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Sachs' Linkless Embedding Conjecture

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We prove Sachs' conjecture that a graph can be embedded in 3-space so that it contains no non-trivial link (in the sense of knot theory) if and only if it contains as a minor none of the seven graphs obtainable from K_6 by $Y-\Delta$ and $\Delta-Y$ exchanges. We also show the following:

(i) A graph admits such a "linkless" embedding if and only if it admits a "panelled" embedding, one such that every circuit of the graph bounds a disc disjoint from the remainder of the graph. This was a conjecture of Böhme.

(ii) An embedding is panelled if and only if for every subgraph, its complement in 3-space has free fundamental group. This extends a theorem of Scharlemann and Thompson, who proved it for planar graphs.

(iii) If two panelled embeddings of the same graph are "different," that is, are not related by an orientation-preserving homeomorphism of the 3-space, then there is a subgraph which is a subdivision of K_5 or $K_{3,3}$ such that the two induced embeddings of this subgraph are still different.

1. INTRODUCTION

Let M be a 3-sphere, that is, a 3-manifold homeomorphic to S^3, with a specified orientation. (We assume throughout the paper that all embeddings

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and all subspace considered are tame.) If $X \subseteq M$, we denote its topological closure by $\bar{X}$. A *line* in M is a subset homeomorphic to the closed unit interval, and its *ends* are defined in the natural way. A *circle* in M is a subset homeomorphic to the unit circle. A *frame* in M is a pair (U, V), where

(i) $U \subseteq M$ is closed, and $V \subseteq U$ is finite

(ii) $U - V$ has only finitely many arc-wise connected components, called *edges*

(iii) for every edge, e, either $|\bar{e} \cap V| = 1$ and $\bar{e}$ is a circle, or $|\bar{e} \cap V| = 2$ and $\bar{e}$ is a line with ends the two members of $\bar{e} \cap V$.

We call V the set of *vertices* of the frame. In other words, a frame is a graph, viewed as a topological space in the usual way, and embedded in M. We use standard graph-theory terms for frames without further definition. If $\Gamma = (U, V)$ is a frame, we write $U(\Gamma) = U$ and $V(\Gamma) = V$, and we denote the set of edges by $E(\Gamma)$. In general, we shall use Γ, Γ', Γ_1 etc. to refer to frames, and G, G', G_1 etc. for "abstract" graphs.

If v is a vertex of a graph G, its *valency* is the number of edges of G incident with it, counting loops twice. A vertex of valency k is called *k-valent*. A frame in which every vertex is 2-valent is called a *link*, and if it has one component it is called a *knot*. Of particular significance in knot theory are the so-called "trivial" links Γ, those in which for every circuit C of Γ there is a disc $\Delta \subseteq M$ with $\Delta \cap U(\Gamma) = U(C) = \mathrm{bd}(\Delta)$. (By a *disc* in M we mean a subset of M homeomorphic to the closed unit disc.) Sachs [9] suggested the study of *linkless* frames, those frames Γ such that every link subgraph of Γ is trivial, and made a conjecture characterizing by excluded minors the graphs isomorphic to linkless frames. The main objective of this paper is to prove Sachs' conjecture and a strengthening of it due to Böhme [1].

To state the result we need some further constructions. If X is a vertex or edge of a graph G, or a set of vertices or edges of G, the graph obtained by deleting X is denoted by $G \backslash X$. (Deleting a vertex implies deleting all edges incident with the vertex.) If $f \in E(G)$, we denote the graph obtained by contracting f by G/f. We say a graph H is a *minor* of G if H can be obtained from a subgraph of G by contracting edges. (Contracting a loop has the same effect as deleting it; while contracting a non-loop edge parallel to k other edges produces k new loops.) A minor H of G is *proper* if $H \neq G$.

Let v be a 3-valent vertex of a graph G, with neighbours u_1, u_2, u_3, where v, u_1, u_2, u_3 are all distinct. Let H be obtained from $G \backslash v$ by adding three new edges with ends u_1u_2, u_2u_3, u_3u_1. We say that H is obtained from G by a *$Y - \Delta$ exchange* (*at* v), and G is obtained from H by a *$\Delta - Y$ exchange* (*at* $u_1u_2u_3$). If G' can be obtained from G by a series of $Y - \Delta$ or $\Delta - Y$ exchanges, we say that G and G' are *$Y - \Delta$ equivalent*. The *Petersen family* is the set of the seven graphs (up to isomorphism) that are $Y - \Delta$ equivalent to K_6. One of these is the Petersen graph. Let us say a graph G is *semi-flat* if it is isomorphic to a linkless frame. Sachs' conjecture, which we shall prove, was the following.

(1.1) *A graph is semi-flat if and only if it has no minor in the Petersen family.*

Conway and Gordon [2] and Sachs [9] proved that K_6 is not semi-flat; indeed, every frame isomorphic to K_6 has two vertex-disjoint circuits with odd "linking number." Sachs [9] also proved that every minor of a semi-flat graph is semi-flat, and that the members of the Petersen family are not semi-flat and all their proper minors are, and in particular, proved the "only if" part of (1.1). We shall prove the "if" part. (Incidentally, a proof of this was announced in 1988 by Motwani *et al.* [4] but the version in [10] has fundamental flaws.)

If $X \subseteq M$ and $F \subseteq X$ is a circle, an *X-panel for F* is a disc $\Delta \subseteq M$ with $\Delta \cap X = F = \mathrm{bd}(\Delta)$. If Γ is a frame and C is a circuit, we abbreviate "$U(\Gamma)$-panel for $U(C)$" to "Γ-panel for C." Let us say a frame Γ in M is *panelled* if for every circuit C of Γ there is a Γ-panel for C. (Circuits have no "repeated" vertices or edges.) Every panelled frame is linkless, but the converse is false—there is a counterexample with one vertex and two edges. Let us say a graph is *flat* if it is isomorphic to a panelled frame. Böhme [1] proposed the following strengthening of Sachs' conjecture, which we shall prove.

(1.2) *For a graph G, the following are equivalent:*

(i) *G is semi-flat*

(ii) *G is flat*

(iii) *G has no minor in the Petersen family.*

Since (ii) clearly implies (i), and Sachs showed that (i) implies (iii), it remains to show that (iii) implies (ii). We shall prove it in this section, using several lemmas that will be proved in the remainder of the paper.

A *separation* of a graph G is a pair (A, B) of subgraphs with $A \cup B = G$ and $E(A \cap B) = \emptyset$; its *order* is $|V(A \cap B)|$. A *k-separation* is a separation of order k, and a $(\leqslant k)$*-separation* has order $\leqslant k$. We say that G is *internally 4-connected* if $|V(G)| \geqslant 4$, G is simple and 3-connected, and $\min(|E(A)|, |E(B)|) \leqslant 3$ for every $(\leqslant 3)$-separation (A, B) of G. We say $X \subseteq V(G)$ is *2-coherent in G* if there is no $(\leqslant 1)$-separation (A, B) of G with

$$|X \cap V(A)|, |X \cap V(B)| > |V(A \cap B)|.$$

If $e \in E(G)$, we define $V(e)$ to be the set of ends of e, and $N(e)$ to be the set of all $v \in V(G) - V(e)$ with a neighbour in $V(e)$.

We say G is *Kuratowski connected* if for every $(\leqslant 3)$-separation (A, B) of G, if there is a component C of $A \backslash V(A \cap B)$, and a component D of $B \backslash V(A \cap B)$, such that every vertex in $V(A \cap B)$ has a neighbour in $V(C)$ and a neighbour in $V(D)$, then one of A, B can be drawn in a disc Δ with $V(A \cap B)$ drawn in $\mathrm{bd}(\Delta)$. $K_{5,5}^-$ denotes the graph obtained from $K_{5,5}$ by deleting five mutually non-adjacent edges. The first lemma we need is the following result of [6].

(1.3) *Let G be a graph, not isomorphic to $K_{5,5}^-$, and with no minor in the Petersen family. Then there is a graph H, $Y-\Delta$ equivalent to G, such that either*

(i) *H is not internally 4-connected, or*

(ii) *H is a proper 4-sum, or*

(iii) *H is disc-reducible, or*

(iv) *$H \backslash v$ is planar for some $v \in V(H)$, or*

(v) *there exists $f \in E(H)$, and $v \in V(H) - (V(f) \cup N(f))$, and $e \in E(H)$ incident with v, such that $H \backslash e/f$, $H/e/f$, and $H \backslash v$ are all Kuratowski connected, and $(H \backslash v)/f$ is non-planar, and $N(f)$ is 2-coherent in $H \backslash (V(f) \cup \{v\})$.*

We have not yet defined the terms used in (1.3)(ii) or (iii), and we postpone this for the moment; the definitions are in section 6. Roughly, (ii) and (iii) both assert that H can be constructed from smaller graphs by means of certain simple constructions.

Let us say that a graph is a *Sachs graph* if it is not flat, but all its proper minors are flat. To prove that (iii) implies (ii) in (1.2), we need to show that every Sachs graph is in the Petersen family, or equivalently, that every Sachs graph has a minor in the Petersen family. The second lemma we need is

(1.4) *Every graph that is $Y-\Delta$ equivalent to a Sachs graph is a Sachs graph.*

This is proved in Section 6. Next, we need

(1.5) *Every Sachs graph is internally 4-connected, and is not a proper 4-sum, and is not disc-reducible.*

This is also proved in section 6.

If M_1, M_2 are 3-spheres, by a *homeomorphism* from M_1 to M_2 we mean a homeomorphism of the topological spaces, not necessarily preserving orientation. A homeomorphism from M_1 to M_2 which is orientation-preserving is called a *flex* of M_1 to M_2 (or just a flex of M_1, if $M_1 = M_2$). The next lemma is:

(1.6) *Let Γ be a non-planar panelled frame in M. There is no orientation-reversing homeomorphism of M to itself fixing $U(\Gamma)$ pointwise.*

Let us emphasize that "planar" and "non-planar" are graph-theoretic properties, and refer to the abstract graph of which Γ is an embedding. Thus, for instance, all knots are planar. (1.6) is proved in Section 7.

If Γ_1, Γ_2 are frames, a *tracing of Γ_1 onto Γ_2* is a homeomorphism of $U(\Gamma_1)$ to $U(\Gamma_2)$ mapping $V(\Gamma_1)$ onto $V(\Gamma_2)$. It might be more appropriate to call this a homeomorphism of Γ_1 to Γ_2, but we prefer to use a different word for clarity, and to emphasize that this map is not necessarily derived from a homeomorphism of the corresponding 3-spheres. A panelled frame Γ in M is *rigid* if for every panelled frame Γ' in M and every tracing γ of Γ onto Γ', there is a homeomorphism of M to itself (not necessarily a flex) extending γ. Intuitively, a panelled frame is rigid if there is no other way to embed the same graph in M as a panelled frame, except for the trivial modifications of the embedding arising from homeomorphisms of M. The next lemma is

(1.7) *Every Kuratowski connected panelled frame is rigid.*

The converse of (1.7), that every rigid panelled frame is Kuratowski connected, is also true but we do not need it. (1.7) is proved in section 10. This is the most difficult of all our lemmas.

If Γ is a frame in M and $e \in E(\Gamma)$, then $\Gamma \backslash e$ is a frame $(U(\Gamma) - e, V(\Gamma))$ in M, and we need to interpret Γ/e as a frame. Let M be a 3-sphere, and let $e \subseteq M$ such that $\bar{e}$ is a line and $\bar{e} - e$ is the set of ends of $\bar{e}$. We define M/e to be the topological space with element set $(M - \bar{e}) \cup \{m\}$ (where m is a new element; we call it simply the *new element* of M/e) with open sets

(i) all sets $X \subseteq M$ open in M with $X \cap \bar{e} = \varnothing$, and

(ii) all sets $(X - \bar{e}) \cup \{m\}$ such that X is open in M and $\bar{e} \subseteq X$;

and with orientation corresponding to that of M. Then M/e is also a 3-sphere (because e is piece-wise linear).

If Γ is a frame in M and $e \in E(\Gamma)$ has distinct ends, we define Γ/e to be the frame in M/e with $U(\Gamma/e) = (U(\Gamma) - \bar{e}) \cup \{m\}$ and $V(\Gamma/e) = (V(\Gamma) - V(e)) \cup \{m\}$, where m is the new element of M/e. It is easy to see that if Γ is panelled, then so is Γ/e. If $X \subseteq E(\Gamma)$ and $X = \{f_1, ..., f_k\}$, and there is no circuit C with $E(C) \subseteq X$, we define $\Gamma/X = (\cdots((\Gamma/f_1)/f_2)/\cdots/f_k)$. The sixth lemma we need is

(1.8) *Let Γ be a frame in M, and let $e \in E(\Gamma)$ have distinct ends. If $\Gamma \backslash e$ and Γ/e are both panelled then Γ is panelled.*

The proof of (1.8) is an application of a recent theorem of Scharlemann and Thompson [11], which we have found very useful. We prove (1.8) in Section 3.

Let Γ, Γ' be frames in M, M' respectively. If γ is a tracing of Γ onto Γ', and $e \in E(\Gamma)$, we denote by $\gamma \backslash e$ the restriction of γ to $U(\Gamma \backslash e)$. Thus, $\gamma \backslash e$ is tracing of $\Gamma \backslash e$ onto $\Gamma' \backslash \gamma(e)$. Similarly, if $v \in V(\Gamma)$, we denote the restriction of γ to $U(\Gamma \backslash v)$ by $\gamma \backslash v$. If $e \in E(\Gamma)$ has distinct ends, γ/e denotes the tracing of Γ/e onto $\Gamma'/\gamma(e)$ defined by $(\gamma/e)(x) = \gamma(x)$ for $x \in U(\Gamma) - \bar{e}$, and $(\gamma/e)(m) = m'$ where m, m' are the new elements of M/e and $M'/\gamma(e)$.

Again, let Γ, Γ' be frames in M, M' respectively. A tracing γ of Γ onto Γ' is *even* if there is a flex of M of M' extending γ. It is *odd* if there is an orientation-reversing homeomorphism of M to M' extending γ. (Thus, a tracing could be both even and odd, and it could be neither. But we shall see later that for rigid non-planar panelled frames, every tracing is either even or odd and not both.) Our last lemma is the following.

(1.9) *Let Γ, Γ' be frames in M, and let γ be a tracing of Γ onto Γ'. Let $f \in E(\Gamma)$, with distinct ends, such that Γ/f is panelled. Let $v \in V(\Gamma) - (V(f) \cup N(f))$, and suppose that*

(i) *$|N(f)| \geqslant 3$, and $N(f)$ is 2-coherent in $\Gamma \backslash (V(f) \cup \{v\})$, and*

(ii) *$\gamma \backslash v$ and γ/f are both even.*

Then γ is even.

This is proved in section 9. Now let us use these lemmas to prove (1.2).

Proof of (1.2). As we have seen, it suffices to prove that every Sachs graph has a minor in the Petersen family. Let G be a Sachs graph, and suppose that it has no such minor. Now G is not isomorphic to $K_{5,5}^-$, since the latter is flat. (Take a 3-cube drawn in a 2-sphere in M, and add two more vertices, one inside the 2-sphere adjacent to the odd vertices of the cube, and one outside adjacent to the even vertices, placing the new edges in the simplest possible way. It is easy to check that this frame is panelled, using (4.5) for example, and hence $K_{5,5}^-$ is flat.)

Consequently, we may choose H as in (1.3). By (1.4), H is also a Sachs graph. By (1.5), H does not satisfy (1.3)(i), (ii), or (iii). Also, H does not satisfy (1.3)(iv), for if $H \backslash v$ is planar then H is flat as is easily seen. In particular, $|V(H)| \geqslant 6$. Hence (1.3)(v) holds. Let e, f, v be as in (1.3)(v).

Since H is a Sachs graph, H/f is flat. We may therefore assume, to simplify the notation, that H is itself a frame Γ in M (not panelled) and that Γ/f is panelled. Let m be the new element of M/f.

(1) *There is a frame Γ_1 in M and a tracing γ_1 of Γ onto Γ_1 such that $\Gamma_1 \backslash \gamma_1(e)$ is panelled and γ_1/f is even.*

For since $H = \Gamma$ is a Sachs graph, there is a panelled frame Γ' in M such that there is a tracing γ of $\Gamma \backslash e$ onto Γ'. Since $\Gamma \backslash e/f$ is panelled and Kuratowski connected, it is rigid by (1.7). Since $\Gamma'/\gamma(f)$ is panelled and γ/f is a tracing of $\Gamma \backslash e/f$ onto $\Gamma'/\gamma(f)$, and $\Gamma \backslash e/f$ is rigid, there is a homeomorphism β: $M/f \to M/\gamma(f)$ extending γ/f. By replacing Γ' by its mirror image if necessary, we may assume that β is a flex. Define $\gamma_1(x) = \gamma(x)$ if $x \in U(\Gamma) - e$, and $\gamma_1(x) = \beta(x)$ if $x \in U(\Gamma) - \bar{f}$. This is well-defined, for if $x \in U(\Gamma) - (e \cup \bar{f})$ then $\beta(x) = (\gamma/f)(x) = \gamma(x)$. Moreover, γ_1 is one-to-one, for if $x \in e$ and $y \in \bar{f}$, then $\gamma_1(x) = \beta(x) \in M - \overline{\gamma(f)}$ and $\gamma_1(y) = \gamma(y) \in \overline{\gamma(f)}$, and so $\gamma_1(x) \neq \gamma_1(y)$. Consequently, γ_1 is a tracing of Γ to some frame Γ_1 in M. Now $\Gamma_1 \backslash \gamma_1(e) = \gamma(\Gamma \backslash e) = \Gamma'$, and so $\Gamma_1 \backslash \gamma_1(e)$ is panelled. Moreover, β extends γ_1/f and β is a flex, and so γ_1/f is even. This proves (1).

Since γ_1 provides an isomorphism between Γ_1 and $\Gamma = H$, we may assume that $\Gamma_1 = \Gamma = H$ and γ_1 is the identity, to further simplify the notation. In other words,

(2) *$\Gamma = H$ is a frame in M, Γ/f is panelled, and $\Gamma \backslash e$ is panelled.*

(3) *There is a frame Γ' in M and a tracing γ of Γ onto Γ', such that $\Gamma'/\gamma(e)$ is panelled and γ/f is even.*

The proof is similar to that of (1), using that $\Gamma/e/f$ is Kuratowski connected.

We claim that the hypotheses of (1.9) are satisfied. For γ is a tracing of Γ onto Γ', and Γ/f is panelled. Also, $v \in V(\Gamma) - (V(f) \cup N(f))$ by (1.3)(v); and (1.9)(i) holds, by (1.3)(v), and since $|N(f)| \geqslant 3$ because Γ is internally 4-connected and hence 3-connected and $|V(\Gamma)| \geqslant 5$. Also, γ/f is even, by (3). Let us verify that $\gamma \backslash v$ is even. By (1.3)(v), $\Gamma \backslash v$ is Kuratowski connected, and hence rigid by (1.7) (since it is panelled because $\Gamma \backslash e$ is panelled by (2)). Let α be a homeomorphism of M to itself extending $\gamma \backslash v$, and let α' be the homeomorphism of M/f to $M/\gamma(f)$ induced by α. Since γ/f is even, there is a flex β of M/f to $M/\gamma(f)$ extending γ/f. Let β' be the homeomorphism of M/f defined by $\beta'(x) = \beta^{-1}(\alpha'(x))$ $(x \in M/f)$. Then β' fixes $U((\Gamma \backslash v)/f)$ pointwise; for if $x \in U((\Gamma \backslash v)/f - \{m\}$ (we recall that m is the new element of M/f) then

$$\beta'(x) = \beta^{-1}(\alpha'(x)) = \beta^{-1}(\alpha(x)) = \beta^{-1}(\gamma(x)) = \beta^{-1}(\beta(x)) = x,$$

and

$$\beta'(m) = \beta^{-1}(\alpha'(m)) = \beta^{-1}(m') = m,$$

where m' is the new element of $M/\gamma(f)$. This proves that β' fixes $U((\Gamma\backslash v)/f)$ pointwise. But $(\Gamma\backslash v)/f$ is non-planar by (1.3)(v), and panelled, and so β' is a flex by (1.6). Hence α' is a flex, since β is a flex, and so α is a flex. Consequently $\gamma\backslash v$ is even, as required.

Thus all the hypotheses of (1.9) hold, and so from (1.9), γ is even. Let γ' be a flex of M extending γ. Now $\Gamma\backslash e$ is panelled by (2), and $\Gamma'/\gamma(e)$ is panelled by (3). But the homeomorphism of M/e to $M/\gamma(e)$ induced by γ' maps Γ/e to $\Gamma'/\gamma(e)$, and so Γ/e is panelled. By (1.8), Γ is panelled. But $\Gamma = H$, and H is a Sachs graph, a contradiction.

Thus our assumption, that G has no minor in the Petersen family, is false, and the result follows. ∎

2. Böhme's Lemma

The remainder of the paper is devoted to proving (1.4)–(1.9). We begin with a useful lemma due to Böhme [1]. We prove it for the reader's convenience.

The *unit crossing* is the triple (B, D_1, D_2) in $\mathbf{R}^3$, where B is the unit closed 3-ball $\{(x, y, z)\colon x^2+y^2+z^2 \leqslant 1\}$, and D_1, D_2 are the discs $\{(x, y, 0)\colon x^2+y^2 \leqslant 1\}$ and $\{(x, 0, z)\colon x^2+z^2 \leqslant 1\}$. A *crossing* in M is a triple (B, D_1, D_2) homeomorphic to the unit crossing. A *ball* in M is a subset homeomorphic to the unit closed 3-ball, and a *sphere* in M is a subset homeomorphic to the unit 2-sphere. Let Δ_1, Δ_2 be discs in M. A point $x \in \Delta_1 \cap \Delta_2$ is a *crossing point* of Δ_1, Δ_2 if there is a ball $B \subseteq M$ with $x \in B - \mathrm{bd}(B)$, such that $(B, B\cap\Delta_1, B\cap\Delta_2)$ is a crossing. It follows that no point of $\mathrm{bd}(\Delta_1)\cup\mathrm{bd}(\Delta_2)$ is a crossing point of Δ_1, Δ_2. We say discs Δ_1, Δ_2, are *in general position* if each point of $\Delta_1\cap\Delta_2 - \mathrm{bd}(\Delta_1)\cap\mathrm{bd}(\Delta_2))$ is a crossing point of Δ_1, Δ_2. This implies that $\Delta_1 \cap \mathrm{bd}(\Delta_2) \subseteq \mathrm{bd}(\Delta_1)$, and $\Delta_2 \cap \mathrm{bd}(\Delta_1) \subseteq \mathrm{bd}(\Delta_2)$. If $X \subseteq M$, a *neighbourhood* of X is an open subset $Y \subseteq M$ with $X \subseteq Y$, such that every arc-wise connected component of Y has non-empty intersection with X.

(2.1) *Let $X \subseteq M$ be closed, let $F_1, F_2 \subseteq X$ be circles, and let Δ_i be an X-panel for F_i $(i=1, 2)$. There is an X-panel Δ'_1 for F_1 such that*

(i) *Δ'_1, Δ_2 are in general position, and*

(ii) *there is a flex of M mapping Δ_1 to Δ'_1 and fixing pointwise X and a neighbourhood of $X - F_1$.*

The proof is standard—see [8] (we recall that all subspaces considered are tame).

(2.2) *Let Δ_1, Δ_2 be discs in M, in general position. Then $(\Delta_1 - \mathrm{bd}(\Delta_1)) \cap (\Delta_2 - \mathrm{bd}(\Delta_2))$ has only finitely many arc-wise connected components, and for each one, L say, either*

(i) *$\bar{L}$ is a line with ends u, v in $\mathrm{bd}(\Delta_1)\cap\mathrm{bd}(\Delta_2)$, and $L = \bar{L} - \{u, v\}$, or*

(ii) *$\bar{L}$ is a circle with one point v in $\mathrm{bd}(\Delta_1)\cap\mathrm{bd}(\Delta_2)$, and $L = \bar{L} - \{v\}$, or*

(iii) *$L = \bar{L}$ is a circle disjoint from $\mathrm{bd}(\Delta_1)\cap\mathrm{bd}(\Delta_2)$.*

Again, the proof is easy using tameness, and we omit it.

Let $\Delta \subseteq M$ be a disc. A *digon* for Δ is a disc $D \subseteq M$ with $D \cap \Delta = \mathrm{bd}\, D$ and with $\mathrm{bd}(D) - \mathrm{bd}(\Delta)$ non-null and arc-wise connected.

(2.3) *Let Δ_1, Δ_2 be discs in M, in general position, with $\mathrm{bd}(\Delta_1)\cap\mathrm{bd}(\Delta_2)$ null or arc-wise connected. Then either $\Delta_1\cap\Delta_2 = \mathrm{bd}(\Delta_1)\cap\mathrm{bd}(\Delta_2)$, or there is a digon D for Δ_1 with $D \subseteq \Delta_2$.*

Proof. Let $L_1, ..., L_k$ be the arc-wise connected components of $(\Delta_1 - \mathrm{bd}(\Delta_1)) \cap (\Delta_2 - \mathrm{bd}(\Delta_2))$. We may assume that $k \geqslant 1$, for otherwise $\Delta_1 \cap \Delta_2 \subseteq \mathrm{bd}(\Delta_1)\cup\mathrm{bd}(\Delta_2)$, and since

$$\Delta_1 \cap \mathrm{bd}(\Delta_2) = \mathrm{bd}(\Delta_1)\cap\Delta_2 = \mathrm{bd}(\Delta_1)\cap\mathrm{bd}(\Delta_2)$$

because Δ_1, Δ_2 are in general position, it would follow that $\Delta_1\cap\Delta_2 = \mathrm{bd}(\Delta_1)\cap\mathrm{bd}(\Delta_2)$.

Suppose first that $\bar{L}_i$ is a circle for some i, and choose i such that the disc $D \subseteq \Delta_2$ bounded by $\bar{L}_i$ is minimal. By (2.2), it follows that $|D \cap \mathrm{bd}(\Delta_2)| \leqslant 1$, and so D includes no L_j satisfying (2.2)(i); and from the minimality of D, it includes no L_j satisfying (2.2)(ii) or (2.1)(iii) with $j \neq i$. Consequently $D \cap \Delta_1 = \mathrm{bd}(D)$. But $\mathrm{bd}(D) - \mathrm{bd}(\Delta_1)$ is non-null and arc-wise connected since $\mathrm{bd}(D) - \mathrm{bd}(\Delta_1) = L_i$, and so D is a digon for Δ_1, as required.

We may assume therefore that $L_1, ..., L_k$ all satisfy (2.2)(i). Since both ends of $\bar{L}_1$ are in $\mathrm{bd}(\Delta_1)\cap\mathrm{bd}(\Delta_2)$, and the latter is arc-wise connected, we may choose a disc $D \subseteq \Delta_2$ with $\mathrm{bd}(D) - \mathrm{bd}(\Delta_1)\cap\mathrm{bd}(\Delta_2) = L_i$ for some i, such that D is minimal. From the minimality of D, and since each L_j satisfies (2.2)(i), it follows that $D \cap \Delta_1 = \mathrm{bd}(D)$, and since $\mathrm{bd}(D) - \mathrm{bd}(\Delta_1) = L_i$, it follows that D is a digon for Δ_1, as required. ∎

Let Δ be a disc in M, and let D, D' be digons for Δ. We say that D' is an *enlargement* of D if

(i) D and D' are on the same side of Δ, in the natural sense (this can easily be made precise because of piecewise linearity)

(ii) $D \cap D' = D' \cap \mathrm{bd}(\Delta) = D \cap \mathrm{bd}(\Delta)$, and

(iii) the disc in Δ bounded by $\mathrm{bd}(D')$ includes $\mathrm{bd}(D)$.

Evidently, we have

(2.4) *If Δ is a disc in M, and D is a digon for Δ, there is an enlargement of D.*

Note, however, that although (2.4) is obvious, it is false without the condition that $\mathrm{bd}(D) - \mathrm{bd}(\Delta)$ is arc-wise connected. Combining these results, we obtain:

(2.5) *Let $X \subseteq M$ be closed, let $F_1, F_2 \subseteq X$ be circles, and let Δ_i be an X-panel for F_i $(i = 1, 2)$, so that Δ_1, Δ_2 are in general position. Suppose that $F_1 \cap F_2$ is null or arc-wise connected, and that $\Delta_1 \cap \Delta_2 \neq F_1 \cap F_2$. Then there is a digon $D \subseteq \Delta_2$ for Δ_1, and an enlargement D' of D, such that $D' \cap (X \cup \Delta_2) = \mathrm{bd}(D') \cap F_1$.*

Proof. By (2.3) there is a digon $D \subseteq \Delta_2$ for Δ_1, and by (2.4) there is an enlargement D' of D. By choosing D' sufficiently close to D, we may choose D' so that $D' \cap (X \cup \Delta_2) = \mathrm{bd}(D') \cap F_1$. ∎

We deduce

(2.6) *Let $X \subseteq M$ be closed, and let $F_1, F_2 \subseteq X$ be circles, such that $F_1 \cap F_2$ is null or arc-wise connected. Let Δ_2 be an X-panel for F_2. If there is an X-panel Δ_1 for F_1 then it can be chosen so that $\Delta_1 \cap \Delta_2 = F_1 \cap F_2$.*

Proof. By (2.1) there is an X-panel Δ_1 for F_1 so that Δ_1, Δ_2 are in general position. Choose Δ_1 so that $\Delta_1 \cap \Delta_2 - F_1 \cap F_2$ has as few arc-wise connected components as possible. If $\Delta_1 \cap \Delta_2 \neq F_1 \cap F_2$, choose D, D' as in (2.5) with $D' \cap (X \cup \Delta_2) = \mathrm{bd}(D') \cap F_1$. Let $\Delta_1' = (\Delta_1 - D_1) \cup D'$, where $D_1 \subseteq \Delta_1$ is the disc bounded by $\mathrm{bd}(D')$. Then Δ_1' is an X-panel for F_1, and Δ_1', Δ_2 are in general position, and every arc-wise connected component of $\Delta_1' \cap \Delta_2 - F_1 \cap F_2$ is also a component of $\Delta_1 \cap \Delta_2 - F_1 \cap F_2$, and $\mathrm{bd}(D) - F_2$ is a component of $\Delta_1 \cap \Delta_2 - F_1 \cap F_2$ that is not a component of $\Delta_1' \cap \Delta_2 - F_1 \cap F_2$. This contradicts the choice of Δ_1, and the result follows. ∎

(2.7) *Let $C_1, ..., C_k$ be circuits of a frame Γ, and for $1 \leqslant i \leqslant k$ let Δ_i be a Γ-panel for C_i, such that $\Delta_i \cap \Delta_j = U(C_i \cap C_j)$ for $1 \leqslant i < j \leqslant k$. Let C_0 be a circuit of Γ such that $C_0 \cap C_i$ is null or connected for $1 \leqslant i \leqslant k$, and such that there is a Γ-panel for C_0. Then there is a Γ-panel Δ_0 for C_0 such that $\Delta_0 \cap \Delta_i = U(C_0 \cap C_i)$ for $1 \leqslant i \leqslant k$.*

Proof. Choose j with $1 \leqslant j \leqslant k+1$, maximum such that there is a Γ-pannel Δ_0 for C_0 such that $\Delta_0 \cap \Delta_i = U(C_0 \cap C_i)$ for $1 \leqslant i < j$. Suppose that $j \neq k+1$, and let $X = U(\Gamma) \cup \Delta_1 \cup \cdots \cup \Delta_{j-1}$. Then Δ_j is an X-panel for C_j, and Δ_0 is an X-panel for C_0. By (2.6) there is an X-panel Δ_0' for C_0 such that $\Delta_0' \cup \Delta_j = U(C_0 \cap C_j)$, contrary to the maximality of j. Hence $j = k+1$, and the result follows. ∎

From (2.7) we have immediately (by induction on k) the following, which is Böhme's lemma:

(2.8) *Let $C_1, ..., C_k$ be circuits of a frame Γ, such that $C_i \cap C_j$ is null or connected for $1 \leqslant i < j \leqslant k$. If there is a Γ-panel for each of $C_1, ..., C_k$, then there exists Γ-panels $\Delta_1, ..., \Delta_k$ for $C_1, ..., C_k$ respectively so that $\Delta_i \cap \Delta_j = U(C_i \cap C_j)$ for $1 \leqslant i < j \leqslant k$.*

We shall also need a more complicated lemma proved by similar techniques, and the following is used for that.

(2.9) *Let $X \subseteq M$ be closed, let $F \subseteq X$ be a circle, and let $y_1, y_2, y_3 \in F$ be distinct, such that $(X - F) \cup \{y_i\}$ is arc-wise connected for $i = 1, 2, 3$. Let Δ_1, Δ_2 be X-panels for F, in general position, with $\Delta_1 \cap \Delta_2 \neq F$. Then there is a digon $D \subseteq \Delta_2$ for Δ_1 such that $X - D \cap F$ is arc-wise connected.*

Proof. Let $L_1, ..., L_k$ be the arc-wise connected components of $\Delta_1 \cap \Delta_2 - F$. Then $k \geqslant 1$. If some $\bar{L}_i$ is a circle, then as in (2.5) there is a digon $D \subseteq \Delta_2$ for Δ_1 with $|D \cap F| \leqslant 1$, and hence $X - D \cap F$ is connected from the hypothesis. We assume then that each $\bar{L}_i$ is a line. Let $\bar{L}_1$ have ends $x_1, x_2 \in F$. From the hypothesis, there exists $y \in F$ with $y \neq x_1, x_2$ such that $(X - F) \cup \{y\}$ is arc-wise connected. But there is a disc $D \subseteq \Delta_2$ with $\mathrm{bd}(D) - F = L_i$ for some i (namely, $i = 1$) with $y \notin D$; let us choose i and D so that D is minimal. Then

$$X - D \cap F = ((X - F) \cup \{y\}) \cup (F - (D \cap F))$$

and since $(X - F) \cup \{y\}$ and $F - (D \cap F)$ are both arc-wise connected and y is in both of them, it follows that $X - D \cap F$ is arc-wise connected, as required. ∎

(2.10) *Let $X \subseteq M$ be closed, let $F \subseteq X$ be a circle and let $y_1, y_2, y_3 \in F$ be distinct, such that $(X - F) \cup \{y_i\}$ is arc-wise connected for $i = 1, 2, 3$. Let Δ_1, Δ_2 be X-panels for F, in general position, with $\Delta_1 \cap \Delta_2 \neq F$. Then there is a digon $D \subseteq \Delta_2$ for Δ_1 and an enlargement D' of D, such that $D' \cap (X \cup \Delta_2) = D \cap F$, and there is a flex of M fixing pointwise X and a neighbourhood of $X - F$, and mapping Δ_1 to $(\Delta_1 - D_1) \cup D'$, where $D_1 \subseteq \Delta_1$ is the disc bounded by* $\mathrm{bd}(D')$.

Proof. By (2.6) there is a digon D with $X - D \cap F$ arc-wise connected. Choose D' as in the proof of (2.5), so that $D' \cap (X \cup \Delta_2) = D \cap F$. Let $D_1 \subseteq \Delta$ be the disc bounded by bd(D'), and let S be the sphere $D_1 \cup D'$. Now $X \cap D' \subseteq D_1$ and $X \cap D_1 = D \cap F$, and so $X \cap S = X \cap (D \cap F)$. Consequently $X - S$ is arc-wise connected, and so there is a ball B bounded by S with $B \cap (X - S) = \emptyset$. Now $\Delta_1 - D_1$ is arc-wise connected and meets $X - S$ since both sets include $F - \mathrm{bd}(D)$. Since $(\Delta_1 - D_1) \cap S = \emptyset$, it follows that $(\Delta_1 - D_1) \cap B = \emptyset$. There is therefore a flex of M mapping D_1 to D' and fixing pointwise X, $\Delta_1 - D_1$, and a neighbourhood of $X - F$, as required. ∎

From (2.10) we deduce

(2.11) *Let $X \subseteq M$ be closed, let $F \subseteq X$ be a circle such that $X - F$ is arc-wise connected, and let $y_1, y_2, y_3 \in F$ be distinct, such that $(X - F) \cup \{y_i\}$ is arc-wise connected for $i = 1, 2, 3$. Let Δ_1, Δ_2 be X-panels for F. Then there is a flex of M fixing X and a neighbourhood of $X - F$ pointwise, and mapping Δ_1 to Δ_2.*

Proof. By (2.1) there is a flex α_1 of M fixing X and a neighbourhood of $X - F$ pointwise, so that $\alpha_1(\Delta_1), \Delta_2$ are in general position. By choosing α_1 to minimize the number of components of $\alpha_1(\Delta_1) \cap \Delta_2 - F$, it follows from (2.10) that $\alpha_1(\Delta_1) \cap \Delta_2 = F$, as in the proof of (2.6). Then $\alpha_1(\Delta_1) \cup \Delta_2$ is a sphere disjoint from $X - F$, and since $X - F$ is arc-wise connected, there is a ball bounded by $\alpha_1(\Delta_1) \cup \Delta_2$ disjoint from $X - F$. The result follows. ∎

No doubt (2.11) is still true without y_1, y_2, y_3, but it is good enough for us as stated, and easier to prove. (2.11) has the following corollary.

(2.12) *Let Γ be a frame, and let $C_1, \ldots, C_k$ be distinct circuits of Γ, such that for $1 \leqslant i \leqslant k$, at least three vertices of C_i are incident with edges of Γ not in C_i and $U(\Gamma) - U(C_i)$ is arc-wise connected. For $1 \leqslant i \leqslant k$, let Δ_i and Δ'_i be Γ-panels for C_i, so that $\Delta_i \cap \Delta_j = \Delta'_i \cap \Delta'_j = U(C_i \cap C_j)$ for $1 \leqslant i < j \leqslant k$. Then there is a flex of Γ fixing $U(\Gamma)$ and a neighbourhood of $U(\Gamma) - U(C_1 \cup \cdots \cup C_k)$ pointwise, and mapping Δ_i to Δ'_i for $1 \leqslant i \leqslant k$.*

Proof. Choose j with $1 \leqslant j \leqslant k+1$, maximum such that there is a flex α of Γ fixing $U(\Gamma)$ and a neighbourhood of $U(\Gamma) - U(C_1 \cup \cdots \cup C_k)$ pointwise, and mapping Δ_i to Δ'_i for $1 \leqslant i < j$. Suppose that $j \neq k+1$, and let $X = U(\Gamma) \cup \Delta'_1 \cup \cdots \cup \Delta'_{j-1}$. Then $\alpha(\Delta_j)$ and Δ'_j are both X-panels, and (2.11) yields a contradiction to the maximality of j. Hence $j = k+1$, and the result follows. ∎

3. The Scharlemann–Thompson Theorem

A frame Γ in M is *spherical* if there is a sphere $S \subseteq M$ with $U(\Gamma) \subseteq S$. A necessary condition for a frame to be spherical is that it should be planar (that is, as an abstract graph) but that is not sufficient; for instance, all knots are planar, but only the trivial knot is spherical. There is a beautiful characterization of spherical frames by Scharlemann and Thompson [11] as follows. (If G, H are graphs, we write $H \subseteq G$ to denote that H is a subgraph of G; and in particular, if Γ, Γ' are frames, the $\Gamma' \subseteq \Gamma$ denotes that $U(\Gamma') \subseteq U(\Gamma)$ and $V(\Gamma') = V(\Gamma) \cap U(\Gamma')$.)

(3.1) *Let Γ be a frame in M, such that Γ is planar. Then Γ is spherical if and only if for every $\Gamma' \subseteq \Gamma$, $M - U(\Gamma')$ has free fundamental group.*

It is curious that while (3.1) will be very important for us, one does not need to know what is meant by a free fundamental group to understand our applications of (3.1), and so we omit the definition. What is important is that $X \subseteq M$ having free fundamental group is a property of X as a topological space, and therefore any other topological space Y homeomorphic to X has free fundamental group if and only if X does. This is important because non-isomorphic frames Γ_1, Γ_2 can have $M - U(\Gamma_1)$ homeomorphic to $M - U(\Gamma_2)$. To see this, observe that an analogous situation occurs in planar graph theory; if Γ_1, Γ_2 are any two connected graphs drawn in a 2-sphere S, both with the same number of regions, then $S - U(\Gamma_1)$ is homeomorphic to $S - U(\Gamma_2)$. Actually, in three dimensions it is harder for two different frames to have homeomorphic complements; for instance, it was recently shown [3] that if Γ_1, Γ_2 are knots and $M - U(\Gamma_1)$ is homeomorphic to $M - U(\Gamma_2)$ then there is a homeomorphism of M to itself taking $U(\Gamma_1)$ to $U(\Gamma_2)$. (This result does not hold for links—see [7].) But we obviously have at least the following.

(3.2) *Let Γ be a frame in M, and let $e \in E(M)$ with distinct ends. Then $M - U(\Gamma)$ is homeomorphic to $M/e - U(\Gamma/e)$.*

We shall prove the following variation on (3.1). (See [14] for an interesting extension of (3.3).)

(3.3) *Let Γ be a frame in M. Then Γ is panelled if and only if for every $\Gamma' \subseteq \Gamma$, $M - U(\Gamma')$ has free fundamental group.*

For this we need the following lemma (see [11]).

(3.4) *If Γ is a frame in M and $e \in E(\Gamma)$ is a loop, and there is a Γ-panel for the circuit C with $E(C) = \{e\}$, and $\Gamma \backslash e$ is spherical, then Γ is spherical.*

We deduce

(3.5) *Let Γ be a frame in M with no edges with distinct ends. Then Γ is panelled if and only if it is spherical.*

Proof. If Γ is spherical it is clearly panelled, and the converse follows by induction on $|E(\Gamma)|$ from (3.4). ∎

From (3.5) we have half of (3.3), the following.

(3.6) *If Γ is a panelled frame in M, then $M - U(\Gamma')$ has free fundamental group for every $\Gamma' \subseteq \Gamma$.*

Proof. Let $\Gamma' \subseteq \Gamma$, and let F be a maximal forest of Γ', with $E(F) = \{f_1, ..., f_k\}$ say. Then $\Gamma_1 = \Gamma'/\{f_1, ..., f_k\}$ is panelled in $M_1 = M/\{f_1, ..., f_k\}$, and so is spherical by (3.5). By (3.1), $M_1 - U(\Gamma_1)$ has free fundamental group; but by (3.2), $M - U(\Gamma')$ is homeomorphic to $M_1 - U(\Gamma_1)$, and so $M - U(\Gamma')$ has free fundamental group, as required. ∎

For the other half of (3.3) we need some further lemmas. The first is an applications of (2.7).

(3.7) *Let Γ be a frame in M, and let $H \subseteq \Gamma$ be planar. Let $k \geqslant 1$ and let $C_0, C_1, ..., C_k$ be circuits of H so that H can be drawn in a sphere with $k+1$ regions bounded by $C_0, C_1, ..., C_k$ respectively. Suppose that*

(i) *for every $v \in V(H) - V(C_0)$, every edge of Γ incident with v is an edge of H, and*

(ii) *for $1 \leqslant i \leqslant k$ there is a Γ-panel for C_i.*

Let Δ_k be a Γ-panel for C_k; then there is a Γ-panel Δ_0 for C_0 with $\Delta_0 \cap \Delta_k = U(C_0 \cap C_k)$.

Proof. We proceed by induction on k. If $k = 1$ then $C_0 = C_1$ and the result is trivial, and so we assume that $k \geqslant 2$. By an elementary property of planar graphs, there is one of $C_1, ..., C_{k-1}$, say C_{k-1}, such that $C_{k-1} \cap C_k$ is a path P with $E(P) \neq \emptyset$. Hence no edge or internal vertex of P belongs to any of $C_0, C_1, ..., C_{k-2}$. Let C'_{k-1} be the circuit such that $C'_{k-1} \cup P = C_{k-1} \cup C_k$, and let Γ' be obtained from Γ by deleting all edges and internal vertices of P. Since $C_{k-1} \cap C_k$ is connected, it follows from (2.7) that there is a Γ-panel Δ_{k-1} for C_{k-1} such that $\Delta_{k-1} \cap \Delta_k = U(P)$. Let $\Delta'_{k-1} = \Delta_{k-1} \cup \Delta_k$; then Δ'_{k-1} is a Γ'-panel for C'_{k-1}. By the inductive hypothesis, there is a Γ'-panel Δ_0 for C_0 with $\Delta_0 \cap \Delta'_{k-1} = U(C_0 \cap C'_{k-1})$. But $U(\Gamma) - U(\Gamma') \subseteq \Delta'_{k-1}$, since no internal vertex of P is incident with an edge of Γ not in P, from (i), and so Δ_0 is a Γ-panel for C_0. Since

$$\Delta_0 \cap \Delta_k \subseteq \Delta_0 \cap \Delta'_{k-1} = U(C_0 \cap C'_{k-1})$$

and $U(C'_{k-1}) \cap \Delta_k \subseteq U(C_k)$, it follows that $\Delta_0 \cap \Delta_k = U(C_0 \cap C_k)$ as required. ∎

If C is a circuit of frame Γ, we say C is *bad* if there is no Γ-panel for C.

(3.8) *Let Γ be a frame in M, such that for every edge e and every bad circuit C, some end of e is in $V(C)$. Then Γ is either panelled or planar.*

Proof. We assume that Γ is not panelled, and so there is a bad circuit C_0; choose it with $|E(C_0)|$ minimum. Let $L_1, ..., L_k$ be the arc-wise connected components of $U(\Gamma) - U(C_0)$, and for $1 \leqslant i \leqslant k$ let $H_i \subseteq \Gamma$ be the frame with $U(H_i) = \bar{L}_i$.

(1) *For $1 \leqslant i \leqslant k$, $|V(H_i) - V(C_0)| \leqslant 1$, and if $|V(H_i \cap C_0)| > 1$ there is a bad circuit $C_i \subseteq H_i \cup C_0$ such that $C_i \cap C_0$ is a path with no internal vertex in $V(H_i)$.*

If $|V(H_i) - V(C_0)| \geqslant 2$ then there is an edge of H_i with no end in $V(C_0)$, contrary to the hypothesis. Thus $|V(H_i) - V(C_0)| \leqslant 1$. If $|V(H_i \cap C_0)| > 1$ then $H_i \cup C_0$ is planar, and can be drawn in a sphere so that every region is bounded by a circuit, and C_0 is one of these circuits, and all the others meet C_0 in a path with no internal vertex in $V(H_i)$. By (3.7) applied to $H_i \cup C_0$, one of these circuits different from C_0 is bad. This proves (1).
Let $I = \{i: 1 \leqslant i \leqslant k,\ |V(H_i \cap C_0)| \geqslant 2$ and $|V(H_i) - V(C_0)| = 1\}$.

(2) *If $i, j \in I$ are distinct then $V(H_i \cap C_0) \subseteq V(C_j \cap C_0)$.*

For if $v \in V(H_i \cap C_0)$ there is an edge e of H_i with ends u, v say where $u \notin V(C_0)$. By hypothesis, $\{u, v\} \cap V(C_j) \neq \emptyset$ since C_j is bad, and so $v \in V(C_j)$, as required.

(3) *For $1 \leqslant i \leqslant k$, if $i \notin I$ then either $|E(H_i)| = 1$, and the edge of H_i is parallel to an edge of C_0, or $|V(H_i \cap C_0)| \leqslant 1$ and $|V(H_i)| \leqslant 2$.*

If $|V(H_i \cap C_0)| \geqslant 2$, then $V(H_i) - V(C_0) = \emptyset$ since $i \notin I$, and so $|E(H_i)| = 1$; let $E(H_i) = \{e\}$ say. Now $|E(C_i)| \leqslant |E(C_0)|$, and so equality holds, from the choice of C_0; and consequently e is parallel to an edge of C_0 as required. If $|V(H_i \cap C_0)| \leqslant 1$, then $|V(H_i)| \leqslant 2$ by (1). This proves (3).
From (2) and (3) it follows by elementary graph theory that Γ is planar (using that for $i \in I$, no internal vertex of $C_i \cap C_0$ is in $V(H_i)$). ∎

Now we complete the proof of (3.3).

Proof of (3.3). By (3.6), it suffices to show that if $M - U(\Gamma')$ has free fundamental group for all $\Gamma' \subseteq \Gamma$, then Γ is panelled. We proceed by induction on $|E(\Gamma)|$. If Γ is planar, then by (3.1) Γ is spherical and hence panelled as required. We assume then that Γ is not planar.

(1) *For every bad circuit C of Γ, every edge of Γ with distinct ends has an end in $V(C)$.*

For suppose that $e \in E(\Gamma)$ has distinct ends, not in $V(C)$. There is no disc in $U(C) \cup (M - U(\Gamma))$ bounded by $U(C)$; but there is a homeomorphism from $U(C) \cup (M - U(\Gamma))$ to $U(C) \cup (M/e - U(\Gamma/e))$ fixing $U(C)$ pointwise, and so there is no disc in $U(C) \cup (M/e - U(\Gamma/e))$ bounded by $U(C)$, that is, C is bad in Γ/e. Hence Γ/e is not panelled, and from the inductive hypothesis there exists $\Gamma'' \subseteq \Gamma/e$ such that $M/e - U(\Gamma'')$ does not have free fundamental group. But by (3.2), there exists $\Gamma' \subseteq \Gamma$ such that $M - U(\Gamma')$ is homeomorphic to $M/e - U(\Gamma'')$ (taking $\Gamma' = \Gamma''$ if $U(\Gamma'')$ does not contain the new element of M/e), and so $M - U(\Gamma')$ does not have free fundamental group, contrary to the hypothesis. This proves (1).

(2) *For every bad circuit C, every edge of Γ has an end in C.*

For let $e \in E(\Gamma)$; by (1) we may assume that e is a loop, incident with $v \in V(\Gamma) - V(C)$. Let C' be the circuit with $E(C') = \{e\}$. If C' is bad then by (1), every edge of $\Gamma\backslash v$ is a loop, and so Γ is planar, a contradiction. Thus there is a Γ-panel Δ for C'. Now C is bad, and so there is no disc in $U(C) \cup (M - U(\Gamma))$ bounded by $U(C)$. In particular, there is no such disc in $U(C) \cup (M - (U(\Gamma) \cup \Delta))$. But there is a homeomorphism of $U(C) \cup (M - (U(\Gamma) \cup \Delta))$ to $U(C) \cup (M - U(\Gamma\backslash e))$ fixing $U(C)$ pointwise, and so C is bad in $\Gamma\backslash e$, contrary to the inductive hypothesis. This proves (2).

From (2) and (3.8), Γ is panelled, as required. ∎

(3.3) has (1.8) as a corollary.

Proof of (1.8). Γ is a frame in M and $e \in E(\Gamma)$ has distinct ends. Suppose that Γ is not panelled; then by (3.3) there is a subgraph $\Gamma' \subseteq \Gamma$ such that $M - U(\Gamma')$ does not have free fundamental group. If $e \notin E(\Gamma')$, then by (3.3), $\Gamma\backslash e$ is not panelled, as required. If $e \in E(\Gamma')$, then by (3.2) and (3.3), Γ/e is not panelled, as required. ∎

4. Some Frame Constructions

(4.1) *Let Γ be a frame in M. If either*

(i) *$v \in V(\Gamma)$ has valency $\leqslant 1$, and $\Gamma\backslash v$ is panelled, or*

(ii) *C is a circuit of Γ with $|E(C)| \leqslant 2$, and there is a Γ-panel for C, and $\Gamma\backslash e$ is panelled for some $e \in E(C)$, or*

(iii) *C is a circuit of Γ with $|E(C)| = 3$, and there is a Γ-panel for C, and $v \in V(C)$ is 3-valent in Γ with a neighbour not in $V(C)$, and $e \in E(C)$ is not incident with v, and $\Gamma\backslash e$ is panelled.*

then Γ is panelled.

Proof. (i) can be proved directly, but let us use (1.8). If v has valency 0 the result is clear, so let $e \in E(\Gamma)$ be incident with v. Then $\Gamma\backslash e$ and Γ/e are both panelled since $\Gamma\backslash v$ is panelled, and so Γ is panelled by (1.8).

For (ii) we proceed by induction on $|E(\Gamma)|$. If there is an edge f of Γ with distinct ends not both in $V(C)$, then $\Gamma\backslash f\backslash e$ is panelled and hence so is $\Gamma\backslash f$, from the inductive hypothesis; and $\Gamma/f\backslash e$ is panelled and hence so is Γ/f, similarly. Hence Γ is panelled from (1.8). We assume then that there is no such edge, and so Γ is planar, and hence $\Gamma\backslash e$ is spherical by (3.1) and (3.3). If $|E(C)| = 1$ it follows from (3.4) that Γ is spherical and hence panelled, and so we assume that $|E(C)| = 2$, $E(C) = \{e, f\}$ say. There is a homeomorphism of M exchanging e and f and fixing $U(\Gamma) - e \cup f$ pointwise, because of the Γ-panel for C. Therefore $\Gamma\backslash f$ is panelled, and so $(\Gamma/e)\backslash f$ is panelled. From the inductive hypothesis, Γ/e is panelled, and so Γ is panelled by (1.8).

For (iii), let v have neighbours u_1, u_2, u_3 in Γ, where e has ends u_1, u_2; and let e_i be the edge with ends vu_i $(i = 1, 2, 3)$. Now for $i = 1, 2$, $\Gamma\backslash e/e_i$ is panelled; and since Γ/e_i is obtained from $\Gamma\backslash e/e_i$ as in (ii), it follows that Γ/e_i is panelled. In particular, Γ/e_2 is panelled, and hence $\Gamma\backslash e_1/e_2$ is panelled. Since v has valency 2 in $\Gamma\backslash e_1$, it follows that $\Gamma\backslash e_1$ is panelled. Hence $\Gamma\backslash e_1$ and Γ/e_1 are both panelled and so Γ is panelled by (1.8). ∎

(4.2) *Let Γ_1, Γ_2 be frames in M and let $e_i \in E(\Gamma_i)$ $(i = 1, 2)$ such that $\Gamma_1\backslash e_1 = \Gamma_2\backslash e_2$ and $\Gamma_1\backslash e_1$ is panelled. Let γ be a tracing of Γ_1 onto Γ_2 fixing $U(\Gamma_1\backslash e_1)$ pointwise. (Hence e_1 and e_2 have the same ends.) For $i = 1, 2$ let C_i be a circuit of Γ_i with $e_i \in E(C_i)$, such that there is a Γ_i-panel for C_i, and either $|E(C_i)| \leqslant 2$, or $|E(C_i)| = 3$ and the vertex of C_i not incident with e_i is 3-valent in Γ_i and has a neighbour not in $V(C_i)$. Then there is a flex of M extending γ and fixing pointwise a neighbourhood of $U(\Gamma_1) - \bar{e}_1$.*

Proof. Let $\Gamma = \Gamma_1\backslash e_1$. If e_1 is a loop of Γ_1 then e_2 is a loop of Γ_2 since e_1 and e_2 have the same ends, and the result is clear. We assume then that e_1 and e_2 both have distinct ends u_1, u_2. For $i = 1, 2$, let Δ_i be a Γ_i-panel for C_i. By perturbing Δ_1 slightly, we may assume that $\Delta_1 \cap e_2$ is finite. Choose a disc $\Delta_3 \subseteq \Delta_1$ with $\Delta_3 \cap U(\Gamma) = \Delta_1 \cap U(\Gamma)$, so small that $\Delta_3 \cap e_2 = \{u_1, u_2\}$, and let $\mathrm{bd}(\Delta_3) - U(\Gamma) = e_3$. Since $e_3 \subseteq \Delta_1$, there is a flex α of M mapping e_1 to e_3 and fixing pointwise a neighbourhood of

$U(\Gamma)-\{u_1, u_2\}$. Now Γ_2 is panelled, by (4.1), since $\Gamma_2\backslash e_2$ is panelled. But $e_2 \cap \Delta_3 = \emptyset$, and so Γ_3 is panelled, by (4.1), where $\Gamma_3 = (U(\Gamma_2) \cup e_3, V(\Gamma_2))$. In particular, the circuit of Γ_3 with edges e_2, e_3 has a Γ_3-panel, and so there is a flex β of M fixing pointwise a neighbourhood of $U(\Gamma)-\{u_1, u_2\}$, and mapping each $x \in \bar{e}_3$ to $\gamma(\alpha^{-1}(x))$. Then the flex $\beta(\alpha(x))$ $(x \in M)$ satisfies the theorem. ▮

Actually, the hypothesis in (4.2) that $\Gamma_1\backslash e_1$ is panelled is not necessary, at least if $C_1\backslash e_1 = C_2\backslash e_2$, but the proof without that hypothesis is arduous, and we do not need the result.

If Γ is a frame in M, and $X \subseteq M$, we say X is *Γ-orthogonal* if for every $e \in E(\Gamma)$, either $\bar{e} \subseteq X$ or $e \cap X = \emptyset$. Under this assumption, $(U(\Gamma) \cap X, V(\Gamma) \cap X)$ is a frame, which we denote by $\Gamma \cap X$.

(4.3) *Let Γ be a frame in M, and let $\Delta \subseteq M$ be a disc with $U(\Gamma) \cap \Delta = V(\Gamma) \cap \mathrm{bd}(\Delta)$. Let Γ_1 be a frame with $U(\Gamma_1) \subseteq \Delta$ and $V(\Gamma) \cap \Delta \subseteq V(\Gamma_1)$. If $\Gamma \cup \Gamma_1$ is not panelled then there is a simple frame Γ_2 with $U(\Gamma_2) \subseteq \Delta$ and $V(\Gamma_2) = V(\Gamma) \cap \Delta$ so that $\Gamma \cup \Gamma_2$ is not panelled.*

Proof. We proceed by induction on $|V(\Gamma_1)| + |E(\Gamma_1)|$. If there is an edge e of Γ_1 with distinct ends not both in $V(\Gamma)$, then by (1.8) one of $\Gamma \cup \Gamma_1\backslash e$, $\Gamma \cup \Gamma_1/e$ is not panelled, and the result follows from the inductive hypothesis. If Γ_1 has a vertex v of valency 0 not in $V(\Gamma)$ then $\Gamma \cup \Gamma_1\backslash v$ is not panelled, and again the result follows by induction. If Γ_1 has a loop e such that $\bar{e}$ bounds a region of Γ_1 in Δ, then $\Gamma \cup \Gamma_1\backslash e$ is not panelled by (4.1)(ii) and the result follows by induction. If two edges e, f of Γ_1 are parallel and $\bar{e} \cup \bar{f}$ bounds a region of Γ_1 in Δ, then $\Gamma \cup \Gamma_1\backslash e$ is not panelled by (4.1)(ii) and the result follows by induction. If none of these occur, then $V(\Gamma_1) = \Delta \cap V(\Gamma)$, and Γ_1 is simple, and so the theorem holds. ▮

A corollary of (4.3) is:

(4.4) *Let Γ be a panelled frame in M, and let C be a circuit of Γ with $|V(C)| = 3$. Let Δ be a Γ-panel for C, and let Γ_1 be a frame isomorphic to K_4 with $U(\Gamma_1) \subseteq \Delta$ and $C \subseteq \Gamma_1$. Then $\Gamma \cup \Gamma_1$ is panelled.*

Proof. Let Γ' be obtained from Γ by deleting the three edges of C. If $\Gamma' \cup \Gamma_1$ $(= \Gamma \cup \Gamma_1)$ is not panelled then by (4.3) there is a simple frame Γ_2 with $U(\Gamma_2) \subseteq \Delta$ and $V(\Gamma_2) = V(C)$ such that $\Gamma' \cup \Gamma_2$ is not panelled. But there is a flex of M fixing $U(\Gamma')$ pointwise and mapping Γ_2 to a subgraph of C, and so $\Gamma' \cup C = \Gamma$ is not panelled, a contradiction. The result follows. ▮

A circuit C of a graph is *induced* if there do not exist distinct $u, v \in V(C)$ adjacent in G but not in C.

(4.5) *Let Γ be a frame in M. If there is a Γ-panel for every induced circuit of Γ then Γ is panelled.*

Proof. We prove that every circuit C has a Γ-panel by induction on $|E(C)|$. If C is induced it has a Γ-panel by hypothesis, and we therefore assume that there is an edge e of Γ with distinct ends $u, v \in V(C)$, non-adjacent in C.

Let C_1, C_2 be the two circuits using e with $C_1\backslash e, C_2\backslash e \subseteq C$. From the inductive hypothesis there are Γ-panels Δ_1, Δ_2 for C_1, C_2 respectively, and by (2.8) they may be chosen so that $\Delta_1 \cap \Delta_2 = \bar{e}$. Then a Γ-panel for C can be found in a neighbourhood of $\Delta_1 \cup \Delta_2$. ▮

(4.6) *Let Γ be a frame in M, and let S be a Γ-orthogonal sphere such that every two vertices of $\Gamma \cap S$ are adjacent in $\Gamma \cap S$. Let S bound balls B_1, B_2. If $\Gamma \cap B_1$ and $\Gamma \cap B_2$ are both panelled then Γ is panelled.*

Proof. First we prove this when $\Gamma \cap S$ is isomorphic to K_4. Let $C_1, ..., C_4$ be the four circuits of $\Gamma \cap S$ of length 3, and let $\Delta_1, ..., \Delta_4$ be the closures of the regions they bound in S. By subdividing each edge of Γ not in $\Gamma \cap S$, we may assume that no edge of Γ is parallel to an edge of $\Gamma \cap S$. By (4.5), it suffices to show that there is a Γ-panel for every induced circuit C of Γ. If $C \subseteq \Gamma \cap S$ then C is one of $C_1, ..., C_4$ and hence has a Γ-panel. We assume therefore that $U(C) \nsubseteq B_2$. Since C is induced, it therefore has $\leqslant 1$ edge in $\Gamma \cap S$, and $\leqslant 2$ vertices in $\Gamma \cap S$; and if it has two vertices in $\Gamma \cap S$ then they are adjacent in C (because no edge of Γ is parallel to an edge of $\Gamma \cap S$). Consequently $U(C) \subseteq B_1$. Moreover, $C \cap C_i$ is null or connected for $i = 1, ..., 4$, and so by (2.7) applied to $\Gamma \cap B_1$, there is a $(\Gamma \cap B_1)$-panel Δ for C with $\Delta \cap \Delta_i = U(C \cap C_i)$ $(1 \leqslant i \leqslant 4)$. In particular, $\Delta \cap S \subseteq \mathrm{bd}(\Delta)$, and so $\Delta \subseteq B_1$, and hence Δ is a Γ-panel for C, as required.

Thus the result holds if $\Gamma \cap S$ is isomorphic to K_4. For the general case, we may assume that no circuit of $\Gamma \cap S$ of length $\leqslant 2$ bounds a region in S, by (4.1)(ii), and consequently $\Gamma \cap S$ is simple. Since every two vertices of $\Gamma \cap S$ are adjacent, it follows that $|V(\Gamma \cap S)| \leqslant 4$ since K_5 is non-planar, and so there exists a frame Γ_0 isomorphic to K_4, with $U(\Gamma_0) \subseteq S$ and $\Gamma \cap S \subseteq \Gamma_0$. Now for $i = 1, 2$, $(\Gamma \cap B_i) \cup \Gamma_0$ is panelled, by repeated use of (4.1)(i), (ii) and (4.4); and so $\Gamma \cup \Gamma_0$ is panelled, by (4.6) with $\Gamma \cap S$ isomorphic to K_4. Consequently Γ is panelled, as required. ▮

5. Constructions for Flat Graphs

Now we apply the results of section 4 to obtain some constructions for flat graphs.

(5.1) *Let G be a graph. If either*

(i) *$v \in V(G)$ has valency $\leqslant 1$, and $G\backslash v$ is flat, or*

(ii) *e is a loop or parallel edge of G, and $G\backslash e$ is flat, or*

(iii) *G is obtained from a flat graph by $Y-\Delta$ or $\Delta-Y$ exchange*

then G is flat.

Proof. (i) follows from (4.1)(i), and (ii) from (4.1)(ii). For (iii), let H be flat, so that G is obtained from H by a $Y-\Delta$ or $\Delta-Y$ exchange. We may assume that H is a panelled frame, $H=\Gamma$ say. If G is obtained by a $Y-\Delta$ exchange at v say, let $v \in V(\Gamma)$ have neighbours u_1, u_2, u_3, where v, u_1, u_2, u_3 are distinct, and let the edge of Γ with ends v, u_i be e_i. Let $\Delta \subseteq M$ be a Γ-orthogonal disc such that $\mathrm{bd}(\Delta) \cap U(\Gamma) = \{u_1, u_2, u_3\}$ and $\Gamma \cap \Delta$ consists of v, u_1, u_2, u_3 and the edges e_1, e_2, e_3. By three applications of (4.1)(iii), $(U(\Gamma) \cup \mathrm{bd}(\Delta), V(\Gamma))$ is panelled, and hence $G \cup H$ is flat, and therefore so is G.

If G is obtained by a $\Delta-Y$ exchange at u_1, u_2, u_3, let $H=\Gamma$ where Γ is a panelled frame, and let C be a circuit of Γ with $V(C)=\{u_1, u_2, u_3\}$. Let Δ be a Γ-panel for C, and let Γ_1 be as in (4.4). Then by (4.4), $\Gamma \cup \Gamma_1$ is panelled, and so $G \cup H$ and hence G is flat. ∎

(5.2) *Let (A, B) be a 4-separation of a graph G with $V(A \cap B)=\{v_1, v_2, v_3, v_4\}$. Let $V(A)-V(B)=\{v_0\}$, and let A have eight edges, with ends $v_0v_1, v_0v_2, v_0v_3, v_0v_4, v_1v_2, v_2v_3, v_3v_4, v_4v_1$. Let (A', B) be a 4-separation of another graph G', where $V(A' \cap B)=V(A \cap B)$, such that A' can be drawn in a disc with v_1, v_2, v_3, v_4 drawn in the boundary in order. If G is flat then G' is flat.*

Proof. We may assume that $G=\Gamma$, where Γ is a panelled frame in M. By (2.8) applied to the four circuits of A of length 3, there is a Γ-orthogonal disc Δ such that $\Gamma \cap \Delta = A$ and $\mathrm{bd}(\Delta)=U(C)$, where C is the circuit of A with vertex set $\{v_1, v_2, v_3, v_4\}$. We may assume also that A' is a frame with $U(A') \subseteq \Delta$ (with $v_1, ..., v_4$ as previously described). Suppose that $A' \cup B$ is not panelled. By (4.3) there is a simple frame Γ' with $U(\Gamma') \subseteq \Delta$ and $V(\Gamma')=\{v_1, ..., v_4\}$ such that $\Gamma' \cup B$ is not panelled. But there is a flex of M fixing $V(B)$ pointwise and mapping $U(\Gamma')$ into $U(A)$, and hence $A \cup B$ is not panelled, a contradiction. Thus $A' \cup B$ is panelled, and so G' is flat as required. ∎

A graph K is *complete* if it is simple and every two distinct vertices are adjacent.

(5.3) *Let (A_1, A_2) be a $(\leqslant 3)$-separation of G, and let K be a complete graph with $V(K)=V(A_1 \cap A_2)$ and $E(K) \cap E(G)=\varnothing$. If $A_1 \cup K$ and $A_2 \cup K$ are flat then G is flat.*

Proof. Let Γ_1 be a panelled frame isomorphic to $A_1 \cup K$, and let $K_1 \subseteq \Gamma_1$ correspond to K. Trivially (if $|V(K)| \leqslant 2$) or since there is a Γ_1-panel for K_1 (if $|V(K)|=3$), there is a Γ_1-orthogonal sphere S with $\Gamma_1 \cap S = K_1$, bounding balls B_1, B_2 with $U(\Gamma_1) \subseteq B_1$. Choose Γ_2 and a sphere similarly for $A_2 \cup K$. By applying a suitable homeomorphism, we may assume that $\Gamma_2 \cap S = \Gamma_1 \cap S$ and $U(\Gamma_2) \subseteq B_2$. By (4.6), we deduce that $\Gamma_1 \cup \Gamma_2$ is panelled, and so $G \cup K$ is flat, and hence so is G. ∎

(5.4) *Let (A_1, A_2) be a 4-separation of G, and let K be a complete graph with $V(K)=V(A_1 \cap A_2)$ and $E(K) \cap E(G)=\varnothing$. If $A_1\backslash V(A_1 \cap A_2)$ and $A_2\backslash V(A_1 \cap A_2)$ are both connected, and $A_1 \cup K$ and $A_2 \cup K$ are both flat, then $G \cup K$ is flat, and hence so is G.*

Proof. By (5.1), we may assume that every edge of G has an end in $V(G)-V(A_1 \cap A_2)$. Let Γ_1 be a panelled frame in M isomorphic to $A_1 \cup K$, and let $K_0 \subseteq \Gamma_1$ correspond to K, and let $C_1, ..., C_4$ be the four circuits of K_1 of length 3. By (2.8) applied to $C_1, ..., C_4$, there is a Γ_1-orthogonal sphere S with $\Gamma_1 \cap S = K_0$, bounding balls B_1, B_2. Since $A_1\backslash V(A_1 \cap A_2)$ is connected and every edge of G has an end in $V(G)-V(A_1 \cap A_2)$, it follows that $U(\Gamma_1)-U(K_0)$ is arc-wise connected, and so we may assume that $U(\Gamma_1) \subseteq B_1$. By a similar argument for $A_2 \cup K$, and applying a suitable homeomorphism of M, we may assume that there is a panelled frame Γ_2 with $U(\Gamma_2) \subseteq B_2$ and $\Gamma_2 \cap S = K_0$, such that $\Gamma_1 \cup \Gamma_2 \cup K_0$ is isomorphic to $G \cup K$. By (4.6), $\Gamma_1 \cup \Gamma_2 \cup K_0$ is panelled, and hence $G \cup K$ and G are flat, as required. ∎

6. Irreducibility of Sachs Graphs

In this section we prove (1.4) and (1.5). We recall that G is *internally 4-connected* if $|V(G)| \geqslant 4$, G is simple and 3-connected, and $\min(|E(A_1)|, |E(A_2)|) \leqslant 3$ for every $(\leqslant 3)$-separation (A_1, A_2) of G. The following was proved by Böhme [1], but we give a proof for the reader's convenience.

(6.1) *Every Sachs graph is internally 4-connected.*

Proof. Let G be a Sachs graph. Since planar graphs are flat, it follows that G is non-planar and so $|V(G)| \geqslant 5$. By (5.1)(ii), G is simple, and by (5.1)(i) G has minimum valency $\geqslant 2$, and hence $\geqslant 3$, since suppressing 2-valent vertices does not affect flatness.

Suppose that (A_1, A_2) is a $(\leqslant 2)$-separation of G, such that $V(A_1)$, $V(A_2) \neq V(G)$. If $|(A_1 \cap A_2)| \leqslant 1$, then since A_1 and A_2 are flat it follows from (5.3) that G is flat, a contradiction. Thus G is 2-connected, and

$|V(A_1 \cap A_2)| = 2$, $V(A_1 \cap A_2) = \{x, y\}$ say. Let K be a complete graph with $V(K) = \{x, y\}$ and with $E(K) \cap E(G) = \varnothing$. Then $A_1 \cup K$ is isomorphic to a minor of G, because there is a path of A_2 between x and y (since G is 2-connected); and this minor is proper since $|V(A_1 \cup K)| < |V(G)|$. Consequently $A_1 \cup K$ is flat, and similarly so is $A_2 \cup K$. Then by (5.3), G is flat, a contradiction. Thus G is 3-connected.

Now suppose that (A_1, A_2) is a $(\leqslant 3)$-separation of G, such that $\min(|E(A_1)|, |E(A_2)|) \geqslant 4$. Since G is simple, not every edge of A_2 has both ends in $V(A_1 \cap A_2)$, and so $V(A_1) \neq V(G)$, and similarly $V(A_2) \neq V(G)$. Hence $|V(A_1 \cap A_2)| = 3$, since G is 3-connected. Let $V(A_1 \cap A_2) = \{x, y, z\}$. Let $v_i \in V(A_i) - V(A_1 \cap A_2)$, and let A'_i be a graph with $V(A'_i) = \{v_i, x, y, z\}$, with three edges, joining v_i to x, y, and z respectively, such that $E(A'_i) \cap E(G) = \varnothing$ $(i = 1, 2)$. Then $A'_1 \cup A_2$ is isomorphic to a minor of G, since there are three paths of G (and hence of A_1) between v_1 and $\{x, y, z\}$, mutually vertex-disjoint except for v_1; and this minor is proper, since $|E(A'_1)| = 3 < |E(A_1)|$. Thus $A'_1 \cup A_2$ is flat, and similarly $A_1 \cup A'_2$ is flat. Let K be a complete graph with $V(K) = \{x, y, z\}$ and $E(K) \cap E(G) = \varnothing$. By (5.1)(iii) applied to $A_1 \cup A'_2$, we deduce that $A_1 \cup K$ is flat, and similarly $A_2 \cup K$ is flat. Hence G is flat by (5.3), a contradiction. The result follows. ∎

We need the following lemma.

(6.2) *Let H be obtained from a graph G by a $\Delta - Y$ exchange, let $e \in E(H)$, and let $H' = H \backslash e$ or H/e. Then there is a proper minor G' of G such that either*

(i) *H' is obtained from G' by a $\Delta - Y$ exchange, or*

(ii) *H' is isomorphic to G', or*

(iii) *there is a 3-valent vertex of H' incident with three edges two of which, f, f' say, are parallel, and $H' \backslash f/f'$ is isomorphic to G', or*

(iv) *there is an edge f of H' incident with a vertex of valency $\leqslant 2$, such that H/f is isomorphic to G'.*

Proof. Let $v \in V(H)$ of valency 3, with neighbours u_1, u_2, u_3 in H, where v, u_1, u_2, u_3 are all distinct. Let the edge of H with ends v, u_i be f_i. Let G be obtained from $H \backslash v$ by adding three new edges e_1, e_2, e_3 with ends u_2u_3, u_3u_1, u_1u_2 respectively. If $e = f_1$ and $H' = H \backslash e$ then (iv) holds with $f = f_2$ and $G' = G \backslash \{e_2, e_3\}$. If $e = f_1$ and $H' = H/e$ then (ii) holds with $G' = G \backslash e_1$. We assume then that $e \neq f_1$, and similarly $e \neq f_2, f_3$. If $H' = H \backslash e$ then (i) holds with $G' = G \backslash e$, and so we assume that $H' = H/e$. If some end of e is not in $\{u_1, u_2, u_3\}$ or if e is a loop, then (i) holds with $G' = G/e$. We may therefore assume that e has ends u_1, u_2. But then (iii) holds with $f = f_1$, $f' = f_2$ and $G' = G \backslash \{e_1, e_2\}/e_3$. ∎

(6.3) *Let H be obtained from G by a $Y - \Delta$ exchange, let $e \in E(H)$, and let $H' = H \backslash e$ or H/e. Then there is a proper minor G' of G such that either*

(i) *H' is obtained from G' by a $Y - \Delta$ exchange, or*

(ii) *H' is isomorphic to G', or*

(iii) *there is an edge f of H', in a circuit of H' of length $\leqslant 2$, such that $H' \backslash f$ is isomorphic to G'.*

The proof is similar to (6.2), and we omit it.

We deduce (1.4), which we restate.

(6.4) *Every graph that can be obtained from a Sachs graph by $Y - \Delta$ or $\Delta - Y$ exchange is also a Sachs graph.*

Proof. Let G be a Sachs graph, and let H be obtained from G by a $Y - \Delta$ or $\Delta - Y$ exchange. By (5.1)(iii), H is not flat. But for every edge e of H, it follows from (6.2), (6.3) and (5.1) that $H \backslash e$ and H/e are flat. Since H has no isolated vertices it follows that every proper minor of H is flat, and so H is a Sachs graph, as required. ∎

Let (A, B) be a separation of order 4 of a graph G, and let K be a complete graph with $V(K) = V(A \cap B)$, such that every edge of G with both ends in $V(A \cap B)$ belongs to K. Suppose that $A \backslash V(A \cap B)$ and $B \backslash V(A \cap B)$ are both connected, and that $A \cup K$ and $B \cup K$ are both $Y - \Delta$ equivalent to proper minors of G. In these circumstances we say G is a *proper* 4-*sum*.

(6.5) *No Sachs graph is a proper* 4-*sum*.

Proof. Let G be a Sachs graph, and suppose that there exist A, B, K as above. By (5.1)(iii), $A \cup K$ and $B \cup K$ are flat. Hence G is flat by (5.1)(ii) and (5.4), a contradiction. ∎

Let (A, B) be a 4-separation of G, such that A can be drawn in a disc with the four vertices v_1, v_2, v_3, v_4 of $V(A \cap B)$ drawn on the boundary in order. Let v_0 be a new vertex, and let A' be a graph with five vertices v_0, v_1, v_2, v_3, v_4 and six new edges with ends $v_0v_1, v_0v_2, v_0v_3, v_0v_4, v_1v_2$ and v_2v_3. If $A' \cup B$ is isomorphic to a proper minor of G we say G is *disc-reducible.*

(6.6) *No Sachs graph is disc-reducible.*

Proof. Let G be a Sachs graph, and suppose that there are A, B, A' as above, where $A' \cup B$ is flat. Add a new edge to A' with ends v_1, v_2 forming A_1 say; by (5.1)(ii), $A_1 \cup B$ is flat. By a $\Delta - Y$ exchange at $v_0v_1v_2$ followed by a $Y - \Delta$ exchange at v_0, we deduce from (5.1)(iii) that $A_2 \cup B$ is flat, where A_2 is obtained from A' by adding an edge with ends v_3v_4. Add a new edge to A_2 with ends v_2v_3, forming A_3; by (5.1)(ii), $A_3 \cup B$ is flat. By a $\Delta - Y$ exchange at $v_0v_2v_3$ and a $Y - \Delta$ exchange at v_0, it follows from (5.1)(iii) that $A_4 \cup B$ is flat, where A_4 is obtained from A_3 by adding an edge with ends v_1v_4. By (5.2), G is flat, a contradiction. ∎

From (6.1), (6.5) and (6.6), we see that (1.5) follows.

7. Rigidity

In this section we prove (1.6) and some related lemmas. We begin with the following. A graph G is *cyclically* 3-*connected* if it is loopless and 2-connected, and has $\geqslant 3$ vertices, and one of A, B is a path for every 2-separation (A, B) with $A, B \neq G$.

(7.1) *Let Γ be a panelled frame in M, and let $\Gamma_0 \subseteq \Gamma$ be planar and cyclically* 3-*connected. Then there is a Γ-orthogonal sphere $S \subseteq M$ with $\Gamma \cap S = \Gamma_0$.*

Proof. Let $C_1, \dots, C_k$ be the circuits of Γ_0 bounding the regions of some drawing of Γ_0 in a sphere. Then $C_i \cap C_j$ is null or connected for $1 \leqslant i < j \leqslant k$, and the result follows from (2.8). ∎

(7.2) *Let Γ be a panelled frame in M, and let S be a Γ-orthogonal sphere with $\Gamma \cap S = \Gamma_0$. Let $r_1, \dots, r_k$ be the regions of Γ_0 in S, and for $1 \leqslant i \leqslant k$ let r_i be bounded by a circuit C_i of Γ_0, where $C_1, \dots, C_k$ are all distinct. For $1 \leqslant i \leqslant k$, let $U(\Gamma) - U(C_i)$ be arc-wise connected, and let $V(C_i)$ contain $\geqslant 3$ vertices with valency $\geqslant 3$ in Γ. Let S' be a Γ-orthogonal sphere with $\Gamma \cap S' = \Gamma_0$, and let $r'_1, \dots, r'_k$ be the regions of Γ_0 in S', where r'_i is bounded by $U(C_i)$ $(1 \leqslant i \leqslant k)$. Then there is a flex of M fixing $U(\Gamma)$ and a neighbourhood of $U(\Gamma) - U(\Gamma_0)$ pointwise and mapping r_i to r'_i for $1 \leqslant i \leqslant k$.*

Proof. This is immediate from (2.12). ∎

We deduce

(7.3) *Let Γ be a panelled frame in M, and let $\Gamma_0 \subseteq \Gamma$ be a cyclically* 3-*connected planar graph with $\geqslant 3$ vertices of valency $\geqslant 3$ in Γ_0. Let S, S' be Γ-orthogonal spheres with $\Gamma \cap S = \Gamma \cap S' = \Gamma_0$, and for each region r of Γ_0 in S let $U(\Gamma) - \bar{r}$ be arc-wise connected. Then there is a flex of M fixing $U(\Gamma)$ and a neighbourhood of $U(\Gamma) - U(\Gamma_0)$ pointwise and mapping S to S'.*

Proof. Let $r_1, \dots, r_k$ be the regions of Γ_0 in S. Since Γ_0 is cyclically 3-connected and has $\geqslant 3$ vertices of valency $\geqslant 3$, it follows that there are distinct circuits $C_1, \dots, C_k$ of Γ_0 bounding $r_1, \dots, r_k$ respectively, and $C_i \cap C_j$ is null or connected for $1 \leqslant i < j \leqslant k$, and each C_i has $\geqslant 3$ vertices of valency $\geqslant 3$ in Γ_0 (and hence in Γ). Since Γ_0 is cyclically 3-connected, it follows from Whitney's theorem [12] that Γ_0 has exactly k regions in S', $r'_1, \dots, r'_k$ say, such that r'_i is bounded by $U(C_i)$ $(1 \leqslant i \leqslant k)$. Then the result follows from (7.2). ∎

We shall need to prove that a number of tracings are even, and to do so we make frequent use of the following trivial lemma.

(7.4) *Let Γ, Γ' be frames in M, and let γ be a tracing of Γ onto Γ'. Let β be a homeomorphism of M, and define γ' by $\gamma'(x) = \beta^{-1}(\gamma(x))$ $(x \in M)$. Then γ' is a tracing from Γ to $\beta^{-1}(\Gamma')$. Moreover, γ' is either even or odd if and only if γ is either even or odd; and if β is a flex then γ' is even if and only if γ is even.*

Proof. Let α' be a homeomorphism of M extending γ', and define $\alpha(x) = \beta(\alpha'(x))$ $(x \in M)$. Then α extends γ. Moreover, if α', β are flexes then α is a flex. The converse is similar. ∎

Now we prove (1.6), which we restate.

(7.5) *If Γ is a non-planar panelled frame in M, then every homeomorphism of M fixing $U(\Gamma)$ pointwise is a flex.*

Proof. We may assume that no proper subgraph of Γ is non-planar; and hence, by Kuratowski's theorem, Γ is isomorphic to a subdivision of K_5 or $K_{3,3}$, and by suppressing 2-valent vertices, we may assume that Γ is isomorphic to K_5 or to $K_{3,3}$. Choose $e \in E(\Gamma)$, and let $\Gamma_0 = \Gamma \backslash e$. Then Γ_0 is cyclically 3-connected, and so there is a Γ-orthogonal sphere S with $\Gamma \cap S = \Gamma_0$, by (7.1). Moreover, Γ_0 has $\geqslant 3$ vertices of valency $\geqslant 3$, and for each region r of Γ_0 in S, $U(\Gamma) - \bar{r}$ is arc-wise connected.

Let α be a homeomorphism of M fixing $U(\Gamma)$ pointwise, and let $S' = \alpha(S)$. By (7.3) there is a flex β of M fixing $U(\Gamma)$ pointwise and mapping S to S'. Define $\alpha'(x) = \beta^{-1}(\alpha(x))$ $(x \in M)$. Then α' fixes $U(\Gamma)$ pointwise, and $\alpha'(S) = S$. Let B be the ball bounded by S with $e \subseteq B$. Since $\alpha'(e) = e$ it follows that $\alpha'(B) = B$. Let Ω be an orientation of S, and let r be a region of Γ_0 in S. Then $\bar{r}$ is a disc, and α' fixes $\mathrm{bd}(\bar{r})$ pointwise, and maps $\bar{r}$ to $\bar{r}$ (for no other region of Γ_0 in S has the same boundary as r). It follows that $\alpha'(\Omega) = \Omega$. But $\alpha'(B) = B$, and so α' is a flex, and therefore so is α, as required. ∎

We sketch another proof of (7.5) due to D. Vertigan (private communication). We assume that $\Gamma = K_{3,3}$; label the vertices of Γ as

$a_0, a_1, a_2, b_0, b_1, b_2$, and direct each edge from a_i to b_j. For a pair (e, e') of edges with ends a_ib_j and $a_{i'}b_{j'}$, define

$$f(e,e')=\begin{cases}0 & \text{if } i=i' \text{ or } j=j'\\ 1 & \text{if } i\neq i' \text{ and } j-i\equiv j'-i' \pmod 3\\ -1 & \text{if } i\neq i',\ j\neq j' \text{ and } j-i\not\equiv j'-i' \pmod 3.\end{cases}$$

Take a regular projection of Γ onto a sphere, and assign to each crossing point of two edges a sign $\pm$, depending whether the "upper" edge at the crossing passes over the lower from left to right or from right to left. Define $F(\Gamma)$ to be the sum, over all crossings, of its sign multiplied by $f(e, e')$, where e, e' are the two edges involved in the crossing. One can show (by Reidemeister moves) that $F(\Gamma)$ does not depend on which projection is chosen, and that if Γ' is the image of Γ under an orientation-reversing homeomorphism, then $F(\Gamma') \neq F(\Gamma)$. The result follows. A similar method (with a different function f) can be found for K_5.

(7.6) *Let Γ be a panelled frame in M, let $e \in E(\Gamma)$, and let $\Gamma_0 = \Gamma\backslash e$. Let Γ_0 be cyclically 3-connected and planar, with $\geqslant 3$ vertices of valency $\geqslant 3$. Let $C_1, ..., C_k$ be distinct circuits of Γ_0, bounding the regions of some drawing of Γ_0 in a sphere. Let C_0 be a circuit of Γ with $e \in E(C_0)$, such that $C_0 \cap C_i$ is null or connected for $1 \leqslant i \leqslant k$. Finally, let Γ be non-planar. Then Γ is rigid.*

Proof. Let γ be a tracing of Γ onto some panelled frame Γ' in M. Let $e' = \gamma(e)$, and $\Gamma'_0 = \gamma(\Gamma_0) = \Gamma'\backslash e'$. Choose a Γ-orthogonal sphere S with $\Gamma \cap S = \Gamma_0$, by (7.1), and choose S' similarly for Γ'. Now Γ_0 in S and Γ'_0 in S' are both drawings in spheres of the same cyclically 3-connected graph, and so by Whitney's theorem [12] there is a homeomorphism of S to S' extending $\gamma\backslash e$. Consequently, there is a flex of M extending $\gamma\backslash e$ and mapping S to S'. We wish to prove that γ is even or odd; and therefore, by (7.4), we may assume that $S = S'$, and $\gamma\backslash e$ is the identity on $U(\Gamma_0)$.

Since $C_0 \cap C_i$ is null or connected for $1 \leqslant i \leqslant k$, there is by (2.7) a Γ-panel Δ_0 for C_0 with $\Delta_0 \cap S = U(C_0 \cap \Gamma_0)$. Thus $\mathrm{bd}(\Delta_0) - S = e$; choose Δ'_0 similarly for Γ'. Let α be a homeomorphism of M fixing S pointwise, mapping Δ_0 to Δ'_0 and mapping x to $\gamma(x)$ for $x \in e$; then α extends γ, and so γ is even or odd. Consequently Γ is rigid, as required. ∎

A *Kuratowski graph* is a graph isomorphic to a subdivision of $K_{3,3}$ or of K_5.

(7.7) *Let Γ be a panelled frame which is a Kuratowski graph, and let γ be a tracing of Γ onto some panelled frame Γ'. Then γ is either even or odd and not both.*

Proof. By (7.6) Γ is rigid, and so γ is either even or odd. If it is both, let α, β be homeomorphisms of M extending γ, one orientation-preserving and the other orientation-reversing, and let $\alpha'(x) = \alpha^{-1}(\beta(x))$ $(x \in M)$. Then α' is an orientation-reversing homeomorphism of M fixing $U(\Gamma)$ pointwise, contrary to (7.5). ∎

We define V_7 to be the graph with seven vertices $a_1, a_2, a_3, b_1, b_2, b_3, c$ in which each a_i is adjacent to each b_j, and c is adjacent to a_1, a_2, b_1 and b_2. We define V_8 to be the graph obtained from an eight-vertex circuit with vertices $v_1, ..., v_8$ in order, by adding edges with ends $v_1v_5, v_2v_6, v_3v_7, v_4v_8$.

(7.8) *Every panelled frame isomorphic to V_7 or V_8 is rigid.*

Proof. In both there is an edge e as in (7.6). To see this, in V_7 as described above, let e have ends a_1b_2; and in V_8 let e have ends v_1v_2. The result follows from (7.6). ∎

We mention the following, which we shall not need.

(7.9) *If Γ is a panelled frame in M, and γ is a tracing of Γ onto a panelled frame Γ', then γ is both even and odd if and only if Γ is spherical. In particular, if Γ is planar then it is rigid.*

Proof. Suppose that Γ is planar. Then so is Γ', and by (3.1) and (3.3), Γ and Γ' are spherical. By Whitney's theorem [13] characterizing the different drawings of a planar graph in a sphere, it follows that γ is even (and hence Γ is rigid). But since Γ is spherical, there is an orientation-reversing homeomorphism of M fixing $U(\Gamma)$ pointwise, and so γ is also odd, as required.

On the other hand, if Γ is not planar, then from (7.5) (or (7.7)) γ is not both even and odd. ∎

(7.10) *If Γ, Γ' are panelled frames in M and γ is tracing of Γ onto Γ', and either*

(i) *$v \in V(\Gamma)$ has valency $\leqslant 1$ and $\gamma\backslash v$ is even, or*

(ii) *e is a loop or parallel edge of Γ, and $\gamma\backslash e$ is even, or*

(iii) *C is a circuit of Γ of length 3, and $v \in V(C)$ is 3-valent and has a neighbour not in $V(C)$, and $e \in E(C)$ is not incident with v, and $\gamma\backslash e$ is even*

then γ is even.

Proof. For (i), since $\gamma\backslash v$ is even, we may assume by (7.4) that $\gamma\backslash v$ is the identity on $U(\Gamma\backslash v)$; but then the claim is obvious. For (ii) and (iii), we may similarly assume that $\gamma\backslash e$ is the identity on $U(\Gamma\backslash e)$; but then the result follows from (4.2). ∎

(7.11) *If Γ is a panelled frame in M, and*

(i) *$v \in V(\Gamma)$ has valency $\leqslant 1$ and $\Gamma\backslash v$ is rigid, or*

(ii) *e is a loop or parallel edge of Γ, and $\Gamma\backslash e$ is rigid, or*

(iii) *C is a circuit of Γ of length 3, and $v \in V(C)$ is 3-valent and has a neighbour not in $V(C)$, and $e \in E(C)$ is not incident with v, and $\Gamma \backslash e$ is rigid*

then Γ is rigid.

The proof is like that of (7.1), and we omit it.

A *hexad* is a graph isomorphic to a subdivision of $K_{3,3}$, and a *pentad* is isomorphic to a subdivision of K_5. Let H_1, H_2 be subgraphs of a graph, both hexads or both pentads. We say that H_1, H_2 are 1-*adjacent* if for $i = 1, 2$ there is a path P_i of $H_1 \cup H_2$ such that

(i) P_i has distinct ends u_i, v_i say, and

(ii) (P_i, H_i) is a 2-separation of $H_1 \cup H_2$ with $V(P_i \cap H_i) = \{u_i, v_i\}$.

If H_1, H_2 are both pentads it follows that $\{u_1, v_1\} = \{u_2, v_2\}$.

(7.12) *Let Γ be a panelled frame in M, and let $H_1, H_2 \subseteq \Gamma$ be both hexads or both pentads, and* 1-*adjacent. Then $H_1 \cup H_2$ is rigid.*

Proof. We proceed by induction on $|V(\Gamma)| + |E(\Gamma)|$, and therefore may assume that $\Gamma = H_1 \cup H_2$. If Γ has a vertex v of valency 2 the result follows from the inductive hypothesis by suppressing v, and so we assume that Γ has minimum valency $\geqslant 3$. If e, f are parallel edges of Γ then exactly one of e, f is in H_1, and we may assume that $\Gamma \backslash e = H_1$; but H_1 is rigid by (7.7), and so Γ is rigid by (7.11)(ii). We assume then that Γ is simple. Consequently H_1, H_2 are both hexads, and there are distinct edges e_1, e_2 so that $H_i = \Gamma \backslash e_i$ $(i = 1, 2)$. Let the six 3-valent vertices of H_1 be $u_1, ..., u_6$, and let the nine paths of H_1 between pairs of these vertices with no internal vertex in $\{u_1, ..., u_6\}$ be L_{ij} $(1 \leqslant i \leqslant 3, 4 \leqslant j \leqslant 6)$, where each L_{ij} has ends u_i, u_j. Let e_1 have ends x, y. If $x = u_1$ and $y = u_4$ then Γ is not simple, and if $x = u_1$ and $y = u_2$ there is no $e_2 \neq e_1$ so that $\Gamma \backslash e_2$ is a hexad. Thus not both x, y are 3-valent in H_1, and so we may assume that $x \in V(L_{14}) - \{u_1, u_4\}$. Since Γ is simple and has minimum valency $\geqslant 3$, it follows that $y \notin V(L_{14})$, and we may therefore assume that $y \in V(L_{15}) - \{u_1\}$ or $y \in V(L_{25}) - \{u_2, u_5\}$. In either case

$$V(\Gamma) = \{u_1, u_2, u_3, u_4, u_5, u_6, x, y\}.$$

Now in the first case e_1 joins two neighbours of the 3-valent vertex u_1, and so Γ is rigid by (7.11)(iii), since $\Gamma \backslash e_1 = H_1$ is rigid. In the second case Γ is isomorphic to V_8, and the result follows (7.8). ▮

If H_1, H_2 are subgraphs of G, and H_1 is a hexad and H_2 is a pentad, we say that H_1, H_2 are 1-*adjacent* if there is a path P of $H_1 \cup H_2$ with distinct ends u, v so that (P, H_2) is a 2-separation of $H_1 \cup H_2$ with $V(P \cap H_2) = \{u, v\}$.

(7.13) *Let Γ be a panelled frame in M, and let $H_1 \subseteq \Gamma$ be a hexad and $H_2 \subseteq \Gamma$ a pentad, so that H_1, H_2 are* 1-*adjacent. Then $H_1 \cup H_2$ is rigid.*

Proof. We proceed by induction on $|V(\Gamma)| + |E(\Gamma)|$. As in (7.12), we may assume that $\Gamma = H_1 \cup H_2$, that Γ is simple, and has minimum valency $\geqslant 3$. Hence there exists $e \in E(\Gamma)$ such that $\Gamma \backslash e = H_2$. Let $u_1, ..., u_5$ be the five 4-valent vertices of H_2, and let the ten paths of H_2 between them with no internal vertices in $\{u_1, ..., u_5\}$ be L_{ij} $(1 \leqslant i < j \leqslant 5)$, where each L_{ij} has ends u_i, u_j. Since Γ is simple, not both $x, y \in \{u_1, ..., u_5\}$, and so we may assume that $x \in V(L_{12}) - \{u_1, u_2\}$. Then $y \notin V(L_{12})$, and we may assume that either $y \in V(L_{13}) - \{u_1\}$ or $y \in V(L_{34}) - \{u_3, u_4\}$. In the second case Γ is isomorphic to V_7 and the result follows from (7.8), and we therefore assume that $y \in V(L_{13}) - \{u_1\}$.

Since the circuit of Γ with vertex set $\{x, y, u_1\}$ is not a circuit of H_1, one of its edges, f say, does not belong to H_1. But $e \in E(H_1)$, since $e \notin E(H_2)$, and so $f \neq e$, and hence f has ends xu_1 or yu_1. If $y = u_3$ then f does not have ends xu_1, because no hexad remains after deleting the edge with ends xu_1; while if $y \neq u_3$ there is symmetry between x and y and so we may assume that f does not have ends xu_1. Thus in either case f has ends yu_1. There is a unique hexad in $\Gamma \backslash f$, and it is obtained by deleting from $\Gamma \backslash f$ the edges with ends u_2u_3 and u_4u_5; and hence this hexad is H_1. But H_1 is rigid, and it follows by three applications of (7.11)(iii) that Γ is rigid, as required. ▮

(7.14) *Let Γ, Γ' be panelled frames in M, and let $H_1, H_2 \subseteq \Gamma$ be* 1-*adjacent Kuratowski graphs. Let γ be a tracing of Γ onto Γ'. Then the restriction of γ to H_1 is even if and only if the restriction if γ to H_2 is even.*

Proof. Let γ_i be the restriction of γ to H_i $(i = 1, 2)$. Since $H_1 \cup H_2$ is rigid by (7.12) and (7.13) there is a homeomorphism of M extending the restriction of γ to $H_1 \cup H_2$. If it is a flex then γ_1 and γ_2 are both even, and otherwise they are both odd, and hence not even by (7.7). ▮

We say hexads H_1, H_2 in G are 2-*adjacent* if there are seven vertices $u_1, ..., u_7$ of G, and thirteen paths L_{ij} of G $(1 \leqslant i \leqslant 4$ and $5 \leqslant j \leqslant 7$, or $(i, j) = (3, 4))$, such that

(i) each path L_{ij} has ends u_i, u_j

(ii) the paths L_{ij} are mutually vertex-disjoint except for their ends, and

(iii) $H_1 = \bigcup (L_{ij}: i = 2, 3, 4$ and $j = 5, 6, 7)$, and $H_2 = \bigcup (L_{ij}: i = 1, 3, 4$, and $j = 5, 6, 7)$.

Note that L_{34} is not used in either hexad.

We need an analogue of (7.14) for 2-adjacent hexads, but unfortunately the analogue of (7.12) and (7.13) is false, and we use a different method.

(7.15) *Let Γ, Γ' be panelled frames in M, and let $H_1, H_2 \subseteq \Gamma$ be 2-adjacent hexads. Let γ be a tracing of Γ onto Γ'. Then the restriction of γ to H_1 is even if and only if the restriction of γ to H_2 is even.*

Proof. Let $u_1, ..., u_7$ and the paths L_{ij} be as in the definition "2-adjacent." We may assume that Γ is the union of all the paths L_{ij}, and has minimum valency $\geqslant 3$. Each L_{ij} therefore has exactly one edge, e_{ij} say, and $H_1 = \Gamma \backslash u_1$ and $H_2 = \Gamma \backslash u_2$. Since u_1 is 3-valent in Γ, there is by (4.1)(iii) a panelled frame Γ_1 with an edge e_{56} with ends $u_5 u_6$, so that $\Gamma_1 \backslash e_{56} = \Gamma$; and similarly there is a panelled frame Γ'_1 with an edge e'_{56} with ends $\gamma(u_5), \gamma(u_6)$, so that $\Gamma' \backslash e'_{56} = \Gamma'$. Let γ' be a tracing from Γ_1 to Γ'_1 extending γ.

Let H_3 be the pentad $\Gamma_1 \backslash \{e_{16}, e_{27}\}$. Then H_1, H_3 are 1-adjacent in Γ_1 because $H_1 \backslash e_{25} \subseteq H_3$. Similarly H_2, H_3 are 1-adjacent. For $i = 1, 2$, the restriction of γ' to H_i is even if and only if the restriction of γ' to H_3 is even, by (7.14). The result follows. ∎

Let H, J be Kuratowski graphs in a graph G. We say that H, J *communicate* in G if there is a sequence

$$H = H_1, H_2, ..., H_k = J$$

of Kuratowski graphs in G such that for $1 \leqslant i < k$, H_i and H_{i+1} are 1- or 2-adjacent. (2-adjacency only applies to pairs of hexads.)

(7.16) *Let Γ, Γ' be panelled frames in M, and let H, J be Kuratowski graphs in Γ that communicate. Let γ be a tracing of Γ onto Γ'. Then the restriction of γ to H is even if and only if the restriction of γ to J is even.*

Proof. Let $H = H_1, ..., H_k = J$ be a sequence of Kuratowski graphs as above. By (7.14) and (7.15), for $1 \leqslant i < k$ the restriction of γ to H_i is even if and only if the restriction of γ to H_{i+1} is even. The result follows. ∎

The following was proved in [5].

(7.17) *If G is Kuratowski connected, then every pair of Kuratowski graphs in G communicate.*

(Actually, the converse of (7.17) is also true.)
We deduce:

(7.18) *Let Γ be a Kuratowski connected panelled frame in M, and let γ be a tracing of Γ onto a panelled frame Γ'. If $H_1, H_2 \subseteq \Gamma$ are Kuratowski graphs, then the restriction of γ to H_1 is even if and only if the restriction of γ to H_2 is even.*

8. Fixing a Ball

The main result of this section is the following.

(8.1) *Let Γ be a panelled frame in M, and $S \subseteq M$ be a Γ-orthogonal sphere bounding a ball B. Let (Γ_1, Γ_2) be a separation of Γ such that*

(i) *$\Gamma_2 = \Gamma \cap B$ and $(\Gamma_1) \cap B \subseteq \mathrm{bd}(B)$*

(ii) *$V(\Gamma_1 \cap \Gamma_2)$ is 2-coherent in Γ_2, and $|V(\Gamma_1 \cap \Gamma_2)| \geqslant 3$, and*

(iii) *Γ_1 is connected, and $U(\Gamma_1) - V(\Gamma_1 \cap \Gamma_2)$ is non-null and arc-wise connected.*

Let α be a flex of M fixing $U(\Gamma)$ pointwise. Then there is a flex β of M fixing B pointwise, such that $\beta(x) = \alpha(x)$ for all x in a neighbourhood of $U(\Gamma) - B$.

To prove (8.1) we need several lemmas. The first, below is standard result—see [8].

(8.2) *Let $\Delta \subseteq M$ be a disc, and let α be a flex of Δ to itself fixing* $\mathrm{bd}(\Delta)$ *pointwise. Let $B \subseteq M$ be a ball with $\Delta \subseteq B$ and $\Delta \cap \mathrm{bd}(B) = \mathrm{bd}(\Delta)$. Then there is a flex of M fixing* $\mathrm{bd}(B)$ *pointwise, mapping B to B and mapping x to $\alpha(x)$ for each $x \in \Delta$.*

Next we need the following.

(8.3) *Let C be a circuit of a frame Γ in M, and let Δ be a Γ-panel for C. Let B_1, B_2 be balls in M with $\Delta \subseteq \mathrm{bd}(B_i)$ $(i = 1, 2)$, on the same side of Δ, and let $U(\Gamma) \cap B_i = U(C)$ $(i = 1, 2)$. Then there is a flex of M fixing $U(\Gamma)$ and a neighbourhood of $U(\Gamma) - U(C)$ pointwise and mapping Δ to Δ and B_1 to B_2.*

Proof. Since B_1, B_2 are on the same side of Δ, there is a ball $B_3 \subseteq B_1 \cap B_2$ with $\Delta \subseteq \mathrm{bd}(B_3)$. The result is clearly true for the pairs B_1, B_3 and B_2, B_3, and hence it is true for B_1, B_2. ∎

We use (8.2) and (8.3) to prove the following.

(8.4) *Let Γ be a panelled frame in M. Let $\Gamma_0 \subseteq \Gamma$ be simple and 2-connected, with $|V(\Gamma_0)| \geqslant 3$, and let every vertex of Γ_0 be incident with an edge of Γ not in $E(\Gamma_0)$. For $i = 1, 2$, let $B_i \subseteq M$ be a Γ-orthogonal ball with $B_i \cap \Gamma = \Gamma_0$ and $U(\Gamma_0) \subseteq \mathrm{bd}(B_i)$. For each region r of Γ_0 in* $\mathrm{bd}(B_1)$, *let $U(\Gamma) - \bar{r}$ be arc-wise connected. Let α be a homeomorphism of B_1 to B_2 fixing $U(\Gamma_0)$ pointwise, orientation-preserving if Γ_0 is a circuit. Then there is a flex of M extending α and fixing pointwise a neighbourhood of $U(\Gamma) - B_1$.*

Proof. Let the regions of Γ_0 in $\mathrm{bd}(B_1)$ be $r_1, ..., r_k$. Since $|V(\Gamma_0)| \geqslant 3$ and Γ_0 is simple and 2-connected, each r_i is bounded by a circuit C_i of Γ_0. Moreover, each C_i has $\geqslant 3$ vertices with valency $\geqslant 3$ in Γ, and each $U(\Gamma) - U(C_i)$ is arc-wise connected.

(1) *There is a flex β of M fixing pointwise $U(\Gamma)$ and a neighbourhood of $U(\Gamma) - B_1$, and mapping r_i to $\alpha(r_i)$ for $1 \geqslant i \geqslant k$.*

To see this we assume first that Γ_0 is a circuit. It follows from (2.12) that there is a flex β' of M fixing pointwise $U(\Gamma)$ and a neighbourhood of $U(\Gamma) - B_1$, and mapping r_1 to $\alpha(r_1)$. Now $\alpha(B_1) = B_2$ and $\beta'(B_1)$ are on the same side of $\bar{r}_1$, because α and β' are both flexes fixing $\mathrm{bd}(\bar{r}_1)$ pointwise and both map r_1 to $\alpha(r_1)$. Hence by (8.3) there is a flex α' of M fixing $U(\Gamma)$ and a neighbourhood of $U(\Gamma) - B_2$ pointwise and mapping $\beta'(B_1)$ to $\alpha(B_1)$ and $\beta'(r_2)$ to $\alpha(r_2)$. Define $\beta(x) = \alpha'(\beta'(x))$ $(x \in M)$; then β satisfies (1).

Now we assume that Γ_0 is not a circuit. Hence $C_1, ..., C_k$ are all distinct; and so the claim follows from (2.12). This proves (1).

(2) *There is a flex β' of M to itself fixing pointwise $U(\Gamma)$ and a neighbourhood of $V(\Gamma) - B_1$, and mapping x to $\alpha(x)$ for all $x \in \mathrm{bd}(B_1)$.*

For let β be as in (1). For $1 \leqslant i \leqslant k$, let D_i be a ball with $\bar{r}_i \subseteq D_i$ and $\bar{r}_i \cap \mathrm{bd}(D_i) = U(C_i)$, chosen so small that $D_i \cap U(\Gamma) = U(C_i)$ $(1 \leqslant i \leqslant k)$ and $D_i \cap D_j = \bar{r}_i \cap \bar{r}_j$ $(1 \leqslant i < j \leqslant k)$. By (8.2) for $1 \leqslant i \leqslant k$ there is a flex δ_i of D_i to itself fixing $\mathrm{bd}(D_i)$ pointwise, with $\delta_i(x) = \beta^{-1}(\alpha(x))$ for $x \in \bar{r}_i$. Define $\beta'(x) = \beta(x)$ if $x \in M - D_1 \cup \cdots \cup D_k$, and $\beta'(x) = \beta(\delta_i(x))$ if $x \in D_i$; then β' satisfies (2).

(3) *β' maps B_1 to B_2.*

For if not, β' maps B_1 to the other ball in M bounded by $\mathrm{bd}(B_2)$. In this case, let $\alpha'(x) = \beta'^{-1}(\alpha(x))$ $(x \in B_1)$; then α' is a flex of B_1 to $\overline{M - B_1}$ fixing $\mathrm{bd}(B_1)$ pointwise, which is impossible.

Let β' be as in (2), and define α' by $\alpha'(x) = \beta'(x)$ $(x \notin B_1)$ and $\alpha'(x) = \alpha(x)$ $(x \in B_1)$. By (3), and since $\beta'(x) = \alpha(x)$ for $x \in \mathrm{bd}(B_1)$, α' is a flex of M satisfying the theorem. ∎

Next we need the following lemma.

(8.5) *Let G be a graph, let $X \subseteq V(G)$ be 2-coherent, and let $v \in V(G) - X$. Then either v has valency 0, or there is an edge e incident with v such that X is 2-coherent in one of $G \backslash e$, G/e.*

(Here we are identifying X with a subset of $V(G/e)$ in the natural way.)

Proof.

(1) *If there is a separation (A, B) of G with $V(A \cap B) = \{v\}$ and with $X \cap V(A)$, $X \cap V(B) \neq \emptyset$, the result is true.*

For choose $a \in X \cap V(A)$, $b \in X \cap V(B)$. Since X is 2-coherent in G we may assume that $X \cap V(A) = \{a\}$. Now $a, b \neq v$ since $v \notin X$, and since X is 2-coherent in G, there is a path P between a and b. Since $V(A \cap B) = \{v\}$, there is an edge $e \in E(P \cap A)$ incident with v. Since $V(A) \cap X = \{a\}$ it follows easily that X is 2-coherent in G/e. This proves (1).

Now let e be any edge of G incident with v (we may assume that there is one). We assume that X is not 2-coherent in either $G \backslash e$ or G/e. Consequently there is a $(\leqslant 1)$-separation (A, B) of $G \backslash e$ with

$$|X \cap V(A)|, |X \cap V(B)| > |V(A \cap B)|$$

and with $v \in V(A) - V(B)$ and $u \in V(B) - V(A)$, where u is the other end of e; and there is a pair (A', B') of subgraphs of G with $A' \cup B' = G$, $E(A' \cap B') = \{e\}$, and $V(A' \cap B') = \{u, v\}$, such that $|X \cap V(A')|$, $|X \cap V(B')| \geqslant 2$. In particular, $A \cap B \cap A' \cap B'$ is null, and since $|V(A \cap B)| \leqslant 1$ we may assume that $A \cap B \cap A'$ is null. Hence $(A \cap A', B \cup B')$ is a 1-separation of G, and since

$$|X \cap V(B \cup B')| \geqslant |X \cap V(B')| \geqslant 2$$

it follows that $|X \cap V(A \cap A')| \leqslant 1$. Similarly, $|X \cap V(B \cap A')| \leqslant 1$. But $|X \cap V(A')| \geqslant 2$, and so

$$|X \cap V(A \cap A')| = |X \cap V(B \cap A')| = 1.$$

Then $(A \cap A', B \cup B')$ is a 1-separation of G as in (1), and the result follows. ∎

Proof of (8.1). We proceed by induction on $|V(\Gamma_2)| + 2|E(\Gamma_2)| - |E(\Gamma_2 \cap S)|$. Let $X = V(\Gamma_1 \cap \Gamma_2)$. We may assume that

(1) *$U(\Gamma_1) \nsubseteq B$, and X is not 2-coherent in $\Gamma_2 \backslash v$ for all $v \in V(\Gamma_2) - S$.*

For if $U(\Gamma_1) \subseteq B$, we may take β to be the identity. If X is 2-coherent in $\Gamma_2 \backslash v$ for some v, the result follows from the inductive hypothesis applied to $\Gamma \backslash v$.

Similarly, we have

(2) *For all $e \in E(\Gamma_2)$, X is not 2-coherent in $\Gamma_2 \backslash e$; and for all $e \in E(\Gamma_2)$ with distinct ends, and with at least one end not in X, X is not 2-coherent in Γ_2/e.*

From (1), (2) and (8.5), we deduce

(3) *Γ_2 is simple and 2-connected, and $X = V(\Gamma_2)$.*

We may assume that

(4) $U(\Gamma_2) \subseteq S$.

For suppose that $e \in E(\Gamma_2)$ and $e \not\subseteq S$. Let e have ends u, v. By (3), $u, v \in X \subseteq S$. Since $u, v \in V(\Gamma_1)$ and Γ_1 is connected, and $x \neq y$ by (3), it follows that u, v are both incident with edges of Γ_1. Since $U(\Gamma_1) - X$ is arc-wise connected, there is a path P of Γ_1 between u and v with $U(P) \cap S = \{u, v\}$. Let C be the circuit of Γ with $E(C) = E(P) \cup \{e\}$. Let Δ be a Γ-panel for C; by perturbing Δ slightly, we may assume that $S \cap (\Delta - \mathrm{bd}(\Delta))$ has only finitely many arc-wise connected components $L_1, ..., L_k$ say, and for each i, either $\bar{L}_i$ is a circle with $|\bar{L}_i \cap \mathrm{bd}(\Delta)| \leqslant 1$, or $\bar{L}_i$ is a line with ends u, v and with $\bar{L}_i \cap \mathrm{bd}(\Delta) = \{u, v\}$. (Recall that $\mathrm{bd}(\Delta) \cap S = U(C) \cap S = \{u, v\}$.)

If each $\bar{L}_i$ is a circle, there is a line L in Δ with one end in $U(P) - \{x, y\}$ and the other in e, disjoint from $S \cap \Delta$; but then $L \cap S = \varnothing$, and L has one end in B and the other in $M - B$, which is impossible. Thus some $\bar{L}_i$ is a line. Let $e' = L_i$, and let $\Gamma' = ((U(\Gamma) - e) \cup e', V(\Gamma))$. Then Γ' is a frame.

Since $e' \subseteq \Delta$ and has the same ends as e, there is a flex δ of M fixing pointwise $U(\Gamma) - e$ and a neighbourhood of $U(\Gamma) - B$ and mapping e to e'. Then $\Gamma' = \delta(\Gamma)$, and so Γ' is panelled. Since $e' \subseteq S$ it follows that S is Γ'-orthogonal.

Define $\alpha'(x) = \delta(\alpha(\delta^{-1}(x)))$ $(x \in M)$; then α' is a flex of M, fixing pointwise $U(\Gamma) - e$ and a neighbourhood of $U(\Gamma) - B$, and fixing e' pointwise. Thus α' fixes $U(\Gamma')$ pointwise, and $\alpha'(x) = \alpha(x)$ for all x in a neighbourhood of $U(\Gamma) - B$. From the inductive hypothesis, there is a flex β of M fixing B pointwise, such that $\beta(x) = \alpha'(x) = \alpha(x)$ for all x in a neighbourhood of $U(\Gamma) - B$. But then β satisfies the theorem. This proves (4).

(5) *For each region r of Γ_2 in S, $U(\Gamma) - \bar{r}$ is arc-wise connected.*

For let Q be the arc-wise connected component of $U(\Gamma) - \bar{r}$ including $U(\Gamma_1) - B$, and suppose that Q' is another arc-wise connected component of $U(\Gamma) - \bar{r}$. For each $v \in V(\Gamma_2)$ with $v \notin \bar{r}$, there is an edge e of Γ_1 incident with v, and so $v \in Q$. Hence $Q' \cap V(\Gamma) = \varnothing$, and so $Q' = e$ for some edge e of Γ_2 with both ends in $\bar{r}$. Since $e \not\subseteq \bar{r}$, there is a circuit C of Γ_2 with $e \notin E(C)$ but with both ends of e in $V(C)$. It follows that X is 2-coherent in $\Gamma_2 \backslash e$, contrary to (2). This proves (5).

From (4), (5) and (8.4), there is a flex α' of M such that $\alpha'(x) = \alpha(x)$ for all $x \in B$, and α' fixes pointwise a neighbourhood of $U(\Gamma) - B$. Define $\beta(x) = \alpha(\alpha'^{-1}(x))$ $(x \in M)$; then β satisfies the theorem. ∎

9. Equators

Let Γ be a frame in M, and let $v \in V(\Gamma)$. A *slice* at v is a disc $\Delta \subseteq M$ with $v \in \Delta - \mathrm{bd}(\Delta)$ and with $U(\Gamma) \cap \Delta = \{v\}$. If Δ is a slice at v, and $\Delta' \subseteq \Delta$ is a disc with $v \in \Delta' - \mathrm{bd}(\Delta')$, then Δ' is also a slice at v, called a *reduction* of Δ. We say two slices Δ_1, Δ_2 at v are *equivalent* if for every neighbourhood N of v, there are reductions Δ_1' of Δ_1 and Δ_2' of Δ_2, with $\Delta_1', \Delta_2' \subseteq N$, and a homeomorphism of M to itself fixing $U(\Gamma)$ and $M - N$ pointwise and mapping Δ_1' to Δ_2'. This is an equivalence relation; we call the equivalence classes the *equators* (of Γ at v).

Let Γ be a frame in M, and let $e \in E(\Gamma)$ with distinct ends. Let $\Delta \subseteq M$ be a disc with $\Delta \cap U(\Gamma) = \{x\}$ say, where $x \in e \cap (\Delta - \mathrm{bd}(\Delta))$ (and in particular, x is not an end of e), such that e passes through Δ at x, in the natural sense. Let m be the new element of M/e. Then $(\Delta - \{x\}) \cup \{m\}$ is a slice at m in M/e, and defines an equator Π of Γ/e at m called the *equator normal to* e in M/e. It is easy to verify that Π does not depend on the choice of Δ. We omit the proof of the following, which is a straightforward application of "regular neighbourhood" methods, and is well-known.

(9.1) *Let Γ_1, Γ_2 be frames in M, and let γ be a tracing of Γ_1 onto Γ_2. Let $e_1 \in E(\Gamma_1)$ with distinct ends, and let $e_2 = \gamma(e_1)$. For $i = 1, 2$, let Π_i be the equator of Γ_i/e_i normal to e_i in M/e_i. Then γ is even if and only if there is a flex of M/e_1 to M/e_2 extending γ/e_1 and mapping Π_1 to Π_2.*

We need the following.

(9.2) *Let Γ be a panelled frame in M, and let (Γ_1, Γ_2) be a separation of Γ such that $U(\Gamma) - U(\Gamma_2)$ is arc-wise connected. Then there is a ball B such that $\mathrm{bd}(B) \cap U(\Gamma) = V(\Gamma_1 \cap \Gamma_2)$ and $\Gamma \cap B = \Gamma_1$.*

Proof. We proceed by induction on $|E(\Gamma_1)|$. If $|E(\Gamma_1)| \leqslant 1$ the result is clear, and we assume that $|E(\Gamma_1)| \geqslant 2$. If there is an edge $e \in E(\Gamma_1)$ with distinct ends both in $V(\Gamma) - V(\Gamma_2)$, the result follows from the inductive hypothesis applied to Γ/e. If $e \in E(\Gamma_1)$ is a loop or a parallel edge of Γ_1, it must have an end not in $V(\Gamma_2)$; and from the inductive hypothesis, there is a ball B such that $\mathrm{bd}(B) \cap U(\Gamma \backslash e) = V(\Gamma_1 \cap \Gamma_2)$ and $(\Gamma \backslash e) \cap B = \Gamma_1$. By (4.2) there is a flex α of M fixing $U(\Gamma \backslash e)$ pointwise and mapping e into B; and then $\alpha^{-1}(B)$ satisfies the theorem. Finally, if none of these things happen, then Γ_1 is a tree, and every vertex in $V(\Gamma_1 \cap \Gamma_2)$ has valency $\leqslant 1$ in Γ_1, and then the result is clear. ∎

From (8.1) and (9.2) we have the following.

(9.3) *Let Γ be a panelled frame in M, let $v \in V(\Gamma)$, and let Π_1, Π_2 be equators at v, such that there is no flex of M fixing $U(\Gamma)$ pointwise and mapping Π_1 to Π_2. Let X be the set of neighbours of v in $V(\Gamma) - \{v\}$, and let $|X| \geqslant 3$.*

(i) *If $u \in V(\Gamma) - (X \cup \{v\})$ and X is 2-coherent in $\Gamma \backslash \{u, v\}$, then there is no flex of M fixing $U(\Gamma \backslash u)$ pointwise and mapping Π_1 to Π_2.*

(ii) *If $e \in E(\Gamma \backslash v)$ and X is 2-coherent in $(\Gamma \backslash v) \backslash e$, then there is no flex of M fixing $U(\Gamma \backslash e)$ pointwise and mapping Π_1 to Π_2.*

(iii) *If $e \in E(\Gamma \backslash v)$, and has distinct ends, and at least one of them is not in X, and X is 2-coherent in $(\Gamma \backslash v)/e$, then there is no flex of M/e fixing $U(\Gamma/e)$ pointwise and mapping Π_1 to Π_2.*

Proof. Let Γ_1 be the subgraph of Γ consisting of $X \cup \{v\}$ and the edges incident with v, and let (Γ_1, Γ_2) be the separation with $V(\Gamma_1 \cap \Gamma_2) = X$. By (9.2), since $U(\Gamma) - U(\Gamma_2)$ is arc-wise connected, there is a sphere S with $S \cap U(\Gamma) = X$ bounding a ball B with $\Gamma \cap B = \Gamma_2$.

To prove (i), let u be as in (i), and suppose that α is a flex of M fixing $U(\Gamma \backslash u)$ pointwise and mapping Π_1 to Π_2. By (8.1), there is a flex β of M fixing B pointwise, such that $\beta(x) = \alpha(x)$ in a neighbourhood of $U(\Gamma) - B$. Consequently, β fixes $U(\Gamma)$ pointwise and maps Π_1 to Π_2, contrary to the hypothesis. This proves (i), and (ii) and (iii) follow similarly. ∎

We recall that $V(f)$ and $N(f)$ were defined in Section 1. Now we prove (1.9), which is case (i) of the following. A tracing is *uneven* if it is not even (we recall that this is different from "odd").

(9.4) *Let Γ, Γ' be frames in M, and let γ be a tracing of Γ onto Γ'. Let $f \in E(\Gamma)$ with distinct ends, such that Γ/f is panelled. Suppose that γ/f is even, and γ is uneven. Let $|N(f)| \geqslant 3$, and let $N(f)$ be 2-coherent in $\Gamma \backslash V(f)$.*

(i) *If $u \in V(\Gamma) - V(f) \cup N(f)$, and $N(f)$ is 2-coherent in $\Gamma \backslash (V(f) \cup \{u\})$, then $\gamma \backslash u$ is uneven*

(ii) *if $e \in E(\Gamma \backslash V(f))$, and $N(f)$ is 2-coherent in $\Gamma \backslash V(f) \backslash e$, then $\gamma \backslash e$ is uneven*

(iii) if *$e \in E(\Gamma \backslash V(f))$ has distinct ends, not both in $N(f)$, and $N(f)$ is 2-coherent in $(\Gamma \backslash V(f))/e$, then γ/e is uneven.*

Proof. Let m be the new element of M/f, and let Π_1 be the equator of Γ/f normal to f in M/f. Let α be a flex of M/f to $M/\gamma(f)$ extending γ/f. let Π_0 be the equator of $\Gamma'/\gamma(f)$ normal to $\gamma(f)$ in $M/\gamma(f)$, and let Π_2 be the equator of Γ/f at m such that $\alpha(\Pi_2) = \Pi_0$.

(1) *There is no flex of M/f fixing $U(\Gamma/f)$ pointwise and mapping Π_1 to Π_2.*

For if β is such a flex, define $\alpha'(x) = \alpha(\beta(x))$ $(x \in M/f)$; then α' extends γ/f (because α does) and $\alpha'(\Pi_1) = \alpha(\Pi_2) = \Pi_0$. From (9.1) we deduce that γ is even, contrary to the hypothesis. This proves (1).

To prove (i), let u be as in (i). Then $u \in V(\Gamma/f) - (N(f) \cup \{m\})$, and so by (9.3)(i) we deduce that there is no flex of M/f fixing $U(\Gamma/f \backslash u)$ pointwise and mapping Π_1 to Π_2. If β is a flex of M/f to $M/\gamma(f)$ extending $(\gamma/f) \backslash u$ and mapping Π_1 to Π_0, define $\alpha'(x) = \alpha^{-1}(\beta(x))$ $(x \in M)$; then α fixes $U(\Gamma/f \backslash u)$ pointwise and maps Π_1 to Π_2, a contradiction. Thus there is no such β. From (9.1), $\gamma \backslash u$ is uneven. This proves (i), and (ii) and (iii) follow similarly. ∎

10. Uneven Tracings

In this section we prove (1.7), thereby completing the proof of (1.2). We begin with the following.

(10.1) *Let Γ, Γ' be panelled frames in M, and let γ be an uneven tracing of Γ to Γ'. Then either*

(i) *Γ has a vertex of valency 0, or*

(ii) *there exists $e \in E(\Gamma)$ such that $\gamma \backslash e$ is uneven, or*

(iii) *there exists $e \in E(\Gamma)$ with distinct ends such that γ/e is uneven, or*

(iv) *Γ is isomorphic to $K_{3,3}$ or to K_5.*

For this we need two lemmas.

(10.2) *Let Γ, Γ' be panelled frames in M, and let γ be an uneven tracing of Γ to Γ'. Let Γ be isomorphic to $K_{3,n}$ where $n \geqslant 4$. Then there exists $e \in E(\Gamma)$ such that $\gamma \backslash e$ is uneven.*

Proof. Let $V(\Gamma) = \{a_1, a_2, a_3, b_1, ..., b_n\}$ where each a_i is adjacent to each b_j, and let e_{ij} be the edge with ends $a_i b_j$. let Δ_1^* be a Γ-orthogonal disc such that $a_1, a_2, a_3 \in \mathrm{bd}(\Delta_1^*)$, and $\Gamma \cap \Delta_1^*$ consists of a_1, a_2, a_3, b_1 and e_{11}, e_{21}, e_{31}. Let $\mathrm{bd}(\Delta_1^*) = F$. By three applications of (4.1)(iii), $\Gamma^+ = (U(\Gamma) \cup F, V(\Gamma))$ is panelled. By (2.8) applied to all the length 3 circuits of Γ^+, there are Γ-orthogonal disc $\Delta_1, ..., \Delta_k$, each with $\mathrm{bd}(\Delta_i) = F$ and otherwise mutually disjoint, such that for $1 \leqslant i \leqslant k$, $\Gamma \cap \Delta_i$ consists of a_1, a_2, a_3, b_i, and e_{1i}, e_{2i}, e_{3i}.

Similarly, let $V(\Gamma') = \{a_1', a_2', a_3', b_1', ..., b_n'\}$, numbered so that $\gamma(a_i) = a_i'$ $(1 \leqslant i \leqslant 3)$ and $\gamma(b_j) = b_j'$ $(1 \leqslant j \leqslant n)$, and choose discs $\Delta_1', ..., \Delta_n'$ similarly for Γ'.

The vertices a_1, a_2, a_3 (in order) determine an orientation of each Δ_i, and since M is also oriented, we may speak of the "positive" and "negative" side of each Δ_i. For each i $(1 \leqslant i \leqslant n)$ there exists a unique $\pi(i)$ $(1 \leqslant \pi(i) \leqslant n)$ such that $\Delta_i \cup \Delta_{\pi(i)}$ bounds a ball $B \subseteq M$ on the positive side of Δ_i with

$$B \cap (\Delta_1 \cup \cdots \cup \Delta_n) = \Delta_i \cup \Delta_{\pi(i)}.$$

Thus π is a cyclic permutation of $1, ..., n$. Define π' similarly, from Γ'. Since γ is not even, it follows that $\pi \neq \pi'$. For $X \subseteq \{1, ..., n\}$, let $\pi | X$ denote the induced cyclic permutation of X (that is, for $x \in X$, $(\pi | X)(x) = \pi^i(x)$ where $i \geqslant 1$ is minimum with $\pi^i(x) \in X$). It follows easily that there exists $X \subseteq \{1, ..., n\}$ with $|X| = 3$ such that $\pi | X \neq \pi' | X$; say $X = \{i, j, k\}$. But then the restriction of γ to $\Gamma \cap (\Delta_i \cup \Delta_j \cup \Delta_k)$ is odd (because it is easy to construct the required homeomorphism) and hence uneven by (7.7). In particular, $\gamma \backslash e$ is uneven for any edge e of Γ with $e \nsubseteq \Delta_i \cup \Delta_j \cup \Delta_k$, and there is such an edge since $n \geqslant 4$. The result follows. ∎

The second lemma we need is purely graph-theoretic. We say G is *complete multipartite* if it is simple and there is a partition $V_1, ..., V_k$ of $V(G)$ such that distinct vertices of G are non-adjacent if and only if they belong to the same V_i. Thus, K_5 and $K_{3,3}$ are complete multipartite. It is an easy exercise (look at the complementary graph) to show that a simple graph is complete multipartite if and only if $N(e) \cup V(e) = V(G)$ for every edge e. For our second lemma we use the following variation.

(10.3) *Let G be a simple graph with minimum valency $\geqslant 3$, such that no 3-valent vertex has two adjacent neighbours. Suppose that for each $e \in E(G)$, either*

(i) *$N(e) \cup V(e) = V(G)$, or*

(ii) *$N(e)$ is not 2-coherent in $G \backslash V(e)$.*

Then G is complete multipartite.

Proof. We proceed by induction on $|V(G)|$. if G is not connected, let C be a component of G. Then for all $e \in E(C)$, $N(e) \cup V(e) \neq V(G)$, and so $N(e)$ is not 2-coherent in $G \backslash V(e)$. Consequently, $N(e)$ is not 2-coherent in $C \backslash V(e)$, since $V(e) \subseteq V(C)$. Thus C satisfies the hypotheses of the theorem, and so from the inductive hypothesis, C is complete multipartite. In particular, $N(e)$ is 2-coherent in $G \backslash V(e)$ for all $e \in E(C)$ (since C has minimum valency $\geqslant 3$) and so $E(C) = \varnothing$, which is impossible.

Hence G is connected. We say $X \subseteq V(G)$ is *connected* if the subgraph of G induced on X is connected, and we define $N(X)$ to be the set of all vertices of G not in X but with a neighbour in X. We say $X \subseteq V(G)$ is *bad* if $X \neq \varnothing$, X is connected, $X \cup N(X) \neq V(G)$, $|N(X)| \leqslant 3$ and if $|N(X)| = 3$ some two vertices in $N(X)$ are adjacent.

We suppose that there is a bad set, and choose a bad set X with X maximal.

(1) $|N(X)| = 1$.

For let $v \in V(G) - X \cup N(X)$. If every neighbour of v is in X then v is 3-valent and has two adjacent neighbours since X is bad, a contradiction. Thus there is an edge e with ends $u, v \in V(G) - X \cup N(X)$. Now $X \cap N(e) = \varnothing$ and so $V(e) \cup N(e) \neq V(G)$, and hence from the hypothesis, $N(e)$ is not 2-coherent in $G \backslash V(e)$. Therefore there is a $(\leqslant 3)$-separation (A, B) of G with $V(e) \subseteq V(A \cap B)$, such that

$$|N(e) \cap V(A)|, |N(e) \cap V(B)| > |V(A \cap B)| - 2.$$

Suppose first that $X \cap V(A \cap B) \neq \varnothing$. Then $V(A \cap B) = \{u, v, x\}$ say where $x \in X$, and $|N(e) \cap V(A)|, |N(e) \cap V(B)| \geqslant 2$. Also, $V(A \cap B) \cap N(X) = \varnothing$, and since $1 \leqslant |N(X)| \leqslant 3$ we may assume that $|N(X) \cap V(A)| \leqslant 1$ and $|N(X) \cap V(B)| \geqslant 1$. Let

$$W = V(e) \cup (N(X) \cap V(A)).$$

Then $|W| \leqslant 3$, and two members of W are adjacent, and $W \cap X = \varnothing$. Let X' be the vertex set of the component of $G \backslash W$ including X. Then $N(X) \cap V(B') \subseteq X'$, and so $|X'| > |X|$, and therefore X' is not bad, from the maximality of X. Consequently, $X' \cup N(X') = V(G)$ and so $V(A) \subseteq V(e) \cup X \cup N(X)$. But $|N(e) \cap V(A)| \geqslant 2$, and $|N(X) \cap V(A)| \leqslant 1$, and $N(e) \cap V(e) = \varnothing$, and $N(e) \cap X = \varnothing$, which is impossible.

Thus $X \cap V(A \cap B) = \varnothing$. Since X is connected, one of $X \cap V(A)$, $X \cap V(B)$ is null, say $X \cap V(A) = \varnothing$. Let X' be the vertex set of the component of $G \backslash V(A \cap B)$ including X. Since $X' \cup N(X') \subseteq V(B) \neq V(G)$ and so X' is bad, it follows from the maximality of X that $X' = X$, that is, $N(X) \subseteq V(A \cap B)$. Hence $|N(X)| = 1$, since $u, v \notin N(X)$. This proves (1).

(2) *There is no bad set.*

For suppose X is a maximal bad set; then by (1), $N(X) = \{w\}$ say. Let C be a component of $G \backslash w$ with $X \cap V(C) = \varnothing$. Then $|V(C)| \geqslant 3$ since G has minimum valency $\geqslant 3$, and so C has a path of length $\geqslant 2$. Let P be a maximum length path of C, with vertices $v_1, ..., v_k$ in order; then $k \geqslant 3$. Extend P to a maximal tree T of C with the property that v_1 is 1-valent and v_2 is 2-valent in T. Suppose that $V(T) \neq V(C)$, and choose $x \in V(C) - V(T)$ with a neighbour in $V(T)$. From the maximality of P, x is not adjacent to

v_1; and from the maximality of T, x is not adjacent to any vertex of T except v_1, v_2. Thus x is adjacent to v_2, and to no other vertex of T. From the choice of P, it follows that x has no neighbour in $V(C)-V(T)$, and so x has valency $\leqslant 2$ in G, a contradiction. Thus $V(T)=V(C)$, and so T is a spanning tree of C. Let e be the edge of T with ends v_1, v_2. Since $X \cap N(e)=\varnothing$, there is a $(\leqslant 3)$-separation (A, B) of G with $V(e) \subseteq V(A \cap B)$ such that

$$|N(e) \cap V(A)|, |N(e) \cap V(B)| > |V(A \cap B)| - 2.$$

We claim that $V(A \cap B) \cap X = \varnothing$. For if not, then $V(A \cap B \cap C) = \{v_1, v_2\}$, and $|V(A \cap B)| = 3$, and so $|N(e) \cap V(A)|, |N(e) \cap V(B)| \geqslant 2$. But $N(e) \subseteq V(T \backslash \{v_1, v_2\}) \cup \{v\}$, and so all members of $N(e) - \{v\}$ are in the same component of $G \backslash V(A \cap B)$, which is impossible. Thus $V(A \cap B) \cap X = \varnothing$. Let X' be the vertex set of the component of $G \backslash V(A \cap B)$ including X; then $v \in X'$, and so $|X'| > |X|$, and yet X' is bad, contrary to the maximality of X. This proves (2).

We deduce

(3) *For each edge e, $V(G) = V(e) \cup N(e)$.*

For if not, there is by hypothesis a $(\leqslant 3)$-separation (A, B) of G with $V(e) \subseteq V(A \cap B)$ and with

$$|N(e) \cap V(A)|, |N(e) \cap V(B)| > |V(A \cap B)| - 2.$$

Let X be the vertex set of a component of $G \backslash V(A \cap B)$. Then X is bad, contrary to (2). This proves (4).

From (3) it follows that G is complete multipartite, as required. ∎

We use (10.3) for our second lemma, the following.

(10.4) *Let G satisfy the hypotheses of* (10.3), *and suppose in addition that G is non-null, and that for every $e \in E(G)$, there is no $f \in E(G \backslash V(e))$ such that $N(e)$ is 2-coherent in $G \backslash V(e) \backslash f$. Then G is isomorphic to K_5 or to $K_{3,n}$ for some $n \geqslant 3$.*

Proof. By (10.3), G is complete multipartite; let the corresponding partition of $V(G)$ be $(V_1, ..., V_k)$, where $|V_1| \geqslant |V_2| \geqslant \cdots \geqslant |V_k|$. Now $k \geqslant 2$ since otherwise $E(G) = \varnothing$ and so G is null, contrary to hypothesis. Let $v_1 \in V_1$, $v_2 \in V_2$, and let e have ends v_1, v_2. Let $H = G \backslash \{v_1, v_2\}$. Then $N(e) = V(H)$, and therefore H is 2-connected, and for each $f \in E(H)$, $H \backslash f$ is not 2-connected. Since H is non-null, it follows easily that H has a vertex of valency $\leqslant 2$. But H is complete multipartite, and so either H is isomorphic to $K_{2,n}$ for some $n \geqslant 2$, or to K_n for some $n \leqslant 3$. In the second case $|V(G)| \leqslant 5$, and since it has no 3-valent vertex with adjacent neighbours, it follows that G is isomorphic to K_5. In the first case, H is isomorphic to $K_{2,n}$ and so $|V_1| \geqslant n \geqslant 2$ and $|V_2| \geqslant 2$, since $|V_1| \geqslant |V_2| \geqslant \cdots \geqslant |V_k|$; and $|V(G)| = n+4$. But $|V_1|, |V_2| \geqslant 2$ and H is bipartite, and so $k=2$. Consequently G is isomorphic to $K_{3,n+1}$ as required. ∎

Proof of (10.1).

If Γ has a vertex of valency $\leqslant 1$, then (10.1)(i) or (ii) holds, by (7.10)(i). If e is a loop or parallel edge of Γ, then by (7.10)(ii) $\gamma \backslash e$ is uneven and (ii) holds. If Γ has a vertex of valency 2 then (iii) holds. We therefore assume that Γ is simple and has minimum valency $\geqslant 3$. If $v \in V(\Gamma)$ is 3-valent and has two adjacent neighbours then (ii) holds by (7.10)(iii), and so we assume there is no such vertex v.

(1) *If $f \in E(\Gamma)$ and $V(\Gamma) \neq V(f) \cup N(f)$, and $N(f)$ is 2-coherent in $\Gamma \backslash V(f)$, then one of* (i), (ii), (iii) *holds.*

For let $u \in V(\Gamma) - V(f) \cup N(f)$. Now $|N(f)| \geqslant 3$, since both ends of f have valency $\geqslant 3$ and no 3-valent vertex has two adjacent neighbours. By (8.5) there is an edge e of Γ incident with u such that $N(f)$ is 2-coherent in one of $\Gamma \backslash V(f) \backslash e$, $\Gamma \backslash V(f)/e$. In the fist case, by (9.4)(ii), either γ/f is uneven or $\gamma \backslash e$ is uneven, and so (ii) or (iii) holds. In the second case, by (9.4)(iii), either γ/f is uneven or γ/e is uneven, and again (ii) or (iii) holds. This proves (1).

From (1) and (10.4), Γ is isomorphic either to K_5 or to $K_{3,n}$ for some $n \geqslant 3$, and we may therefore assume that Γ is isomorphic to $K_{3,n}$ where $n \geqslant 4$, for otherwise (iv) holds. But then (ii) holds, by (10.2). The result follows. ∎

Now we deduce a strengthening of (10.1), as follows.

(10.5) *Let Γ, Γ' be panelled frames in M, and let γ be an uneven tracing of Γ onto Γ'. Then there is a Kuratowski graph $H \subseteq \Gamma$ such that the restriction of γ to H is uneven.*

Proof. We proceed by induction on $|V(\Gamma)| + |E(\Gamma)|$. We may assume that for every proper subgraph H of Γ, the restriction of γ to H is even. Consequently, by (7.10), Γ has minimum valency $\geqslant 2$ and is simple, and no 3-valent vertex has two adjacent neighbours. Also we may assume that Γ has no 2-valent vertices, and so has minimum valency $\geqslant 3$.

We assume (for a contradiction) that Γ is not a Kuratowski graph. By (10.1) there is an edge $e \in E(\Gamma)$ so that γ/e is uneven. From the inductive hypothesis, there is a Kuratowski graph $H' \subseteq \Gamma/e$ such that the restriction of γ/e to H' is uneven. Hence $\gamma/e \backslash f$ is uneven for all $f \in E(\Gamma/e) - E(H')$, and hence $\gamma \backslash f$ is uneven for all such f; for a flex of M extending $\gamma \backslash f$ yields a flex of M/e extending $\gamma/e \backslash f$. Thus there are no such edges f, and so $H' = \Gamma/e$, since Γ/e has no isolated vertices. Hence Γ/e is a Kuratowski graph. But Γ has minimum valency $\geqslant 3$, and hence so does Γ/e, and therefore Γ/e is isomorphic to K_5 or to $K_{3,3}$. Let $m \in V(\Gamma/e)$ be the new element of M/e. Now m has valency $\geqslant 4$ in Γ/e, and so Γ/e is not isomorphic to $K_{3,3}$, and hence it is isomorphic to K_5. Consequently all neighbours of m in Γ/e are mutually adjacent, and m has valency 4; and so both ends of e are 3-valent in Γ, and have adjacent neighbours, a contradiction. Thus Γ is a Kuratowski graph, as required. ∎

Consequently we have (1.7), which we restate.

(10.6) *Every Kuratowski connected panelled frame is rigid.*

Proof. Let Γ be a Kuratowski connected panelled frame, and let γ' be a tracing of Γ onto some panelled frame Γ'. Let α be a orientation-reversing homeomorphism of M and let Γ'' be $\alpha(\Gamma')$; and define $\gamma''(x) = \alpha(\gamma'(x))$ $(x \in U(\Gamma))$. Then γ'' is a tracing of Γ onto Γ''. If for every Kuratowski graph $H \subseteq G$, the restriction of γ' to H is even, then by (10.5) γ' is even as required. By (7.18) we may therefore assume that for every Kuratowski graph $H \subseteq G$, the restriction of γ' to H is uneven, and therefore odd, by (7.7). Hence the restriction of γ'' to each H is even, and so by (10.5), γ'' is even. Consequently γ' is odd, as required. ∎

This completes the proof of our main theorem (1.2).

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