UNIQUENESS OF HIGHLY REPRESENTATIVE SURFACE EMBEDDINGS

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ABSTRACT

Let Σ be a (connected) surface of "complexity" κ; that is, Σ may be obtained from a sphere by adding either $\frac{1}{2}\kappa$ handles or κ crosscaps. Let $\rho \geq 0$ be an integer, and let Γ be a "ρ-representative drawing" in Σ; that is, a drawing of a graph in Σ so that every simple closed curve in Σ that meets the drawing in $< \rho$ points bounds a disc in Σ. Now let Γ' be another drawing, in another surface Σ' of complexity κ', so that Γ and Γ' are isomorphic as abstract graphs. We prove that

(i) If $\rho \geq 100 log\kappa / loglog\kappa$ (or $\rho \geq 100$ if $\kappa \leq 2$) then $\kappa' \geq \kappa$, and if $\kappa' = \kappa$ and Γ is simple and 3-connected there is a homeomorphism from Σ to Σ' taking Γ to Γ'

(ii) if Γ is simple and 3-connected and Γ' is 3-representative, and $\rho \geq min(320, 5log\kappa)$, then either there is a homeomorphism from Σ to Σ' taking Γ to Γ', or $\kappa' \geq \kappa + 10^{-4}\rho^2$.

1. INTRODUCTION

A useful theorem of Whitney [6] says that, if G is a simple, 3-connected planar graph, it can be drawn in a sphere in essentially only one way; more precisely, for any two drawings of it in a sphere Σ, there is a homeomorphism of Σ to itself carrying the first drawing to the second.

It is natural to ask whether this result has some analogue for higher surfaces. Certainly 3-connectivity is no longer sufficient. For instance, let G consist of two disjoint circuits with vertex sets $\{a_1, ..., a_{100}\}$ and $\{b_1, ..., b_{100}\}$ in order, together with an edge a_ib_i for $1 \leq i \leq 100$; and embed G in a cylinder $\{(x, y) : 1 \leq x^2 + y^2 \leq 2\}$ in 3-space so that the two disjoint circuits are the two ends of the cylinder. Now "thicken" the wall of the cylinder so that it becomes a torus. Then for each of the edges a_ib_i, there are ≥ 2 ways to add it to the drawing (above and below the original plane of the cylinder), and all these choices can be made independently; and so we have about 2^{100} different drawings of G in the torus, almost all not related by homeomorphisms of the torus.

We need some further requirement in addition to 3-connectivity; and one that works nicely is to require that the drawing is sufficiently "representative" of the surface, meaning that every simple closed curve in the surface with only a small number ($< \rho$, say) of points in common with the drawing bounds a disc in the surface. Robertson and Vitray [4] showed that, if ρ is large enough (depending on Σ) and Σ is orientable then G has no other drawing in Σ, up to homeomorphisms of Σ; and indeed, G has no other drawing in any simpler orientable surface either. (We shall state this more precisely later.) The question arises, what does "large enough" mean here? How should the lower bound on ρ depend on Σ? The bound of Robertson and Vitray was linear in the genus of Σ, but no example was known to show this was necessary; the best known example was due to Archdeacon [1], who gave an example with ρ slightly less than the logarithm of the genus of Σ. We shall show that Archdeacon's example is close to best possible, that every example has $\rho \leq O\left(\frac{log(g)}{loglog(g)}\right)$ where g is the genus. Now let us be more precise.

A *surface* (in this paper) means a connected, compact 2-manifold without boundary. Every surface is homeomorphic (denoted $\cong$) to exactly one of $\Sigma_0, \Sigma_1, \Sigma_2, ..., \tilde{\Sigma}_1, \tilde{\Sigma}_2...$ where Σ_k denotes the orientable surface obtained from a sphere by adding k handles, and $\tilde{\Sigma}_k$ denotes the non-orientable surface obtained from a sphere by adding k crosscaps. We define the *complexity* $\kappa(\Sigma)$ of a surface Σ by

$$\kappa(\Sigma) = \begin{cases} 2k & \text{if} \quad \Sigma \cong \Sigma_k \\ k & \text{if} \quad \Sigma \cong \tilde{\Sigma}_k. \end{cases}$$

Thus, $\kappa(\Sigma)$ is $2 - \chi(\Sigma)$, where $\chi(\Sigma)$ is the Euler characteristic of Σ.

A *drawing* in a surface Σ is a pair (U, V) where $U \subseteq \Sigma$ is closed, $V \subseteq U$ is finite, $U - V$ has only finitely many arc-wise connected components, called *edges*, and for every edge e, either

(i) $|\bar{e} - e| = 1$, and $\bar{e}$ is homeomorphic to a circle, or

(ii) $|\bar{e} - e| = 2, \bar{e} - e = \{u, v\}$ say, and $\bar{e}$ is homeomorphic to the closed interval [0, 1] (and consequently $(\bar{e}, e)$ is homeomorphic to ([0, 1], (0, 1))).

If $\Gamma = (U, V)$ is a drawing, we write $U = U(\Gamma), V = V(\Gamma)$, and denote its set of edges by $E(\Gamma)$. Every drawing Γ in Σ is therefore a graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$, and we use standard graph-theory terminology for drawings without further explanation. (In particular, by an *isomorphism* between drawings Γ and Γ' we mean an isomorphism regarding Γ and Γ' as abstract graphs, and by a *simple drawing* we mean a drawing that is a simple abstract graph.)

Let Γ be a drawing in a surface Σ, and let $\rho \geq 0$ be an integer. We say that Γ is *ρ-representative* if for every $F \subseteq \Sigma$ homeomorphic to a circle, if F is not the boundary of some closed disc in Σ then $|F \cap U(\Gamma)| \geq \rho$. Robertson and Vitray proved [3]

(1.1) *Let Γ be a simple 3-connected drawing in a surface Σ, let Γ' be a drawing in an orientable surface Σ', and let $\alpha : \Gamma \to \Gamma'$ be an isomorphism. If Γ is $(4\kappa(\Sigma) + 3)$-representative then $\kappa(\Sigma') \geq \kappa(\Sigma)$, and if equality holds then there is a homeomorphism from Σ to Σ' extending α.*

We shall prove the following, which strengthens (1.1) for large κ. (Another strengthening of (1.1) is given in [2].)

(1.2) *Let Σ be a surface of complexity κ, let $\sigma \geq 2$ be an integer, and let $\epsilon = 24\sigma^{-1}\kappa^{1/\sigma}$. Let Γ be a (2σ)-representative drawing in Σ, let Γ' be a drawing in a surface Σ', and let $\alpha : \Gamma \to \Gamma'$ be an isomorphism.*

(i) *If $\epsilon \leq \frac{1}{2}$ (that is, $(\sigma/48)^\sigma \geq \kappa$) then $\kappa(\Sigma') \geq \kappa$*

(ii) *If $\epsilon \leq \frac{1}{2}$ and Γ is simple, 3-connected and $(2\sigma+2)$-representative, and $\kappa(\Sigma') = \kappa$, then there is a homeomorphism $\Sigma \to \Sigma'$ extending α.*

(iii) *If Γ is simple and 3-connected and Γ' is 3-representative, and $\epsilon \leq 1$, then either there is a homeomorphism $\Sigma \to \Sigma'$ extending α, or $\kappa(\Sigma') \geq \kappa + \frac{2}{3}\epsilon^{-2}(1 - \epsilon)$.*

From (1.2), the results stated in the abstract follow easily, and we omit those details. We shall prove (1.2)(i) in section 3, and the easier results (1.2)(ii) and (iii) in section 4.

2. SOME PRELIMINARY LEMMAS

Let Γ be a drawing in a surface Σ. A *region* of Γ is an arc-wise connected component of $\Sigma - U(\Gamma)$, and we denote the set of all regions by $R(\Gamma)$. We say Γ is a 2-*cell* drawing in Σ if every region is homeomorphic to an open disc.

To prove (1.2) we need several lemmas. The first is just Euler's formula.

(2.1) *Let Γ be a non-null drawing in a surface Σ. Then*

$$|V(\Gamma)| - |E(\Gamma)| + |R(\Gamma)| \geq 2 - \kappa(\Sigma)$$

with equality if Γ is 2-cell.

Next, we need the following. (If Γ is a drawing, $\Gamma \backslash X$ is the drawing obtained by deleting X; hence X may be a vertex or an edge, or a set of vertices or edges.)

(2.2) *Let Γ be a non-null simple drawing in a surface Σ. Let $\delta = |E(\Gamma)|/|V(\Gamma)|$, and let $\sigma \geq 2$ be an integer such that every circuit of Γ of length $< 2\sigma$ bounds a disc in Σ. Then either $\delta < 3$ or $\kappa(\Sigma) \geq (\delta/3 - 1)^{\sigma}$; and consequently either $\kappa(\Sigma) = 0$ or $\delta \leq 6\kappa(\Sigma)^{1/\sigma}$.*

Proof. The second conclusion follows from the first, because if $\kappa(\Sigma) > 0$ and $\delta \leq 6$ then $\delta \leq 6\kappa(\Sigma)^{1/\sigma}$, and if $\delta \geq 6$ and $\kappa(\Sigma) \geq (\frac{\delta}{3} - 1)^{\sigma}$ then $\kappa(\Sigma)^{1/\sigma} \geq \frac{\delta}{3} - 1 \geq \frac{\delta}{6}$. We prove the first conclusion by induction on $|V(\Gamma)|$. Certainly we may assume that $\delta \geq 3$. If there exists $v \in V(\Gamma)$ with valency $\leq \delta$, let $\Gamma' = \Gamma \backslash v$; then Γ' is non-null, since Γ is simple and $\delta > 0$, and so from the inductive hypothesis, either $\delta' < 3$ or $\kappa(\Sigma) \geq (\delta'/3 - 1)^{\sigma}$ where $\delta' = |E(\Gamma')|/|V(\Gamma')|$. But

$$|E(\Gamma')| \geq |E(\Gamma)| - \delta \geq \delta|V(\Gamma)| - \delta = \delta|V(\Gamma')|$$

and so $\delta' \geq \delta$, and the result follows. We may assume, therefore, that every vertex has valency $> \delta$. We may also assume that Γ is connected.

Choose $v_0 \in V(\Gamma)$. For all $i \geq 0$ let N_i be the set of all $v \in V(\Gamma)$ such that the shortest path of Γ between v_0 and v has i edges. Thus, $N_0 = \{v_0\}$, and the sets $N_0, N_1, N_2, ...$ are mutually disjoint, and partition $V(\Gamma)$. For each $i \geq 0$, let Γ_i be $\Gamma \backslash (N_{i+1} \cup N_{i+2} \cup ...)$; that is, Γ_i is the restriction of Γ to $N_0 \cup ... \cup N_i$.

(1) *For* $0 \leq i < \sigma$, $|E(\Gamma_i)| \leq 3|V(\Gamma_i)|$.

Subproof. Choose a spanning tree T of Γ_i such that for every $j \leq i$ and every $v \in N_j$, the path of T between v_0 and v has j edges. It follows that for every edge $e \in E(\Gamma_i) - E(T)$, the unique circuit C of Γ_i with $E(C) \subseteq E(T) \cup \{e\}$ has $\leq 2i + 1 < 2\sigma$ edges, and hence bounds a disc in Σ. An easy homotopy argument implies that, consequently, every circuit of Γ_i bounds a disc in Σ, and so (for example, by [3, theorem (11.10)]) there is a closed disc $\Delta \subseteq \Sigma$ with $U(\Gamma_i) \subseteq \Delta$. Hence Γ_i is planar. But it is simple, and so

$$|E(\Gamma_i)| \leq 3|V(\Gamma_i)| - 6$$

unless $|V(\Gamma_i)| \leq 2$, by the usual application of Euler's formula. In particular, $|E(\Gamma_i)| \leq 3|V(\Gamma_i)|$. This proves (1).

(2) *For* $0 \leq i < \sigma$, $|E(\Gamma_{i+1})| \geq (\frac{\delta}{3} - 1)|E(\Gamma_i)|$.

Subproof. Every vertex in $N_0 \cup ... \cup N_i$ has valency $> \delta$, and every edge incident with such a vertex belongs to $E(\Gamma_{i+1})$. Since an edge is incident with two such vertices if and only if it belongs to $E(\Gamma_i)$, we deduce from (1) that

$$2|E(\Gamma_i)| + |E(\Gamma_{i+1}) - E(\Gamma_i)| \geq \delta|N_0 \cup ... \cup N_i| \geq \frac{\delta}{3}|E(\Gamma_i)|,$$

and so $|E(\Gamma_{i+1})| \geq \left(\frac{\delta}{3} - 1\right)|E(\Gamma_i)|$. This proves (2).

Since $|E(\Gamma_1)| \geq \delta$ because v_0 has valency $\geq \delta$, and since $\delta \geq 3$, we deduce by repeated application of (2) that

$$|E(\Gamma_i)| \geq (\delta/3 - 1)^{i-1}\delta$$

for $1 \leq i < \sigma$. In particular, since $\sigma \geq 2$, we have

$$|E(\Gamma_{\sigma-1})| \geq (\delta/3 - 1)^{\sigma-2}\delta.$$

Hence from (1),

$$|V(\Gamma)| \geq |V(\Gamma_{\sigma-1})| \geq \frac{1}{3}|E(\Gamma_{\sigma-1})| \geq (\delta/3-1)^{\sigma-1}.$$

Now since Γ is simple and $|E(\Gamma)| \geq 2$, it follows that every region of Γ is incident with ≥ 3 edges (counting an edge twice if the same region is incident on both sides of it). Thus,

$$2|E(\Gamma)| \geq 3|R(\Gamma)|,$$

and so

$$|E(\Gamma)| - |R(\Gamma)| = \frac{1}{3}(|E(\Gamma)| + (2|E(\Gamma)| - 3|R(\Gamma)|)) \geq \frac{1}{3}|E(\Gamma)| = \frac{\delta}{3}|V(\Gamma)|.$$

From (2.1),

$$\kappa(\Sigma) - 2 \geq |E(\Gamma)| - |R(\Gamma)| - |V(\Gamma)| \geq (\delta/3-1)|V(\Gamma)| \geq (\delta/3-1)^{\sigma}$$

as required. ■

(2.3) *Let Γ be a simple 3-connected (2σ)-representative drawing in a surface Σ, where $\kappa(\Sigma) > 0$ and $\sigma \geq 2$ is an integer. Let $\mathcal{S} \subseteq R(\Gamma)$, and let $\mathcal{F}$ be the set of all pairs (v,r) such that $v \in V(\Gamma)$, $r \in \mathcal{S}$, and v is incident with r and with at least one other member of $\mathcal{S}$. Then*

$$|\mathcal{F}| \leq 24\kappa(\Sigma)^{1/\sigma}|\mathcal{S}|.$$

Proof. Let Z be the set of all $v \in V(\Gamma)$ incident with ≥ 2 members of $\mathcal{S}$. If $Z = \emptyset$ then $\mathcal{F} = \emptyset$ and the result is true, and so we may assume that $Z \neq \emptyset$ and hence $|R(\Gamma)| \geq 2$. Since Γ is simple, 2-connected and 2-representative, every region of Γ is bounded by a circuit of Γ.

For each $r \in \mathcal{S}$, choose a point $v_r \in r$. Let Γ_1 be a drawing with $V(\Gamma_1) = Z \cup \{v_r : r \in \mathcal{S}\}$ and $|E(\Gamma_1)| = |\mathcal{F}|$, in which for each $(v, r) \in \mathcal{F}, v$ is adjacent to v_r in Γ_1, joined by an edge of Γ_1 within r. Then Γ_1 is bipartite and simple, and each $v \in Z$ has valency ≥ 2 in Γ_1.

For each $v \in V(\Gamma)$, choose a closed disc $\Delta(v) \subseteq \Sigma$ such that $\Delta(v) - bd(\Delta(v))$ contains v and all edges of Γ_1 incident with v, every vertex of Γ_1 adjacent to v belongs to $bd(\Delta(v))$, and for all distinct $v, v' \in V(\Gamma)$, $\Delta(v) \cap \Delta(v')$ is the set of common neighbours of v and v' in Γ_1. Let Γ_2 be the drawing with

$$\begin{aligned} U(\Gamma_2) &= \bigcup(bd(\Delta(v)) : v \in Z) \\ V(\Gamma_2) &= \{v_r : r \in \mathcal{S}\}. \end{aligned}$$

Now Γ_2 is loopless, since each $v \in Z$ has valency ≥ 2 in Γ_1; it may have multiple edges, however. Let $r, r' \in \mathcal{S}$ be distinct, and let $e \in E(\Gamma_2)$ with ends $v_r, v_{r'}$. It follows that for some $v \in Z$, v is incident with both r and r', and e is a subset of $bd(\Delta(v))$. Since Γ is 3-connected, 3-representative and simple, there are at most two possibilities for v. For any one of them, $bd(\Delta(v))$ contains at most two edges of Γ_2 with ends v_r and $v_{r'}$. Consequently, there are at most four edges of Γ_2 with ends $v_r, v_{r'}$; and so there is a simple subdrawing Γ_3 of Γ_2 with $V(\Gamma_3) = V(\Gamma_2)$ and $|E(\Gamma_3)| \geq \frac{1}{4}|E(\Gamma_2)|$. But $|E(\Gamma_2)| = |E(\Gamma_1)| = |\mathcal{F}|$, and so $|E(\Gamma_3)| \geq \frac{1}{4}|\mathcal{F}|$. Let C be a circuit of Γ_3 which is non-null-homotopic in Σ. Then there is a circuit C' of Γ_1, non-null-homotopic in Σ, with

$$V(C') \cap \{v_r : r \in \mathcal{S}\} \subseteq V(C).$$

Since Γ is (2σ)-representative it follows that $|U(C') \cap U(\Gamma)| \geq 2\sigma$. But

$$U(C') \cap U(\Gamma) \subseteq U(\Gamma_1) \cap U(\Gamma) \subseteq Z$$

and so $|V(C') \cap Z| \geq 2\sigma$. Moreover, since Γ_1 is bipartite,

$$|V(C') \cap Z| = |V(C') \cap \{v_r : r \in \mathcal{S}\}|$$

and so $|V(C)| \geq 2\sigma$. Let

$$\delta = |E(\Gamma_3)|/|V(\Gamma_3)| \geq |\mathcal{F}|/(4|\mathcal{S}|).$$

Since $\kappa(\Sigma) > 0$, we deduce from (2.2) that $\delta \leq 6\kappa(\Sigma)^{1/\sigma}$, and so

$$|\mathcal{F}| \leq 24\kappa(\Sigma)^{1/\sigma}|\mathcal{S}|,$$

as required. ■

(2.4) *Let Γ be a simple 3-connected (2σ)-representative drawing in a surface Σ, where $\kappa(\Sigma) > 0$ and $\sigma \geq 2$ is an integer. Let $\mathcal{S} \subseteq R(\Gamma)$, and let $\mathcal{C}$ be a set of circuits of Γ such that*

(i) *for each $C \in \mathcal{C}$, there is no closed disc in Σ bounded by C, and every edge of C is incident with a region of Γ in $\mathcal{S}$, and*

(ii) *every edge of Γ belongs to at most two members of $\mathcal{C}$.*

Then $|\mathcal{C}| \leq 24\sigma^{-1}\kappa(\Sigma)^{1/\sigma}\ |\mathcal{S}|$.

Proof. Define $\mathcal{F}$ and Z as in (2.3). Now let $C \in \mathcal{C}$. Since every edge of C is incident with a member of $\mathcal{S}$, and since Γ is (2σ)-representative, there are at least 2σ distinct regions in $\mathcal{S}$ incident with an edge of C. For each one of them, say r, there are at least two vertices v of C such that exactly one (f, say) of the two edges of C incident with v is not incident with r. Since f is therefore incident with a member of $\mathcal{S} - \{r\}$ it follows that $v \in Z$ and therefore that $(v, r) \in \mathcal{F}$. Consequently, there are at least 4σ members $(v, r) \in \mathcal{F}$ such that $v \in V(C)$ and r is incident with one of the edges of C incident with v. By summing over all $C \in \mathcal{C}$, we deduce that there are at least $4\sigma|\mathcal{C}|$ triples (v, r, C) such that $(v, r) \in \mathcal{F}, C \in \mathcal{C}$, $v \in V(C)$, and r is incident with an edge of C.

On the other hand, for any given $(v, r) \in \mathcal{F}$, there are at most four $C \in \mathcal{C}$ such that (v, r, C) is such a triple, since every edge of Γ is in ≤ 2 members of $\mathcal{C}$ and there are only two edges incident with v and with r. Consequently, there are at most $4|\mathcal{F}|$ such triples, and so $4\sigma|\mathcal{C}| \leq 4|\mathcal{F}|$. The result follows from (2.3). ■

3. THE MAIN THEOREM

In this section we prove (1.2)(i), but first we need some more definitions. Let Γ be a drawing in a surface Σ, and let $v \in V(\Gamma)$. A triple (e, r, f) is an *angle of* Γ *at* v if $e, f \in E(\Gamma)$ are distinct and both incident with v, and $r \in R(\Gamma)$, and there is a closed disc $\Delta \subseteq r \cup e \cup f \cup \{v, x, y\}$, where x and y are the other ends of e and f, with $\Delta \cap U(\Gamma)$ equal to the union of e, f and their ends (or, in less formal terms, if e, f are consecutive in the cyclic order of edges at v and r is the region between them).

(3.1) *Let Γ be a loopless drawing in a surface Σ, and let $v \in V(\Gamma)$. Let $e_1, ..., e_k$ be distinct edges all incident with v where $k \geq 2$, and for $1 \leq i \leq k$ let (e_{i-1}, r_i, e_i) be an angle at v, where $e_0 = e_k$. Then either v has valency k, or $k = 2$ and $r_1 = r_2$.*

The proof is clear.

A *walk* W in a drawing Γ is a sequence $v_0, e_1, v_1, ..., e_k, v_k$ where $v_0, v_1, ..., v_k \in V(\Gamma)$, $e_1, ..., e_k \in E(\Gamma)$, and for $1 \leq i \leq k$, e_i has ends v_{i-1} and v_i. Its *length* is k. If $0 \leq a \leq b \leq k$, the walk

$$v_a, e_{a+1}, v_{a+1}, ..., e_b, v_b$$

is a *subwalk* of W. We say W is *closed* if $v_k = v_0$. It is *simple* if $v_0, v_1, ..., v_k$ are all distinct (except possibly $v_k = v_0$) and $e_1, ..., e_k$ are all distinct. (The last condition is redundant unless $k = 2$ and $v_2 = v_0$.) If W is a walk in $\Gamma, \Gamma|W$ denotes the subdrawing of Γ formed by the vertices and edges in W.

If r is a region of a 2-cell drawing Γ in Σ, then $bd(r)$ is connected since r is homeomorphic to an open disc; and there is a closed walk of Γ tracing the boundary of r in the natural sense. (Indeed, there are several, depending on the choice of initial vertex of the walk, and on whether the boundary is being followed in a clockwise or anticlockwise direction.) We call such a walk W a *perimeter walk* of r. Every edge e incident with r occurs once (if e is also incident with another region) or twice in W; and if W is $v_0, e_1, v_1, e_2, ..., v_{k-1}, e_k, v_k = v_0$, then each triple (e_i, r, e_{i+1}) is an angle at v_i; and if all vertices of Γ have valency ≥ 3, then all these angles are distinct. (Note that the perimeter walk of a region need not be simple.)

Let us restate (1.2)(i).

(3.2) *Let Γ be a drawing in a surface Σ, let $\sigma \geq 2$ be an integer so that $(\sigma/48)^\sigma \geq \kappa(\Sigma)$, and let Γ be (2σ)-representative. There is no drawing isomorphic to Γ in any surface of complexity $< \kappa(\Sigma)$.*

Proof. We define the *magnitude* of Γ to be $|V(\Gamma)| + |E(\Gamma)| + 2|R(\Gamma)|$, and we proceed by induction on the magnitude of Γ.

(1) *We may assume that, for every closed disc $\Delta \subseteq \Sigma$ with $bd(\Delta) \cap U(\Gamma) \subseteq V(\Gamma)$ and $|bd(\Delta) \cap U(\Gamma)| \leq 3$, such that each vertex in $bd(\Delta)$ is incident with an edge included in Δ:*

(i) *if $|bd(\Delta) \cap U(\Gamma)| \leq 1$ then $\Delta \cap U(\Gamma) = \emptyset$*

(ii) *if $|bd(\Delta) \cap U(\Gamma)| = 2$ then $\Gamma \cap \Delta$ consists of a single edge with ends the two vertices in $bd(\Delta)$*

(iii) *if $|bd(\Delta) \cap U(\Gamma)| = 3$, then either $\Gamma \cap \Delta$ consists of the three vertices in*

$bd(\Delta) \cap U(\Gamma)$ and two or three edges joining pairs of them, or there is a unique vertex x in $\Delta - bd(\Delta)$ and $\Gamma \cap \Delta$ contains precisely three edges, each with one end x and the other end in $bd(\Delta)$.

Subproof. If (i) is false for some Δ, the result follows from the inductive hypothesis applied to the drawing obtained from Γ by deleting the part of Γ in $\Delta - bd(\Delta)$. We assume then that (i) holds for all Δ. If (ii) is false for some Δ, then by (i) there is a path P in $\Gamma \cap \Delta$ between the two vertices in $bd(\Delta)$; and hence the result follows from the inductive hypothesis, applied to the drawing obtained by deleting all of Γ in $\Delta - bd(\Delta)$ except P, and contracting all edges of P except one. We assume then that (ii) holds for all Δ.

If (iii) is false for some Δ, let $bd(\Delta) \cap U(\Gamma) = \{a, b, c, \}$. Since (iii) is false, there is a vertex $d \in \Delta - bd(\Delta)$; and by (i) and (ii), there are three paths P_1, P_2, P_3 of $\Gamma \cap \Delta$ from d to a, b, c respectively, vertex-disjoint except for d. Let Γ' be obtained from Γ by deleting all of Γ inside of $\Delta - bd(\Delta)$ except $P_1 \cup P_2 \cup P_3$, and for $i = 1, 2, 3$ contracting all edges of P_i except one. It is easy to see that Γ is (2σ)-representative, and therefore the result follows from the inductive hypothesis applied to Γ'. This proves (1).

Now suppose for a contradiction that Γ' is a drawing in a surface Σ' with $\kappa(\Sigma') < \kappa$, and that $\alpha : \Gamma \to \Gamma'$ is an isomorphism.

(2) *Γ is simple and 3-connected; and we may assume that for every circuit C of Γ of length 3, C bounds a region of Γ, and $\alpha(C)$ does not bound a region of Γ'.*

Subproof. If Γ is not simple, it has a circuit C of length ≤ 2. Since Γ is 3-representative, C bounds a disc in Σ, contrary to (1). Thus Γ is simple. By (1), Γ is 2-cell and hence connected. By (1)(i), every region of Γ is bounded by a circuit, and so Γ is 2-connected. If Γ is not 3-connected, it is an easy exercise to show that there is a disc violating (1)(ii), because Γ is 3-representative; and so Γ is 3-connected. Let C be a circuit of Γ of length

3. Since Γ is 4-representative, C bounds a disc in Σ, and hence by (1)(iii), C bounds a region r in Σ. Let Γ_1 be obtained from Γ by deleting the three edges of C, and adding a new vertex d and three new edges joining d to the three vertices of C, so that d and the new edges are all drawn within r. Then Γ_1 is (2σ)-representative, as may easily be verified, and it has magnitude less than that of Γ. Suppose that $\alpha(C)$ bounds a region r' of Γ'. Then, similarly let Γ'_1 be obtained from Γ' by deleting the edges of $\alpha(C)$ and adding a new vertex adjacent to the vertices of $\alpha(C)$, all drawn within r'. The result follows from the inductive hypothesis applied to Γ_1 and Γ'_1. This proves (2).

Let us choose Γ' and Σ' so that $\kappa(\Sigma')$ is as small as possible. It follows that

(3) *Γ' is 2-cell in Σ'.*

Subproof. Since Γ is connected and Γ is isomorphic to Γ', it follows that Γ' is connected. Consequently, if Γ' is not 2-cell, there is a subset $F \subseteq \Sigma'$, homeomorphic to a circle, disjoint from $U(\Gamma')$, such that F bounds no closed disc in Σ'. Let Σ'' be obtained by: cutting Σ along F; taking the connected component of the result that contains $U(\Gamma')$ (this exists since Γ' is connected); and pasting one or two discs onto the boundary of this component, to obtain a surface Σ'' with $\kappa(\Sigma'') < \kappa(\Sigma')$. This contradicts the minimality of $\kappa(\Sigma')$. Hence (3) is true.

Let $\mathcal{C}$ be the set of all circuits of Γ that bound regions of Γ. Let $\mathcal{A}$ be the set of all $C \in \mathcal{C}$ such that $\alpha(C)$ bounds a region (necessarily an open disc, by (3)) of Γ' in Σ'. For fixed Γ, Σ and Σ', let us choose Γ' and α to maximize $|\mathcal{A}|$.

(4) *Let W be a simple closed walk in Γ, such that $\Gamma|W$ bounds a disc in Σ. Suppose that $\alpha(W)$ is a subwalk of a perimeter walk of a region of Γ'. Then $\Gamma|W$ bounds a region of Γ.*

Subproof. Let W be

$$v_0, e_1, v_1, ..., e_n, v_n = v_0 \ ;$$

then $n \geq 3$, by (1). Let $\Delta \subseteq \Sigma$ be a closed disc bounded by $\Gamma|W$, and let $\mathcal{D}$ be the set of all $C \in \mathcal{C}$ with $U(C) \subseteq \Delta$. Let r' be a region of Γ' so that $\alpha(W)$ is a subwalk of a perimeter walk of r'. Since $\alpha(v_1), ..., \alpha(v_n)$ are all distinct and $\alpha(e_2), ..., \alpha(e_n)$ are all distinct, there is a closed disc $\Delta' \subseteq r' \cup bd(r')$ such that

$$bd(\Delta') \cap U(\Gamma') = \{\alpha(v_1), ..., \alpha(v_n)\} \cup \alpha(e_2) \cup ...\alpha(e_n) \ .$$

Let $\gamma : \Delta \to \Delta'$ be a homeomorphism so that $\gamma(v_i) = \alpha(v_i)$ for $1 \leq i \leq n$, and $\gamma(e_i) = \alpha(e_i)$ for $2 \leq i \leq n$. Let Γ'' be obtained from Γ' by replacing $\alpha(x)$ by $\gamma(x)$, for every vertex and edge x of Γ that belongs to $\Delta - bd(\Delta)$; and let $\beta : \Gamma \to \Gamma''$ be the isomorphism defined by $\beta(x) = \gamma(x)$ if x belongs to $\Delta - bd(\Delta)$, and $\beta(x) = \alpha(x)$ otherwise. Note that since Γ'' is a drawing in Σ' isomorphic to Γ, it is necessarily 2-cell, by (3). Let $\mathcal{B}$ be the set of all $C \in \mathcal{C}$ such that $\beta(C)$ bounds a region of Γ''. From the choice of Γ', $|\mathcal{A}| \geq |\mathcal{B}|$, and so

$$|\mathcal{A} - \mathcal{B}| \geq |\mathcal{B} - \mathcal{A}| \ .$$

Let C_1 be the circuit of Γ bounding the region of Γ included in Δ and incident with e_1. From the choice of Γ'', $\mathcal{D} - \{C_1\} \subseteq \mathcal{B}$.

Now suppose that $C \in \mathcal{A} - \mathcal{B}$; we shall investigate the possibilities for C. Since $C \in \mathcal{A}$, there is a closed disc $D \subseteq \Sigma'$ bounded by $\alpha(C)$, with $D \cap U(\Gamma') \subseteq bd(D)$. Since $D - bd(D)$ is not a region of Γ'' (because $C \notin \mathcal{B}$), either $U(C) \cap (\Delta - bd(\Delta)) \neq \emptyset$, or $(D - bd(D)) \cap \Delta' \neq \emptyset$. In the first case $U(C) \subseteq \Delta$, and since $C \in \mathcal{A}$ and hence $C \in \mathcal{D}$ it follows that $C = C_1$ (for every other circuit of $\mathcal{D}$ belongs to $\mathcal{B}$). In the second case, $\alpha(C)$ bounds r', and so since $\alpha(W)$ is a subwalk of some perimeter walk of r', it follows that $\Gamma|W = C$. But $C \in \mathcal{C}$, and so $\Gamma|W$ bounds a region of Γ, as required. We have shown, therefore, that we may assume that $\mathcal{A} - \mathcal{B} \subseteq \{C_1\}$.

Choose $C^* \in \mathcal{D}$ as follows. If $C_1 \notin \mathcal{A} - \mathcal{B}$, let $C^* = C_1$. If $C_1 \in \mathcal{A} - \mathcal{B}$, choose $C^* \in \mathcal{D}$ so that

$$(\mathcal{B} - \mathcal{A}) \cap \mathcal{D} \subseteq \{C^*\}$$

(this is possible since $|\mathcal{B} - \mathcal{A}| \leq 1$). We claim that in either case, $\mathcal{D} \subseteq \mathcal{A} \cup \{C^*\}$. For if $C_1 \notin \mathcal{A} - \mathcal{B}$ then $\mathcal{A} - \mathcal{B} = \emptyset$ and hence $\mathcal{B} - \mathcal{A} = \emptyset$, that is, $\mathcal{A} = \mathcal{B}$; but $\mathcal{D} - \{C_1\} \subseteq \mathcal{B}$, and so $\mathcal{D} \subseteq \mathcal{A} \cup \{C^*\}$. On the other hand, if $C_1 \in \mathcal{A} - \mathcal{B}$, then since $\mathcal{D} - \{C_1\} \subseteq \mathcal{B}$ and

$$(\mathcal{B} - \mathcal{A}) \cap \mathcal{D} \subseteq \{C^*\}$$

it follows that $\mathcal{D} - \{C_1\} \subseteq \mathcal{A} \cup \{C^*\}$; and since in this case $C_1 \in \mathcal{A}$ we have again that $\mathcal{D} \subseteq \mathcal{A} \cup \{C^*\}$.

Since Γ is 3-connected and (we may assume) $U(\Gamma) \cap (\Delta - bd(\Delta)) \neq \emptyset$, there is a vertex $v \in bd(\Delta) - V(C^*)$ with a neighbour in Γ not in Δ. Let the edges of Γ in Δ incident with v be $g_1, ..., g_k$, in their cyclic order according to Γ, where g_1 and g_k belong to $E(\Gamma|W)$. For $1 \leq i < k$, let $F_i \in \mathcal{C}$ contain g_i and g_{i+1}. Since each $F_i \in \mathcal{D} \subseteq \mathcal{A} \cup \{C^*\}$, and $F_i \neq C^*$ since $v \notin V(C^*)$, it follows that $F_i \in \mathcal{A}$, and hence there is a region r_i of Γ' bounded by $\alpha(F_i)$ such that $(\alpha(g_i), r_i, \alpha(g_{i+1}))$ is an angle of Γ'. From (3.1) applied to the sequence

$$(\alpha(g_1), r_1, \alpha(g_2)), ..., (\alpha(g_{k-1}), r_{k-1}, \alpha(g_k)), (\alpha(g_k), r', \alpha(g_1))$$

we deduce that one of the following holds:

(i) $k = 2$ and $r_1 = r'$

(ii) $\alpha(v)$ has valency k in Γ', or

(iii) $(\alpha(g_k), r', \alpha(g_1))$ is not an angle of Γ' at $\alpha(v)$.

First let us dispose of (i). Since $F_1 \in \mathcal{C}$ and hence F_1 is an induced subgraph of Γ (because Γ is simple and 3-connected), and since we may assume that $\Gamma|W \notin \mathcal{C}$, it follows that $V(\Gamma|W) \not\subseteq V(F_1)$. Choose $u \in V(\Gamma|W) - V(F_1)$. Then $\alpha(u)$ is a vertex of $\alpha(W)$ and hence belongs to $bd(r')$, and $\alpha(u) \notin V(\alpha(F_1))$. Consequently, $\alpha(F_1)$ does not bound r'. Since $\alpha(F_1)$ bounds r_1, it follows that $r_1 \neq r'$, and so (i) is false. Now (ii) is false, because v has valency $> k$ in Γ from the choice of v. It follows that (iii) holds. We recall that W is the simple closed walk

$$v_0, e_1, v_1, ..., e_n, v_n = v_0 \ ,$$

and that $\alpha(W)$ is a subwalk of a perimeter walk W' of r'. If $v = v_i$ where $1 \leq i < n$, then since $\alpha(W)$ is a subwalk of W', the walk

$$\alpha(v_{i-1}), \alpha(e_i), \alpha(v_i), \alpha(e_{i+1}), \alpha(v_{i+1})$$

is also a subwalk of W', and hence $(\alpha(e_i), r', \alpha(e_{i+1}))$ is an angle of Γ' at $\alpha(v_i) = \alpha(v)$; and since $\{e_i, e_{i+1}\} = \{g_1, g_k\}$, this contradicts (iii). Consequently $v = v_0 = v_n$.

We have shown therefore that v_0 is the only member of $V(\Gamma) \cap \Delta$ that is not in $V(C^*)$ and that has a neighbour not in Δ; and in particular $v_0 \notin V(C^*)$ and $C^* \neq C_1$, and therefore $C_1 \in \mathcal{A} - \mathcal{B}$. It follows easily that there is a closed disc $D \subseteq \Sigma$ with $bd(D) \cap U(\Gamma) \subseteq V(\Gamma)$, $|bd(D) \cap V(\Gamma)| = 3$, and with

$$D \cap U(\Gamma) = \bigcup (U(C) : C \in \mathcal{D} - \{C^*\}).$$

By (1), $|\mathcal{D}| = 2$ and $C^* = C_2$ (since $C^* \neq C_1$), and C_1 has length 3. But $C_1 \in \mathcal{A} \cap \mathcal{C}$, contrary to (2). This proves (4).

Now let $(\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3)$ be a partition of $R(\Gamma')$, defined as follows. Let $\mathcal{R}_1$ be the set of all $r' \in R(\Gamma')$ such that r' is bounded by $\alpha(C)$ for some $C \in \mathcal{C}$. Let $\mathcal{R}_2$ be the set of all $r' \in R(\Gamma') - \mathcal{R}_1$ with a perimeter walk W' such that for every simple closed subwalk W'' of W' of length ≥ 1, $\alpha^{-1}(\Gamma'|W'')$ bounds a disc in Σ. Let $\mathcal{R}_3 = R(\Gamma) - (\mathcal{R}_1 \cup \mathcal{R}_2)$.

An *envelope* is a pair (C, r') such that $C \in \mathcal{C}, r' \in R(\Gamma')$, and there is a simple closed walk W in Γ with $\Gamma|W = C$ such that $\alpha(W)$ is a subwalk of some perimeter walk of r'.

Now let $r' \in \mathcal{R}_1 \cup \mathcal{R}_2$. We claim there is at least one envelope (C, r'), and more than one if $r' \in \mathcal{R}_2$. Let W' be a perimeter walk of r'. If W' is a simple closed walk, then since $\alpha^{-1}(\Gamma'|W')$ bounds a disc in Σ, it also bounds a region of Γ by (4), and so $r' \in \mathcal{R}_1$, and $(\alpha^{-1}(\Gamma'|W'), r')$ is the desired envelope. We assume therefore W' is not simple. Let W' be

$$v_0, e_1, v_1, ..., e_n, v_n = v_0$$

say. Since W' is not simple, we may assume that $v_0 = v_k = v_n$ for some k with $1 \le k \le n-1$. Choose a_1, b_1 with $0 \le a_1 < b_1 \le k$ and with $b_1 - a_1$ minimum, such that $v_{a_1} = v_{b_1}$. It follows that $b_1 \ge a_1 + 3$ since Γ' is simple and 2-connected; and

$$v_{a_1}, e_{a_1+1}, v_{a_1+1}, ..., e_{b_1}, v_{b_1}$$

is a simple closed subwalk W_1 say of W'. Choose a_2, b_2 with $k \le a_2 < b_2 \le n$ and with $v_{a_2} = v_{b_2}$, similarly, and let W_2 be the corresponding subwalk of W'. Since $r' \in \mathcal{R}_1 \cup \mathcal{R}_2$, $\alpha^{-1}(\Gamma'|W_1)$ bounds a disc in Σ, and hence by (4), $\alpha^{-1}(\Gamma'|W_1) \in \mathcal{C}$ and $(\alpha^{-1}(\Gamma'|W_1), r')$ is an envelope; and so is $(\alpha^{-1}(\Gamma'|W_2), r')$. They are distinct, as we see as follows.

As we observed earlier, for $1 \le i \le n$ (e_i, r', e_{i+1}) is an angle at v_i (where $e_{n+1} = e_1$), and by (2) all these angles are distinct. Consequently, there is only one value of i with $1 \le i \le n$ such that $\{e_i, e_{i+1}\} = \{e_{a_1+1}, e_{a_1+2}\}$, namely $i = a_1 + 1$. In particular, the walk

$$v_{a_1}, e_{a_1+1}, v_{a_1+1}, e_{a_1+2}, v_{a_1+2}$$

is not a subwalk of W_2, and neither is its reverse. Hence $\Gamma'|W_1 \neq \Gamma'|W_2$, and so $(\alpha^{-1}(\Gamma'|W_1), r')$, $(\alpha^{-1}(\Gamma'|W_2), r')$ are distinct envelopes, as required. We deduce that the number of distinct envelopes is at least $|\mathcal{R}_1| + 2|\mathcal{R}_2|$.

On the other hand, for each $C \in \mathcal{C}$ there is at most one $r' \in \mathcal{R}_1 \cup \mathcal{R}_2$ such that (C, r') is an envelope. For let $r'_1, r'_2 \in \mathcal{R}_1 \cup \mathcal{R}_2$, such that (C, r'_1) and (C, r'_2) are envelopes; we shall show that $r'_1 = r'_2$. Let W_1 be a simple closed walk of Γ with $\Gamma|W_1 = C$ such that $\alpha(W_1)$

is a subwalk of a perimeter walk of r'_1, and define W_2 similarly. Since $|V(C)| \geq 3$, there exists $v \in V(C)$ which is not the initial vertex of W_1 and not the initial vertex of W_2. Let e_1, e_2 be the two edges of C incident with v; then $(\alpha(e_1), r'_1, \alpha(e_2))$ and $(\alpha(e_1), r'_2, \alpha(e_2))$ are both angles of Γ' at $\alpha(v)$. Since $\alpha(v)$ has valency ≥ 3 in Γ' (because Γ' is 3-connected) it follows from (3.1) that $r'_1 = r'_2$, as required. Hence the number of envelopes is at most $|\mathcal{C}|$. Consequently,

$$|\mathcal{R}_1| + 2|\mathcal{R}_2| \leq |\mathcal{C}| \ .$$

Now let us use (2.4) to get an upper bound for $|\mathcal{R}_3|$. For each $r' \in \mathcal{R}_3$, let $W_{r'}$ be a simple closed subwalk of a perimeter walk of r', with length ≥ 1, such that the circuit $C_{r'} = \alpha^{-1}(\Gamma'|W_{r'})$ does not bound a disc in Σ. Now for all distinct $r'_1, r'_2 \in \mathcal{R}_3$, the walks $W_{r'_1}, W_{r'_2}$ have no common subwalk of length 2 (since Γ' has minimum valency ≥ 3) and so the circuits $C_{r'}$ are all distinct. Let $\mathcal{C}' = \{C_{r'} : r' \in \mathcal{R}_3\}$; then $|\mathcal{C}'| = |\mathcal{R}_3|$.

Now every edge of Γ belongs to at most two members of $\mathcal{C}'$, because every edge of Γ' is incident with at most two regions of Γ'. Moreover, let $\mathcal{S}$ be the set of all $r \in R(\Gamma)$, bounded by a circuit $C \in \mathcal{C}$ such that $\alpha(C)$ does not bound a region of Γ'. We claim that for each $r' \in \mathcal{R}_3$, every edge $e \in E(C_{r'})$ is incident with a member of $\mathcal{S}$. For let r_1, r_2 be the regions of Γ incident with e; they are distinct, since Γ is 2-connected and 2-representative. For $i = 1, 2$, let C_i be the circuit of Γ bounding r_i. If $r_1, r_2 \notin \mathcal{S}$ then $\alpha(C_1)$ and $\alpha(C_2)$ both bound regions of Γ' incident with $\alpha(e)$, and these regions are distinct since $C_1 \neq C_2$. But r' is also a region of Γ' incident with $\alpha(e)$, and so r' is bounded by one of $\alpha(C_1), \alpha(C_2)$, contradicting that $r' \in \mathcal{R}_3$. This proves our claim that each edge of $E(C_{r'})$ is incident with a member of $\mathcal{S}$.

From (2.4), we deduce that

$$|\mathcal{R}_3| = |\mathcal{C}'| \leq 24\sigma^{-1}\kappa(\Sigma)^{1/\sigma}|\mathcal{S}| \leq \frac{1}{2}|\mathcal{S}| \ .$$

But $|\mathcal{S}| + |\mathcal{R}_1| = |\mathcal{C}|$ and $|\mathcal{R}_1| + 2|\mathcal{R}_2| \le |\mathcal{C}|$, and so

$$2|\mathcal{R}_3| + (|\mathcal{S}| + |\mathcal{R}_1|) + (|\mathcal{R}_1| + 2|\mathcal{R}_2|) \le |\mathcal{S}| + 2|\mathcal{C}|.$$

Hence $|\mathcal{R}_1| + |\mathcal{R}_2| + |\mathcal{R}_3| \le |\mathcal{C}|$. But by (2.1),

$$|R(\Gamma')| - |R(\Gamma)| = \kappa(\Sigma) - \kappa(\Sigma') > 0\ ,$$

and so $|\mathcal{R}_1| + |\mathcal{R}_2| + |\mathcal{R}_3| - |\mathcal{C}| > 0$, a contradiction.

Thus there is no such Γ', Σ', α, as required. ■

4. 3-REPRESENTATIVE DRAWINGS

In this section, we prove (1.2)(ii) and (1.2)(iii). Let us restate the latter:

(4.1) *Let Γ be a simple, 3-connected, (2σ)-representative drawing in a surface Σ, where $\kappa(\Sigma) > 0$ and $\sigma \ge 2$ is an integer. Let α be an isomorphism from Γ to a 3-representative drawing Γ' in a surface Σ'. Define $\epsilon = 24\sigma^{-1}\kappa(\Sigma)^{1/\sigma}$, and let $\epsilon \le 1$. Then either there is a homeomorphism $\Sigma \to \Sigma'$ extending α, or $\kappa(\Sigma') \ge \kappa(\Sigma) + \frac{2}{3}\,\epsilon^{-2}(1 - \epsilon)$.*

Proof. The proof is similar to that of (3.2). Define $\mathcal{C}, \mathcal{S}, \mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$ as in the proof of (3.2).

(1) $\mathcal{R}_2 = \emptyset$.

Subproof. Suppose that $r' \in \mathcal{R}_2$. Since Γ' is 2-connected and 2-representative, there is a circuit C' of Γ' bounding r'. Since Γ' is 3-connected and 3-representative, C' is induced. Let $C = \alpha^{-1}(C')$. Since $r' \notin \mathcal{R}_3$, C bounds a closed disc Δ in Σ. Since $r' \notin \mathcal{R}_1$, $U(\Gamma) \cap (\Delta - bd(\Delta)) \ne \emptyset$. Since $\kappa(\Sigma) > 0$ and Γ is 1-representative and hence 2-cell,

$U(\Gamma) \not\subseteq \Delta$. Let

$$X = \{v \in V(\Gamma) : v \in \Delta - bd(\Delta)\}$$
$$Y = \{v \in V(\Gamma) : v \notin \Delta\};$$

then $X, Y \neq \emptyset$ and $(X, Y, V(C))$ is a partition of $V(\Gamma)$. Let $X' = \alpha(X), Y' = \alpha(Y)$; then $X' \cup Y' = V(\Gamma') - V(C')$. Since C' bounds a region of Γ' in Σ' (and deleting the closure of this region does not disconnect Σ'), and C' is an induced subgraph of Γ', there is a region r'' of Γ' incident with a vertex x in X' and with a vertex y in Y'. Let C'' be the circuit of Γ' bounding r''. Since every path of Γ from X to Y meets $V(C)$, and hence every path of Γ' from X' to Y' meets $V(C')$, it follows that both components of $C''\backslash\{x, y\}$ have a vertex in $V(C')$. Hence there exist distinct $u, v \in V(C') \cap V(C'')$, non-adjacent in C''. But C' and C'' both bound regions of Γ', and the latter is simple, 3-connected and 3-representative, which is impossible. This proves (1).

Now let Q be the set of all quadruples (C_1', C_2', e_1, e_2) such that $C_1', C_2' \in \mathcal{R}_3$, $C_1' \neq C_2'$, $e_1 \in E(C_1'), e_2 \in E(C_2')$, $e_1 \neq e_2$, and e_1 and e_2 have a common end. For any choice of distinct $C_1', C_2' \in \mathcal{R}_3$ there are at most six choices for (e_1, e_2) so that $(C_1', C_2', e_1, e_2) \in Q$, because $|V(C_1' \cap C_2')| \leq 2$, and if equality holds then C_1' and C_2' have an edge in common; and so $|Q| \leq 6|\mathcal{R}_3|^2$. On the other hand, for each $r \in \mathcal{S}$, if $e \in E(\Gamma)$ is incident with r, then not both the regions of Γ' incident with $\alpha(e)$ are in $\mathcal{R}_1$, and so by (1) $\alpha(e)$ is incident with a member of $\mathcal{R}_3$; and since no member of $\mathcal{R}_3$ is incident with $\alpha(e)$ for every $e \in E(\Gamma)$ incident with r, it follows that there are at least four members $(C_1', C_2', e_1, e_2) \in Q$ with e_1, e_2 both incident with r. Hence $|Q| \geq 4|\mathcal{S}|$. Consequently,

$$4|\mathcal{S}| \leq 6|\mathcal{R}_3|^2 .$$

Writing $s = |\mathcal{S}|$ and $t = |\mathcal{R}_3|$, we have $s \leq 3t^2/2$, and as in the proof of (3.2), $t \leq \epsilon s$. Hence

$$s \leq 3t^2/2 \leq 3\epsilon^2 s^2/2 .$$

If $s = 0$ then $t = 0$ and $\mathcal{R}_1 = R(\Gamma')$ and the conclusion of the theorem holds, and so we may assume that $s > 0$. Hence $s \geq \frac{2}{3}\epsilon^{-2}$. By (2.1), and since $\epsilon \leq 1$,

$$\kappa(\Sigma') - \kappa(\Sigma) = s - t = s(1 - t/s) \geq \frac{2}{3}\epsilon^{-2}(1 - \epsilon)$$

as required. ■

Here is an example, due to Thomassen [5], that shows that (4.1) is not far from best possible. Let Σ be a torus, let $\sigma \geq 2$, and let Γ be a drawing in Σ of a toroidal grid with $4\sigma^2$ vertices $\{(i,j) : 1 \leq i,j \leq 2\sigma\}$, where (i,j) and (i',j') are adjacent if either $i = i'$ and $|j' - j| = 1$, or $i = i'$ and $|j' - j| = 2\sigma - 1$, or $j = j'$ and $|i' - i| = 1$, or $j = j'$ and $|i' - i| = 2\sigma - 1$. It can be shown that Γ is 4-connected and (2σ)-representative. Now take a black and white chessboard colouring of the regions of Γ. Then the boundaries of the white squares, together with the circuits with vertex sets $\{(i,j) : 1 \leq i \leq 2\sigma\}$ (for all fixed j) and $\{(i,j) : 1 \leq j \leq 2\sigma\}$ (for all fixed i) form the set of region boundaries of a 4-representative drawing of the same graph in another surface Σ', with $\kappa(\Sigma') = 2\sigma^2 - 4\sigma + 2$. Our theorem (4.1) implies that for σ large, any such surface Σ' must satisfy $\kappa(\Sigma') \geq C\sigma^2$, where $C > 0$ is a constant; and so the quadratic complexity of Thomassen's example is necessary.

Here is another example, to show that in (4.1), Γ' being 3-representative cannot be replaced by being 2-representative. Let Γ be the toroidal grid, as before. Let C_1 be the circuit with vertex set

$$(1,1),(1,2),(2\sigma,2),(2\sigma,3),(1,3),(1,4),...,(1,2\sigma-1),(1,2\sigma),(2\sigma,2\sigma),(2\sigma,1),(1,1)$$

and let C_2 be the circuit with vertex set

$$(1,1),(2\sigma,1),(2\sigma,2),(1,2),(1,3),(2\sigma,3),...,(2\sigma,2\sigma),(1,2\sigma),(1,1)\,.$$

The regions boundaries of Γ that use no edge of the form $(1,i) - (2\sigma,i)$, together with C_1 and C_2, form the region boundaries of a 2-representative drawing of the same graph in a surface of complexity $2\sigma - 2$; thus, the conclusion of (4.1) does not hold.

Finally, let us deduce (1.2)(ii).

Proof of (1.2)(ii).

Let $\Gamma, \Sigma, \kappa, \sigma, \Gamma', \Sigma', \alpha$ be as in (1.2), let $\kappa(\Sigma') = \kappa$, let $\epsilon \leq \frac{1}{2}$, and let Γ be simple, 3-connected, and $(2\sigma + 2)$-representative. Suppose first that Γ' is not 3-representative. There exists $X' \subseteq V(\Gamma')$ with $|X'| \leq 2$ and a subset $F \subseteq \Sigma$ homeomorphic to a circle, such that $F \cap U(\Gamma' \backslash X') = \emptyset$ and F bounds no disc in Σ'. However, Γ' is 3-connected and so $\Gamma' \backslash X'$ is connected, and there is therefore a component of $\Sigma' - F$ including $U(\Gamma' \backslash X')$. Let $X = \alpha^{-1}(X')$; it follows that $\Gamma \backslash X$ is isomorphic to a drawing in a surface of complexity $< \kappa$. Yet $\Gamma \backslash X$ is (2σ)-representative in Σ, contrary to (3.2). We deduce that Γ' is 3-representative; but then the result follows from (4.1). ■

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