the determinant of a related matrix. While no such procedure is known in general (and probably does not exist), the following question was asked by Pólya in 1913. If $A$ is an $n \times n$ 0-1 matrix $A$, under what conditions does there exist a matrix $B$ obtained from $A$ by changing some of the 1's to $-1$'s in such a way that the permanent of $A$ equals the determinant of $B$? For the purpose of this note let us say that $B$ (when it exists) is a *Pólya matrix* for $A$. The complexity status of the decision problem whether an input matrix has a Pólya matrix remained open until the present. In this paper we describe a solution of the problem. Specifically, we first give a structural characterization of matrices that have a Pólya matrix. Roughly speaking, they can all be obtained by piecing together "planar" matrices and one sporadic non-planar matrix. We then use the characterization to design a polynomial-time algorithm that given an input matrix $A$ outputs either a Pólya matrix for $A$, or a certain "obstruction" submatrix $A$ whose presence implies that $A$ has no Pólya matrix. The algorithm easily extends to matrices with non-negative entries, as pointed out by Vazirani and Yannakakis [15].

Our results are best stated and proved in terms of bipartite graphs. By a *graph* we mean a simple undirected graph, that is, one with no loops or parallel edges. Let $G$ be a graph, and let $C$ be a circuit in $G$. (*Paths* and *circuits* have no "repeated" vertices.) We say that $C$ is *alternating* if it has even length and $G\backslash V(C)$ has a perfect matching. Let $D$ be an orientation of $G$, and let $C$ be a circuit of $G$ of even length. We say that $C$ is *oddly oriented* (in $D$) if $C$ contains an odd number of edges that are directed (in $D$) in the direction of each orientation of $C$. We say that $D$ is a *Pfaffian orientation* of $G$ if every alternating circuit of $G$ is oddly oriented in $D$.

With every $n \times n$ 0-1 matrix $A$ we associate a bipartite graph $G$ as follows. There is a vertex of $G$ corresponding to every row and every column of $A$, and two vertices of $G$ are adjacent if and only if one represents a row, say $r$, and the other represents a column, say $c$, such that the entry of $A$ in row $r$ and column $c$ is non-zero. Vazirani and Yannakakis [15] proved the following.

**(1.1)** *Let $A$ be an $n \times n$ 0-1 matrix, and let $G$ be the* *associated bipartite graph. Then A has a Pólya matrix if and only if G has a Pfaffian orientation.*

In fact, there is a correspondence between Pólya matrices for $A$ and Pfaffian orientations of $G$.

To state our main result we need some definitions. Let $G_0$ be a graph, let $C$ be a circuit of $G_0$ of length four, and let $G_1, G_2$ be two subgraphs of $G_0$ such that $G_1 \cup G_2 = G_0$, $G_1 \cap G_2 = C$, $V(G_1) - V(G_2) \neq \emptyset$ and $V(G_2) - V(G_1) \neq \emptyset$. (The intersection and union of two subgraphs of a graph is defined in the natural way.) Let $G$ be obtained from $G_0$ by deleting some (possibly none) of the edges of $C$. In these circumstances we say that $G$ is a *sum* of $G_1$ and $G_2$ (along $C$). The *Heawood graph* is the bipartite graph associated with the incidence matrix of the Fano plane. A graph $G$ is *$k$-extendable*, where $k \geq 0$ is an integer, if every matching of size at most $k$ can be extended to a perfect matching. A 2-extendable connected bipartite graph is called a *brace*. It is easy to see that the problem of finding Pfaffian orientations of bipartite graphs can be reduced to braces. The following is our main result [7].

**(1.2)** *A brace has a Pfaffian orientation if and only if it is either isomorphic to the Heawood graph, or can be obtained by repeated application of the sum operation, starting from planar braces.*

By [15] this also solves the even circuit problem for directed graphs (see [9, 10, 12, 13]), by [8] it solves the problem of determining which hypergraphs with $n$ vertices and $n$ hyperedges are minimally non-bipartite, and by [5] it solves the problem of determining which real $n \times n$ matrices are sign non-singular. See also [1] for variations of sign-singularity.

## 2. OUTLINE OF PROOF

We say that a graph $G$ is a *subdivision* of a graph $H$ if $G$ is obtained from $H$ by replacing the edges of $H$ by internally disjoint paths, each containing at least one edge. We say that $G$ is an *even subdivision* of $H$ if $G$ is obtained from $H$ by replacing the edges of $H$ by internally disjoint paths, each containing an even number of vertices and at least one edge. We say that a graph $G$ *contains* a graph $H$ and that $H$ is *contained* in $G$ if some even subdivision of $H$ is isomorphic to a subgraph $K$ of $G$, and $G \backslash V(K)$ has a perfect matching. We first characterize containment-minimal non-planar 1-extendable bipartite graphs.

To state this characterization we need to define several classes of graphs. Let $k \geq 2$ be an integer, and let $C$ be a circuit with vertices $u_1, u_2, \ldots, u_{2k}$ listed in their order on $C$. Let $G$ be the graph obtained from $C$ by adding $k$ vertices $v_1, v_2, \ldots, v_k$, where, for $k = 1, 2, \ldots, k$, $v_i$ is adjacent to $u_i$ and $u_{i+k}$. If $k \geq 4$ is even we say that $G$ is a *stem*. Let $H$ be the graph obtained from $C$ by adding $2k+2$ vertices $w_1, w_2, \ldots, w_{2k}$, $v_1$ and $v_2$, an edge joining $v_1$ and $v_2$, and for $i = 1, 2, \ldots, k$ edges with ends $v_1$ and $w_{2i-1}$, $w_{2i-1}$ and $u_{2i-1}$, $v_2$ and $w_{2i}$, and $w_{2i}$ and $u_{2i}$, respectively. We say that $H$ is a *flower*. Let $(\{a_1, a_2, a_3\}, \{b_1, b_2, b_3\})$ be the bipartition of $K_{3,3}$. The graph *Uno* is obtained from $K_{3,3}$ by subdividing every edge incident with $a_1$ exactly once, and adding a two-edge path joining $a_2$ and $a_3$. We define *Duo* to be the graph obtained from $K_{3,3}$ by adding a two-edge path joining $a_2$ and $a_3$, and a two-edge path joining $b_2$ and $b_3$.

**(2.1)** *Let G be a non-planar 1-extendable bipartite graph. Then G contains one of the following graphs:*

*(i)* $K_{3,3}$,
*(ii)* *a stem,*
*(iii)* *a flower,*
*(iv)* *Uno, or*
*(v)* *Duo.*

As a next step we use (2.1) to find all containment-minimal non-planar braces, as follows.

**(2.2)** *Let G be a non-planar brace. Then G contains one of the following graphs:*

*(i)* $K_{3,3}$,
*(ii)* *the Heawood graph, or*
*(iii)* *Rotunda.*

The graph Rotunda is defined as follows. Let $C$ be a circuit of length four, and let $H$ be obtained from $C$ by adding four new vertices of degree one, each adjacent to a different vertex of $C$. Let the new vertices be $a, b, c, d$ listed in the order of their neighbors on $C$. Let $H_1, H_2, H_3$ be three isomorphic copies of $H$, and let $a_i, b_i, c_i, d_i$ $(i = 1, 2, 3)$ be the vertices corresponding to $a, b, c, d$, respectively. Let $G$ be obtained by identifying $a_1, a_2, a_3$ into $a_0$, identifying $b_1, b_2, b_3$ into $b_0$ and so on. Then $G$ is Rotunda.

Finally, we derive our main theorem (1.2) from (2.2) as follows. It is easy to see that if $G$ is the Heawood graph, or can be obtained as a sum of two braces that have Pfaffian orientations, then $G$ has a Pfaffian orientation. The difficulty is to prove the converse. Let $G$ be a brace with a Pfaffian orientation. If $G$ is planar, then the theorem holds. We may therefore assume that $G$ is not planar. It follows that $G$ does not contain $K_{3,3}$, because $K_{3,3}$ does not have a Pfaffian orientation, and containment preserves that property. Thus, by (2.2), $G$ contains either the Heawood graph or Rotunda. We show that if a brace contains the Heawood graph, then it is either isomorphic to it, or it contains $K_{3,3}$. Thus if $G$ contains the Heawood graph, then it is isomorphic to it, and the theorem holds. We may therefore assume that $G$ contains Rotunda. But then it can be shown that the vertices corresponding to the center of Rotunda form a cut of $G$, and then it is easy to see that

$G$ can be obtained as a sum of two smaller braces that have Pfaffian orientations, and so the theorem follows by induction.

Let us mention the following corollary of our main theorem.

**(2.3)** *Every brace with $n$ vertices and more than $2n-4$ edges contains $K_{3,3}$, and hence does not have a Pfaffian orientation.*

## 3. AN ALGORITHM

**(3.1) Algorithm.**
Input. *A bipartite graph $G$ on $n$ vertices.*
Output. *Either a Pfaffian orientation of $G$, or a subgraph of $G$ isomorphic to an even subdivision $K$ of $K_{3,3}$ in such a way that $G \backslash V(K)$ has a perfect matching (in which case $G$ has no Pfaffian orientation).*
Running time. $O(n^3)$.

*Description* (outline).
**Step 1.** We use the algorithm of Hopcroft and Karp [2] to find a perfect matching $M$ in $G$. If $G$ has no perfect matching, then every orientation of $G$ is Pfaffian. In that case we output an arbitrary orientation of $G$ and stop. Otherwise we go to step 2.
**Step 2.** We use $M$ to delete edges of $G$ that belong to no perfect matching of $G$, and then consider each component of the resulting graph separately. Given $M$ this is equivalent to finding strongly connected components of a directed graph (see [6]), and hence can be done in time $O(n^2)$.
**Step 3.** If $G$ is a brace we go to step 4; otherwise we decompose $G$ into two smaller graphs, and call step 3 for both of the smaller graphs. If both of these calls result in Pfaffian orientations, then those can be combined to give a Pfaffian orientation of $G$. If one of the calls yields a $K_{3,3}$ containment, then that can be easily converted to a $K_{3,3}$ containment in $G$. Decomposing $G$ is equivalent to testing strong 2-connectivity for digraphs, and hence can be done in time $O(n^3)$.
**Step 4.** If $G$ is planar we use Kasteleyn's algorithm [3, 4] (see also [6]) to output a Pfaffian orientation of $G$ and return. Otherwise we go to step 5.
**Step 5.** We use the (algorithmic) proof of (2.1) to find one of the graphs of (2.1) contained in $G$. This can be done in quadratic time.
**Step 6.** We use the proof method of (2.2) to find one of the graphs of (2.2) contained in $G$. This is equivalent to applying the network flow algorithm to find an augmenting path (and repeating a bounded number of times), and so can again be done in linear time.
**Step 7.** If step 6 produces a $K_{3,3}$ contained in $G$, we output the corresponding subgraph of $G$. If step 6 produces the Heawood graph, then if $G$ is isomorphic to it we output a Pfaffian orientation of $G$, and otherwise we again apply the proof method of the main theorem to find $K_{3,3}$ contained in $G$. Finally, if step 6 gives Rotunda contained in $G$, we again use the proof method of the main theorem to either find $K_{3,3}$ contained in $G$, or to express $G$ as a sum of two smaller braces. In the former case we ouput the corresponding subgraph of $G$ and return. In the latter case we call step 4 for both of the smaller braces. □


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The main result of the paper was first discovered by the first author, although he did not publically announce it. (His proof is not yet available — it is longer, and uses a different method.) The proof outlined here was discovered by the last three authors, who were not aware at the time of the prior work of the first author.


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