Spanning Paths in Infinite Planar Graphs

Nathaniel Dean

AT&T, ROOM 2C-415
600 MOUNTAIN AVENUE
MURRAY HILL, NEW JERSEY 07974-0636, USA

Robin Thomas*
Xingxing Yu†

SCHOOL OF MATHEMATICS
GEORGIA INSTITUTE OF TECHNOLOGY
ATLANTA, GEORGIA 30332-0160, USA

ABSTRACT

Let G be a 4-connected infinite planar graph such that the deletion of any finite set of vertices of G results in at most one infinite component. We prove a conjecture of Nash-Williams that G has a 1-way infinite spanning path. © 1996 John Wiley & Sons, Inc.

1. INTRODUCTION

Tutte [7] has shown that every 4-connected finite planar graph is hamiltonian. (In this paper graphs have no loops or multiple edges, but may be infinite.) To generalize Tutte's result to infinite graphs one can ask if every 4-connected infinite planar graph has a 1-way infinite spanning path, but that is clearly false. Indeed, no graph can have such a path if the deletion of some finite set of vertices leaves more than one infinite component. However, Nash-Williams [2] conjectured that this is the only way the generalization can fail. Nash-Williams' conjecture was partially confirmed by Jung [1] who proved it for triangulations. (Our thanks to R. Halin for bringing this reference to our attention.) We prove the conjecture in general, but we need some definitions before we can state our main result precisely.

Let P be the graph with vertex-set $\{v_1, v_2, v_3, \ldots\}$ and edge-set $\{v_1v_2, v_2v_3, \ldots\}$. Every graph isomorphic to P is called a 1-*way infinite path*, and

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v_1 is called the *end* of P. Likewise, a 2-*way infinite path* is any graph isomorphic to the graph with vertex-set $\{\ldots, v_{-2}, v_{-1}, v_0, v_1, v_2, \ldots\}$ and edge-set $\{\ldots, v_{-2}v_{-1}, v_{-1}v_0, v_0v_1, v_1v_2, \ldots\}$. A graph H is a *spanning* graph of a graph G if $V(H) = V(G)$. If G is a graph, and X is a vertex, an edge, a set of vertices or a set of edges we denote by $G\backslash X$ the graph obtained from G by deleting X. Our main result is as follows.

(1.1) *Let G be a 4-connected planar graph such that $G\backslash X$ has exactly one infinite component for every finite set $X \subseteq V(G)$. Then G has a 1-way infinite spanning path.*

In fact, we prove a slightly stronger result, which we now state. Let G be a graph, and let H be a subgraph of G. By an *H-bridge* of G we mean a subgraph B of G such that either B is isomorphic to K_2 (the complete graph on two vertices) and its vertices but not its edge belong to H, or B consists of a component of $G\backslash V(H)$ together with all edges from this component to H and all ends of these edges. The vertices of $V(H) \cap V(B)$ are called the *attachments* of the H-bridge B. We say that a graph G is *cohesive* if for every finite set $X \subseteq V(G)$, $G\backslash X$ has only finitely many components, exactly one of which is infinite. It is easy to see that if G is as in (1.1) then G is cohesive. Indeed, if G is a 3-connected planar graph, and $X \subseteq V(G)$ has $n \geq 3$ elements, then $G\backslash X$ has at most $2\binom{n}{3}$ components. (Proof. If not, then there are three distinct elements $x_1, x_2, x_3 \in X$ and three distinct components C_1, C_2, C_3 of $G\backslash X$ such that for $i, j = 1, 2, 3$, x_i has a neighbor in C_j, in which case G is easily seen to contain a subgraph isomorphic to a subdivision of $K_{3,3}$, contrary to planarity.) Thus the following implies (1.1).

(1.2) *Let G be a cohesive planar graph. Then G has a 1-way infinite path P such that every P-bridge of G is finite and has at most three attachments.*

Theorem (1.2) is proved in Section 3. In Section 2 we develop some structural lemmas about cohesive planar graphs. Let us mention the following conjecture, also made by Nash-Williams [2].

(1.3) Conjecture. *Let G be a 4-connected planar graph such that $G\backslash X$ has at most two infinite components for every finite set $X \subseteq V(G)$. Then G has a 2-way infinite spanning path.*

In the rest of this section we outline the proof of (1.2). Let G be a cohesive plane graph. We first prove that G has at most two vertices of infinite valency. Then we show in (2.2) that if G is 2-connected and has no vertex of infinite valency, then it has a "net," a sequence of circuits that "exhausts" the whole graph. There are two types of nets – "radial nets" and "ladder nets" as we call them, and most of our arguments need to be done separately for each type. Lemmas (2.3) and (2.4) are technical devices that permit us to reduce the problem to locally finite graphs, and (2.5) and (2.6) show that if a graph has a net, then it has a tight one; that is, the circuits forming the net are packed as tightly as possible.

In Section 3 we first state results from other papers that we need, and then prove that a finite portion N_n of a net N contains a "forward" path P_n such that every P_n-bridge of N_n has at most three attachments. (In fact, since G may have up to two vertices of infinite valency, we need a stronger condition, but let us ignore that now.) This is done in (3.4) and (3.6). By König's lemma the sequence $P_1, P_2, \ldots$ of finite paths has a "cluster point" P, and it is relatively easy to show (using the fact that each P_n is forward) that P is as desired.

2. NETS

If G_1, G_2 are subgraphs of a graph G we define $G_1 \cup G_2$ to be the subgraph of G with vertex-set $V(G_1) \cup V(G_2)$ and edge-set $E(G_1) \cup E(G_2)$. The graph $G_1 \cap G_2$ is defined analogously. If G and H are graphs we write $G \subseteq H$ to mean that G is a subgraph of H.

A *plane* graph is a graph embedded in the plane (without crossings). If G is a cohesive plane graph and C is a circuit (circuits have no "repeated" vertices) of G, then C divides the plane into two closed sets exactly one of which, say Δ, contains finitely many vertices of G. We denote by $I(C)$ (or by $I_G(C)$ if the graph is not clear from context) the subgraph of G consisting of all vertices and edges of G contained in Δ (and so C is a subgraph of $I(C)$).

(2.1) *Let G be a cohesive plane graph. Then G has at most two vertices of infinite valency.*

Proof. Suppose for a contradiction that x, y, z are distinct vertices of G of infinite valency. Since G is cohesive it follows that x and y are in the same component of $G \backslash X$ for every finite set $X \subseteq V(G) - \{x, y\}$. Hence there are infinitely many paths $P_1, P_2, \ldots$ in $G \backslash z$ between x and y, vertex-disjoint except for their ends. By cohesiveness, there is a path Q_1 in $G \backslash \{u, v\}$ between z and $V(P_1) \cup V(P_2) \cup \ldots$. We may assume that the other end of Q_1 belongs to $V(P_1)$. Similarly, there exists a path Q_2 in $G \backslash (V(P_1) \cup V(Q_1) - \{z\})$ between z and $V(P_2) \cup V(P_3) \cup \ldots$. We may assume that the other end of Q_2 belongs to $V(P_2)$. Finally, there exists a path Q_3 in $G \backslash (V(P_1) \cup V(P_2) \cup V(Q_1) \cup V(Q_2) - \{z\})$ between z and $V(P_3) \cup V(P_4) \cup \ldots$. We may assume that the other end of Q_3 belongs to $V(P_3)$. Then $P_1 \cup P_2 \cup P_3 \cup Q_1 \cup Q_2 \cup Q_3$ is a subgraph of G isomorphic to a subdivision of $K_{3,3}$, contrary to the planarity of G. □

Let G be a cohesive plane graph. By a *net* in G we mean a sequence $N = (C_1, C_2, \ldots)$ of circuits such that

(i) $I(C_i)$ is a subgraph of $I(C_{i+1})$ for all $i = 1, 2, \ldots,$
(ii) $\bigcup_{i=1}^{\infty} I(C_i) = G$, and either
(iii) $C_1, C_2, \ldots$ are pairwise disjoint, or
(iii′) for every $i = 1, 2, \ldots$, $C_i \cap C_{i+1}$ is a non-null path, it is a subgraph of $C_{i+1} \cap C_{i+2}$, and no end of $C_i \cap C_{i+1}$ is an end of $C_{i+1} \cap C_{i+2}$.

If (iii) holds we say that N is a *radial net*, and if (iii′) holds we say that N is a *ladder net*. It is easy to see that if a cohesive plane graph G has a net then G is locally finite (that is, every vertex has finite valency).

(2.2) *Let G be a locally finite 2-connected cohesive plane graph. Then G has a net.*

Proof. Since G is locally finite and connected, it has countably many edges; let $E(G) = \{e_1, e_2, \ldots\}$. We say that a circuit C of G is *saturated* if every finite C-bridge of G is a subgraph of $I(C)$. If C, C' are circuits of G we denote by $J(C, C')$ the set of all edges of $I(C') \backslash E(C')$ that are incident with a vertex of $I(C)$. It is easy to see that

(1) for every circuit C of G there exists a saturated circuit C' of G with $I(C) \subseteq I(C')$,
(2) if C_1, C_2, C_3 are circuits of G with $I(C_1) \subseteq I(C_2) \subseteq I(C_3)$, then $J(C_1, C_2) \subseteq J(C_1, C_3)$, and
(3) for every circuit C there exists a constant k such that $|J(C, C')| \leq k$ for every circuit C'.

Since G is 2-connected we may therefore choose an infinite sequence C_1, $C_2, \ldots$ of circuits of G such that

(i) C_i is saturated and $e_i \in E(I(C_i))$ for all $i = 1, 2, \ldots$,
(ii) $I(C_1) \subseteq I(C_2) \subseteq I(C_3) \subseteq \cdots$, and
(iii) for every $i = 1, 2, \ldots,$ if C is a saturated circuit of G with $e_{i+1} \in E(I(C))$ and $I(C_i) \subseteq I(C)$, then $|J(C_i, C)| \leq |J(C_i, C_{i+1})|$.

By Ramsey's theorem [3] we may assume (by choosing a subsequence of $C_1, C_2, \ldots$) that either $V(C_i) \cap V(C_{i+1}) = \emptyset$ for all $i = 1, 2, 3, \ldots$, or that $V(C_i) \cap V(C_{i+1}) \neq \emptyset$ for all $i = 1, 2, \ldots$. In the former case $C_1, C_2, \ldots$ is a radial net, and so we assume the latter. We claim that then $N = (C_1, C_2, \ldots)$ is a ladder net.

We first show that $C_i \cap C_{i+1}$ is a path for every $i = 1, 2, \ldots$. Suppose to the contrary that $C_i \cap C_{i+1}$ has at least two components; then C_{i+1} has at least two $C_i \cap C_{i+1}$-bridges, say H_1 and H_2. But C_i is saturated, and so $H_1 \backslash V(C_i)$, $H_2 \backslash V(C_i)$ are non-null subgraphs of the same component of $G \backslash V(C_i)$, and hence there is a path P between $V(H_1)$ and $V(H_2)$ in $G \backslash V(C_i)$. The graph $C_{i+1} \cup P$ contains a circuit C' with $P \cup I(C_{i+1}) \subseteq I(C')$, and hence there exists, by (1), a saturated circuit C with $I(C') \subseteq I(C)$. The graph H_1 is a path with both its end edges incident with vertices of C_i, and exactly one of these end edges, say e, belongs to $I(C') \backslash E(C')$. Then by (2) $e \in J(C_i, C') - J(C_i, C_{i+1}) \subseteq J(C_i, C) - J(C_i, C_{i+1})$ and $J(C_i, C_{i+1}) \subseteq J(C_i, C)$, contrary to (iii). This proves that $C_i \cap C_{i+1}$ is a path.

If $C_i \cap C_{i+1}$ is not a subpath of $C_{i+1} \cap C_{i+2}$, or if an end of $C_i \cap C_{i+1}$ is an end of $C_{i+1} \cap C_{i+2}$, then there exists an edge $e \in E(C_{i+1}) - E(C_{i+2})$ incident to a vertex of C_i, and hence $e \in J(C_i, C_{i+2}) - J(C_i, C_{i+1})$, while $J(C_i, C_{i+1}) \subseteq J(C_i, C_{i+2})$ by (2), again contrary to (iii). Thus N is a ladder net, as desired. □

Let $N = (C_1, C_2, \ldots)$ be a net in a cohesive plane graph G, and let P be the graph $\bigcup_{i=1}^{\infty} (C_i \cap C_{i+1})$. We say that P is the *boundary* of N. Thus the boundary is the null graph if N is a radial net, and it is a 2-way infinite path if N is a ladder net. We denote the boundary of N by ∂N. Let us recall that a *block* is either a 2-connected graph, or a graph isomorphic to K_2. A *block of a graph* G is a subgraph of G that is a block and is maximal with this property. Let H be a subgraph of a graph G, and let L be a subgraph of H. We say that L is H*-shielded* by a set $X \subseteq V(G)$ if there is no path in $G \backslash X$ with one end in $V(L)$, the other end in $V(H) - V(L)$, and otherwise disjoint (that is, both vertex- and edge-disjoint) from H. If H is a subgraph of a graph G, and $S \subseteq V(G)$, we denote by $H + S$ the subgraph of G with vertex-set $V(H) \cup S$ and edge-set $E(H)$.

(2.3) *Let G be a cohesive plane graph, let S be the set of (at most two) vertices of G of infinite valency, and assume that $G \backslash S$ has an infinite block. Then there exists a 2-connected cohesive subgraph H of G and a net N in H with boundary Q such that*

- *(i) $S \subseteq V(Q)$,*
- *(ii) every H-bridge of G is finite and has at most three attachments,*
- *(iii) if an H-bridge of G has at least two attachments, then all its attachments belong to $V(Q)$, and*
- *(iv) every subpath of $Q \backslash S$ is H-shielded by a subset of S of size at most one.*

(We remark that (ii) is implied by (iii) and (iv), but is included for convenience.)

Proof. Let G and S be as stated; then $|S| \leq 2$ by (2.1). Let H' be an infinite block of $G \backslash S$. Then every $(H' + S)$-bridge B of G has at most three attachments – at most two in S and at most one in $V(H')$, and therefore

B is finite. The graph H' is locally finite, 2-connected and cohesive, and so by (2.2) has a net $N' = (C'_1, C'_2, \ldots)$. Let Q' be the boundary of N'. We claim the following.

(1) *If an $(H'+S)$-bridge of G has at least two attachments, then they are contained in $V(Q') \cup S$.*

To prove (1) suppose for a contradiction that an $(H'+S)$-bridge B of G has at least two attachments and an attachment $v \in V(H') - (V(Q') \cup S)$. Hence $v \in V(I_G(C'_i)) - V(C'_i)$ for some $i = 1, 2, \ldots$. But then $B \subseteq I_G(C'_i)$, and, in particular, a member of S belongs to $V(I_G(C'_i)) - V(C'_i)$, contrary to the finiteness of $I_G(C'_i)$. This contradiction proves (1).

We may assume (by omitting an initial segment of $C'_1, C'_2, \ldots$) that for every $x \in S$, $V(C'_1)$ contains at least two vertices u such that some $(H'+S)$-bridge of G has x and u as attachments. This is possible, because vertices in S have infinite valency, and every $(H'+S)$-bridge of G is finite. Let L denote the union of $I_{H'}(C'_1)$ and all $(H'+S)$-bridges of G with at least two attachments such that their attachments are contained in $S \cup V(C'_1)$. Then L is finite and 2-connected, and so has a circuit D such that $I_G(D) \supseteq L$. Let X denote the set of interior vertices of the (possibly null) path $C'_1 \cap C'_2$. We claim that

(2) *$S \subseteq V(D)$ and $C'_1 \backslash X$ is a subgraph of D.*

Indeed, if $x \in S - V(D)$, then $x \in I_G(D) \backslash V(D)$, contrary to the finiteness of $I_G(D)$. To prove the other inclusion we may assume that N' is a ladder net, because otherwise $S = \emptyset$, $X = \emptyset$ and $C'_1 = D$. Let a, b be the ends of the path $C'_1 \backslash X$. Certainly $a, b \in V(D)$, because if say $a \in V(I(D)) - V(D)$, then a 1-way infinite subpath of Q' with end a is a subgraph of $I_G(D)$, contrary to the finiteness of $I_G(D)$. Since $I_G(D)$ is finite there exists an integer $i \geq 2$ such that $C'_i \backslash (X \cup \{a, b\})$ is disjoint from $I_G(D)$ (because the former is either a subgraph of the latter, or disjoint from it). Let C denote the circuit $(C'_i \backslash X) \cup (C'_1 \backslash X)$. Certainly $C'_1 \backslash X \subseteq I_G(D)$; if $C'_1 \backslash X \nsubseteq D$ then

$$\emptyset \neq E(D) \cap E(I_{H'}(C)) - E(C) \subseteq E(I_{H'}(C'_1)) \cap E(I_{H'}(C)) - E(C) = \emptyset,$$

a contradiction. This proves (2).

Let $H = H' \cup L$, and for $i = 1, 2, \ldots$ let $C_i = (C'_i \backslash X) \cup (D \backslash (V(C_1) - V(C_2)))$. Then H is 2-connected and cohesive, and $N = (C_1, C_2, \ldots)$ is a net in H. We claim that N is as desired. Let Q denote the boundary of N; then $S \subseteq V(Q)$ by (2). Let B be an H-bridge of G. Then B is an $(H'+S)$-bridge of G, and (ii) follows. To prove (iii) assume that B has at least two attachments. Then all its attachments are contained in $S \cup (V(Q') - V(C'_1)) \subseteq V(Q)$ by (1) and the choice of L. This proves (iii).

To prove (iv) let R be a subpath of $Q \backslash S$, and suppose for a contradiction that it is H-shielded by no subset of S of size at most 1. Since every H-

bridge has at most one attachment not in S, we deduce that there are distinct vertices $x_1, x_2 \in S$ and (not necessarily distinct) vertices $x_1', x_2' \in V(R)$ such that for $i = 1, 2$ there is a path P_i in G between x_i and x_i', disjoint from H except for its ends. It follows from the choice of L that R is a subgraph of an infinite component of $Q \backslash S$. From the symmetry we may assume that x_1, x_2, x_1' occur on Q in this order. Choose an integer $i \geq 1$ such that $x_1, x_2, x_1' \in V(C_i)$; since $P_1 \not\subseteq I_G(C_i)$ we deduce that a circuit contained in $P_1 \cup C_i$ separates x_2 from $Q \backslash V(I(C_i))$, contrary to the cohesiveness of G. This proves (iv), and hence the theorem. □

Let H be a cohesive plane graph such that there exist finite blocks $B_1, B_2, \ldots$ of H and vertices $v_1, v_2, \ldots$ of H such that $v_i \in V(B_i) \cap V(B_{i+1})$ for $i = 1, 2, \ldots$ and $H = \bigcup_{i=1}^{\infty} B_i$. We say that H is a *linear graph* and that $(B_1, v_1, B_2, v_2, \ldots)$ is a *block-decomposition* of H. Let $i \geq 1$ be an integer. If B_i is isomorphic to K_2 we define $D_i = B_i$; otherwise we define D_i to be the unique circuit D_i of B_i such that $I_H(D_i) \supseteq B_i$. We define the *boundary* of $(B_1, v_1, B_2, v_2, \ldots)$ to be $\bigcup_{i=1}^{\infty} D_i$.

(2.4) *Let G be a cohesive plane graph, let S be the set of (at most two) vertices of G of infinite valency, and assume that $G \backslash S$ has no infinite block. Then G has a linear subgraph H with a block-decomposition $(B_1, v_1, B_2, v_2, \ldots)$ such that if Q denotes the boundary of this decomposition, then*

(i) $S \subseteq V(B_1) - \{v_1\}$,
(ii) every H-bridge of G is finite and has at most three attachments,
(iii) if an H-bridge of G has at least two attachments, then all its attachments belong to $V(Q)$, and
(iv) every subpath of $Q \backslash (S \cup \{v_1, v_2, \ldots\})$ is H-shielded by a subset of S of size at most one.

Proof. Let G and S be as stated. Since $G \backslash S$ is locally finite it follows that there exist distinct blocks $B_2, B_3, \ldots$ and distinct vertices $v_2, v_3, \ldots$ of $G \backslash S$ such that $v_i \in V(B_i) \cap V(B_{i+1})$ for $i = 2, 3, \ldots$. Let $H' = \bigcup_{i=2}^{\infty} B_i$. If $S = \emptyset$ then H' and $(B_2, v_2, B_3, v_3, \ldots)$ satisfy the theorem, because every H'-bridge of G has at most one attachment. We therefore assume that $S \neq \emptyset$. We may assume (by omitting an initial segment of the sequence $B_2, v_2, B_3, v_3, \ldots$) that G has an $(H' + S)$-bridge B_1 with set of attachments $S \cup \{v_1\}$, where B_1 is a block and $v_1 \in V(B_2) - \{v_2\}$. (This can be done because every $x \in S$ has infinite valency and every $(H' + S)$-bridge of G is finite.) Let $H = \bigcup_{i=1}^{\infty} B_i$. It follows similarly as in (2.3) that H and $(B_1, v_1, B_2, v_2, \ldots)$ satisfy the theorem. □

Let $N = (C_1, C_2, \ldots)$ be a net in a cohesive plane graph G. For $i = 1, 2, 3, \ldots$ we define C_i° to be the graph obtained from C_i by deleting $C_i \cap C_{i+1}$ except its endpoints. Thus if N is a radial net then $C_i^\circ = C_i$, and if N is a ladder net then C_i° is a path with both ends on the boundary of N and otherwise disjoint from it. We say that N is *tight* if

(a) $I(C_1) = C_1$,
(b) $C_1 \cap C_2$ is either null or contains at least one edge, and
(c) for every $i = 1, 2, \ldots$, every C_{i+1}°-bridge of $I(C_{i+1})\backslash V(I(C_i))$ has at most one attachment.

(2.5) *Let G be a 2-connected cohesive plane graph with a ladder net N, and let u be a vertex of the boundary of N. Then there exists a tight net $(C_1, C_2, \ldots)$ in G with $u \in V(C_1) \cap V(C_2)$ and with the same boundary as N.*

Proof. Let P be the boundary of N, and let e be an edge of P incident with u. Let P_1 be a path with both ends on P and otherwise disjoint from P such that if C_1 denotes the unique circuit of $P \cup P_1$ then C_1 contains e and subject to that $|E(I(C_1))|$ is minimum. Assume that we have already constructed disjoint paths $P_1, P_2, \ldots, P_k$ such that

(i) each P_i has both ends on P and otherwise is disjoint from P,
(ii) if for $i = 1, 2, \ldots, k$, C_i denotes the unique circuit in $P \cup P_i$, then $I(C_1) \subseteq I(C_2) \subseteq \cdots \subseteq I(C_k)$,
(iii) for every $i = 1, 2, \ldots, k-1$, every P_{i+1}-bridge of $I(C_{i+1})\backslash V(I(C_i))$ has at most one attachment.

Let P_{k+1} be a path disjoint from $I(C_k)$ with both ends on P and otherwise disjoint from P and subject to that with $|E(I(C_{k+1}))|$ minimum, where C_{k+1} is the unique circuit of $P \cup P_{k+1}$. It follows that $P_1, P_2, \ldots, P_k, P_{k+1}$ satisfy (i), (ii) and (iii), and so the construction is complete. Now $(C_1, C_2, \ldots)$ is the desired tight net. □

(2.6) *Let G be a 2-connected cohesive plane graph with a radial net, and let $u \in V(G)$. Then G has a tight radial net $(C_1, C_2, \ldots)$ with $u \in V(C_1)$.*

Proof. Let C_1 be a circuit of G bounding a face of G with $u \in V(C_1)$. Assume that we have already constructed circuits $C_1, C_2, \ldots, C_k$, and let H denote the graph $G\backslash V(I(C_k))$. Since G has a radial net and $I(C_k)$ is finite it follows that H contains a circuit C such that $I(C_k)$ is a subgraph of $I(C)$. Choose such a circuit C_{k+1} with $|E(I(C_{k+1}))|$ minimum. This completes the construction of an infinite sequence $N = (C_1, C_2, \ldots)$ of circuits. It follows that N is a desired net. □

3. FORWARD PATHS

The following theorem was proved by Thomassen [6].

(3.1) *Let G be a 2-connected finite plane graph with a circuit C bounding the infinite region. Let $v \in V(C)$, let $e \in E(C)$, and let $u \in V(G) - \{v\}$. Then there exists a path P in G between u and v with $e \in E(P)$ such that*

(i) every P-bridge of G has at most three attachments, and

(ii) every P-bridge of G that contains an edge of C has at most two attachments.

The next two theorems are [4, Theorem (2.4)] and [4, Theorem (2.6)], respectively. Recall that $Q + S$ was defined prior to (2.3).

(3.2) *Let G be a finite plane graph, and let P be a path in G with ends u, v such that every vertex and every edge of P is incident with the infinite region. Let $S \subseteq V(G) - V(P)$ be a set of at most two vertices, all incident with the infinite region. Then there exists a path Q in $G \backslash S$ between u and v such that*

(i) every $(Q + S)$-bridge of G has at most three attachments, and

(ii) if a $(Q + S)$-bridge of G contains an edge of P, then it has at most two attachments.

If C is a circuit in a plane graph G and $x, y \in V(C)$, then by the clockwise xy-segment of C, denoted by xCy we mean the subpath of C obtained by traversing C clockwise from x to y. Thus xCx is the graph with vertex-set $\{x\}$ and no edges.

(3.3) *Let G be a 2-connected plane graph with a circuit C bounding the infinite region, let $u, v \in V(C)$ be distinct, let $e, f \in E(C)$, and assume that u, v, e, f occur on C in this clockwise order. Then there exists a path P between u and v in G with $e, f \in E(P)$ and such that*

(i) every P-bridge of G has at most three attachments, and

(ii) if a P-bridge of G contains an edge of the clockwise vu-segment of C, then it has at most two attachments.

Let G be a connected graph, and let P be a path in G with ends u, v. Let $v_1, v_2, \ldots v_{n-1}$ be all the cutvertices of G that belong to $V(P) - \{u, v\}$, let $v_0 = u$, $v_n = v$ and assume that $v_0, v_1, \ldots, v_n$ occur on P in this order. For $i = 1, 2, \ldots, n$ let B_i be the subgraph of G induced by all vertices $v \in V(G)$ with the property that there is a vertex $w \in V(P)$ such that v_{i-1}, w, v_i are distinct and occur on P in this order and there is a path in G between v and w vertex-disjoint from $P \backslash w$. We call the sequence $v_0, B_1, v_1, B_1, \ldots, v_{n-1}, B_n, v_n$ the *structure sequence* of G relative to P (it is unique up to reversal). Let G be a cohesive plane graph with a net $N = (C_1, C_2, \ldots)$, and let P be a path in G with ends u and v. We say that P is a *forward path from* u *to* v (with respect to N) if there do not exist an integer $i \in \{1, 2, \ldots\}$ and vertices $x, y \in V(P)$ such that $x \in V(C_{i+2}) - V(C_{i+1})$, $y \in V(C_i)$ and u, x, y, v occur on P in the order listed.

(3.4) *Let G be a 2-connected cohesive plane graph with a tight radial net $N = (C_1, C_2, \ldots)$, and let $u \in V(C_1)$. Then for every $n \in \{1, 2, \ldots\}$ there exist a vertex $w \in V(C_n)$ and a forward path P from u to w in $I(C_n)$ such that*

(i) every P-bridge of $I(C_n)$ has at most three attachments, and

(ii) every P-bridge of $I(C_n)$ that contains an edge of C_1 has at most two attachments.

Proof. We proceed by induction on n. If $n = 1$ then $w = u$ and the path with vertex-set $\{u\}$ satisfy the lemma. Let us assume that $n > 1$ and that the lemma holds for all graphs G, all tight radial nets N, all vertices $u \in V(C_1)$ and all integers $n' < n$. Let H be the block of $I(C_n)\backslash V(C_1)$ containing C_n with its embedding inherited from G. Since N is tight it follows that C_2 bounds a face of H. Let $w_1, w_2, \ldots, w_m$ be all the vertices of C_2 that are attachments of some $(H \cup C_1)$-bridge of $I(C_n)$, listed in their clockwise cyclic order on C_2. For $i = 1, 2, \ldots, m$ let $x_i, y_i \in V(C_1)$ be such that

(a) every $(H \cup C_1)$-bridge of $I(C_n)$ with w_i as an attachment has all its other attachments in $x_i C_1 y_i$,

(b) $x_i C_1 y_i$ is minimal subject to (a), and

(c) $x_1 C_1 y_1, x_2 C_1 y_2, \ldots, x_m C_1 y_m$ are edge-disjoint.

Such a choice is possible because G is a plane graph and N is tight. We may assume (by renumbering $x_1, y_1, x_2, y_2, \ldots, x_m, y_m$ and/or reversing the orientation) that $u \in V(x_1 C_1 x_2) - \{x_1\}$. The sequence $(C_2, C_3, \ldots)$ is a tight net in H, and so by the induction hypothesis there exist a vertex $w \in V(C_n)$ and a forward path R from w_1 to w satisfying (i) and (ii) with C_2 in place of C_1. Let T' be the union of $x_1 C_1 x_2$ and all $(H \cup C_1)$-bridges of $I(C_n)$ whose set of attachments is contained in $x_1 C_1 x_2 \cup \{w_1\}$, and let T be obtained from T' by adding an edge with ends $x_2 w_1$, or $T = T'$ if x_2 and w_1 are adjacent in T'. By (3.1) there exists a path Q in T between x_1 and u containing the edge $x_2 w_1$ and such that every Q-bridge of T has at most three attachments, and if a Q-bridge of T contains an edge of $x_1 C_1 x_2$, then it has at most two attachments. We put $L_0 = Q \backslash x_2 w_1$; then L_0 is a disjoint union of two paths, one with ends x_1 and w_1, and the other with ends u_1 and x_2. Let J be the union of $x_2 C_1 x_1$, all $(H \cup C_1)$-bridges of $I(C_n)$ that were not included in T', and all R-bridges of H that contain a vertex of $\{w_2, w_3, \ldots, w_m\} - V(R)$. Let $x_2 = v_0$, $B'_1, v_1, \ldots, v_{n-1}, B'_n$, $v_n = x_1$ be the structure sequence of $J \backslash V(R)$ relative to $x_2C_1x_1$. Since every R-bridge of H has at most two attachments we deduce that for every $i = 1, 2, \ldots, n$, there is a set $S \subseteq V(R) \cap V(C_2)$ with $|S| \leq 2$ such that $\{v_{i-1}, v_i\} \cup S$ separates $V(B'_i)$ from $V(G) - V(B'_i)$ in G. Moreover, if B_i is the graph obtained from B'_i by adding the vertices of S and all edges of G from S to $V(B_i)$ then B_i has the structure as described in (3.2). Let L_i be a path in B_i between v_{i-1} and v_i as in (3.2).

Let $P = L_0 \cup L_1 \cup L_2 \cup \ldots \cup L_n \cup R$; we claim that P is as desired. It follows from the construction that P is a path, and it also follows (either directly or from [4, Theorem (2.3)]) that every P-bridge has at most three attachments, and that every P-bridge that contains an edge of C_1 has at most two attachments. (Notice that $\bigcup_{i=1}^{n} B'_i$ need not be the whole of J, but the P-bridges of G that were "left out" this way contain no edge of C_1 and have at most three attachments – one in $V(C_1)$ and at most two in $V(C_2)$.) It remains to show that P is a forward path from u to w. To prove this assume that u, x, y, w are distinct vertices of P, that they occur on P in the order listed, and that $x \in V(C_{i+2})$ for some $i = 1, 2, \ldots$. Since $H \cap (L_0 \cup L_1 \cup L_2 \cup \ldots \cup L_n)$ is contained in the union of R-bridges of H that contain an edge of C_2, and since each such bridge has two attachments, we deduce that $L_0 \cup L_1 \cup L_2 \cup \ldots \cup L_n$ is disjoint from C_{i+2}. Thus $x \in V(R)$, and hence $y \in V(R)$. It follows that $y \notin V(C_i)$, as desired, because R is forward and $V(R) \cap V(C_1) = \emptyset$. □

Let G be a cohesive plane graph with a ladder net $N = (C_1, C_2, \ldots)$, let P be the boundary of N, and let i, n be integers with $1 \leq i \leq n$. We define the (i, n)-*truncation* of G *relative* to N to be the graph H obtained from $I(C_n)$ by deleting the vertices of $I(C_i) \backslash V(C_i^\circ)$. The ends of C_i° are called the *tips* of H, and the *frontier* of H is defined to be $(P \cup C_i^\circ) \cap H$. Thus the frontier of H is a path.

(3.5) *Let G be a 2-connected cohesive plane graph with a tight ladder net $N = (C_1, C_2, \ldots)$, let i, n be integers with $1 \leq i \leq n$, let H be the (i, n)-truncation of G relative to N, let u_0, u_1 be the tips of H, and let e be an edge of the intersection of the frontier of H with the boundary of N. Then for $j = 0, 1$ there exist a vertex $w \in V(C_n^\circ)$ and a path P in H between u_j and w with $u_{1-j} \in V(P)$ such that*

(i) every P-bridge of H has at most 3 attachments,

(ii) every P-bridge of H containing an edge of the frontier of H has at most 2 attachments, and

(iii) P is a forward path of G from u_j to w.

Moreover, there exists an integer $j \in \{0,1\}$ such that there exist a vertex $w \in V(C_n^\circ)$ and a path P as above with the additional property that $e \in E(P)$.

Proof. We will prove the second statement, and it will become clear from the proof that if we do not insist on $e \in E(P)$ then the argument applies to $1-j$ as well.

We proceed by induction on $n-i$. If $n=i$ then C_n° satisfies the conclusion of the theorem, and so we assume that $i<n$. Let H, u_0, u_1 and e be as in the statement of the lemma, and let K be the $(i+1,n)$-truncation of G relative to N. Let $w_1, w_2, \ldots, w_m$ be all the attachments of $(K \cup C_i^\circ)$-bridges of H in the order they appear on C_{i+1}°. Thus $m \geq 2$ and w_1 and w_m are the tips of K.

If $e \notin E(K)$ we may assume that w_1, u_0, u_1, e, w_m occur on the frontier of H in this order. By the induction hypothesis there exist a vertex $w \in V(C_n^\circ)$ and a path R in K between w_m and w with $w_1 \in V(R)$ satisfying conditions (i), (ii), (iii) of the theorem. If $e \in E(K)$ we may assume that w_1, u_0, u_1, w_m occur on the frontier of H in this order. By the induction hypothesis there exist an integer $j' \in \{1,m\}$, a vertex $w \in V(C_n^0)$ and a path R in K between $w_{j'}$ and w with $w_{m+1-j'} \in V(R)$ and $e \in E(R)$ satisfying conditions (i), (ii), (iii) of the theorem. From the symmetry we may assume that $j'=m$.

Let J be the union of C_i°, all $(K \cup C_i^\circ)$-bridges of H, and all R-bridges B of K with $\{w_1, w_2, \ldots, w_m\} \cap V(B) - V(R) \neq \emptyset$. Let $u_0 = v_0, B_1', v_1, \ldots, v_{p-1}, B_p', v_p = u_1$ be the structure sequence of $J \backslash V(R)$ relative to C_i°. For $l = 1, 2, \ldots, p-1$, let B_l and L_l be defined similarly as in the proof of (3.4). For $l=p$ we argue as follows. Let B_p' be obtained from B_p by adding w_m and all edges of J with one end in w_m and the other end in B_p, and let f be an edge of the frontier of H incident with u_1. By (3.3) there exists a path L_p in B_p' between v_{p-1} and w_m with $f \in E(L_p)$, with $e \in E(L_p)$ if $e \notin E(K)$ and such that every L_p-bridge of B_p has at most three attachments, and if an L_p-bridge of B_p contains an edge of the frontier of H, then it has at most two attachments. Then $j=0$, w and $P = L_1 \cup L_2 \cup \ldots \cup L_p \cup R$ satisfy the theorem by a similar argument as in (3.4). □

(3.6) *Let G be a 2-connected cohesive plane graph with a tight ladder net $N = (C_1, C_2, \ldots)$, and let $u \in V(C_1) \cap V(C_2)$. Then for every $n \in \{1, 2, \ldots\}$ and every $v \in V(C_n) \cap V(\partial N)$ there exists a forward path P in $I(C_n)$ from some vertex of $C_1 \cap C_2$ to some vertex of C_n° with $u, v \in V(P)$ such that*

- *(i) every P-bridge of $I(C_n)$ has at most three attachments, and*
- *(ii) every P-bridge of $I(C_n)$ that contains an edge of the boundary of N has at most two attachments.*

Proof. Let G, N, u, v and n be as stated in the lemma, let H be the $(1,n)$-truncation of G relative to N, and let u_0, u_1 be the tips of H. If $v \in V(H)$

choose an edge e incident with v in the intersection of the frontier of H with the boundary of N, and let j and P be as in the second assertion of (3.5). If $v \notin V(H)$ choose $j \in \{0,1\}$ arbitrarily and let P be as in the first assertion of (3.5). Then $P \cup (C_1 \cap C_2 \backslash u_{1-j})$ is as desired, because $I(C_1) = C_1$. □

Let P be a path with end u_0, let P' be a subpath of P and let P' have vertex-set $\{u_0, u_1, \ldots, u_k\}$ (in this order). We say that $u_0, u_1, \ldots, u_k$ is an *initial segment* of P.

(3.7) *Let G be a 2-connected cohesive plane graph with a net $N = (C_1, C_2, \ldots)$. Let $u = v \in V(G)$ be arbitrary if N is a radial net, and let u, v be vertices of the boundary of N if N is a ladder net. Then there exists a* 1*-way infinite path P in G with $u, v \in V(P)$ such that*

(i) every P-bridge of G is finite and has at most three attachments, and

(ii) every P-bridge of G that contains an edge of the boundary of N has at most two attachments.

Proof. By (2.5) or (2.6) we may assume that N is tight and that $u \in V(C_1)$ if N is radial and $u \in V(C_1) \cap V(C_2)$ if N is a ladder net. If N is a radial net, then for $n = 1, 2, \ldots$ let P_n be a forward path from u to some vertex of C_n satisfying (i) and (ii) of (3.4). If N is a ladder net let P_n be a forward path from some vertex of $C_1 \cap C_2$ to some vertex of C_n with $u, v \in V(P)$ satisfying (i) and (ii) of (3.6). There exist a vertex $u_0 \in V(C_1)$ and an infinite set $A_0 \subseteq \{1, 2, \ldots\}$ such that u_0 is an end of P_n for every $n \in A_0$. (In fact $u = u_0$ if N is a radial net.) Suppose that for some $m = 0, 1, \ldots$ we have already constructed a sequence $u_0, u_1, \ldots, u_m$ of distinct vertices of G and infinite sets $A_0 \supseteq A_1 \supseteq \cdots \supseteq A_m$ such that $u_0, u_1, \ldots, u_m$ is an initial segment of P_n for every $n \in A_m$. Since G is locally finite there is an infinite set $A_{m+1} \subseteq A_m$ and a vertex $u_{m+1} \in V(G)$ such that $u_0, u_1, \ldots, u_m, u_{m+1}$ is an initial segment of P_n for every $n \in A_{m+1}$. This completes the construction of an infinite sequence $u_0, u_1, \ldots$ of vertices of G; let P be the 1-way infinite path of G with this vertex-set (in the order listed). We claim that P is as desired. We first prove the following.

(1) *Every P-bridge of G is a subgraph of some $I(C_n)$.*

To prove (1) suppose for a contradiction that B is a P-bridge of G that is contained in no $I(C_n)$. Then there exists an integer $i \geq 2$ such that $V(B) \cap V(C_j) - V(C_{j-1}) \neq \emptyset$ for $j = i, i+1, i+2, i+3$. Let $u_0, u_1, \ldots, u_k$ be an initial segment of P such that $u_k \in V(C_{i+5}) - V(C_{i+4})$, and let $n \geq 1$ be an integer such that $u_0, u_1, \ldots, u_k$ is an initial segment of P_n. Since P_n is a forward path we deduce that $V(P_n) \cap V(I(C_{i+3})) \subseteq \{u_0, u_1, \ldots, u_k\}$, and hence $B \cap I(C_{i+3})$ is a subset of a P_n-bridge of $I(C_{i+3})$, say B'. But B' has at least four attachments (at least one in $V(C_j) - V(C_{j-1})$ for $j = i, i+1, i+2, i+3$), contrary to the choice of P_n. This proves (1).

(2) *Every P-bridge of G is a P_n-bridge of $I(C_n)$ for some $n \geq 1$.*

To prove (2) let B be a P-bridge of G. By (1) there exists an integer $n' \geq 3$ such that B is a subgraph of $I(C_{n'-2})$. Choose an initial segment $u_0, u_1, \ldots, u_k$ of P such that $u_k \in V(C_{n'}) - V(C_{n'-1})$, and let $n \geq 1$ be an integer such that $u_0, u_1, \ldots, u_k$ is an initial segment of P_n. Then $V(P_n) \cap V(I(C_{n'-2})) \subseteq \{u_0, u_1, \ldots, u_k\}$ because P_n is a forward path, and hence B is a P_n-bridge of $I(C_n)$. This proves (2).

The result now follows from (2), (3.4) and (3.6). □

(3.8) *Let H be a linear graph with a block-decomposition $(B_1, v_1, B_2, v_2, \ldots)$ with boundary Q, and let $u, v \in V(Q) \cap V(B_1) - \{v_1\}$. Then there exists a 1-way infinite path P in H with $u, v \in V(P)$ such that*

(i) every P-bridge of H has at most three attachments, and

(ii) every P-bridge of H that contains an edge of Q has at most two attachments.

Proof. For $i = 1, 2, \ldots$ let D_i be as in the definition of boundary of a block-decomposition. Let $v_0 = v$, and let $i \geq 1$. By (3.1) there exists a path P_i in B_i between v_{i-1} and v_i with $u \in V(P_i)$ if $i = 1$ such that every P_i-bridge of B_i has at most three attachments, and if a P_i-bridge of B_i contains an edge of D_i then it has at most two attachments. Then $P_1 \cup P_2 \cup P_3 \cup \cdots$ satisfies the theorem. □

Now we are ready to prove (1.2).

(3.9) *Let G be a cohesive plane graph. Then there exists a 1-way infinite path P in G such that every P-bridge of G is finite and has at most three attachments.*

Proof. Let S be the set of vertices of G of infinite valency, and let H, N, Q or H, $(B_1, v_1, B_2, v_2, \ldots)$, Q be as in (2.3) or (2.4), respectively. If H is a block and N is a radial net we choose $u = v \in V(H)$ arbitrarily (in this case $S = \emptyset$), otherwise let $u, v \in V(Q)$ be such that $S \subseteq \{u, v\}$, and $\{u, v\} \subseteq V(B_1) - \{v_1\}$ if H is a linear graph. Let P be as in (3.7) or (3.8); we claim that P satisfies the theorem. To prove this let B be a P-bridge of G. If B contains no edge of H then B is an H-bridge of G, and so is finite and has at most three attachments by the properties of H. Otherwise $B \cap H$ is a P-bridge of H, because $S \subseteq V(P)$ and every H-bridge has at most one attachment not in S by (2.3iv) or (2.4iv). If $B \cap H$ contains no edge of Q, then $B = B \cap H$ by (2.3iii) or (2.4iii) and it has at most three attachments by the choice of P. We may therefore assume that $B \cap H$ contains an edge of Q, and so it has at most two attachments. Let $Q' = Q$ if H is a block, and let $Q' = Q \backslash \{v_1, v_2, \ldots\}$ otherwise. Then $B \cap Q$ is a subset of a component of $Q' \backslash S$, and hence B has at most three attachments by (2.3iv) or (2.4iv) because $S \subseteq V(P)$. □

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