Reprinted from JOURNAL OF COMBINATORIAL THEORY, Series B
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Vol. 64, No. 2, July 1995
Printed in Belgium

Petersen Family Minors

NEIL ROBERTSON*

*Department of Mathematics, Ohio State University,
231 West 18th Avenue, Columbus, Ohio 43210*

PAUL SEYMOUR

*Bellcore, 445 South Street,
Morristown, New Jersey 07960*

AND

ROBIN THOMAS†

*School of Mathematics, Georgia Institute of Technology,
Atlanta, Georgia 30332*

Received May 11, 1992

We show that the only graphs with certain connectivity and planarity properties are the Petersen graph and some other more trivial graphs. Then this is used to show that every graph with no minor in the Petersen family (the seven graphs that can be obtained from the Petersen graph by $Y-\Delta$ and $\Delta-Y$ exchanges) is either decomposable in some sense, or it is a 1-vertex extension of a planar graph, or it has high connectivity in a sense we can use, or it can be converted to one of these by $Y-\Delta$ and $\Delta-Y$ exchanges. This is a lemma for use in a subsequent paper, where we show that a graph has a "linkless" embedding in 3-space if and only if it has no minor in the Petersen family. © 1995 Academic Press, Inc.

1. INTRODUCTION

This is the second of two papers establishing graph-theoretic lemmas towards a proof of Sachs' conjecture, that a graph has a "linkless" embedding in 3-space if (and only if, but this is easy) it contains none of seven specific graphs as a minor. Here we show that every graph with certain

* This research was carried out under a consulting agreement with Bellcore, and with support from NSF Grant DMS 8903132 and from ONR Grant N00014-911-J-1905.

† This research was carried out while visiting the DIMACS Center at Rutgers University and with support from NSF Grant DMS 9103480.

0095-8956/95 $12.00
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connectivity/planarity properties contains one of the seven graphs; and the proof of the conjecture will be completed in another paper, by showing that any minor-minimal graph with no linkless embedding has these connectivity/planarity properties.

Let v be a vertex of a graph G (graphs in this paper are finite, possibly with loops or multiple edges). Let v be incident with exactly three edges of G, and let these edges have ends v and x_i $(1 \leqslant i \leqslant 3)$, where v, x_1, x_2, x_3 are all distinct. Let H be obtained from G by deleting v and adding three new edges with ends x_1x_2, x_2x_3, and x_3x_1, respectively. We say that H is obtained from G by a *$Y-\Delta$ exchange* (at v). The inverse operation is called a *$\Delta - Y$ exchange (at $x_1x_2x_3$)*. We say that graphs G, G' are *$\Delta - Y$ equivalent* if there is a sequence $H_1, H_2, ..., H_k$ of graphs such that H_1 is isomorphic to G, H_k is isomorphic to G', and for $1 \leqslant i < k$, H_{i+1} is obtained from H_i by either a $Y-\Delta$ exchange or a $\Delta - Y$ exchange.

We say G is *expanded* if for every graph H that is $\Delta - Y$ equivalent to G

(i) H is simple and has minimum valency $\geqslant 3$

(ii) $|V(H)| \leqslant |V(G)|$

(iii) if $|V(H)| = |V(G)|$ then G has at least as many 3-valent vertices as H.

We use \ to denote the result of deletion; thus, if $v \in V(G)$, $G\backslash v$ is the graph obtained by deleting v and all incident edges, while if $e \in E(G)$, $G\backslash e$ is the graph obtained by deleting e. If $X \subseteq E(G)$, G/X denotes the graph obtained from G by contracting all edges in X. If $f \in E(G)$, we write G/f for $G/\{f\}$.

A *separation* of G is a pair (A, B) of subgraphs with $A \cup B = G$ and $E(A \cap B) = \varnothing$, and its *order* is $|V(A \cap B)|$. A separation of order k is called a *k-separation*, and one of order $\leqslant k$ is called a *$(\leqslant k)$-separation*. We say G is *basically 5-connected* if

(i) G is simple and 3-connected, and $|V(G)| \geqslant 4$

(ii) if (A, B) is a 3-separation of G then $\min(|E(A)|, |E(B)|) \leqslant 3$, and

(iii) if (A, B) is a 4-separation of G then $\min(|E(A)|, |E(B)|) \leqslant 6$.

A separation (A, B) is a *cutset* if there are components C, D of $A\backslash V(A \cap B)$ and $B\backslash V(A \cap B)$, respectively, such that every vertex in $V(A \cap B)$ has a neighbour in $V(C)$ and a neighbour in $V(D)$. A separation (A, B) of G is *non-planar* if it is a cutset, it has order $\leqslant 3$, and neither A nor B can be drawn in a disc with $V(A \cap B)$ on the boundary. We say G is *Kuratowski connected* if it has no non-planar separation. In other words, G is Kuratowski connected if for every $(\leqslant 3)$-separation (A, B) of G, if there is a component C of $A\backslash V(A \cap B)$ and a component D of $B\backslash V(A \cap B)$, such that every vertex in $V(A \cap B)$ has a neighbour in $V(C)$ and a neighbour in $V(D)$, then one of A, B can be drawn in a disc Δ with $V(A \cap B)$ drawn in $bd(\Delta)$.

If $f \in E(G)$, $V(f)$ denotes the set of ends of f, and $N(f)$ denotes the set of vertices in $V(G) - V(f)$ adjacent to an end of f. We say $X \subseteq V(G)$ is *2-coherent* in G if there is no $(\leqslant 1)$-separation (A, B) of G such that $|X \cap V(A)|, |X \cap V(B)| > |V(A \cap B)|$. We say that G is *straight* if for all e, $f \in E(G)$ and every end u of e, one of the following holds:

(i) $u \in V(f) \cup N(f)$, or

(ii) $N(f)$ is not 2-coherent in $G\backslash (V(f) \cup \{u\})$, or

(iii) $G\backslash e/f$ is not Kuratowski connected, or

(iv) $G/\{e, f\}$ is not Kuratowski connected, or

(v) $(G\backslash u)/f$ is planar.

This peculiar definition arises because this is what we can prove about a minimal counterexample to Sachs' conjecture and we therefore want to study what follows from it.

We say G is *apex* if $G\backslash v$ is planar for some vertex v. For $m, n \geqslant 0$, $K_{m,n}$ is the complete bipartite graph with vertex set $\{x_1, ..., x_m, y_1, ..., y_n\}$ and edges x_iy_j $(1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n)$, and $K_{5,5}^-$ is the graph obtained from $K_{5,5}$ by deleting five mutually non-adjacent edges. Our first goal is the following.

(1.1) *Let G be expanded, basically 5-connected, straight, and not apex. Then G is isomorphic to either the Petersen graph, or $K_{5,5}^-$, or $K_{m,n}$ for some $m, n \geqslant 5$.*

The proof of (1.1) falls into two parts, depending whether G has a 3-valent vertex or not. We see that (1.1) follows immediately from the following two results. (A graph is *triangle-free* if it has no circuit of length $\leqslant 3$. Clearly every expanded graph is triangle-free; and every triangle-free graph with minimum valency $\geqslant 4$ is expanded.)

(1.2) *Let G be triangle-free, basically 5-connected, and straight, with no vertex of valency 3. Then G is isomorphic to either $K_{5,5}^-$ or to $K_{m,n}$ for some $m, n \geqslant 5$.*

(1.3) *Let G be expanded, basically 5-connected and straight, with a vertex of valency 3. Then either G is apex or G is isomorphic to the Petersen graph.*

After some preliminary lemmas in Section 2, we prove (1.2) in Sections 3 and 4, and (1.3) in Section 5. Finally in Section 6, we give a complementary result about graphs that are not basically 5-connected and combine it with (1.1) to obtain our main result. The latter needs several more definitions before it can be stated, and so we postpone its precise statement to Section 6. Roughly, it says that every graph (except one) with no minor $Y-\Delta$ equivalent to the Petersen graph, is $Y-\Delta$ equivalent to a graph that either is decomposable in some convenient way, or it is apex, or it is expanded, basically 5-connected, and not straight.

2. Some Connectivity Lemmas

In this section we prove some lemmas that will be repeatedly used through the paper.

(2.1) *If G is expanded and $u, v \in V(G)$ are distinct and both have valency 3, then they have at most one common neighbour.*

Proof. Suppose that $x, y \in V(G)$ are distinct and both x and y are adjacent to both u and v. Then u, v are not adjacent, since G is triangle-free. By performing two $Y-\Delta$ exchanges, one at u and one at v, we produce a graph with two parallel edges, contradicting that G is expanded. ∎

(2.2) *Let G be expanded, and let (A, B) be a $(\leqslant 4)$-separation of G, with $|E(A)| \leqslant 6$. Then $|V(A)-V(A\cap B)| \leqslant 2$. Moreover, if $u, v \in V(A)-V(A\cap B)$ are distinct, then $|V(A\cap B)|=4$, u and v are adjacent, and both have valency 3.*

Proof. Let $|V(A)-V(A\cap B)|=n$, and let there be k edges of G with both ends in $V(A)-V(A\cap B)$. Then $|E(A)|+k \geqslant 3n$, since every vertex in $V(A)-V(A\cap B)$ has valency $\geqslant 3$; but $|E(A)| \leqslant 6$, and so $3n \leqslant k+6$. Since $k \leqslant |E(A)| \leqslant 6$ it follows that $3n \leqslant 12$ and so $n \leqslant 4$. Hence $k \leqslant 4$ since G is triangle-free, and so $3n \leqslant 10$ and $n \leqslant 3$. Hence $k \leqslant 2$ since G is triangle-free, but $3n \leqslant k+6$, and so $n \leqslant 2$. Let $u, v \in V(A)-V(A\cap B)$ be distinct. Suppose first that u, v are not adjacent. Then u and v both have $\geqslant 3$ neighbours in $V(A\cap B)$, and so they have $\geqslant 2$ common neighbours, and by (2.1) one of u, v, say u, has valency $\geqslant 4$, contradicting that $|E(A)| \leqslant 6$.

This proves that u, v are adjacent. Since each of them has $\geqslant 2$ neighbours in $V(A\cap B)$, and they have no common neighbour in $V(A\cap B)$ (because G is triangle-free) and $|V(A\cap B)| \leqslant 4$, it follows that $|V(A\cap B)|=4$ and u, v both have valency 3, as required. ∎

(2.3) *If G is expanded and basically 5-connected, and $u, v \in V(G)$ are distinct and both have valency $\leqslant 4$, then some vertex $w \neq u, v$ of G is adjacent to v and not to u.*

Proof. If u, v are adjacent this is clear, since G is triangle-free. We assume that u, v are not adjacent, and every neighbour of v is a neighbour of u. By (2.1), u is 4-valent; let its neighbours be a, b, c, d. Let B be the subgraph of G with vertex set $\{u, v, a, b, c, d\}$ and edge set all edges incident with u or v; then $|E(B)| \geqslant 7$. Let $A = G\backslash\{u, v\}$. Then (A, B) is a 4-separation of G, and so $|E(A)| \leqslant 6$, and $|V(A)-V(A\cap B)| \leqslant 2$ by (2.2). Since no two of a, b, c, d are adjacent since G is triangle-free, there is a $(\leqslant 4)$-separation (C, D) of G with $V(C)-V(C\cap D)=\{a, b\}$, $V(D)-V(C\cap D)=\{c, d\}$; and hence we may assume that $|E(C)| \leqslant 6$. By (2.2), a and b are adjacent, a contradiction. The result follows. ∎

(2.4) *If G is expanded and basically 5-connected, then $|V(G)| \geqslant 10$.*

Proof. Let w be a vertex of G with maximum valency, k say, and let the neighbours of w be $v_1, ..., v_k$. If $k=3$ then G is a cubic graph, with no circuit of length $\leqslant 4$ by (2.1), and hence $|V(G)| \geqslant 10$ as required. We assume then that $k \geqslant 4$, and we suppose for a contradiction that $|V(G)| \leqslant 9$. Let (A, B) be a separation of G with $v_1, v_2 \in V(A)-V(A\cap B)$ and $v_3, v_4, ..., v_k \in V(B)-V(A\cap B)$; this exists, for no two of $v_1, ..., v_k$ are adjacent since G is triangle-free. Since v_1, v_2 and v_3, v_4 are not adjacent, it follows from (2.2) that (A, B) has order $\geqslant 5$, and so $k=4$ and $|V(G)|=9$. Let

$$V(G)=\{w, v_1, v_2, v_3, v_4, u_1, u_2, u_3, u_4\}.$$

Suppose that u_1 is adjacent to $\leqslant 2$ of v_1, v_2, v_3, v_4; say, u_1 is not adjacent to v_3 or v_4. Let (C, D) be a separation of G with

$$V(C\cap D)=\{w, u_2, u_3, u_4\},$$
$$V(C)-V(C\cap D)=\{u_1, v_1, v_2\}$$
$$V(D)-V(C\cap D)=\{v_3, v_4\}.$$

By (2.2), v_3 and v_4 are adjacent, a contradiction. Thus u_1, and similarly each u_i, has $\geqslant 3$ neighbours in $\{v_1, v_2, v_3, v_4\}$, and since G is triangle-free, we deduce that $u_1, ..., u_4$ are mutually non-adjacent. But this contradicts (2.3), and the result follows. ∎

We deduce the following.

(2.5) *If G is expanded and basically 5-connected, and (A, B) is a $(\leqslant 4)$-separation of G with $|V(A)-V(A\cap B)| \leqslant 3$, then $|E(A)| \leqslant 6$ and $|V(A)-V(A\cap B)| \leqslant 2$, and if $u, v \in V(A)-V(A\cap B)$ are distinct then they are adjacent and both 3-valent.*

Proof. Since $|V(A\cap B)| \leqslant 4$ and $|V(A)-V(A\cap B)| \leqslant 3$, it follows from (2.4) that $|V(B)-V(A\cap B)| \geqslant 3$. By (2.2), $|E(B)| \geqslant 7$, and so $|E(A)| \leqslant 6$ since G is basically 5-connected. The result follows from (2.2). ∎

(2.6) *Let (A, B) be a non-planar separation of G. Then $|V(A)-V(A\cap B)| \geqslant 2$, and if equality holds then $|V(A\cap B)|=3$ and both vertices in $V(A)-V(A\cap B)$ are adjacent to every vertex in $V(A\cap B)$.*

The proof is clear.

If A is a subgraph of G and e is an edge of G, we denote by $A+e$ the subgraph of G with vertex set the union of $V(A)$ and the ends of e, and with edge set $E(A) \cup \{e\}$.

(2.7) *Let G be expanded and basically 5-connected, and let $e \in E(G)$. Then $G \backslash e$ is Kuratowski connected.*

Proof. Suppose not, and let (A, B) be a non-planar separation of $G\backslash e$. Since G is clearly Kuratowski connected, it follows that e has ends a, b, where $a \in V(A) - V(A \cap B)$ and $b \in V(B) - V(A \cap B)$. Let $X = V(A) - V(A \cap B)$, $Y = V(B) - V(A \cap B)$. If $|X|, |Y| \geqslant 4$ then the $(\leqslant 4)$-separation $(A, B+e)$ contradicts that G is basically 5-connected, by (2.2). We assume therefore that $|X| \leqslant 3$. If $|X| = 3$ then the $(\leqslant 4)$-separation $(A+e, B)$ violates (2.5), since $|V(A+e) - V((A+e) \cap B)| = 3$. Hence $|X| \leqslant 2$, and so $|X| = 2$ by (2.6), and $X = \{a, a'\}$ say. By (2.5) applied to $(A+e, B)$, a and a' are adjacent and both 3-valent in G, contrary to (2.6). The result follows. ∎

(2.8) *Let G be expanded and basically 5-connected, and let $f \in E(G)$. Then G/f is Kuratowski connected.*

Proof. Suppose not; then there are subgraphs A, B of G with $A \cup B = G$ and $E(A \cap B) = \{f\}$, such that $|V(A \cap B)| \leqslant 4$, and $(A/f, B/f)$ is a non-planar separation of G/f. Since G is basically 5-connected we may assume that $|E(A/f)| \leqslant 6$. By (2.2), $|V(A) - V(A \cap B)| \leqslant 2$, and so by (2.6), equality holds, and $V(A) - V(A \cap B) = \{a_1, a_2\}$ say. By (2.2), a_1, a_2 are adjacent in G and both have valency 3 in G, contrary to (2.6). The result follows. ∎

(2.9) *Let G be expanded and basically 5-connected, and let $f \in E(G)$ with ends w, x such that either G has no 3-valent vertex or w is 3-valent. Let $u \in V(G)$, not adjacent to w or x, and suppose that $N(f)$ is not 2-coherent in $G \backslash (V(f) \cup \{u\})$. Then w is 3-valent, and there are adjacent vertices a, $b \neq u$, w, x such that a is 3-valent and adjacent to u and to w, and b is adjacent to x.*

Proof. Since $N(f)$ is not 2-coherent in $G\backslash(V(f) \cup \{u\})$, there is a $(\leqslant 4)$-separation (A, B) of G with $u, w, x \in V(A \cap B)$, such that

$$|N(f) \cap V(A)|,\ |N(f) \cap V(B)| > |V(A \cap B)| - 3.$$

If $V(A \cap B) = \{u, w, x\}$, we may assume that $|V(A) - V(A \cap B)| \leqslant 1$; but $N(f) \cap V(A) \neq \varnothing$, and $u, w, x \notin N(f)$, and so $|V(A) - V(A \cap B)| = 1$. Let $V(A) - V(A \cap B) = \{a\}$. Then a has valency $\geqslant 3$, and hence is adjacent to both w and x, contradicting that G is triangle-free.

Thus $|V(A \cap B)| = 4$, $V(A \cap B) = \{u, b, w, x\}$ say, and consequently $|N(f) \cap V(A)|, |N(f) \cap V(B)| \geqslant 2$. Since G is basically 5-connected, we may assume that $|E(A)| \leqslant 6$ and hence $|V(A) - V(A \cap B)| \leqslant 2$, by (2.2). If $a_1, a_2 \in V(A) - V(A \cap B)$ are distinct, then they are adjacent and both 3-valent by (2.2), and one is adjacent to w and the other to x, since G is triangle-free. But then w is 3-valent since G has a 3-valent vertex, and w has two common neighbours with one of a_1, a_2, contrary to (2.1).

Thus $|V(A) - V(A \cap B)| \leqslant 1$. Since $|N(f) \cap V(A)| \geqslant 2$ and $u, w, x \notin N(f)$, it follows that $|V(A) - V(A \cap B)| = 1$, $V(A) - V(A \cap B) = \{a\}$, say. Every neighbour of a is in $V(A \cap B)$, and a is not adjacent to both w and x since G is triangle-free; and so a is 3-valent (and hence so is w), and a is adjacent to u, b and to exactly one of w, x. Since $|N(f) \cap V(A)| \geqslant 2$, it follows that $b \in N(f)$. If a is adjacent to x then w is adjacent to b, contrary to (2.1) since a, w have two common neighbours, namely b and x. Thus a is adjacent to w and b to x, as required. ∎

(2.10) *Let G be expanded, and let $w \in V(G)$ be 3-valent, with neighbours x_1, x_2, x_3. There do not exist distinct $u_1, u_2 \in V(G) - \{w\}$ such that u_i is adjacent to x_i and x_3 $(i = 1, 2)$.*

Proof. Suppose that such u_1, u_2 exist. Then $u_1, u_2, w, x_1, x_2, x_3$ are all distinct. By a $Y-\Delta$ exchange at w, a $\Delta - Y$ exchange at $u_1 x_1 x_3$, and a $\Delta - Y$ exchange at $u_2 x_2 x_3$, we obtain a graph with more vertices than G, contradicting that G is expanded. ∎

3. Separations of Order 5

In this section and the next we prove (1.2). The main result of this section is the following.

(3.1) *Let G be basically 5-connected and triangle-free, with no vertex of valency 3. Let $v \in V(G)$; then there is a neighbour u of v such that for every $(\leqslant 5)$-separation (A, B) of G with $u, v \in V(A \cap B)$, one of $A \backslash V(A \cap B)$, $B \backslash V(A \cap B)$ has no edges.*

We need the following lemma.

(3.2) *Let G be basically 5-connected and triangle-free, with no vertex of valency 3, and let (A, B) be a separation of G:*

(i) *If (A, B) has order $\leqslant 4$, then either $|V(A) - V(A \cap B)| \leqslant 1$ or $|V(B) - V(A \cap B)| \leqslant 1$;*

(ii) *if (A, B) has order $\leqslant 5$ and $E(A \backslash V(A \cap B)) \neq \varnothing$ then $|V(A) - V(A \cap B)| \geqslant 4$.*

Proof. Since no vertex has valency 3, it follows that G is expanded, and (i) is immediate from (2.2). For (ii), let $v, w \in V(A) - V(A \cap B)$ be adjacent. Since v, w have no common neighbour since G is triangle-free, one of them, w say, has $\leqslant 2$ neighbours in $V(A \cap B)$. Since w has valency $\geqslant 4$, there exists $u \in V(A) - V(A \cap B)$ adjacent to w with $u \neq v$. Suppose that $V(A) - V(A \cap B) = \{u, v, w\}$. Since w has valency $\geqslant 4$, there exists distinct $a, b \in V(A \cap B)$ adjacent to w, and hence not adjacent to u or v. Since u, v both have valency $\geqslant 4$, it follows that u, v both have valency 4 and are adjacent to w and to the three vertices in $V(A \cap B)$ different from a, b. But this contradicts (2.3). The result follows. ∎

Proof of (3.1). Let $v \in V(G)$, and choose a neighbour u of v. We may assume that the theorem does not hold, and so there is a ($\leqslant 5$)-separation (A, B) of G with $u, v \in V(A \cap B)$, such that both $A \backslash V(A \cap B)$ and $B \backslash V(A \cap B)$ have an edge. Let us choose the vertex u and the separation (A, B) so that

(1) $|V(A) - V(A \cap B)|$ *is minimum.*

We claim

(2) $|V(A) - V(A \cap B)| \geqslant 4$, $|V(B) - V(A \cap B)| \geqslant 4$, *and* $A \backslash V(A \cap B)$ *is connected.*

The first two statements follow from (3.2)(ii). For the third, let C be a component of $A \backslash V(A \cap B)$ with an edge. Let A' be the subgraph with vertex set $V(C) \cup V(A \cap B)$ and edge set all edges of G with an end in C, and let $B' = G \backslash V(C)$. Then (A', B') is a separation of G with $A' \cap B' = A \cap B$. By (1),

$$V(A) - V(A \cap B) = V(A') - V(A' \cap B') = V(C)$$

and so $A \backslash V(A \cap B)$ is connected.

(3) $|V(A \cap B)| = 5$, *and each vertex in* $V(A \cap B)$ *has a neighbour in* $V(A) - V(A \cap B)$.

For by (3.2)(i) and (2), $|V(A \cap B)| = 5$. Let $w \in V(A \cap B)$. If w has no neighbour in $V(A) - V(A \cap B)$, then $(A \backslash w, B^+)$ is a separation of G, where B^+ is obtained from B by adding to it all edges of A incident with w. But $(A \backslash w, B^+)$ has order 4, contrary to (3.2)(i), by (2). This proves (3).

Let $u' \in V(A) - V(A \cap B)$ be incident with v. (This exists, by (3).) We suppose, for a contradiction, that u' does not satisfy the theorem, and so there exists a ($\leqslant 5$)-separation (A', B') of G such that $u', v \in V(A' \cap B')$, and both $A' \backslash V(A' \cap B')$, $B' \backslash V(A' \cap B')$ have an edge. Now

$$(V(A) - V(A \cap B), V(A \cap B), V(B) - V(A \cap B))$$

is a partition of $V(G)$ and so is $(V(A') - V(A' \cap B'), V(A' \cap B'), V(B') - V(A' \cap B'))$. We may therefore choose nine subsets X_{ij} of $V(G)$, mutually disjoint and with union $V(G)$, so that

$$V(A) - V(A \cap B) = X_{11} \cup X_{12} \cup X_{13}$$
$$V(A \cap B) = X_{21} \cup X_{22} \cup X_{23}$$
$$V(B) - V(A \cap B) = X_{31} \cup X_{32} \cup X_{33}$$
$$V(A') - V(A' \cap B') = X_{11} \cup X_{21} \cup X_{31}$$
$$V(A' \cap B') = X_{12} \cup X_{22} \cup X_{32}$$
$$V(B') - V(A' \cap B') = X_{13} \cup X_{23} \cup X_{33}.$$

For $1 \leqslant i, j \leqslant 3$ let $|X_{ij}| = x_{ij}$. We have that $x_{21} + x_{22} + x_{23} = 5$ by (3), and $x_{12} + x_{22} + x_{32} = 5$ similarly, and $x_{11} + x_{12} + x_{13}$, $x_{31} + x_{32} + x_{33}$, $x_{11} + x_{21} + x_{31}$, $x_{13} + x_{23} + x_{33} \geqslant 4$, and $u' \in X_{12}$, and $v \in X_{22}$.

Suppose first that $x_{12} + x_{22} + x_{21} \geqslant 6$. Hence $x_{23} + x_{22} + x_{32} \leqslant 4$, since the sum of these equals the sum of $x_{21} + x_{22} + x_{23}$ and $x_{12} + x_{22} + x_{32}$. Every edge with one end in X_{33} has its other end in $X_{23} \cup X_{22} \cup X_{32} \cup X_{33}$, and so by (3.2)(i), $x_{33} \leqslant 1$, since $x_{11} + x_{12} + x_{13} + x_{21} + x_{31} \geqslant 2$. If $x_{23} = 0$, then since $B' \backslash V(A' \cap B')$ has an edge, it follows that X_{13} includes a component of $B' \backslash V(A' \cap B')$ with an edge. Then setting $A'' = B' \backslash X_{33}$ and letting B'' be the union of A', X_{33} and all edges of G with an end in X_{33}, we obtain a 5-separation (A'', B'') of G, with $u', v \in V(A'' \cap B'')$ and with $V(A'') - V(A'' \cap B'') = X_{13}$; yet $A'' \backslash V(A'' \cap B'')$ and $B'' \backslash V(A'' \cap B'')$ both have edges, contrary to (1) (since $x_{12} \neq 0$). Thus $x_{23} \geqslant 1$. But $x_{12} + x_{22} + x_{21} \geqslant 6$ by assumption, and $x_{21} + x_{22} + x_{23} = 5$, and so $x_{12} > x_{23} \geqslant 1$; hence $x_{12} \geqslant 2$. Also, since $x_{23} \geqslant 1$ it follows that $x_{21} + x_{22} \leqslant 4$. If $x_{32} = 0$ then $x_{31} \leqslant 1$ (by (3.2)(i), because every vertex with a neighbour in X_{31} belongs to $X_{21} \cup X_{22} \cup X_{32} \cup X_{31}$), which is impossible since $B \backslash V(A \cap B)$ has an edge and $x_{33} \leqslant 1$. Thus $x_{32} \geqslant 1$. Since $x_{12} + x_{22} + x_{21} > x_{12} + x_{22} + x_{32}$, it follows that $x_{21} > x_{32} \geqslant 1$ and so $x_{21} \geqslant 2$. Since $x_{21} + x_{22} + x_{23} = 5$ and $x_{22} \geqslant 1$, it follows that $x_{23} \leqslant 2$. But $x_{13} + x_{23} + x_{33} \geqslant 4$, and $x_{33} \leqslant 1$, and so $x_{13} \geqslant 1$. If $x_{12} + x_{22} + x_{23} \leqslant 4$, then $x_{13} = 1$ by (3.2)(i), and the unique vertex in X_{13} is adjacent to every vertex in $X_{12} \cup X_{22} \cup X_{23}$, a contradiction since G is triangle-free and $u' \in X_{12}$ and $v \in X_{22}$. Thus $x_{12} + x_{22} + x_{23} \geqslant 5$. Since $x_{23} + x_{22} + x_{32} \leqslant 4$, it follows that $x_{12} \geqslant x_{32} + 1$. But $x_{12} + x_{22} + x_{32} = 5$ and $x_{22} \geqslant 1$, and so $2x_{32} + 2 \leqslant 5$, that is, $x_{32} \leqslant 1$. Hence $x_{32} = 1$, for we already showed that $x_{32} \geqslant 1$. But $x_{31} + x_{32} + x_{33} \geqslant 4$ and $x_{33} \leqslant 1$, and so $x_{31} \geqslant 2$. By (3.2)(i), $x_{21} + x_{22} + x_{32} \geqslant 5$, and so $x_{21} + x_{22} \geqslant 4$. But $x_{23} \geqslant 1$ and $x_{21} + x_{22} + x_{23} = 5$, and so $x_{23} = 1$. Since $x_{12} + x_{22} + x_{32} = 5$ and $x_{23} = x_{32} = 1$, it follows that $x_{12} + x_{22} + x_{23} = 5$. From (1), no edge of G has both ends in X_{13}, and so every vertex in X_{13} has all its neighbours in $X_{12} \cup X_{22} \cup X_{23}$. Since $|X_{12} \cup X_{22} \cup X_{23}| = 5$ and u', v both belong to this set, it follows that each vertex in X_{13} is 4-valent and is joined to one of u', v, and all the three other vertices in $X_{12} \cup X_{22} \cup X_{23}$. By (2.3), $x_{13} \leqslant 2$. But $x_{13} + x_{23} + x_{33} \geqslant 4$, and so $x_{13} = 2$, and $x_{33} = 1$. Since $x_{33} = 1$ and the vertex in X_{33} has at least four neighbours, it follows that $x_{23} + x_{22} + x_{32} \geqslant 4$, and so $x_{22} \geqslant 2$. Hence $x_{12} = x_{22} = 2$ since $x_{12} + x_{22} + x_{32} = 5$ and we already showed that $x_{12} \geqslant x_{32} + 1 = 2$. Also, $x_{21} = 2$ since $x_{21} + x_{22} + x_{23} = 5$. Let $X_{13} = \{a, b\}$, $X_{23} = \{c\}$, $X_{33} = \{d\}$, $X_{12} = \{p, u'\}$, $X_{22} = \{q, v\}$, $X_{32} = \{r\}$. Then a, b are both adjacent to p, q, c and one of u', v as we saw, and by (2.3) one of a, b is adjacent to u' and the other to v. Consequently c is adjacent to none of p, u', q, v since G is triangle-free. Also, d is adjacent to c, q, v, r, and so c is not adjacent to r. Hence the only neighbours of c are a, b, and d, and c is 3-valent, a contradiction.

We have proved then that $x_{12} + x_{22} + x_{21} \leqslant 5$, and similarly $x_{12} + x_{22} + x_{23} \leqslant 5$. By (1), no edge of G has both ends in X_{11}, and no edge has both ends in X_{13}. Suppose that $x_{12} = 1$, that is, $X_{12} = \{u'\}$. Since $x_{11} + x_{12} + x_{13} \geqslant 4$, we may assume by the symmetry between A' and B' that $x_{11} \geqslant 2$; let $x, y \in X_{11}$ be distinct. Since $A \backslash V(A \cap B)$ is connected by (2), it follows that x, y are both adjacent to u' and hence not to v. Since $x_{12} + x_{22} + x_{21} \leqslant 5$, this contradicts (2.3). Hence $x_{12} \geqslant 2$. Since $x_{12} + x_{22} + x_{23} \leqslant 5 = x_{21} + x_{22} + x_{23}$, it follows that $x_{21} \geqslant x_{12}$, and similarly $x_{23} \geqslant x_{12}$. But $x_{22} \geqslant 1$, and $x_{21} + x_{22} + x_{23} = 5$, so $x_{21} = x_{23} = x_{12} = 2$ and $x_{22} = 1$. Now $u \in V(A \cap B) - \{v\}$ and so we may assume from the symmetry that $u \in X_{21}$. Let $X_{12} = \{u', p\}$, $X_{21} = \{u, q\}$, $X_{23} = \{r, s\}$. Every vertex in X_{11} has four neighbours in $\{p, u', v, u, q\}$, and since v is adjacent to u and to u' it follows that every vertex in X_{11} has neighbour set $\{p, u', u, q\}$. By (2.3), $x_{11} \leqslant 1$. Since $x_{11} + x_{12} + x_{13} \geqslant 4$, it follows that $x_{13} \geqslant 1$. Now every vertex in X_{13} is adjacent to p, r, s and to one of u', v, and so $x_{13} \leqslant 2$, by (2.3); and p is not adjacent to r or s, since $x_{13} \neq 0$ and G is triangle-free. Suppose that $x_{11} \neq 0$. Then $x_{11} = 1$, and p is not adjacent to q, u, or u' since G is triangle-free. Hence all neighbours of p lie in $X_{11} \cup X_{22} \cup X_{13}$, and since $x_{11} + x_{13} \leqslant 3$ and p has valency $\geqslant 4$, we deduce that p is adjacent to v. It follows that no vertex in X_{13} is adjacent to both p and v, and so $x_{13} \leqslant 1$ by (2.3), and p has valency $\leqslant 3$, a contradiction. This shows that $x_{11} = 0$, and so $x_{13} \geqslant 2$, since $x_{11} + x_{12} + x_{13} \geqslant 4$. Hence some member of X_{13} is adjacent to u' (for they do not all have neighbour set $\{p, v, r, s\}$, by (2.3)), and so u' is not adjacent to p, r, or s. Thus the only neighbours of u' are v, one vertex in X_{13}, and possibly q, for u' is not adjacent to u since v is adjacent to u and u'. It follows that u' has valency $\leqslant 3$, a contradiction.

We deduce that there is no such (A', B'), and hence the theorem holds. ▮

4. Minimum Valency $\geqslant 4$

Now we complete the proof of (1.2). We begin with the following.

(4.1) *Let G be triangle-free, basically 5-connected and straight, with no vertex of valency 3. Then for every $u \in V(G)$ there is an edge $f \in E(G)$ incident with u such that $(G/f) \backslash v$ is planar for every v not adjacent to either end of f.*

Proof. By (3.1) there is an edge f of G incident with u, such that for every ($\leqslant 5$)-separation (A, B) of G, if both ends of f are in $V(A \cap B)$ then one of $A \backslash V(A \cap B)$, $B \backslash V(A \cap B)$ has no edges. Let $v \in V(G)$, not adjacent to either end of f, and let e be an edge incident with v. Let the ends of f be u, u'.

(1) *$G \backslash e/f$ is Kuratowski connected.*

For suppose not. By (2.7) and (2.8), $G \backslash e$ and G/f are Kuratowski connected. Hence there is a pair of subgraphs (A, B) of $G \backslash e$ such that $A \cup B = G \backslash e$, $E(A \cap B) = \{f\}$, $|V(A \cap B)| \leqslant 4$, and e has ends a, b, where $a \in V(A) - V(A \cap B)$ and $b \in V(B) - V(A \cap B)$; and $(A/f, B/f)$ is a non-planar separation of $G \backslash e/f$. Let $a' \in V(A) - V(A \cap B)$ with $a' \neq a$ (this exists, since $|V(A) - V(A \cap B)| \geqslant 2$). Since a' has valency $\geqslant 4$ and is not adjacent to both ends of f, there is an edge incident with a' with its other end not in $V(A \cap B)$; that is, $(A + e) \backslash V((A + e) \cap B)$ has an edge. But $(A + e, B \backslash f)$ is a ($\leqslant 5$)-separation of G, and $u, u' \in V((A + e) \cap (B \backslash f))$; and so, from the choice of f, $(B \backslash f) \backslash V((A + e) \cap B)$ has no edge. In other words, for every vertex in $V(B) - (V(A) \cup \{b\})$, all its neighbours are in $V(A \cap B) \cup \{b\}$. By (2.3), there are $\leqslant 2$ such vertices since u and u' are adjacent and G is triangle-free; and similarly $V(A) - (V(A \cap B) \cup \{a\})$ contains $\leqslant 2$ vertices and they are non-adjacent. We recall that $a' \in V(A) - V(B)$ and $a' \neq a$. It follows that a' is adjacent to a, for it has no neighbour in $V(A) - V(B)$ except a, and a' is adjacent to $\leqslant 3$ vertices in $V(A \cap B)$. Since a, a' have no common neighbours and both have valency $\geqslant 4$, we deduce that one of them is adjacent to u and the other to u'. In particular, a is adjacent to an end of f, and similarly so is b. But one of a, b equals v, contrary to the choice of v. This proves (1).

(2) *$G/\{e, f\}$ is Kuratowski connected.*

By (2.8), if (2) is false there are subgraphs A, B of G with $E(A \cap B) = \{e, f\}$, $A \cup B = G$ and $|V(A \cap B)| \leqslant 5$, such that $(A \backslash \{e, f\}, B/\{e, f\})$ is a non-planar separation of $G/\{e, f\}$. We may therefore assume that $A \backslash V(A \cap B)$ has no edges, by the choice of f. Let $a \in V(A) - V(B)$. Since a has valency $\geqslant 4$, it is adjacent to $\geqslant 4$ of the vertices in $V(A \cap B)$; and hence either to both ends of e or to both of f. In either case this contradicts that G is triangle-free, as required.

Since G is straight and G has no 3-valent vertex, it follows from (1), (2), and (2.9) that $G\backslash v/f$ is planar, as required. ∎

(4.2) *Let G be triangle-free, basically 5-connected and straight, with no vertex of valency 3. Then either every vertex has valency 4, or G is isomorphic to $K_{m,n}$ for some $m, n \geqslant 5$.*

Proof. We denote the valency of $v \in V(G)$ by $d(v)$. Let $\Delta = \max(d(v) : v \in V(G))$, and choose a pair u, w of vertices, adjacent, so that $d(w) = \Delta$ and, subject to that, $d(u)$ is minimum. By (4.1) there is a neighbour u' of u such that, if A and A' denote the sets of neighbours of u and u', respectively, and $X = V(G) - (A \cup A')$, then for each $v \in X$, $G\backslash\{u, v\}$ is planar and $G\backslash\{u', v\}$ is planar. (We observe that both these graphs are subgraphs of $G\backslash v/f$, where f has ends u, u'.)

Suppose first that $X = \emptyset$. Then $A \cup A' = V(G)$ and $A \cap A' = \emptyset$. Since $w \in A$, and every neighbour of w is in A', it follows that $|A'| \geqslant \Delta$; but $|A'| = d(u') \leqslant \Delta$, and so $d(u') = \Delta$. Hence every vertex in A' has a neighbour of valency Δ, and so from the choice of the pair u, w, it follows that every vertex in A' has valency $\geqslant d(u)$. But every vertex in A' has all its neighbours in A, and $|A| = d(u)$; and consequently G is a complete bipartite graph, with bipartition (A, A'). Since $d(u) \geqslant 4$ it follows that $|A| \geqslant 4$, and since G is basically 5-connected we deduce that $|A| \geqslant 5$. Since $|A| \leqslant |A'|$ it follows that $|A'| \geqslant 5$ as well, and the theorem holds. We may therefore assume that $X \neq \emptyset$.

Choose $v \in X$ with minimum valency. Choose $x \in V(G) - \{u, u', v\}$ with $d(x)$ maximum, and choose $y \in V(G) - \{u, u', v, x\}$ with $d(y)$ maximum. (This is possible by (2.4).)

(1) $|d(u') - d(u)| + d(x) + d(y) \leqslant d(v) + 4$.

For $G\backslash\{u, v\}$ and $G\backslash\{u', v\}$ are both planar, with $n-2$ vertices, where $|V(G)| = n$, and both triangle-free. Since $n - 2 \geqslant 3$ it follows from Euler's formula that both these graphs have $\leqslant 2(n-2) - 4$ edges. Consequently,

$$|E(G)| - d(u) - d(v),\ |E(G)| - d(u') - d(v) \leqslant 2n - 8,$$

that is,

$$|E(G)| \leqslant 2n - 8 + \min(d(u), d(u')) + d(v).$$

But $2\,|E(G)|$ is the sum of the valencies of the vertices of G, and hence

$$2\,|E(G)| \geqslant d(u) + d(u') + d(v) + d(x) + d(y) + 4(n-5).$$

Consequently,

$$4n - 16 + 2\min(d(u), d(u')) + 2d(v) \geqslant d(u) + d(u') + d(v) + d(x) + d(y) + 4(n-5).$$

Since

$$2\min(d(u), d(u')) + |d(u') - d(u)| = d(u) + d(u'),$$

this proves (1).

Now let us suppose that $\Delta > 4$. There are two cases, depending whether $u' = w$ or not. Suppose first that $u' \neq w$. Then $w \neq u, u', v$, for $v \in X$ and $w \notin X$, and so $d(x) = \Delta$ by the choice of x. From (1),

$$|d(u') - d(u)| + \Delta + d(y) \leqslant d(v) + 4;$$

but $\Delta \geqslant d(v)$, $d(y) \geqslant 4$, and $|d(u') - d(u)| \geqslant 0$ and so we have equality throughout. In particular, $d(v) = \Delta$, $d(y) = 4$, and $d(u') = d(u)$. Since $d(y) = 4$, it follows that every vertex except x, u, u', v has valency 4; and in particular $x = w$, since $d(w) = \Delta \geqslant 5$ and $w \neq u, u', v$. Since $d(v) = \Delta$, every vertex in X has valency $\Delta > 4$. Consequently, $X \subseteq \{w, u, u', v\}$; but $w \notin X$, and $u, u' \notin X$, and so $X = \{v\}$. Since there is a vertex of valency $4 < \Delta$, it follows from the choice of u that $d(u) < \Delta$. Since $|A| = d(u) = d(u') = |A'|$ and $|A| + |A'| + |X| = n$, it follows that $n = 2d(u) + 1$. Since $d(u) < \Delta$, we deduce that $2\Delta \geqslant n + 1$. Now $w \in A$, and, since G is triangle-free, w is adjacent to no other member of A, and so it is adjacent to Δ vertices in $A' \cup \{v\}$. Since $|A'| = d(u) < \Delta$, we deduce that v, w are adjacent. But they both have valency Δ, and so $n \geqslant 2\Delta$ since G is triangle-free, a contradiction.

Now suppose that $u' = w$. From (1), we have

$$d(w) - d(u) + d(x) + d(y) \leqslant d(v) + 4.$$

But $d(w) \geqslant d(v)$, and so $d(x) + d(y) \leqslant d(u) + 4$. Now there are $\geqslant 2$ neighbours of w different from u and v, since $\Delta \geqslant 4$, and every such neighbour has valency $\geqslant d(u)$, from the choice of u. Hence $d(x), d(y) \geqslant d(u)$ from the choice of x and y, and so $d(x) = d(y) = d(u) = 4$ and $d(w) = d(v)$. We deduce that every vertex has valency 4 except w and v, and again $X = \{v\}$. Now $|A'| = \Delta$ and $|A| = 4$; let $A = \{a, b, c, w\}$. Since $d(v) = \Delta$ and v is not adjacent to u or w, it follows that v is adjacent to one of a, b, c, say a. If $a' \in A'$, then $d(a') \geqslant 4$, and a' is not adjacent to any other vertex in A'; and so a' has 4 neighbours in $\{a, b, c, w, v\}$. Since it is not adjacent to both a and v, it follows that a' is adjacent to w, b, c and to one of a, v. Hence every vertex in A' is adjacent to b, and so $d(b) = \Delta$, a contradiction, since every vertex has valency 4 except w and v.

In either case we have obtained a contradiction. Thus our assumption that $\Delta > 4$ was false, as required. ∎

Proof of (1.2). By (4.2) we may assume that every vertex of G has valency 4.

(1) *For every $u \in V(G)$ there is an edge $f \in E(G)$ with ends u, u' say, such that for every $v \in V(G)$ not adjacent to u or u', $(G/f)\backslash v$ is planar, simple and 3-connected, and in a planar drawing of this graph, every region incident with the vertex w of G/f formed by identifying u and u' has perimeter circuit of length 3, and every other region has perimeter of length 4.*

For let $u \in V(G)$ and choose an edge f incident with u as in (4.1). Let f have ends u, u', let $v \in V(G)$, not incident with u or u', and let $H = G\backslash v/f$. By (4.1), H is planar, and simple since G is triangle-free. If H is not 3-connected, then G/f is not 4-connected, and so there are subgraphs A, B of G with $A \cup B = G$, $E(A \cap B) = \{f\}$ and $|V(A \cap B)| \leqslant 4$, such that $V(A)$, $V(B) \neq V(G)$. By (3.2)(i) we may assume that $|V(A) - V(A \cap B)| = 1$; let $a \in V(A) - V(A \cap B)$. Then all neighbours of a are in $V(A \cap B)$, and so a has valency $\leqslant 3$ since G is triangle-free and both ends of f are in $V(A \cap B)$, a contradiction. This proves that H is 3-connected.

Let $|V(G)| = n$; then $|E(G)| = 2n$ since every vertex has valency 4. Let w be the vertex of H formed by identifying u and u'. Since G is triangle-free, w lies in every circuit of H of length 3. Take a planar drawing of H; then w is incident with six regions of this drawing, since w has valency 6 in H, and every other region is bounded by a circuit of length $\geqslant 4$. Now $|V(H)| = n-2$, and $|E(H)| = 2n-5$, and so by Euler's formula, the drawing of H has $n-1$ regions. Summing over the lengths of the perimeters of all regions, we deduce that

$$2\,|E(H)| \geqslant 3 \cdot 6 + 4(n-7) = 2(2n-5),$$

and so we have equality throughout, and (1) follows.

(2) *For any two distinct vertices u, v of G, either u, v are adjacent, or there are $\geqslant 3$ vertices adjacent to both u and v, or $G\backslash\{u, v\}$ is planar and G is bipartite.*

For let $u, v \in V(G)$ be distinct, and let f, u', w be as in (1). By (2.4) there is a vertex v' not adjacent to u or to u', since $d(u) = d(u') = 4$. Now $(G/f)\backslash v'$ is planar, by the choice of f. Let C be the circuit of $(G/f)\backslash v'$ bounding the disc formed by the closures of the six triangular regions incident with w; then the vertices of C are alternatively neighbours in G of u and u', since G is triangle-free. If $v = u'$ or $v \in V(C)$ then either u, v are adjacent or there are $\geqslant 3$ vertices adjacent to both u and v, as required. We assume therefore that $v \neq u'$ and v is not adjacent to w in $(G/f)\backslash v'$. Consequently, v is not

adjacent to u or u' in G, and we may therefore assume that $v' = v$; so $(G/f)\backslash v$ is planar, and hence $G\backslash\{u, v\}$ is planar. Since every region of $(G/f)\backslash v$ has perimeter of length 4 except those incident with w, it follows that $G\backslash\{u, u', v\}$ is bipartite; and since the neighbours of u and of u' alternate on C, it follows that $G\backslash v$ is bipartite. From the symmetry, so is $G\backslash u$. But $G\backslash\{u, v\}$ is connected and u, v are not adjacent, and so G is bipartite as required.

Choose $a_1 \in V(G)$. Let b_2 be a neighbour of a_1 such that (1) holds with $u = a_1$ and $u' = b_2$. Let the neighbours of a_1 be b_2, b_3, b_4, b_5, and let the neighbours of b_2 be a_1, a_3, a_4, a_5. As we saw in the proof of (1), these can be ordered so that a_3 is adjacent to b_4 and b_5, a_4 is adjacent to b_3 and b_5, and a_5 to b_3 and b_4.

From (2.4), there is a vertex v not adjacent to a_1 or b_2. If it has $\geqslant 3$ common neighbours with a_1, then v is adjacent to all of b_3, b_4, b_5, and hence does not have $\geqslant 3$ common neighbours with b_2. From the symmetry between a_1 and b_2 we may therefore assume that v does not have three common neighbours with a_1. By (2) with u, v replaced by a_1, v, it follows that G is bipartite. Let (A, B) be a 2-colouring of G with $a_1, a_3, a_4, a_5 \in A$ and $b_2, b_3, b_4, b_5 \in B$.

(3) *If $a \in A$ and $b \in B$ and a, b are not adjacent then $G\backslash\{a, b\}$ is planar.*

This follows from (2), because no vertex is adjacent to both a and b since (A, B) is a 2-colouring.

Let $X = \{a_1, a_3, a_4, a_5, b_2, b_3, b_4, b_5\}$ and $Y = V(G) - X$. Since G is bipartite and every vertex has valency 4, it follows that $|A| = |B|$, and hence $|A \cap Y| = |B \cap Y|$. Let $|A \cap Y| = k$ say. Since the restriction $G \mid X$ of G to X is non-planar, it follows from (3) that every vertex in $A \cap Y$, and so $G \mid Y$ is isomorphic to $K_{k,k}$. By (3), if $b \in B \cap Y$ then $G\backslash\{a_1, b\}$ is planar and so $(G \mid Y)\backslash b$ is planar, and hence $k \leqslant 3$. Suppose that $k = 3$. Then $G\backslash\{a_3, b_3\}$ is not planar, since $G \mid Y$ is not planar, and so by (3), a_3 is adjacent to b_3, and similarly, a_4 to b_4 and a_5 to b_5. But then no edge has one end in X and the other in Y, a contradiction. Hence $k \leqslant 2$. Suppose that $k = 2$, and let $b, b' \in B \cap Y$ be distinct. Since $G \mid X$ is non-planar and b, b' are not adjacent or equal, it follows from (2) that there are three vertices of G adjacent to b and to b'. These vertices all belong to A, and one of them, a say, is not in Y since $|A \cap Y| = k = 2$. Hence $a \in \{a_1, a_3, a_4, a_5\}$, and so a has $\geqslant 3$ neighbours in X, and hence has valency $\geqslant 5$, a contradiction. Thus $k \leqslant 1$. But $k \neq 0$ since $|V(G)| \geqslant 10$ by (2.4), and so $k = 1$. Let $A \cap Y = \{a_2\}$, $B \cap Y = \{b_1\}$. Since a_2, b_1 both have valency 4, it follows that a_2 is adjacent to b_1, b_3, b_4, b_5, and b_1 to a_2, a_3, a_4, a_5. Then G is isomorphic to $K_{5,5}^-$, as required. ∎

5. Using a 3-Valent Vertex

In this section we prove (1.3). We shall have several occasions to discuss the following situation, so let us give it a name for convenience.

Hypothesis J. *Let G be expanded, basically 5-connected and straight. Let $w \in V(G)$ have valency 3, let the neighbours of w be v, x_1, x_2, and let the corresponding edges incident with w be g, f_1, f_2. Let $u \in V(G)$ be adjacent to v with $u \neq w$, and let e be the edge with ends u, v.*

(5.1) *Under Hypothesis J, $G/\{e, f_1\}$ is Kuratowski connected.*

Proof. Suppose not. Since G/e and G/f_1 are both Kuratowski connected by (2.8), it follows that there are two subgraphs A, B of G with $A \cup B = G$ and $E(A \cap B) = \{e\ f_1\}$, such that $|V(A \cap B)| \leqslant 5$ and $(A/\{e, f_1\}, B/\{e, f_1\})$ is a non-planar separation of $G/\{e, f_1\}$. By exchanging A and B if necessary, we may assume that $f_2 \in E(B)$ and hence $x_2 \in V(B)$. Since w has valency 3, and its neighbours v, x_1, x_2 all belong to $V(B)$, it follows that $(A\backslash\{w, e\}, B+g)$ is a separation of G of order $\leqslant 4$. Since $(A/\{e, f_1\}, B/\{e, f_1\})$ is non-planar, it follows from (2.6) that $|V(A) - V(A \cap B)| \geqslant 2$ and $|V(B) - V(A \cap B)| \geqslant 2$. In particular, there are $\geqslant 5$ edges of B with at least one end not in $V(A)$, and so $|E(B+g)| \geqslant 8$. Since G is basically 5-connected, it follows that $|E(A\backslash\{w, e\})| \leqslant 6$, and by (2.2), $|V(A) - V(A \cap B)| = 2$, $V(A) - V(A \cap B) = \{a_1, a_2\}$ say, and a_1, a_2 are adjacent in G and both have valency 3 in G. This contradicts that $(A/\{e, f_1\}, B\backslash\{e, f_1\})$ is non-planar, and the result follows. ∎

(5.2) *Under Hypothesis J, suppose that $G\backslash e/f_1$ is not Kuratowski connected. Then v has valency 4, x_1 has valency 3, and x_1 has a 3-valent neighbour different from w.*

Proof. Since $G\backslash e/f_1$ is not Kuratowski connected, and by (2.7) and (2.8) $G\backslash e$ and G/f_1 are Kuratowski connected, it follows that there are subgraphs A, B of G with $A \cup B = G\backslash e$, $E(A \cap B) = \{f_1\}$ and $|V(A \cap B)| \leqslant 4$, such that $v \in V(A) - V(A \cap B)$, $u \in V(B) - V(A \cap B)$, and $(A/f_1, B/f_1)$ is non-planar.

(1) $x_2 \notin V(A)$.

For suppose that $x_2 \in V(A)$, and let $X = V(A) - V(A \cap B)$. By (2.6), $|X| \geqslant 2$. Since $(A+e+f_2, B\backslash w)$ is a $(\leqslant 4)$-separation of G and

$$V(A+e) - V((A+e) \cap (B\backslash w)) = X \cup \{w\},$$

it follows from (2.5) that $|X \cup \{w\}| > 3$ and so $|X| \geqslant 3$. Since $(A+f_2, (B\backslash w)+e)$ is a $(\leqslant 4)$-separation of G,

$$V(A+f_2) - V((A+f_2) \cap ((B\backslash w)+e)) = (X - \{v\}) \cup \{w\},$$

and $|(X - \{v\}) \cup \{w\}| \geqslant 3$, it follows from (2.2) that there are $\leqslant 2$ vertices of $(B\backslash w)+e$ not in $V(A+f_2) = V(A)$; that is, $|V(B) - V(A \cap B)| \leqslant 2$. Since $(A/f_1, B/f_1)$ is non-planar, it follows by (2.6) that equality holds, and both members of $V(B) - V(A \cap B)$ have $\geqslant 3$ neighbours in $V(A \cap B)$, contrary to (2.2) applied to $((B\backslash w)+e, A+f_2)$. This proves (1).

(2) *If $|V(A) - V(A \cap B)| \leqslant 2$ the theorem is true.*

For then by (2.6), $|V(A) - V(A \cap B)| = 2$; let $V(A) - V(A \cap B) = \{v, a\}$ say. By (2.6) again, in $G\backslash e/f_1$ both v and a have $\geqslant 3$ neighbours in $V((A/f_1) \cap (B/f_1))$; that is, $|V(A \cap B)| = 4$, $V(A \cap B) = \{w, x_1, y, z\}$ say, and v is adjacent in G to w, y, and z, and a is adjacent to x_1, y, z (for a is not adjacent to w since w is 3-valent). Hence v has valency 4, and a is a 3-valent neighbour of x_1 different from w. We claim that x_1 has valency 3. For if not, perform a $Y-\Delta$ exchange at a, and then, a $\Delta - Y$ exchange at vyz; we obtain a new graph with the same number of vertices as G and with more vertices of valency 3, a contradiction since G is expanded. Thus x_1 has valency 3, and the theorem holds. This proves (2).

From (1), we deduce that $(A\backslash w, B+e+g)$ is a $(\leqslant 4)$-separation of G. There are $\geqslant 3$ vertices of $B+e+g$ not in $V(A\backslash w)$ (namely u, w, and x_2) and so from (2.2), there are $\leqslant 2$ vertices of $A\backslash w$ not in $V(B+e+g)$. By (2), we may assume that there are exactly two, a_1 and a_2 say. By (2.2), a_1 and a_2 are adjacent, and both have valency 3 in G. Therefore neither of them is adjacent to both v and x_1, for otherwise it would have $\geqslant 2$ common neighbours with w, contrary to (2.1). Hence $|V(A \cap B)| = 4$, $V(A \cap B) = \{w, x_1, y, z\}$ say, and a_1 is adjacent to x_1, y, and a_2 is adjacent to v, z. Since $(A/f_1, B/f_1)$ is non-planar, it follows that v is adjacent to y. Let $A' = A\backslash z$, and let B' be obtained from B by adding a_2 and all edges incident with z. Then $A' \cup B' = G\backslash e$, $E(A' \cap B') = \{f_1\}$, $|V(A' \cap B')| = 4$, $v \in V(A') - V(B')$, $u \in V(B') - V(A')$, and $(A'/f_1, B'/f_1)$ is non-planar. Thus (A', B') has all the defining properties of (A, B), and, moreover, $|V(A') - V(A' \cap B')| = 2$. By (2) then theorem follows. ∎

We deduce

(5.3) *Under Hypothesis J, suppose that u is not adjacent to x_1, and either*

(i) *x_1 is 3-valent and v is not 4-valent, or*

(ii) *x_1 is 3-valent and has no 3-valent neighbour except w, or*

(iii) *x_1 is not 3-valent and v is not 3-valent, or*

(iv) *x_1 is not 3-valent and w is the only common neighbour of v and x_1.*

Then $G\backslash u/f_1$ is planar.

Proof. Since G is straight, it suffices by (2.9) and (5.1) to show that $G\backslash e/f_1$ is Kuratowski connected and there is no a, b as in (2.9). If $G\backslash e/f_1$ is not Kuratowski connected then by (5.2), v has valency 4, x_1 has valency 3, and x_1 has a 3-valent neighbour different from w; but then none of (i)–(iv) hold, a contradiction. Suppose that there exists a, b as in (2.9); that is, there are adjacent vertices $a, b \neq u, w, x_1$ such that a is 3-valent and adjacent to u and to w, and b is adjacent to x_1. Suppose first that $a \neq v$. Then $a = x_2$ since w is 3-valent. But b is adjacent to x_1 and x_2, and u is adjacent to v and x_2, contrary to (2.10). Hence $a = v$ and so v is 3-valent, and v, x_1 have $\geqslant 2$ common neighbours, namely w and b. Thus (iii) and (iv) do not hold, and so one of (i), (ii) holds, and hence x_1 is 3-valent. But v and x_1 have $\geqslant 2$ common neighbours, contrary to (2.1). Hence there is no such (A, B), and the result follows. ∎

(5.4) *Let G be expanded, basically 5-connected and straight, and let $g \in E(G)$ with ends v, w, such that v, w are both 3-valent and g belongs to no circuit of length 4. Then either G is isomorphic to the Petersen graph or G is apex.*

Proof. Let w have neighbours v, x_1, x_2, and let $u \neq w$ be a neighbour of v. Let e, f_1, f_2 be as in Hypothesis J. Let $H = G\backslash\{u, v\}$.

(1) *H is 2-connected, and there is no 2-separation (C, D) of H so that C and D both have circuits.*

For otherwise there is a $(\leqslant 4)$-separation (A, B) of G, with $V(A)$, $V(B) \neq V(G)$ and $u, v \in V(A \cap B)$, such hat either $|V(A \cap B)| \leqslant 3$ or $A\backslash\{u, v\}$, $B\backslash\{u, v\}$ both have circuits. Since G is basically 5-connected, we may assume that $|E(A)| \leqslant 6$, and hence $|V(A) - V(A \cap B)| \leqslant 2$ by (2.2). Suppose that $a_1, a_2 \in V(A) - V(A \cap B)$ are distinct. By (2.2) they are adjacent and both have valency 3, with two neighbours each in $V(A \cap B)$. In particular, one, a_1 say, is adjacent to u and the other, a_2, to v, since u, v are adjacent. But then a_1 and v have two common neighbours (namely, a_2 and u) contrary to (2.1). Hence $|V(A) - V(A \cap B)| = 1$. Let $a \in V(A) - V(A \cap B)$. Then $|V(A \cap B)| = 4$, and a is adjacent to one of u, v, and to the other two members of $V(A \cap B)$. Consequently, $A\backslash\{u, v\}$ has no circuit, a contradiction. This proves (1).

(2) *For $i = 1, 2$, $G\backslash u/f_i$ is planar.*

Since v has valency 3, this follows from (5.3)(i) if x_i is 3-valent, and from (5.3)(iv) if x_i is not 3-valent, since g is in no 4-circuit.

(3) *$(G\backslash u)\backslash g$ is planar, and hence H is planar.*

For $((G\backslash u)\backslash g)/f_1$ is planar by (2); but $(G\backslash u)\backslash g$ is isomorphic to a subdivision of $((G\backslash u)\backslash g)/f_1$ since w has valency 2 in $(G\backslash u)\backslash g$. Hence $(G\backslash u)\backslash g$ is planar. Since g is incident with v, it follows that $G\backslash\{u, v\} = H$ is planar.

Let the neighbours of v in G be u, w, and z.

(4) *There is a separation (A, B) of $G\backslash\{v, w\}$ with $V(A \cap B) = \{u, x_1, x_2, z\}$, such that $A\backslash u$ and $B\backslash u$ can both be drawn in a disc with x_1, x_2, z on the boundary.*

For let M be a drawing of H in a sphere Σ. Since $(G\backslash u)/f_1$ is planar, it follows that there is a drawing of $((G\backslash u)/f_1)\backslash v$ in Σ such that some region is incident with z and f_2 (the region in which v was drawn). By subdividing f_2 in this drawing, we obtain a drawing of H in which some region is incident with z and x_1. But by Whitney's theorem [3] and (1), any two drawings of H in Σ are related by a homeomorphism of Σ, and consequently there is a region of M incident with z and x_1. Similarly some region of M is incident with z and x_2. Consequently there is a separation (C, D) of $H\backslash w = G\backslash\{u, v, w\}$ with $V(C \cap D) = \{x_1, x_2, z\}$ such that C and D can both be drawn in a disc with x_1, x_2, z on the boundary. Hence (4) follows.

Let (A, B) be as in (4). Neither v nor w is adjacent to any vertex in $V(A)$ or $V(B)$ except u, x_1, x_2, z. Let C be the subgraph of G with vertex set $\{u, v, w, x_1, x_2, z\}$ and edge set all edges of G incident with v or w. If $V(B) = \{u, x_1, x_2, z\}$ then $G\backslash u$ is planar, and so G is apex as required. We assume then that $|V(B)| \geqslant 5$ and similarly $|V(A)| \geqslant 5$. But $(A, B \cup C)$ is a 4-separation of G, and there are $\geqslant 3$ vertices of $B \cup C$ not in $V(A)$. By (2.2) it follows that there are $\leqslant 2$ vertices of A not in $V(B \cup C)$. Similarly there are at most two vertices of B not in $V(A \cup C)$. Since $V(A \cap B) \subseteq V(C)$, we deduce that

$$\begin{aligned}|V(G)| &= |V(C)| + |V(A) - V(B \cup C)| + |V(B) - V(A \cup C)| \\ &\leqslant 6 + 2 + 2 = 10\end{aligned}$$

and we therefore have equality throughout, by (2.4). In particular, there are two vertices a_1, a_2 of A not in $V(B \cup C)$, and by (2.2) they are adjacent and both have valency 3. Neither a_1 nor a_2 is adjacent to both x_1 and x_2, since it has $\leqslant 1$ common neighbour with w by (2.1); and so we may assume that a_1 is adjacent to z and x_1, and a_2 is adjacent to u and x_2. Similarly, there are exactly two vertices b_1, b_2 of B not in $V(A \cup C)$, and b_1, b_2 are adjacent and both 3-valent, and b_i is adjacent to x_i $(i = 1, 2)$, and one of b_1, b_2 is adjacent to u and the other to z. Since u is not adjacent to x_1 or x_2, it follows that every vertex of G has valency 3. Since a_1, b_1 are both 3-valent, they have $\leqslant 1$ common neighbour by (2.1), and so b_1 is not adjacent to z. Hence b_1 is adjacent to u and b_2 to z. Since u is not adjacent to x_1 or x_2 or z, it follows that G is isomorphic to the Petersen graph, as required. ∎

(5.5) *Let G be expanded, and let $w, x_1, x_2 \in V(G)$ all be 3-valent, such that w is adjacent to x_1 and to x_2. Let $g_i \in E(G)$ have ends w, x_i $(i=1, 2)$. Then one of g_1, g_2 belongs to no circuit of length 4.*

Proof. By (2.1), no vertex is adjacent to x_1 and x_2 except w. Let $x_3 \neq x_1, x_2$ be the third neighbour of w. If g_i is in a circuit of length 4, there is a vertex u_i adjacent to x_i and to x_3 $(i=1, 2)$; and $u_1 \neq u_2$, since x_1, x_2 have no common neighbour except w. But this contradicts (2.10). ∎

(5.6) *Let G be expanded and basically 5-connected, such that every edge of G with both ends 3-valent is in a circuit of length 4. Let $w \in V(G)$ be 3-valent. Then there exist u, v, x_1, x_2 as in hypothesis J, such that x_1, x_2 are not adjacent to u, and v has valency $\geqslant 4$, and for $i=1, 2$ if x_i has valency 3 then it has no 3-valent neighbour except w.*

Proof. Let the neighbours of w be x_1, x_2, x_3. Suppose first that x_1, x_2, x_3 all have valency $\geqslant 4$. Let $a \neq w$ be a neighbour of x_3; we may assume that a is adjacent to one of x_1, x_2, say x_1, for otherwise we may take $u=a$ and $v=x_3$ to satisfy Hypothesis J. Let b be a neighbour of x_2 with $b \neq a, w$; then similarly we may assume that b is adjacent to one of x_1, x_3, say x_1. But this contradicts (2.10).

We may assume therefore that x_1 is 3-valent. Let f_1 be the edge with ends w, x_1. By (5.5) x_2 and x_3 both have valency $\geqslant 4$; and by (5.5) again, x_1 has no 3-valent neighbour except w. By hypothesis, f_1 is in a circuit of length 4, and so there is a vertex $a \neq w$ adjacent to x_1 and to one of x_2, x_3, say x_2. Let u be a neighbour of x_3 with $u \neq w, a$. Since a is adjacent to x_1 and x_2, and u is adjacent to x_3, it follows from (2.10) that u is adjacent to neither x_1 nor x_2. Thus, the theorem holds if we set $v=x_3$. ∎

Now we prove (1.3), thereby completing the proof of (1.1).

Proof of (1.3). If some edge of G with both ends 3-valent is not in any circuit of length 4, the result follows from (5.4). We assume therefore that every edge of G with both ends 3-valent is in a circuit of length 4. Let $w \in V(G)$ be 3-valent, and choose u, v, x_1, x_2 as in (5.6). Let e, g, f_1, f_2 be as in Hypothesis J.

(1) *For $i=1, 2$, $(G\backslash u)/f_i$ is planar.*

This follows from (5.3)(ii) if x_i is 3-valent and from (5.3)(iii) if x_i is not 3-valent.

Let $H=(G\backslash u)\backslash g$.

(2) *H is 2-connected, and there is no 2-separation (C, D) of H such that C, D both have circuits.*

For otherwise there is a $(\leqslant 3)$-separation (A, B) of $G\backslash g$ with $u \in V(A \cap B)$, such that $V(A), V(B) \neq V(G)$, and either $|V(A \cap B)| \leqslant 2$ or $A\backslash u, B\backslash u$ both have circuits. If both ends of g are in $V(A)$, then $(A+g, B)$ is a $(\leqslant 3)$-separation of G, with $V(A+g), V(B) \neq V(G)$; hence it has order exactly 3, and so one of $V(A)=V(A+g), V(B)$ has cardinality 4; but both $A\backslash u$ and $B\backslash u$ have circuits and G is triangle-free, which is impossible. Thus, not both ends of g are in $V(A)$, or in $V(B)$, similarly; and so we may assume that $v \in V(A)-V(A \cap B)$ and $w \in V(B)-V(A \cap B)$. If $V(B)-V(A \cap B)=\{w\}$, then $x_1, x_2 \in V(A \cap B)-\{u\}$, and so $|V(A \cap B)|=3$ and $B\backslash u$ has no circuit, a contradiction. Thus, $|V(B)-V(A \cap B)| \geqslant 2$. If $|V(B)-V(A \cap B)|=2$, then by (2.2) applied to the separation $(B+g, A)$, we deduce that w has a neighbour in $V(B)$ adjacent to u, contradicting that u is not adjacent to x_1 or x_2. Thus, $|V(B)-V(A \cap B)| \geqslant 3$. But $|V(B)-V(A \cap B)| \neq 3$ by (2.5) applied to $(B+g, A)$, and so $|V(B)-V(A \cap B)| \geqslant 4$. Since G is basically 5-connected, we deduce from (2.2) and the separation $(A+g, B)$ that $|V(A)-V(A \cap B)| \leqslant 2$, and equality does not hold by (2.2) since v is 4-valent. Thus $V(A)-V(A \cap B)=\{v\}$, and $|V(A \cap B)|=3$, and v is adjacent to every vertex in $V(A \cap B)$; but then $A\backslash u$ has no circuit, a contradiction. This proves (2).

As in the proof of (5.4), it follows that $H=(G\backslash u)\backslash g$ is planar, and from (1) and (2) that there is a planar drawing of H with for $i=1, 2$, a region incident with v and x_i. Consequently there is a separation (A, B) of $G\backslash w$ with $V(A \cap B)=\{u, v, x_1, x_2\}$, such that $A\backslash u$ and $B\backslash u$ can both be drawn in a disc with v, x_1, x_2 on the boundary. Let C be the subgraph with vertex set $\{u, v, w, x_1, x_2\}$ and edge set $\{f_1, f_2, g\}$. Since $|V(G)| \geqslant 10$ by (2.4), it follows that $|V(A \cup B)| \geqslant 9$ and since $|V(A \cap B)|=4$, we may assume that $|V(B)| \geqslant 7$, and so $|V(B)-V(A \cap B)| \geqslant 3$. From (2.2) applied to the 4-separation $(A \cup C, B)$, it follows that $|V(A \cup C)-V((A \cup C) \cap B)| \leqslant 1$, since w has no neighbour in $V(A)-V(A \cap B)$; that is, $V(A)=\{u, v, x_1, x_2\}$. Hence $G\backslash u$ is planar, and so G is apex, as required. ∎

6. Graphs That Are Not Basically 5-Connected

A graph H is a *minor* of a graph G if H can be obtained from a subgraph of G by contracting edges; and H is a *proper minor* if $H \neq G$. As we said earlier, our use of (1.1) will be to show that there are no minor-minimal counterexamples to Sachs' conjecture. In order to do so, we need to prove that a minor-minimal counterexample to Sach's conjecture is basically 5-connected, and our final objective is to prove another lemma which will be used to show that.

A graph is *complete* if it is simple and every two distinct vertices are adjacent. Let us say that a graph G is a *proper 4-sum* if there is a 4-separation (A, B) of G, such that

(i) $A \backslash V(A \cap B)$ and $B \backslash V(A \cap B)$ are both non-null and connected, and the restriction of G to $A \cap B$ is simple, and

(ii) let K be a complete graph with $V(K) = V(A \cap B)$, containing every edge of G with both ends in $V(A \cap B)$; then $A \cup K$ and $B \cup K$ are both $Y-\Delta$ equivalent to graphs isomorphic to proper minors of G.

Let us say that G is *disc-reducible* if there is a 4-separation (A, B) of G such that $V(A \cap B)$ can be enumerated as $\{x_1, x_2, x_3, x_4\}$ in such a way that A can be drawn in a disc with x_1, x_2, x_3, x_4 on the boundary in order; and, moreover, the graph G' is isomorphic to a proper minor of G, where G' is obtained from B by adding an edge joining x_1, x_2, an edge joining x_2, x_3, and a new 4-valent vertex adjacent to x_1, x_2, x_3, x_4.

The *Petersen family* is the set of the seven graphs $Y-\Delta$ equivalent to K_6 (up to isomorphism). They are shown in Fig. 1.

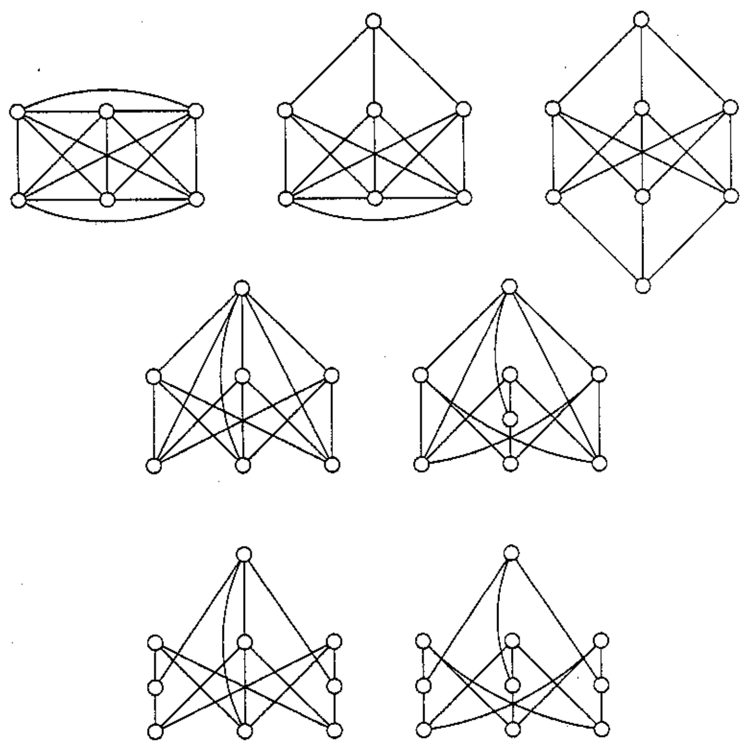

FIG. 1. The Petersen family.

Let us say that that G is *internally* 4-*connected* if G is simple and 3-connected, and $|V(G)| \geq 4$, and $\min(|E(A)|, |E(B)|) \leq 3$ for every 3-separation (A, B). The main result of this section is the following.

(6.1) *Let G be a graph which is not basically 5-connected, with no minor in the Petersen family. Then G is $Y-\Delta$ equivalent to a graph H such that either*

(i) *H is not internally 4-connected, or*

(ii) *H is a proper 4-sum, or*

(iii) *H is disc-reducible.*

To prove (6.1) we begin with the following. A *hexad* in a graph G is a subgraph isomorphic to a subdivision of $K_{3,3}$. It has six vertices of valency 3, called its *nodes*, and nine paths with distinct ends both nodes and with no internal vertex a node, called *arcs*.

(6.2) *Let G be a 3-connected graph with a hexad, and let $v \in V(G)$. Then v is a node of some hexad.*

Proof. Let H be a hexad using as many edges incident with v as possible. Let H have nodes $u_1, ..., u_6$ and arcs L_{ij} $(1 \leq i \leq 3,\ 4 \leq j \leq 6)$, where L_{ij} has ends u_i and u_j.

Suppose that $v \notin V(H)$. Let P_1, P_2, P_3 be paths of G between v and $V(H)$, mutually vertex-disjoint except for v. Let P_i have ends v, w_i $(i = 1, 2, 3)$. If $\{w_1, w_2, w_3) = \{u_1, u_2, u_3\}$, let H' be the hexad obtained from H by deleting all vertices and edges of L_{i4} except u_i $(1 \leq i \leq 3)$ and adding $P_1 \cup P_2 \cup P_3$; this contradicts the choice of H. Thus $\{w_1, w_2, w_3\} \neq \{u_1, u_2, u_3\}$ and, similarly, $\{w_1, w_2, w_3\} \neq \{u_4, u_5, u_6\}$. If w_1, w_2, w_3 are all nodes of H, we may therefore assume that $w_1 = u_1$ and $w_2 = u_4$; but then the hexad obtained by deleting the interior of L_{14} and adding $P_1 \cup P_2$ contradicts the choice of H. It follows that one of w_1, w_3 is not a node of H, and so we may assume that $w_1 \in V(L_{14}) - \{u_1, u_4\}$. If $w_2 \in V(L_{14})$, the hexad obtained by deleting the part of L_{14} strictly between w_1 and w_2, and adding $P_1 \cup P_2$, contradicts the choice of H. Thus $w_2 \notin V(L_{14})$ and we may therefore assume that $w_2 \in V(L_{15}) - \{u_1, u_5\}$ or $w_2 \in V(L_{25}) - \{u_2\}$. In the first case, the hexad obtained by deleting the part of L_{15} strictly between u_1 and w_2, and adding $P_1 \cup P_2$, contradicts the choice of H; and in the second case, the hexad obtained by deleting all of L_{15} except its ends, and adding $P_1 \cup P_2$, contradicts the choice of H. This proves that $v \in V(H)$.

We may therefore assume that $v \in V(L_{14})$, and we suppose for a contradiction that $v \neq u_1, u_4$. Let J be obtained from H by deleting all of L_{14} except its ends. Since there are two paths of G between v and $V(J)$, mutually vertex-disjoint except for v, with no vertices in $V(J)$ except their ends u_1, u_4, and since G is 3-connected, it follows by an augmenting paths argument that there are three paths P_1, P_2, P of G between v and $V(J)$, mutually vertex-disjoint except for v, with no vertices in $V(J)$ except their ends (w_1, w_2, w, respectively, say) such that $w_1 = u_1$ and $w_2 = u_4$. By replacing L_{14} by $P_1 \cup P_2$ we may therefore assume that there is a path P of G from v to $w \in V(J) - \{u_1, u_4\}$ with no internal vertex in $V(H)$. From the symmetry we may assume that either $w \in V(L_{15}) - \{u_1, u_5\}$ or $w \in V(L_{25}) - \{u_2\}$. In the first case the hexad obtained by deleting the part of L_{15} strictly between u_1 and w, and adding P, contradicts the choice of H; and in the second case the hexad obtained by deleting all of L_{15} except its ends, and adding P, contradicts the choice of H.

This proves that $v = u_1$ or $v = u_4$, and the result follows. ∎

Let G be a graph and $X \subseteq V(G)$ with $|X| = 4$; we call (G, X) a *rooted graph*. (It happens that we only need consider graphs with four roots.) If (G, X), (H, Y) are rooted graphs, the second is a *minor* of the first if it can be obtained from a subgraph G' of G with $X \subseteq V(G')$, by contracting edges of G' so that no two members of X become identified and so that X corresponds to Y. More precisely, let us say that (H, Y) is *isomorphic to a minor of* (G, X) if there is a function ϕ with domain $V(H) \cup E(H)$ such that

(i) for each $v \in V(H)$, $\phi(v)$ is a non-null connected subgraph of G; and for distinct $v_1, v_2 \in V(H)$, $\phi(v_1) \cap \phi(v_2)$ is null

(ii) for each $e \in E(H)$, $\phi(e) \in E(G)$, and for distinct $e_1, e_2 \in E(H)$, $\phi(e_1) \neq \phi(e_2)$

(iii) for each $e \in E(H)$ and $v \in V(H)$, $\phi(e) \notin E(\phi(v))$

(iv) if $e \in E(H)$ has ends v_1, v_2 then $\phi(e)$ has one end in $V(\phi(v_1))$ and the other in $V(\phi(v_2))$

(v) for each $v \in Y$, $|V(\phi(v)) \cap X| = 1$.

Proper minors of rooted graphs are defined in the natural way. The main lemma needed to prove (6.1) is the following.

(6.3) *Let (G, X) be a rooted graph, such that*

(i) *$|V(G) - V(B)| = 1$ and $|V(A \cap B)| = 3$ for every separation (A, B) of G of order $\leqslant 3$ with $X \subseteq V(B)$ and $V(B) \neq V(G)$*

(ii) *G is simple*

(iii) *$G \backslash X$ is non-null and connected, and*

(iv) *G cannot be drawn in a disc with the members of X drawn in the boundary in some order.*

Then (G, X) has a minor isomorphic to one of the rooted graphs of Fig. 2.

Proof. We begin with the following.

(1) *For every $(\leqslant 1)$-separation (A, B) of G, one of $|V(A) \cap X|$, $|V(B) \cap X| \leqslant 1$.*

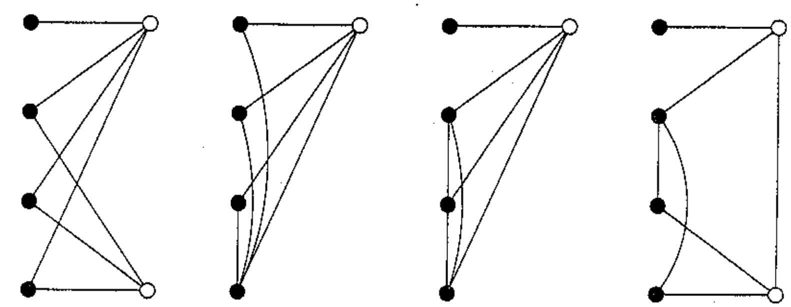

FIGURE 2

For suppose that $X = \{x_1, x_2, x_3, x_4\}$, and $x_1, x_2 \in V(A)$ and $x_3, x_4 \in V(B)$. Let B^+ be the graph with $V(B^+) = V(B) \cup X$ and $E(B^+) = E(B)$. Then (A, B^+) has order $\leqslant 3$, and $X \subseteq V(B^+)$, and so by hypothesis, $|V(G) - V(B^+)| \leqslant 1$; that is,

$$|V(A) - (V(B) \cup \{x_1, x_2\})| \leqslant 1.$$

Since $|V(A) \cap (V(B) \cup \{x_1, x_2\})| \leqslant 3$, it follows that A can be drawn in a disc with the vertices in $V(A) \cap (V(B) \cup \{x_1, x_2\})$ on the boundary. Similarly, B can be drawn in a disc with the vertices in $V(B) \cap (V(A) \cup \{x_3, x_4\})$ on the boundary. Since

$$|(V(A) \cap (V(B) \cup \{x_1, x_2\})) \cap (V(B) \cap (V(A) \cup \{x_3, x_4\}))| \leqslant 1,$$

this contradicts (iv). Hence (1) holds.

Let J be the graph obtained from G by adding a new vertex j of valency 4, adjacent to the four vertices in X. From (iv), J is not planar.

(2) *If (A, B) is a $(\leqslant 2)$-separation of J with $V(A), V(B) \neq V(J)$, then $|V(A \cap B)| = 2$ and one of A, B is a path joining the two vertices in $V(A \cap B)$.*

If $j \notin V(A)$ then $X \subseteq V(B)$, and $(A, B \backslash j)$ is a $(\leqslant 2)$-separation of G with $V(B \backslash j) \neq V(G)$, contrary to (i). Thus $j \in V(A)$ and similarly $j \in V(B)$. Hence $(A \backslash j, B \backslash j)$ is a $(\leqslant 1)$-separation of G, and so by (1), one of $|V(A \backslash j) \cap X|$, $|V(B \backslash j) \cap X| \leqslant 1$. We assume the former without loss of generality. Let B' be obtained from B by deleting j and adding any vertex in $X - V(B)$ as an isolated vertex. Then $(A \backslash j, B')$ is a $(\leqslant 2)$-separation of G, and $X \subseteq V(B')$, and so $V(B') = V(G)$ from the hypothesis. Hence there is unique vertex $x \in V(J) - V(B)$, and $x \in X$. Since every neighbour of x is in $V(A \cap B)$ and $|V(A \cap B)| \leqslant 2$, it follows that x has $\leqslant 1$ neighbour in G. But x has a neighbour in $V(G) - X$, for otherwise $|V(G) - X| \leqslant 1$ by hypothesis (i), and

(iv) is contradicted. Thus, x has precisely two neighbours in J, j and y, say, where $y \in V(G) - X$. Hence $V(A) = \{j, x, y\}$ and A is a path with ends j, y, since J is simple and j is not adjacent to y. This proves (2).

(3) *If J is isomorphic to a subdivision of K_5 then the theorem holds.*

For from (i), every vertex of J not in $X \cup \{j\}$ has valency ≥ 3, and so if J is a subdivision of K_5 then its five vertices of valency 4 are j, u_1, u_2, u_3, u_4, say, where u_1, u_2, u_3, u_4 are mutually adjacent in G. If $\{u_1, u_2, u_3, u_4\} = X$ then $V(G) = X$ contrary to (iii); and so some $x \in X$ has valency 2 in J. But then (G, X) has the third graph of Fig. 2 as a minor as required.

By (3), we may assume that J is not isomorphic to a subdivision of K_5. Since J is non-planar, it follows from (2) and a theorem of Hall [1] that J has a hexad. From (2), and since J has ≥ 6 vertices of valency ≥ 3 (because it has a hexad) it follows that J is a subdivision of a 3-connected graph. By (6.2), there is a hexad H in J such that j is a node of H. We deduce the following.

(4) *There are distinct y_1, $y_2 \in V(G)$. and three paths Q_1, Q_2, Q_3 of G between y_1 and y_2, mutually disjoint except for y_1 and y_2, such that there are three mutually vertex-disjoint paths P_1, P_2, P_3 of G between X and $V(Q_1 \cup Q_2 \cup Q_3)$, each with no vertex in $V(Q_1 \cup Q_2 \cup Q_3)$ except one end, and such that for $1 \leq i \leq 3$ the end of P_i in $V(Q_1 \cup Q_2 \cup Q_3)$ is an internal vertex of Q_i.*

(The paths P_i may have only one vertex.) Now there are four mutually vertex-disjoint paths of G between X and $V(Q_1 \cup Q_2 \cup Q_3)$, by (i), for $|V(Q_1 \cup Q_2 \cup Q_3)| \geq 5$. Hence, by an augmenting paths argument, we may choose P_1, P_2, P_3 as in (4), such that there is a path P_4 from X to $V(Q_1 \cup Q_2 \cup Q_3)$ with no vertex in $V(Q_1 \cup Q_2 \cup Q_3)$ except one end, such that P_1, P_2, P_3, P_4 are mutually vertex-disjoint.

Now if $E(P_4) \neq \emptyset$, it follows that (G, X) has a minor isomorphic to the first graph of Fig. 2; and if $E(P_1)$, $E(P_2)$, or $E(P_3) \neq \emptyset$, (G, X) has a minor isomorphic to the fourth graph of Fig. 2. We may assume then that $X \subseteq V(Q_1 \cup Q_2 \cup Q_3)$. Let $x_i \in X$ be an internal vertex of Q_i $(1 \leq i \leq 3)$, and let $X = \{x_1, x_2, x_3, x_4\}$. By (iii), there is a path of $G \backslash \{x_1, x_2, x_3\}$ between the two components of

$$Q_1 \cup Q_2 \cup Q_3 \backslash \{x_1, x_2, x_3\};$$

but then (G, X) has a minor isomorphic to the second graph of Fig. 2. This completes the proof. ∎

Proof of (6.1). We proceed by induction on $|V(G)|$. Since G is not basically 5-connected, and we may assume that it is internally 4-connected, there is a 4-separation (A, B) of G with $|E(A)|$, $|E(B)| \geq 7$. Let $V(A \cap B) = X$.

(1) *We may assume that every $v \in V(G) - X$ has valency ≥ 4.*

For suppose that $v \in V(G) - X$ has valency ≤ 3. Since G is internally 4-connected, v has valency 3. Let $v \in V(A)$ say. Then $v \notin V(B)$, and so the three edges incident with v belong to $E(A)$. Let A' be obtained from A by a $Y - \Delta$ exchange at v, and let $G' = A' \cup B$. Since $|E(A')| = |E(A)|$, it follows that G' is not basically 5-connected; but $|V(G')| < |V(G)|$ and G' is $Y - \Delta$ equivalent to G and hence has no minor in the Petersen family, and so the result follows from the inductive hypothesis.

Let the components of $G \backslash X$ be $C_1, ..., C_k$. Since $|E(A)| \geq 7$ and G is simple, it follows that $V(A) \neq X$, and so $A \backslash X$ contains one of $C_1, ..., C_k$. Similarly, so does $B \backslash X$, and so $k \geq 2$.

(2) *For $1 \leq i \leq k$, each vertex in X has a neighbour in $V(C_i)$.*

For if $x \in X$ has no neighbour in $V(C_1)$ say, let (A_1, B_1) be a 3-separation with $V(A_1 \cap B_1) = X - \{x\}$ and $A_1 \backslash (X - \{x\}) = C_1$. Since $x \in V(B_1) - V(A_1 \cap B_1)$ and $k \geq 2$, it follows that $|V(B_1) - V(A_1 \cap B_1)| \geq 2$. But G is internally 4-connected, and so $|V(A_1) - V(A_1 \cap B_1)| \leq 1$. Hence $|V(C_1)| = 1$, $V(C_1) = \{v\}$ say, and v has valency 3, contrary to (1). This proves (2).

For $1 \leq i \leq k$, let A_i be the subgraph of G with vertex set $V(C_i) \cup X$, and edge set all edges of G with an end in $V(C_i)$. Let A_0 have vertex set X and edge set all edges with both ends in X. Then $A_0, A_1, ..., A_k$ are mutually edge-disjoint, and have union G.

(3) *We may assume that for $1 \leq i \leq k$, if $|V(C_i)| \geq 2$ then $(A_i \cup A_0, X)$ has a minor isomorphic to one of the rooted graphs of Fig. 2.*

For let $i = 1$, say. If $A_1 \cup A_0$ cannot be drawn on a disc Δ with X on the boundary in some order, the claim follows from (6.3). We assume then that it can be drawn in this way. Let $X = \{x_1, x_2, x_3, x_4\}$, where x_1, x_2, x_3, x_4 are drawn in the boundary of the disc Δ in this order. Choose $v \in V(C_1)$. Since G is internally 4-connected and v has valency ≥ 4 by (1), it follows since $k \geq 2$ that there are four paths P_1, P_2, P_3, P_4 of A_i between v and X, mutually vertex-disjoint except for v, where P_i has ends v, x_i $(1 \leq i \leq 4)$. If there is no path of $(A_0 \cup A_1) \backslash v$ between $\{x_1, x_2\}$ and $\{x_3, x_4\}$, there is a 1-separation (C, D) of $A_0 \cup A_1$ with $V(C \cap D) = \{v\}$, x_1, $x_2 \in V(C)$ and x_3, $x_4 \in V(D)$; but then it follows from (1), since G is internally 4-connected, that $V(C) = \{v, x_1, x_2\}$ and $V(D) = \{v, x_3, x_4\}$, and hence $V(C_1) = \{v\}$, contradicting that $|V(C_1)| \geq 2$. Thus there is a path of $(A_0 \cup A_1) \backslash v$ between $\{x_1, x_2\}$ and $\{x_3, x_4\}$, and hence from the planarity of $A_0 \cup A_1$, we may assume (exchanging x_1 with x_2 and x_3 with x_4 if necessary) that there is a path Q_1 of $A_0 \cup A_1$ from $V(P_1)-\{v\}$ to $V(P_4)-\{v\}$ with no internal vertex in $V(P_1 \cup P_2 \cup P_3 \cup P_4)$. Similarly, we may assume that there is a path Q_2 from $V(P_1)-\{v\}$ to $V(P_2)-\{v\}$ with no internal vertex in $V(P_1 \cup P_2 \cup P_3 \cup P_4)$, and hence $Q_1 \cap Q_2 \subseteq P_1$. We deduce that $(A_0 \cup A_1, X)$ has a minor isomorphic to (L, X), where L is the graph with five vertices v, x_1, x_2, x_3, x_4, in which v is adjacent to x_1, x_2, x_3, x_4 and x_1 is adjacent to x_2 and to x_4. Since $|V(C_1)| \geq 2$, this is a proper minor. But then G is disc-reducible as required.

(4) *We may assume that* $k=2$.

For if $k \geq 4$, by contracting each edge of $C_1, ..., C_4$ and deleting $V(C_5) \cup \cdots \cup V(C_k)$ we obtain by (2) an eight-vertex graph with a subgraph isomorphic to $K_{4,4}$. By deleting one edge of $K_{4,4}$ we obtain a member of the Petersen family, a contradiction. Thus $k \leq 3$; now suppose that $k=3$. If $|V(C_1)| \geq 2$, then by (3) $(A_0 \cup A_1, X)$ has a minor isomorphic to one of the graphs of Fig. 2. But for each of these graphs (H, Y), say, if we add to H two more 4-valent vertices both adjacent to every vertex in Y, we obtain a member of the Petersen family; and consequently if $|V(C_1)| \geq 2$ then G has a minor in the Petersen family, a contradiction. It follows that $|V(C_i)|=1$ $(1 \leq i \leq 3)$. If there are two edges in A_0 with no common ends, say x_1x_2 and x_3x_4, then performing a $\Delta-Y$ exchange at vx_1x_2 and a $\Delta-Y$ exchange at vx_3x_4 (where $V(C_1)=\{v\}$) yields a graph with a vertex of valency 2, which is therefore not internally 4-connected, as required. We may assume then every two edges of A_0 have a common end. Since G is not in the Petersen family, A_0 has no circuit of length 3 or 3-valent vertex, and so $|E(A_0)| \leq 2$. Let $X=\{x_1, x_2, x_3, x_4\}$ and $V(G)-X=\{v_1, v_2, v_3\}$, where x_1 has valency 0 in A_0, and x_2 is not adjacent to x_3. Since G is internally 4-connected, $G\backslash\{v_1, v_2, v_3\}$ has ≤ 2 components, and so x_4 is adjacent to x_2 and to x_3. Performing a $Y-\Delta$ exchange at x_1, a $\Delta-Y$ exchange at $v_1v_2x_2$, and a $\Delta-Y$ exchange at $v_3x_2x_4$, yields a graph with a vertex of valency 2, as required. This proves (4).

Let (L_1, X_1) be the rooted graph obtained from K_4 by taking $L_1=K_4$ and $X_1=V(L_1)$. Let (L_2, X_2) be the rooted graph obtained from $K_{2,3}$ by taking $L_2=K_{2,3}$ and $X_2=V(L_2)-\{v\}$, where v has valency 3. Each of the graphs of Fig. 2 has a proper minor isomorphic to one of (L_1, X_1), (L_2, X_2).

(5) *We may assume that for* $i=1, 2$ $(A_i \cup A_0, X)$ *has a proper minor isomorphic to one of* (L_1, X_1), (L_2, X_2).

For let $i=1$, say. If $|V(C_1)| \geq 2$ this follows from (3). Let $V(C_1)=\{v\}$, say. Since $|E(A)|$, $|E(B)| \geq 7$, it follows that $|E(A_0)| \geq 3$. But if two edges in A_0 have no common end, say x_1x_2 and x_3x_4, then performing a $\Delta-Y$ exchange at vx_1x_2 and a $\Delta-Y$ exchange at vx_3x_4 yields a graph with a vertex of valency 2, as required. We may assume then that all pairs of edges of A_0 have a common end; and so $|E(A_0)|=3$, and $(A_1 \cup A_0, X)$ is isomorphic to the second or third graph of Fig. 2. In either case the claim is true.

Let K be a complete graph with $V(K)=V(A_0)$ and $A_0 \subseteq K$. If $(A_1 \cup A_0, X)$ has a proper minor isomorphic to (L_1, X_1) then G has a proper minor isomorphic to $K \cup A_2$; and if $(A_1 \cup A_0, X)$ has a proper minor isomorphic to (L_2, X_2) then G has a proper minor $Y-\Delta$ equivalent to $K \cup A_2$. Since $k=2$ and the same holds for $(A_2 \cup A_0, X)$, we deduce that G is a proper 4-sum. This completes the proof. ∎

(6.4) *Let G be expanded and basically 5-connected, and let $u \in V(G)$. Then $G\backslash u$ is Kuratowski connected.*

Proof. Suppose not; then there is a (≤ 4)-separation (A, B) of G, with $u \in V(A \cap B)$, such that $(A\backslash u, B\backslash u)$ is non-planar. Since G is basically 5-connected, we may assume that $|E(A)| \leq 6$, and hence $|V(A)-V(A \cap B)| \leq 2$, by (2.2). By (2.6), there are two vertices $a_1, a_2 \in V(A)-V(A \cap B)$, both with three neighbours in $V(A \cap B)-\{u\}$. By (2.2) a_1, a_2 are adjacent and both have valency 3, a contradiction.

By combining (6.1), (6.4), and (1.1), we obtain:

(6.5) *Let G be a graph, not isomorphic to $K_{5,5}^-$, and with no minor in the Petersen family. Then there is a graph H, $Y-\Delta$ equivalent to G, such that either*

(i) *H is not internally 4-connected, or*

(ii) *H is a proper 4-sum, or*

(iii) *H is disc-reducible, or*

(iv) *H is apex, or*

(v) *H is expanded and basically 5-connected, and there is an edge f of H with ends w, x, and a vertex u of H not adjacent to w or x, and an edge e incident with u, such that $H\backslash e/f$, $H/\{e, f\}$, and $H\backslash u$ are all Kuratowski connected, and $(H\backslash u)/f$ is non-planar, and $N(f)$ is 2-coherent in $H\backslash(V(f) \cup \{u\})$.*

Proof. Choose G', $Y-\Delta$ equivalent to G and expanded; this exists unless (i) holds. If G' is not basically 5-connected, then one of (i), (ii), (iii) holds by (6.1). If G' is basically 5-connected and not straight then (v) holds with $H=G'$ by (6.4). If G' is basically 5-connected and straight, then by (1.1) either G' is apex, or G' is isomorphic to $K_{5,5}^-$ (for $K_{4,4}$ has a subgraph in the Petersen family). In the first case (iv) holds, and in the second case G is also isomorphic to $K_{5,5}^-$ (because $K_{5,5}^-$ has no 3-valent vertices or circuits of length 3 and so $G'=G$), contrary to the hypothesis. ∎

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Printed by Catherine Press, Ltd., Tempelhof 41, B-8000 Brugge, Belgium