The Strong Perfect Graph Theorem

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Abstract

A graph G is *perfect* if for every induced subgraph H, the chromatic number of H equals the size of the largest complete subgraph of H, and G is *Berge* if no induced subgraph of G is an odd cycle of length at least five or the complement of one.

The "strong perfect graph conjecture" (Berge, 1961) asserts that a graph is perfect if and only if it is Berge. A stronger conjecture was made recently by Conforti, Cornuéjols and Vušković — that every Berge graph either falls into one of a few basic classes, or admits one of a few kinds of separation (designed so that a minimum counterexample to Berge's conjecture cannot have either of these properties).

In this paper we prove both these conjectures.

1 Introduction

We begin with definitions of some of the terms we use which may be nonstandard. All graphs in this paper are finite and simple. The *complement* $\overline{G}$ of a graph G has the same vertex set as G, and distinct vertices u, v are adjacent in $\overline{G}$ just when they are not adjacent in G. A *hole* of G is an induced subgraph of G which is a cycle of length at least 4. An *antihole* of G is an induced subgraph of G whose complement is a hole in $\overline{G}$. A graph G is *Berge* if every hole and antihole of G has even length.

A *clique* in G is a subset X of $V(G)$ such that every two members of X are adjacent. A graph G is *perfect* if for every induced subgraph H of G, the chromatic number of H equals the size of the largest clique of H. The study of perfect graphs was initiated by Claude Berge, partly motivated by a problem from information theory (finding the "Shannon capacity" of a graph — it lies between the size of the largest clique and the chromatic number, and so for a perfect graph it equals both). In particular, in 1961 Berge [1] proposed two celebrated conjectures about perfect graphs. Since the second implies the first, they were known as the "weak" and "strong" perfect graph conjectures respectively, although both are now theorems, the following:

1.1 *The complement of every perfect graph is perfect.*

1.2 *A graph is perfect if and only if it is Berge.*

The first was proved by Lovász [16] in 1972. The second, the strong perfect graph conjecture, received a great deal of attention over the past 40 years, but remained open until now, and is the main theorem of this paper.

It is easy to see that every perfect graph is Berge, and so to prove 1.2 it remains to prove the converse. By a *minimum imperfect graph* we mean a counterexample to 1.2 with as few vertices as possible (in particular, any such graph is Berge and not perfect). Much of the published work on 1.2 falls into two classes: proving that the theorem holds for graphs with some particular graph excluded as an induced subgraph, and investigating the structure of minimum imperfect graphs. For the latter, linear programming methods have been particularly useful; there are rich connections between perfect graphs and linear and integer programming (see [5, 20] for example).

But a third approach has been developing in the perfect graph community over a number of years; the attempt to show that every Berge graph either belongs to some well-understood basic class of (perfect) graphs, or admits some feature that a minimum imperfect graph cannot admit. Such a result would therefore prove that no minimum imperfect graph exists, and consequently prove 1.2. Our main result is of this type, and our first goal is to state it.

Thus, let us be more precise, and we start with two definitions. We say that G is a *double split graph* if $V(G)$ can be partioned into four sets $\{a_1, \ldots, a_m\}$, $\{b_1, \ldots, b_m\}$, $\{c_1, \ldots, c_n\}$, $\{d_1, \ldots, d_n\}$ for some $m, n \geq 2$, such that:

- a_i is adjacent to b_i for $1 \leq i \leq m$, and c_j is nonadjacent to d_j for $1 \leq j \leq n$
- there are no edges between $\{a_i, b_i\}$ and $\{a_{i'}, b_{i'}\}$ for $1 \leq i < i' \leq m$, and all four edges between $\{c_j, d_j\}$ and $\{c_{j'}, d_{j'}\}$ for $1 \leq j < j' \leq n$
- there are exactly two edges between $\{a_i, b_i\}$ and $\{c_j, d_j\}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$, and these two edges have no common end.

(The name is because such a graph can be obtained from what is called a “split graph” by doubling each vertex). The *line graph* $L(G)$ of a graph G has vertex set the set $E(G)$ of edges of G, and $e, f \in E(G)$ are adjacent in $L(G)$ if they share an end in G. Let us say a graph G is *basic* if either G or $\overline{G}$ is bipartite or is the line graph of a bipartite graph, or is a double split graph. (Note that if G is a double split graph then so is $\overline{G}$.) It is easy to see that all basic graphs are perfect. (For bipartite graphs it is trivial; for line graphs of bipartite graphs it is a theorem of König [15]; for their complements it follows from Lovász’ theorem 1.1, although originally these were separate theorems of König; and for double split graphs we leave it to the reader.)

Now we turn to the various kinds of “features” that we will prove exist in every Berge graph that is not basic. They are all decompositions of one kind or another, so henceforth we call them that. If $X \subseteq V(G)$ we denote the subgraph of G induced on X by $G|X$. First, a special case of the “2-join” due to Cornuéjols and Cunningham [13] — a *proper 2-join* in G is a partition (X_1, X_2) of $V(G)$ such that there exist disjoint nonempty $A_i, B_i \subseteq X_i$ $(i = 1, 2)$ satisfying:

- every vertex of A_1 is adjacent to every vertex of A_2, and every vertex of B_1 is adjacent to every vertex of B_2,
- there are no other edges between X_1 and X_2,
- for $i = 1, 2$, every component of $G|X_i$ meets both A_i and B_i, and
- for $i = 1, 2$, if $|A_i| = |B_i| = 1$ and $G|X_i$ is a path joining the members of A_i and B_i, then it has odd length ≥ 3.

(Thanks to Kristina Vušković for pointing out that we could include the “odd length” condition above with no change to the proof.)

If $X \subseteq V(G)$ and $v \in V(G)$, we say v is *X-complete* if v is adjacent to every vertex in X (and consequently $v \notin X$), and v is *X-anticomplete* if v has no neighbours in X. If $X, Y \subseteq V(G)$ are disjoint, we say X is *complete* to Y (or the pair (X, Y) is *complete*) if every vertex in X is Y-complete; and being *anticomplete* to Y is defined similarly. Our second decomposition is a slight variation of the “homogeneous pair” of Chvátal and Sbihi [7] — a *proper homogeneous pair* in G is a pair of disjoint nonempty subsets (A, B) of $V(G)$, such that, if A_1, A_2 respectively denote the sets of all A-complete vertices and all A-anticomplete vertices in $V(G)$, and B_1, B_2 are defined similarly, then:

- $A_1 \cup A_2 = B_1 \cup B_2 = V(G) \setminus (A \cup B)$ (and in particular, every vertex in A has a neighbour in B and a nonneighbour in B, and vice versa)
- the four sets $A_1 \cap B_1, A_1 \cap B_2, A_2 \cap B_1, A_2 \cap B_2$ are all nonempty.

A *path* in G is an *induced* subgraph of G which is non-null, connected, not a cycle, and in which every vertex has degree ≤ 2 (this definition is highly nonstandard, and we apologise, but it avoids writing “induced” about 600 times), and an *antipath* is an induced subgraph whose complement is a path. The *length* of a path is the number of edges in it (and the length of an antipath is the number of edges in its complement). We therefore recognize paths and antipaths of length 0. If P is a path, P^* denotes the set of internal vertices of P, called the *interior* of P; and similarly for antipaths. Let A, B be disjoint subsets of $V(G)$. We say the pair (A, B) is *balanced* if there is no odd path between nonadjacent vertices in B with interior in A, and there is no odd antipath between adjacent vertices in A with interior in B. A set $X \subseteq V(G)$ is *connected* if $G|X$ is connected (so $\emptyset$ is connected); and *anticonnected* if $\overline{G}|X$ is connected.

The third kind of decomposition we use is due to Chvátal [6] — a *skew partition* in G is a partition (A, B) of $V(G)$ such that A is not connected and B is not anticonnected. Despite their elegance, skew partitions pose a difficulty that the other two decompositions do not, for it has not been shown that a minimum imperfect graph cannot admit a skew partition; indeed, this is a well-known open question, first raised by Chvátal [6], the so-called "skew partition conjecture". We get around it by confining ourselves to balanced skew partitions, which do not present this difficulty. (Another difficulty posed by skew partitions is that they are not really "decompositions" in the sense of being the inverse of a composition operation, but that does not matter for our purposes.)

We shall prove the following (the proof is the contents of sections 2 - 24).

1.3 *For every Berge graph G, either G is basic, or one of G, $\overline{G}$ admits a proper 2-join, or G admits a proper homogeneous pair, or G admits a balanced skew partition.*

There is in fact only one place in the entire proof that we use the homogeneous pair outcome (in the proof of 13.4), and it is natural to ask if homogeneous pairs are really needed. In fact they can be eliminated; one of us (Chudnovsky) showed in her PhD thesis [3, 4] that the following holds:

1.4 *For every Berge graph G, either G is basic, or one of G, $\overline{G}$ admits a proper 2-join, or G admits a balanced skew partition.*

But the proof of 1.4 is very long (it consists basically of reworking the proof of this paper for more general structures than graphs where the adjacency of some pairs of vertices is undecided) and cannot be given here, so in this paper we accept proper homogeneous pairs.

All nontrivial double split graphs admit skew partitions, so if we delete "balanced" from 1.3 then we no longer need to consider double split graphs as basic — four basic classes suffice. Unfortunately, nontrivial double split graphs do not admit balanced skew partitions, and general skew partitions are not good enough for the application to 1.2, so we have to do it the way we did.

Let us prove that 1.3 implies 1.2. For that, we need one lemma, the following. (A maximal connected subset of a nonempty set $A \subseteq V(G)$ is called a *component* of A, and a maximal anticonnected subset is called an *anticomponent* of A.) The lemma following is related to results of [14] that were used by Roussel and Rubio in their proof [23] of 2.1.

1.5 *If G is a minimum imperfect graph, then G admits no balanced skew partition.*

Proof. Suppose that (A, B) is a balanced skew partition of G, and let B_1 be an anticomponent of B. Let G' be the graph obtained from G by adding a new vertex z with neighbour set B_1.

(1) *G' is Berge.*

For suppose not. Then in G' there is an odd hole or antihole using z. Suppose first that there is an odd hole, C say. Then the neighbours of z in C (say x, y) belong to B_1, and no other vertex of B_1 is in C. No vertex of $B \setminus B_1$ is in C since it would be adjacent to x, y and C would have length 4; so $C \setminus z$ is an odd path of G, with ends in B_1 and with interior in A, contradicting that (A, B) is balanced. So we may assume there is no such C. Now assume there is an odd antihole D in G', again using z. Then exactly two vertices of $D \setminus z$ are nonadjacent to z, so all the others belong to B_1. Hence in G there is an odd antipath Q of length ≥ 3, with ends $x, y \notin B_1$ and with interior in B_1. Since both x and y have nonneighbours in the interior of Q it follows that $x, y \notin B$; and so $x, y \in A$, again contradicting that (A, B) is balanced. This proves (1).

For a subset X of $V(G)$, we denote the size of the largest clique in X by $\omega(X)$. Let $\omega(B_1) = s$, and $\omega(A \cup B) = t$. Since G is minimum imperfect it cannot be t-coloured. Let $A_1, \ldots, A_m$ be the components of A.

(2) *For $1 \leq i \leq m$ there is a subset $C_i \subseteq A_i$ such that $\omega(C_i \cup B_1) = s$ and*

$$\omega((A_i \setminus C_i) \cup (B \setminus B_1)) \leq t - s.$$

For let $H = G'|(B \cup A_i \cup \{z\})$; then H is Berge, by (1). Now by [6], there are at least two vertices of G not in H (all the vertices in $A \setminus A_i$), and since H has only one new vertex it follows that $|V(H)| < |V(G)|$. From the minimality of $|V(G)|$ we deduce that H is perfect. Now a theorem of Lovász [16] shows that replicating a vertex of a perfect graph makes another perfect graph; so if we replace z by a set Z of $t - s$ vertices all complete to B_1 and to each other, and with no other neighbours in $A_i \cup B$, then the graph we make is perfect. From the construction, the largest clique in this graph has size $\leq t$, and so it is t-colourable. Since Z is a clique of size $t - s$, we may assume that colours $1, \ldots, s$ do not occur in Z, and colours $s+1, \ldots, t$ do. Since B_1 is complete to Z, colours $s+1, \ldots, t$ do not occur in B_1, and so only colours $1, \ldots, s$ occur in B_1; and since $\omega(B_1) = s$, all these colours do occur in B_1. Since B_1 is complete to $B \setminus B_1$, none of colours $1, \ldots, s$ occur in $B \setminus B_1$. Let C_i be the set of vertices $v \in A_i$ with colours $1, \ldots, s$. Then $C_i \cup B_1$ has been coloured using only s colours, and so $\omega(C_i \cup B_1) = s$; and the remainder of $H \setminus z$ has been coloured using only colours $s+1, \ldots, t$, and so

$$\omega((A_i \setminus C_i) \cup (B \setminus B_1)) \leq t - s.$$

This proves (2).

Now let $C = B_1 \cup C_1 \cup \cdots \cup C_m$ and $D = V(G) \setminus C$. Since there are no edges between different A_i's, it follows from (2) that $\omega(C) = s$, and similarly $\omega(D) \leq t - s$. Since $|C|, |D| < |V(G)|$ it follows that $G|C, G|D$ are both perfect; so they are s-colourable and $(t-s)$-colourable, respectively. But then G is t-colourable, a contradiction. (Alternatively we could apply lemma 2.2 of [14].) Thus there is no such (A, B). This proves 1.5. ∎

Proof of 1.2, assuming 1.3.
Suppose that there is a minimum imperfect graph G. Thus G is Berge and not perfect. Every basic graph is perfect, and so G is not basic. It is shown in [13] that G does not admit a proper 2-join. From Lovász's theorem 1.1, it follows that $\overline{G}$ is also a minimum imperfect graph, and therefore $\overline{G}$ also does not admit a proper 2-join. It is shown in [7] that G does not admit a proper homogeneous pair, and G does not admit a balanced skew partition by 1.5. It follows that G violates 1.3, and therefore there is no such graph G. This proves 1.2. ∎

There were a series of statements like 1.3 conjectured over the past twenty years (although they were mostly unpublished, and were unknown to us when we were working on 1.3.) Let us sketch the course of evolution, kindly furnished to us by a referee. A *star cutset* is a skew partition (A, B) such that some vertex of B is adjacent to all other vertices of B. An *even pair* means a pair of vertices u, v in a graph such that every path between them has even length. It was known [2, 6, 18] that no minimum imperfect graph admits a star cutset or an even pair, and the earlier versions of 1.3 involved these concepts. For instance, in Reed's PhD thesis [19], the following conjecture appears:

1.6 Conjecture: *For every perfect graph G, either one of $G, \overline{G}$ is a line graph of a bipartite graph, or one of them has a star cutset or an even pair.*

Reed also studied the same question for Berge graphs, and researchers at that time were considering using general skew partitions instead of star cutsets (although this would not by itself imply 1.2, since the skew partition conjecture was still open).

A counterexample to all these versions of the conjecture was obtained in the early 1990's by Irena Rusu. At about the same time, Conforti, Cornuéjols and Rao [9] proved a statement analogous to 1.3 for the class of bipartite graphs in which every induced cycle has length a multiple of four, and their theorem involved 2-joins. Since Cornuéjols and Cunningham [13] had already proved that no minimum imperfect graph admits a 2-join, it was natural to add 2-joins to the arsenal.

At a conference in Princeton in 1993, Conforti and Cornuéjols gave a series of talks on their work; and in working sessions at the conference (particularly one in which Irena Rusu presented her counterexample), new variants of the conjecture were discussed, including the following:

1.7 Conjecture: *For every Berge graph G, either*

- *one of $G, \overline{G}$ is a line graph of a bipartite graph, or*
- *one of $G, \overline{G}$ admits a 2-join, or*
- *G admits a skew partition, or*
- *one of $G, \overline{G}$ has an even pair.*

More recently, Conforti, Cornuéjols and Vušković [10] proposed a similar conjecture, with the "even pair" alternative replaced by "one of $G, \overline{G}$ is bipartite", although without explicitly listing a proposed set of decompositions. Our result 1.3 is essentially a version of this conjecture, except that we only accept skew partitions that are balanced (and therefore need a fifth basic class) and also we include homogeneous pairs.

How can we prove a theorem of the form of 1.3? There are several other theorems of this kind in graph theory — for example, [7, 10, 17, 21, 22, 24, 25] and others. All these theorems say that "every graph (or matroid) not containing an object of type X either falls into one of a few basic classes or admits a decomposition". And for each of these theorems, the proof is basically a combination of the same two methods (below, we say "graph" and "subgraph", although the objects and containment relations vary depending on the context):

- We judiciously choose an explicit X-free graph H (*X-free* means not containing a subgraph of type X) that does not fall into any of the basic classes; check that it has a decomposition of the kind it is supposed to have; and show that this decomposition extends to a decomposition of every bigger X-free graph containing H. That proves that the theorem is true for all X-free graphs that contain H, so now we may focus on the X-free graphs that do not contain H.

- We choose a graph J, in one of the basic classes and "decently-connected", whatever that means in the circumstances. Let G be a bigger X-free graph containing J that we still need to understand. Enlarge J to a maximal subgraph K of G that is still decently-connected and belongs to the same basic class as J. We can assume that $K \neq G$, for otherwise G satisfies the theorem. Making use of the maximality of K, we prove that the way the remainder of G attaches to K is sufficiently restricted that we can infer a decomposition of G. Now we may focus on the X-free graphs that do not contain J.

It turns out that these two methods can be used for Berge graphs, in just the same way. We need about twelve iterations of this process.

The paper is organized as follows. The next three sections develop tools that will be needed all through the paper. Section 2 concerns a fundamental lemma of Roussel and Rubio; we give several variations and extensions of it, and more in section 3, of a different kind. In section 4 we develop some features of skew partitions, to make them easier to handle in the main proof, which we begin in section 5. Sections 5-8 prove that every Berge graph containing a "substantial" line graph as an induced subgraph, satisfies 1.3 ("substantial" means a line graph of a bipartite subdivision of a 3-connected graph J, with some more conditions if $J = K_4$). Section 9 proves the same thing for line graphs of subdivisions of K_4 that are not "substantial" — this is where double split graphs come in. In section 10 we prove that Berge graphs containing an "even prism" satisfy 1.3. (To prove this we may assume we are looking at a Berge graph that does not contain the line graph of a subdivision of K_4, for otherwise we could apply the results of the earlier sections. The same thing happens later — at each step we may assume the current Berge graph does not contain any of the subgraphs that were handled in earlier steps.) Sections 11-13 do the same for "long odd prisms", and section 14 does the same for a subgraph we call the "double diamond". Section 15 is a break for resharpening the tools we proved in the first four sections, and in particular, here we prove Chvátal's skew partition conjecture [6], that no minimum imperfect graph admits a skew partition. (Or almost – Chvátal actually conjectured that no *minimal* imperfect graph admits a skew partition, and we only prove it here for *minimum* imperfect graphs. But that is all we need, and of course the full conjecture of Chvátal follows from 1.2.) Section 16 proves that any Berge graph containing what we call an "odd wheel" satisfies 1.3, in sections 17-23 we prove the same for wheels in general, and finally in section 24 we handle Berge graphs not containing wheels.

These steps are summarized more precisely in the next theorem, which we include now in the hope that it will be helpful to the reader, although some necessary definitions have not been given yet — for the missing definitions, the reader should see the appropriate section(s) later. Let $\mathcal{F}_1, \ldots, \mathcal{F}_{11}$ be the classes of Berge graphs defined as follows (each is a subclass of the previous class):

- $\mathcal{F}_1$ is the class of all Berge graphs G such that for every bipartite subdivision H of K_4, every induced subgraph of G isomorphic to $L(H)$ is degenerate

- $\mathcal{F}_2$ is the class of all graphs G such that $G, \overline{G} \in \mathcal{F}_1$ and no induced subgraph of G is isomorphic to $L(K_{3,3})$

- $\mathcal{F}_3$ is the class of all Berge graphs G such that for every bipartite subdivision H of K_4, no induced subgraph of G or of $\overline{G}$ is isomorphic to $L(H)$

- $\mathcal{F}_4$ is the class of all $G \in \mathcal{F}_3$ such that no induced subgraph of G is an even prism

- $\mathcal{F}_5$ is the class of all $G \in \mathcal{F}_3$ such that no induced subgraph of G or of $\overline{G}$ is a long prism
- $\mathcal{F}_6$ is the class of all $G \in \mathcal{F}_5$ such that no induced subgraph of G is isomorphic to a double diamond
- $\mathcal{F}_7$ is the class of all $G \in \mathcal{F}_6$ such that G and $\overline{G}$ do not contain odd wheels
- $\mathcal{F}_8$ is the class of all $G \in \mathcal{F}_7$ such that G and $\overline{G}$ do not contain pseudowheels
- $\mathcal{F}_9$ is the class of all $G \in \mathcal{F}_8$ such that G and $\overline{G}$ do not contain wheels
- $\mathcal{F}_{10}$ is the class of all $G \in \mathcal{F}_9$ such that, for every hole C in G of length ≥ 6, no vertex of G has three consecutive neighbours in C, and the same holds in $\overline{G}$
- $\mathcal{F}_{11}$ is the class of all $G \in \mathcal{F}_{10}$ such that every antihole in G has length 4.

1.8 *(The steps of the proof of 1.3)*

1. *For every Berge graph G, either G is a line graph of a bipartite graph, or G admits a proper 2-join or a balanced skew partition, or $G \in \mathcal{F}_1$; and (consequently) either one of $G, \overline{G}$ is a line graph of a bipartite graph, or one of $G, \overline{G}$ admits a proper 2-join, or G admits a balanced skew partition, or $G, \overline{G} \in \mathcal{F}_1$.*
2. *For every G with $G, \overline{G} \in \mathcal{F}_1$, either $G = L(K_{3,3})$, or G admits a balanced skew partition, or $G \in \mathcal{F}_2$.*
3. *For every $G \in \mathcal{F}_2$, either G is a double split graph, or one of $G, \overline{G}$ admits a proper 2-join, or G admits a balanced skew partition, or $G \in \mathcal{F}_3$.*
4. *For every $G \in \mathcal{F}_1$, either G is an even prism with $|V(G)| = 9$, or G admits a proper 2-join or a balanced skew partition, or $G \in \mathcal{F}_4$.*
5. *For every G such that $G, \overline{G} \in \mathcal{F}_4$, either one of $G, \overline{G}$ admits a proper 2-join, or G admits a proper homogeneous pair, or G admits a balanced skew partition, or $G \in \mathcal{F}_5$.*
6. *For every $G \in \mathcal{F}_5$, either one of $G, \overline{G}$ admits a proper 2-join, or G admits a balanced skew partition, or $G \in \mathcal{F}_6$.*
7. *For every $G \in \mathcal{F}_6$, either G admits a balanced skew partition, or $G \in \mathcal{F}_7$.*
8. *For every $G \in \mathcal{F}_7$, either G admits a balanced skew partition, or $G \in \mathcal{F}_8$.*
9. *For every $G \in \mathcal{F}_8$, either G admits a balanced skew partition, or $G \in \mathcal{F}_9$.*
10. *For every $G \in \mathcal{F}_9$, either G admits a balanced skew partition, or $G \in \mathcal{F}_{10}$.*
11. *For every $G \in \mathcal{F}_{10}$, either $G \in \mathcal{F}_{11}$ or $\overline{G} \in \mathcal{F}_{11}$.*
12. *For every $G \in \mathcal{F}_{11}$, either G admits a balanced skew partition, or G is complete or bipartite.*

The twelve statements of 1.8 are proved in 5.1, 5.2, 9.6, 10.6, 13.4, 14.3, 16.3, 18.7, 23.2, 23.4, 23.5, and 24.1 respectively.

2 The Roussel-Rubio lemma

There is a beautiful and very powerful theorem of [23] which we use many times throughout the paper. (We proved it independently, in joint work with Carsten Thomassen, but Roussel and Rubio found it earlier.) Its main use is to show that in some respects, the common neighbours of an anticonnected set of vertices (in a Berge graph) act like or almost like the neighbours of a single vertex.

If $X \subseteq V(G)$, we say an edge uv is *X-complete* if u, v are both X-complete. Let P be a path in G (we remind the reader that this means P is an induced subgraph which is a path), of length ≥ 2, and let the vertices of P be $p_1, \ldots, p_n$ in order. A *leap* for P (*in* G) is a pair of nonadjacent vertices a, b of G such that there are exactly six edges of G between a, b and $V(P)$, namely $ap_1, ap_2, ap_n, bp_1, bp_{n-1}, bp_n$.

The Roussel-Rubio lemma (slightly reformulated for convenience) is the following.

2.1 *Let G be Berge, let $X \subseteq V(G)$ be anticonnected, and P be a path in $G \setminus X$ with odd length, such that both ends of P are X-complete. Then either:*

1. *some edge of P is X-complete, or*
2. *P has length ≥ 5 and X contains a leap for P, or*
3. *P has length 3 and there is an odd antipath with interior in X, joining the internal vertices of P.*

This has a number of corollaries that again we shall need throughout the paper, and in this section we prove some of them.

2.2 *Let G be Berge, let X be an anticonnected subset of $V(G)$, and P be a path in $G \setminus X$ with odd length, such that both ends of P are X-complete, and no edge of P is X-complete. Then every X-complete vertex of G has a neighbour in P^*.*

Proof. Let v be X-complete. Certainly P has length > 1, since its ends are X-complete and therefore nonadjacent. Suppose first it has length > 3. Then by 2.1, X contains a leap, and so there is a path Q with ends in X and with $Q^* = P^*$. Then v is adjacent to both ends of Q, and since $G|(V(Q) \cup \{v\})$ is not an odd hole, it follows that v has a neighbour in $Q^* = P^*$, as required. Now suppose P has length 3, and let its vertices be $p_1\text{-}\cdots\text{-}p_4$ in order. By 2.1, there is an odd antipath Q between p_2 and p_3 with interior in X. Since Q cannot be completed to an odd antihole via $p_3\text{-}v\text{-}p_2$, it follows that v is adjacent to one of p_2, p_3, as required. ∎

Here is another easy lemma that gets used enough that it is worth stating separately.

2.3 *Let G be Berge, let $X \subseteq V(G)$ be anticonnected, and let P be a path or hole in $G \setminus X$. Let Q be a subpath of P (and hence of G) with both ends X-complete. Then either the number of X-complete edges in Q has the same parity as the length of Q, or the ends of Q are the only X-complete vertices in P. In particular, if P is a hole, then either there are an even number of X-complete edges in P, or there are exactly two X-complete vertices and they are adjacent.*

Proof. The second assertion follows from the first. For the first, we use induction on the length of Q. If some internal vertex of Q is X-complete then the result follows from the inductive hypothesis, so we may assume not. If Q has length 1 or even then the theorem holds, so we may assume its length is ≥ 3 and odd. We may assume that there is an X-complete vertex v say of P that is not an end of Q, and therefore does not belong to Q; and since P is a path or hole, it follows that v has no neighbour in Q^*, contrary to 2.2. This proves 2.3. ∎

A *triangle* in G is a set of three vertices, mutually adjacent. We say a vertex v can be *linked* onto a triangle $\{a_1, a_2, a_3\}$ (via paths P_1, P_2, P_3) if:

- the three paths P_1, P_2, P_3 are mutually vertex-disjoint
- for $i = 1, 2, 3$ a_i is an end of P_i
- for $1 \leq i < j \leq 3$, $a_i a_j$ is the unique edge of G between $V(P_i)$ and $V(P_j)$
- v has a neighbour in each of P_1, P_2 and P_3.

The following is well-known and quite useful:

2.4 *Let G be Berge, and suppose v can be linked onto a triangle $\{a_1, a_2, a_3\}$. Then v is adjacent to at least two of a_1, a_2, a_3.*

Proof. Let v be linked via paths P_1, P_2, P_3. For $1 \leq i \leq 3$, v has a neighbour in P_i; let P_i be the path from v to a_i with interior in $V(Q_i)$. At least two of Q_1, Q_2, Q_3 have lengths of the same parity, say Q_1, Q_2; and since $G|(V(Q_1) \cup V(Q_2))$ is not an odd hole, it is a cycle of length 3, and the claim follows. ∎

A variant of 2.2 is sometimes useful, the following:

2.5 *Let G be Berge, let $X \subseteq V(G)$, and let P be a path in $G \setminus X$ of odd length, with vertices be p_1-$\cdots$-p_n, such that p_1, p_n are X-complete, and no edge of P is X-complete. Let $v \in V(G)$ be X-complete. Then either v is adjacent to one of p_1, p_2, or the only neighbour of v in P^* is p_{n-1}.*

Proof. By 2.2, v has a neighbour in P^*, and we may assume that p_{n-1} is not its only such neighbour, so v has a neighbour in $\{p_2, \ldots, p_{n-2}\}$. If P has length ≤ 3 then the result follows, so we may assume its length is at least 5. By 2.1, there is a leap a, b for P in X; so there is a path a-p_2-$\cdots$-p_{n-1}-b. Now $\{p_1, p_2, a\}$ is a triangle, and v can be linked onto it via the three paths b-p_1, $P \setminus \{p_1, p_{n-1}, p_n\}$, a; and so v has two neighbours in the triangle, by 2.4, and the claim follows. ∎

2.6 *If G is Berge and $A, B \subseteq V(G)$ are disjoint, and $v \in V(G) \setminus (A \cup B)$, and v is complete to B and anticomplete to A, then (A, B) is balanced.*

The proof is clear.

2.7 *Let (A, B) be balanced in a Berge graph G. Let $C \subseteq V(G) \setminus (A \cup B)$. Then :*

1. *if A is connected and every vertex in B has a neighbour in A, and A is anticomplete to C, then (C, B) is balanced*

2. *if B is anticonnected and no vertex in A is B-complete, and B is complete to C, then (A, C) is balanced.*

Proof. The first statement follows from the second by taking complements, so it suffices to prove the second. Suppose $u, v \in A$ are adjacent and joined by an odd antipath P with interior in C. Since B is anticonnected and u, v both have non-neighbours in B, they are also joined by an antipath Q with interior in B, which is even since (A, B) is balanced. But then u-P-v-Q-u is an odd antihole, a contradiction. Now suppose there are nonadjacent $u, v \in C$, joined by an odd path P with interior in A. Then P has length ≥ 5, since otherwise its vertices could be reordered to be an odd antipath of the kind we already handled. The ends of P are B-complete, and no internal vertex is B-complete, and so B contains a leap for P by 2.1; and hence there is an odd path with ends in B and interior in A, which is impossible since (A, B) is balanced. This proves 2.7. ∎

We already said what we mean by linking a vertex onto a triangle, but now we do the same for an anticonnected set. We say an anticonnected set X can be *linked* onto a triangle $\{a_1, a_2, a_3\}$ (via paths P_1, P_2, P_3) if:

- the three paths P_1, P_2, P_3 are mutually vertex-disjoint
- for $i = 1, 2, 3$ a_i is an end of P_i
- for $1 \leq i < j \leq 3$, $a_i a_j$ is the unique edge of G between $V(P_i)$ and $V(P_j)$
- each of P_1, P_2 and P_3 contains an X-complete vertex.

There is a corresponding extension of 2.4, the following:

2.8 *Let G be Berge, let X be an anticonnected set, and suppose X can be linked onto a triangle $\{a_1, a_2, a_3\}$ via P_1, P_2, P_3. For $i = 1, 2, 3$ let P_i have ends a_i and b_i, and let b_i be the unique vertex of P_i that is X-complete. Then either at least two of P_1, P_2, P_3 have length 0 (and hence two of a_1, a_2, a_3 are X-complete) or one of P_1, P_2, P_3 has length 0 and the other two have length 1 (say P_3 has length 0); and in this case, every X-complete vertex in G is adjacent to one of a_1,a_2.*

Proof. Some two of P_1, P_2, P_3 have lengths of the same parity, say P_1 and P_2. Hence the path $Q = b_1$-P_1-a_1-a_2-P_2-b_2 (with the obvious meaning - we shall feel free to specify paths by whatever notation is most convenient) is odd, and its ends are X-complete, and none of its internal vertices are X-complete. If Q has length 1 then the theorem holds, so we assume it has length ≥ 3. By 2.2, every X-complete vertex has a neighbour in Q^*, and since b_3 is X-complete, it follows that $b_3 = a_3$. Hence we may assume both P_1 and P_2 have length ≥ 1 for otherwise the claim holds. Suppose that Q has length 3. Then P_1 and P_2 have length 1, and the claim holds again. So we may assume (for a contradiction) that Q has length ≥ 5, and from the symmetry we may assume P_1 has length ≥ 2. Since b_3 is not adjacent to the end b_1 of Q or to its neighbour in Q, and yet it has at least two neighbours in Q^* (namely a_1 and a_2), this contradicts 2.5. This proves 2.8. ∎

As we said earlier, the main use of 2.1 is to show that the common neighbours of an anticonnected set behave in some respects like the neighbours of a single vertex. From this point of view, 2.1 itself tells us something about when there can be an odd "pseudohole", in which one "vertex" is actually an anticonnected set. We also need a version of this when there are two such vertices, the following.

2.9 *Let G be Berge, and let X,Y be disjoint nonempty anticonnected subsets of $V(G)$, complete to each other. Let P be a path in $G \setminus (X \cup Y)$ with even length > 0, with vertices $p_1, \ldots, p_n$ in order, such that p_1 is the unique X-complete vertex of P and p_n is the unique Y-complete vertex of P. Then either:*

1. *P has length ≥ 4 and there are nonadjacent $x_1, x_2 \in X$ such that x_1-p_2-$\cdots$-p_n-x_2 is a path, or*
2. *P has length ≥ 4 and there are nonadjacent $y_1, y_2 \in Y$ such that y_1-p_1-$\cdots$-p_{n-1}-y_2 is a path, or*
3. *P has length 2 and there is an antipath Q between p_2 and p_3 with interior in X, and an antipath R between p_1 and p_2 with interior in Y, and exactly one of Q,R has odd length.*

In each case, either $(V(P \setminus p_1), X)$ or $(V(P \setminus p_n), Y)$ is not balanced.

Proof. It follows from the hypotheses that X,Y and $V(P)$ are mutually disjoint. If P has length 2, choose an antipath Q between p_2 and p_3 with interior in X, and an antipath R between p_1 and p_2 with interior in Y. Then p_2-Q-p_3-p_1-R-p_2 is an antihole, and so exactly one of Q,R has odd length and the theorem holds. So we may assume P has length ≥ 4. We may assume that $V(G) = V(P) \cup X \cup Y$, by deleting any other vertices. Let G' be obtained from $G \setminus Y$ by adding a new vertex y with neighbour set $X \cup \{p_n\}$. Let P' be the path p_1-$\cdots$-p_n-y of G'. Then P' has odd length ≥ 5. If G' is Berge then by 2.1 there is a leap for P' in X, and the result follows. So we may assume G' is not Berge.

Assume first that there is an odd hole C of length ≥ 7 in G'. It necessarily uses y, and the neighbours of y in C are Y-complete, and no other vertices of $C \setminus y$ are Y-complete. Hence there is an odd path Q in $G \setminus Y$ of length ≥ 5, with both ends Y-complete and no internal vertices Y-complete. So the ends of Q belong to $X \cup \{p_n\}$ and its interior to $V(P) \setminus \{p_n\}$. By 2.1 Y contains a leap for Q; so there is an odd path R of length ≥ 5 with ends (y_1, y_2 say) in Y and with interior in $V(P) \setminus \{p_n\}$. Since R cannot be completed to a hole via y_2-p_n-y_1 it follows that p_n has a neighbour in R^*, and so p_{n-1} belongs to R. If also p_1 belongs to R then the theorem holds, so we may assume it does not. Since R is odd and P is even it follows that p_2 also does not belong to R, and so p_1 has no neighbour in R^*; yet the ends of R are X-complete and its internal vertices are not, contrary to 2.2. This completes the case when there is an odd hole in G' of length ≥ 7.

Since an odd hole of length 5 is also an odd antihole, we may assume that there is an odd antihole in G', say D. Again D must use y, and uses exactly two nonneighbours of y; so in G there is an odd antipath Q between adjacent vertices of $P \setminus p_n$ (say u and v), and with interior in $X \cup \{p_n\}$. Since u and v are not Y-complete, they are also joined by an antipath R with interior in Y, and R must also be odd since its union with Q is an antihole. Since R cannot be completed to an antihole via v-p_n-u it follows that p_n is adjacent to one of u,v, and hence we may assume that $u = p_{n-2}$ and $v = p_{n-1}$. Since P has length ≥ 4 it follows that u,v are also joined by an antipath with interior in X, say S, and again S is odd since its union with R is an antihole. But S can be completed to an antihole via v-p_1-u, a contradiction. This proves 2.9. ∎

Next we need a version of 2.1 for holes. Let C be a hole in G, and let $e = uv$ be an edge of it. A *leap* for C (*in* G, *at* uv) is a leap for the path $C \setminus e$ in $G \setminus e$. A *hat* for C (*in* G, *at* uv) is a vertex of G adjacent to u and v and to no other vertex of C.

2.10 *Let G be Berge, let $X \subseteq V(G)$ be anticonnected, let C be a hole in $G \setminus X$ with length > 4, and let $e = uv$ be an edge of C. Assume that u, v are X-complete and no other vertex of C is X-complete. Then either X contains a hat for C at uv, or X contains a leap for C at uv.*

Proof. Let the vertices of C be $p_1, \ldots, p_n$ in order, where $u = p_1$ and $v = p_n$. Let $G_1 = G|(V(C) \cup X)$, and let $G_2 = G_1 \setminus e$. If G_2 is Berge, then from 2.1 applied to the path $C \setminus e$ in G_2 it follows that X contains a leap for C at uv. So we may assume that G_2 is not Berge. Consequently it has an odd hole or antihole D say, and since D is not an odd hole or antihole in G_1 it must use both p_1 and p_n. Suppose first that D is an odd hole. Since every vertex in X is adjacent to both p_1 and p_n it follows that at most one vertex of X is in D; and since $G_2 \setminus X$ has no cycles, there is exactly one vertex of X in D, say x. Hence $D \setminus x$ is a path of $G_2 \setminus X$ between p_1 and p_n, and so $D \setminus x = C \setminus e$; and since D is a hole of G_2 it follows that x has no neighbours in $\{p_2, \ldots, p_{n-1}\}$, and therefore is a hat as required. Next assume that D is an antihole. Since it uses both p_1 and p_n, and they are nonadjacent in G_2, it follows that they are consecutive in D, so the vertices of D can be numbered $d_1, \ldots, d_m$ in order, where $d_1 = p_1$ and $d_m = p_n$, and therefore $m \geq 5$. Consequently, both d_2 and d_{m-1} are not in X, since they are not complete to $\{p_1, p_n\}$, and therefore d_1, d_2, d_{m-1}, d_m are vertices of C. Yet $d_1 d_{m-1}, d_{m-1} d_2, d_2 d_m$ are edges of G_1, which is impossible since $n \geq 6$. This proves 2.10. ∎

There is an analogous version of 2.9, as follows.

2.11 *Let G be Berge, and let X, Y be disjoint nonempty anticonnected subsets of $V(G)$, complete to each other. Let P be a path in $G \setminus (X \cup Y)$ with even length ≥ 4, with vertices $p_1, \ldots, p_n$ in order, such that p_1 is the unique X-complete vertex of P, and p_1, p_n are the only Y-complete vertices of P. Then either:*

1. *there exists $x \in X$ non-adjacent to all of $p_2, \ldots, p_n$, or*
2. *there are nonadjacent $x_1, x_2 \in X$ such that x_1-p_2-$\cdots$-p_n-x_2 is a path.*

Proof. The proof is similar to that of 2.9. We may assume $V(G) = V(P) \cup X \cup Y$. Let G' be obtained from $G \setminus Y$ by adding a new vertex y with neighbour set $X \cup \{p_1, p_n\}$. If G' is Berge then the result follows from 2.10, so we may assume G' is not Berge. Assume first that there is an odd hole C of length ≥ 7 in G'. Hence there is an odd path Q in $G \setminus Y$ of length ≥ 5, with both ends Y-complete and no internal vertices Y-complete. So the ends of Q belong to $X \cup \{p_1, p_n\}$ and its interior to $V(P^*)$. By 2.1 Y contains a leap for Q; so there is an odd path R of length ≥ 5 with ends (y_1, y_2 say) in Y and with interior in $V(P^*)$. Since R is odd and R^* is a subpath of the even path P^*, it follows that not both p_2 and p_{n-1} belong to R; but then R can be completed to an odd hole via one of y_2-p_n-y_1 , y_2-p_1-y_1, a contradiction. This completes the case when there is an odd hole in G' of length ≥ 7, so now we may assume that there is an odd antihole in G', say D. Again D must use y, and uses exactly two nonneighbours of y; so in G there is an odd antipath Q between adjacent vertices of P^* (say u and v), and with interior in $X \cup \{p_n\}$. Since u and v are not Y-complete, they are also joined by an antipath R with interior in Y, and R must also be odd since its union with Q is an antihole. Since one of p_1,p_n is nonadjacent to both of u, v, we may complete R to an odd antihole via one of u-p_1-v,u-p_n-v, a contradiction. This proves 2.11. ∎

3 Paths and antipaths meeting

Another class of applications of 2.1 is to the situation when a long path or hole meets a long antipath or antihole. In this section we prove a collection of useful lemmas of this type. First, a neat application of 2.1 (we include this only because it is striking — in fact we do not use it at all).

3.1 *Let G be Berge, let C be a hole in G, and D an antihole in G, both of length ≥ 8. Then $|V(C) \cap V(D)| \leq 3$.*

Proof. It is easy to see that $|V(C) \cap V(D)| \leq 4$, without using that G is Berge. Suppose that $|V(C) \cap V(D)| = 4$; then $V(C) \cap V(D)$ is the vertex set of a 3-edge path. Let C have vertices $p_1, \dots, p_m$ in order, and D have vertices $q_1, \dots, q_n$ in order, where $m, n \geq 8$ and $p_1 = q_2, p_2 = q_4, p_3 = q_1, p_4 = q_3$. Let P be the path p_4-p_5-$\cdots$-p_m-p_1, and Q the antipath q_4-q_5-$\cdots$-q_n-q_1. Let X be the interior of Q. Then p_1 and p_4 are X-complete (since D is an antihole), and P is a path with length odd and ≥ 5 between these two vertices. If some vertex p_i say in the interior of P is X-complete, then since p_i is nonadjacent to both p_2 and p_3 we can complete Q to an odd antihole via q_1-p_i-q_4, a contradiction. So by 2.1 X contains a leap for P; so there exists i with $5 \leq i < n$ and a path P' joining q_i and q_{i+1} with the same interior as P. Since $n \geq 8$, either $i > 5$ or $i + 1 < n$ and from the symmetry we may assume the first. But then P' can be completed to an odd hole via q_{i+1}-p_2-q_i, a contradiction. This proves 3.1. ∎

The next two lemmas are results of the same kind:

3.2 *Let p_1-$\cdots$-p_m be a path in a Berge graph G. Let $2 \leq s \leq m-2$, and let p_s-q_1-$\cdots$-q_n-p_{s+1} be an antipath, where $n \geq 2$ is odd. Assume that each of $q_1, \dots, q_n$ has a neighbour in $\{p_1, \dots, p_{s-1}\}$ and a neighbour in $\{p_{s+2}, \dots, p_m\}$. Then either:*

- *$s \geq 3$ and the only nonedges between $\{p_{s-2}, p_{s-1}, p_s, p_{s+1}, p_{s+2}\}$ and $\{q_1, \dots, q_n\}$ are $p_{s-1}q_n$, $p_sq_1, p_{s+1}q_n$, or*
- *$s \leq m-3$ and the only nonedges between $\{p_{s-1}, p_s, p_{s+1}, p_{s+2}, p_{s+3}\}$ and $\{q_1, \dots, q_n\}$ are $p_sq_1, p_{s+1}q_n, p_{s+2}q_1$.*

Proof. The antipath p_s-q_1-$\cdots$-q_n-p_{s+1} is even, of length ≥ 4; all its vertices have neighbours in $\{p_1, \dots, p_{s-1}\}$ except p_{s+1}, and they all have neighbours in $\{p_{s+2}, \dots, p_m\}$ except p_s. Since the sets $\{p_1, \dots, p_{s-1}\}, \{p_{s+2}, \dots, p_m\}$ are both connected and are anticomplete to each other, it follows from 2.9 applied in $\overline{G}$ and the symmetry that we may assume that there are adjacent vertices $u, v \in \{p_1, \dots, p_{s-1}\}$ such that u-p_s-q_1-$\cdots$-q_n-v is an antipath. Since v is adjacent to p_s and to u it follows that $s \geq 3$, $v = p_{s-1}$ and $u = p_{s-2}$. Since p_{s-2}-p_s-q_1-$\cdots$-q_n-p_{s-1} is an odd antipath of length ≥ 5, and its ends are anticomplete to $\{p_{s+1}, \dots, p_m\}$ and its internal vertices are not, it follows from 2.1 applied in $\overline{G}$ that there are adjacent $w, x \in \{p_{s+1}, \dots, p_m\}$ such that w-p_s-q_1-$\cdots$-q_n-x is an antipath. Since x is adjacent to p_s and to w it follows that $x = p_{s+1}$ and $w = p_{s+2}$. But then the theorem holds. This proves 3.2. ∎

3.3 *Let G be Berge, let C be a hole in G of length ≥ 6, with vertices $p_1, \dots, p_m$ in order, and let Q be an antipath with vertices $p_1, q_1, \dots, q_n, p_2$, with length ≥ 4 and even. Let $z \in V(G)$, complete to $V(Q)$ and with no neighbours among $p_3, \dots, p_m$. There is at most one vertex in $\{p_3, \dots, p_m\}$ complete to either $\{q_1, \dots, q_{n-1}\}$ or $\{q_2, \dots, q_n\}$, and any such vertex is one of p_3, p_m.*

Proof. It follows that none of $q_1, \dots, q_n$ belong to C, since they are all adjacent to z. Let $X = \{q_1, \dots, q_n\}$, and let Y_1, Y_2 be the sets of vertices in $\{p_3, \dots, p_m\}$ complete to $X \setminus \{q_n\}, X \setminus \{q_1\}$ respectively.

(1) *$Y_1 \subseteq Y_2 \cup \{p_m\}$, and $Y_2 \subseteq Y_1 \cup \{p_3\}$.*

For suppose some $p_i \in Y_1$, and is not in Y_2; then since the odd antipath $Q \setminus p_2$ cannot be completed to an odd antihole via q_n-p_i-p_1, it follows that $i = m$. This proves (1).

(2) *If $Y_1 \not\subseteq \{p_m\}$ then $p_3 \in Y_1 \cap Y_2$, and if $Y_2 \not\subseteq \{p_3\}$ then $p_m \in Y_1 \cap Y_2$.*

For assume $Y_1 \not\subseteq \{p_m\}$, and choose i with $3 \leq i \leq m-1$ minimum so that $p_i \in Y_1$. By (1), $p_i \in Y_2$, so we may assume $i > 3$, for otherwise the claim holds. If i is odd, then the path p_2-p_3-$\cdots$-p_i is odd and between $X \setminus \{q_n\}$-complete vertices, and no internal vertex is $X \setminus \{q_n\}$-complete, and yet the $X \setminus \{q_n\}$-complete vertex z does not have a neighbour in its interior, contrary to 2.2. So i is even. The path p_i-$\cdots$-p_m-p_1 is therefore odd, and has length ≥ 3, and its ends are $X \setminus \{q_1\}$-complete, and the $X \setminus \{q_1\}$-complete vertex z does not have a neighbour in its interior; so by 2.2 some vertex v of its interior is in Y_2, and therefore in $Y_1 \cap Y_2$ by (1). But the path z-$p_2 \cdots$-p_i is odd, and between X-complete vertices, and has no more such vertices in its interior, and v has no neighbour in its interior, contrary to 2.2. This proves (2).

Now not both p_3, p_m are in $Y_1 \cap Y_2$, for otherwise Q could be completed to an odd antihole via p_2-p_m-p_3-p_1. Hence we may assume $p_3 \notin Y_1 \cap Y_2$, and so from (2), $Y_1 \subseteq \{p_m\}$. By (1), $Y_2 \subseteq \{p_3\} \cup Y_1$, and so $Y_1 \cup Y_2 \subseteq \{p_3, p_m\}$. We may therefore assume that $Y_1 \cup Y_2 = \{p_3, p_m\}$, for otherwise the theorem holds. In particular, $p_3 \in Y_2$. If also $p_m \in Y_2$, then p_3-p_4-$\cdots$-p_m is an odd path between $X \setminus \{q_1\}$-complete vertices, and none of its internal vertices are $X \setminus \{q_1\}$-complete, and yet the $X \setminus \{q_1\}$-complete vertex z does not have a neighbour in its interior, contrary to 2.2. So $p_m \notin Y_2$, and so $p_m \in Y_1$; but then p_3-q_1-q_2-$\cdots$-q_n-p_m-p_3 is an odd antihole, a contradiction. This proves 3.3. ∎

4 Skew partitions

In the main proof (which starts in the next section), it happens quite frequently that we can identify a skew partition in G, and what we really want is to show that G admits a balanced skew partition. In this section we prove several lemmas to facilitate that process.

4.1 *Let G be Berge, and suppose that G admits a skew partition (A, B) such that either some component of A or some anticomponent of B has only one vertex. Then G admits a balanced skew partition.*

Proof. By taking complements if necessary we may assume that for some $a_1 \in A$, $\{a_1\}$ is a component of A. Let N be the set of vertices of G adjacent to a_1; so $N \subseteq B$. Assume first that N is not anticonnected. Then $(V(G) \setminus N, N)$ is a skew partition of G, and it is easy to check that it is balanced, as required. So we may assume that N is anticonnected. Consequently N is a subset of some anticomponent of B, say B_1. Choose $b_2 \in B \setminus B_1$. Then $N' = N \cup \{b_2\}$ is not anticonnected, and so $(V(G) \setminus N', N')$ is a skew partition of G, and once again it is easily checked to be balanced. This proves 4.1. ∎

Let us say a skew partition (A, B) of G is *loose* if either some vertex in B has no neighbour in some component of A, or some vertex in A is complete to some anticomponent of B. In the main proof later in the paper, many of the skew partitions we construct are loose, and so the next lemma is very useful.

4.2 *If G is Berge, and admits a loose skew partition, then it admits a balanced skew partition.*

Proof. Let (A, B) be a loose skew partition of G. By taking complements if necessary, we may assume that some vertex in B has no neighbour in some component of A. With G fixed, let us choose the skew partition (A, B) and a component A_1 of A and an anticomponent B_1 of B with $|B| - 2|B_1|$ minimum, such that some vertex in B_1 (say b_1) has no neighbour in A_1. (We call this property the "optimality" of (A, B).) Let the other components of A be $A_2, \ldots, A_m$, and the other anticomponents of B be $B_2, \ldots, B_n$. By 4.1 we may assume that no $|A_i|$ or $|B_j| = 1$, and in this case we shall show that the skew partition (A, B) is balanced.

(1) *For $2 \leq j \leq n$, no vertex in A is B_j-complete and not B_1-complete, and every vertex in $B \setminus B_1$ has a neighbour in A_1.*

For the first claim, assume some vertex $v \in A$ is B_2-complete and not B_1-complete, say. Let $A_1' = A_1$ if $v \notin A_1$, and let A_1' be a maximal connected subset of $A_1 \setminus \{v\}$ otherwise. (So A_1' is nonempty since we assumed $|A_1| \geq 2$.) Let $A' = A \setminus \{v\}$ and $B' = B \cup \{v\}$; then B_2 is still an anticomponent of B', so (A', B') is a skew partition, violating the optimality of (A, B) (for since v is not B_1-complete, there is an anticomponent of B' including $B \cup \{v\}$). For the second claim, assume that some vertex $v \in B_2$ say has no neighbour in A_1. Then since $|B_2| \geq 2$, it follows that $(A \cup \{v\}, B \setminus \{v\})$ is a skew partition of G, again violating the optimality of (A, B). This proves (1).

By 2.6, the pair (A_1, B_j) is balanced, for $2 \leq j \leq n$, since b_1 is complete to B_j and has no neighbours in A_1. By (1) and 2.7.1, it follows that (A_i, B_j) is balanced for $2 \leq i \leq m$ and $2 \leq j \leq n$. It remains to check all the pairs (A_i, B_1). Let $1 \leq i \leq m$, and let A_i' be the set of vertices in A_i that are not B_1-complete. By (1), no vertex in A_i' is B_2-complete, and (A_i', B_2) is balanced, and hence by 2.7.2, so is (A_i', B_1), and consequently so is (A_i, B_1). This proves that (A, B) is balanced, and so completes the proof of 4.2. ∎

4.3 *Let (A, B) be a skew partition of a Berge graph G. If either:*

- *there exist $u, v \in B$ joined by an odd path with interior in A, and joined by an even path with interior in A, or*
- *there exist $u, v \in A$ joined by an odd antipath with interior in B, and joined by an even antipath with interior in B,*

then (A, B) is loose and therefore G admits a balanced skew partition.

Proof. By taking complements we may assume that the first case of the theorem applies. There is an even path P_1 and an odd path P_2 joining u, v, both with interior in A. Let A_1 be the component of A including the interior of P_1. Since $P_1 \cup P_2$ is not a hole, it follows that P_2 also has interior in A_1. Let A_2 be another component of A. If u, v are joined by a path with interior in A_2, then its union with one of P_1, P_2 would be an odd hole, a contradiction; so there is no such path. Hence one of u, v has no neighbours in A_2, and hence (A, B) is loose, and the theorem follows from 4.2. This proves 4.3. ▮

If (A, B) is a skew partition of G, and A' is a component of A, and B' is an anticomponent of B, we call the pair (A', B') a *path pair* if there is an odd path in G with ends nonadjacent vertices of B' and with interior in A'; and (A', B') is an *antipath pair* if there is an odd antipath in G with ends adjacent vertices of A' and with interior in B'.

4.4 *Let (A, B) be a skew partition of a Berge graph G, and let $A_1, \dots, A_m$ be the components of A, and $B_1, \dots, B_n$ the anticomponents of B. Then either:*

- *(A, B) is loose or balanced, or*
- *(A_i, B_j) is a path pair for all i, j with $1 \le i \le m$ and $1 \le j \le n$, or*
- *(A_i, B_j) is an antipath pair for all i, j with $1 \le i \le m$ and $1 \le j \le n$.*

Proof. We may assume (A, B) is not loose and not balanced.

(1) *If for some i, j there is an odd path of length ≥ 5 with ends in B_j and interior in A_i, then the theorem holds.*

For assume there is such a path for $i = j = 1$ say. Let this path, P_1 say, have vertices b_1-p_1-p_2-$\dots$-p_n-b'_1, where $b_1, b'_1 \in B_1$ and $p_1, \dots, p_n \in A_1$. Let $2 \le j \le n$. Then P_1 is an odd path of length ≥ 5 between common neighbours of B_j, and no internal vertex of it is B_j-complete since (A, B) is not loose. By 2.1, B_j contains a leap; so there exist nonadjacent $b_j, b'_j \in B_j$ such that b_j-p_1-p_2-$\dots$-p_n-b'_j is a path. Hence (A_1, B_j) is a path pair. Now let $2 \le i \le m$ and $1 \le j \le n$. Since (A, B) is not loose, b_j and b'_j both have neighbours in A_i, and so there is a path P_2 say joining them with interior in A_i; it is odd by 4.3, and so (A_i, B_j) is a path pair. This proves (1).

From (1) we may assume that for all i, j, every odd path of length > 1 with ends in B_j and interior in A_i has length 3; and similarly every odd antipath of length > 1 with ends in A_i and interior in B_j has length 3. Consequently, every path pair is also an antipath pair (because a path of length 3 can be reordered to be an antipath of length 3). We may assume that (A_1, B_1) is a path pair, and so there exist $b_1, b'_1 \in B_1$ and $a_1, a'_1 \in A_1$ such that b_1-a_1-a'_1-b'_1 is a path P_1 say. Let $2 \le i \le m$. Since b_1 and b'_1 both have neighbours in A_i, they are joined by a path with interior in A_i, odd by 4.3 ; and so by (1) it has length 3. Hence there exist $a_i, a'_i \in A_i$ such that b_1-a_i-a'_i-b'_1 is a path. By the same argument in the complement, it follows that for all $1 \le i \le m$ and $2 \le j \le n$, there exist $b_j, b'_j \in B_j$ such that b_j-a_i-a'_i-b'_j is a path. So every pair (A_i, B_j) is both a path and antipath pair. This proves 4.4. ▮

We can reformulate the previous result in a form that is easier to apply, as follows.

4.5 *Let G be Berge. Suppose that there is a partition of $V(G)$ into four nonempty sets X, Y, L, R, such that there are no edges between L and R, and X is complete to Y. If either:*

- *some vertex in $X \cup Y$ has no neighbours in L or no neighbours in R, or*
- *some vertex in $L \cup R$ is complete to X or complete to Y, or*
- *(L, Y) is balanced*

then G admits a balanced skew partition.

Proof. Certainly $(L \cup R, X \cup Y)$ is a skew partition, so by 4.2 we may assume it is not loose, and therefore neither of the first two alternative hypotheses holds. So we assume the third hypothesis holds. Let $A_1, \ldots, A_m$ be the components of $L \cup R$, and let $B_1, \ldots, B_n$ be the anticomponents of $X \cup Y$. Since X, Y, L, R are all nonempty we may assume that $A_1 \subseteq L$, and $B_1 \subseteq X$. By hypothesis, (A_1, B_1) is not a path or antipath pair, and so by 4.4 the skew partition is balanced. This proves 4.5. ∎

Let (A, B) be a skew partition of G. We say that an anticonnected subset W of B is a *kernel* for the skew partition if some component of A contains no W-complete vertex.

4.6 *Let (A, B) be a skew partition of a Berge graph G, and let W be a kernel for it. Let A_1 be a component of A, and suppose that*

- *every pair of nonadjacent vertices of W with neighbours in A_1 are joined by an even path with interior in A*
- *every pair of adjacent vertices of A_1 with nonneighbours in W are joined by an even antipath with interior in B.*

Then G admits a balanced skew partition.

Proof. By 4.2 we may assume (A, B) is not loose. Let the components of A be $A_1, \ldots, A_m$, and the anticomponents of B be $B_1, \ldots, B_n$.

(1) *(A_i, W) is balanced for $1 \leq i \leq m$.*

For this is true by 4.3 if $i = 1$, so assume $i > 1$. From 4.3 there is no odd path between nonadjacent vertices of W with interior in A_i. Suppose there is an odd antipath Q of length > 1, with ends in A_i and interior in W. Then it has length ≥ 5, for otherwise it can be reordered to be an odd path that we have already shown impossible. Now the ends of Q have no neighbours in the connected set A_1, and its internal vertices all have neighbours in A_1; and so by 2.1 in the complement, there is a leap in the complement; that is, there is an antipath with ends in A_1 and with the same interior as Q, which is impossible. This proves (1).

Since W is anticonnected, we may assume that $W \subseteq B_1$. Since (1) restores the symmetry between $A_1, \ldots, A_m$, we may assume that there is no W-complete vertex in A_1. By 4.4 we may assume (A_1, B_2) is a path or antipath pair. Suppose first that it is an antipath pair. Then there is an odd antipath Q_1 of length ≥ 3 with ends in A_1 and interior in B_2. Since its ends both have nonneighbours in W, its ends are also joined by an antipath Q_2 with interior in W, odd by 4.3, contrary to (1). So there is no such Q_1. Hence there is an odd path P with ends in B_2 and interior in A_1, necessarily of length ≥ 5 (since we already did the antipath case). Since the interior of P contains no W-complete vertex, 2.1 implies that W contains a leap; and so there is a path with ends in W with the same interior as P, a contradiction. This proves 4.6. ▮

5 Small attachments to a line graph

We come now to the first of the major steps of the proof. Suppose that G is Berge, and contains as an induced subgraph a substantial line graph $L(H)$. Then in general, G itself can only be basic by being a line graph, so 1.3 would imply that either G is a line graph, or it has a decomposition in accordance with 1.3. Proving a result of this kind is our first main goal, but exactly how it goes depends on what we mean by "substantial". To make the theorem as powerful as possible, we need to weaken what we mean by "substantial" as much as we can; but when $L(H)$ gets very small, all sorts of bad things start to happen. One is that the theorem is not true any more. For instance, when $H = K_{3,3}$ or $K_{3,3} \setminus e$ (the graph obtained from $K_{3,3}$ by deleting one edge), then $L(H)$ is not only a line graph but also the complement of a line graph (indeed, it is isomorphic to its own complement). So $L(H)$ can live happily inside bigger graphs that are complements of line graphs, without inducing any kind of decomposition. The best we can hope for, when $L(H)$ is so small, is therefore to prove that either G is a line graph or the complement of a line graph, or has a decomposition of the kind we like. This works for $L(K_{3,3})$, but for $L(K_{3,3} \setminus e)$ the situation is even worse, because this graph is basic in *three* ways — it is a line graph, the complement of a line graph, and a double split graph. So for Berge graphs G that contain $L(K_{3,3} \setminus e)$, the best we can hope is that either G is a line graph or the complement of one or a double split graph, or it has a decomposition. And that turns out to be true, but it also explains why the small cases will be something of a headache, as the reader will see.

The best way to partition these cases appears to be as follows. If H is a bipartite subdivision of K_4, we say that $L(H)$ is *degenerate* if there is a cycle of H of length four containing the four vertices of H that have degree three in H, and *nondegenerate* otherwise. First we prove the following.

5.1 *Let G be Berge, and assume some nondegenerate $L(H)$ is an induced subgraph of G, where H is a bipartite subdivision of K_4. Then either G is a line graph, or G admits a proper 2-join, or G admits a balanced skew partition. In particular, 1.8.1 holds.*

Now we consider the case when G contains $L(H)$ for some bipartite subdivision H of a 3-connected J, and yet 5.1 does not apply. It turns out that then either $H = K_{3,3}$, or H is a subdivision of K_4 and $L(H)$ is degenerate. The case when $H = K_{3,3}$ is handled by the next theorem.

5.2 *Let G be Berge, and assume it contains $L(K_{3,3})$ as an induced subgraph. Then either:*

- *$G = L(K_{3,3})$, or*
- *for some bipartite subdivision H of K_4, $L(H)$ is nondegenerate and is an induced subgraph of one of $G, \overline{G}$, or*

- *G admits a balanced skew partition.*

In particular, 1.8.2 holds.

The proofs of these two theorems are similar, and we prove them both together. The remaining case, when H is a subdivision of K_4 and $L(H)$ is degenerate, seems to have a different character, and is best handled by a separate argument later.

The proof of the two theorems above is roughly as follows. We choose a 3-connected graph J, as large as possible such that G contains $L(H)$ for some bipartite subdivision H of J. (For the the first theorem, we also assume that $L(H)$ includes some nondegenerate $L(H')$ where H' is a bipartite subdivision of K_4, and for the second theorem, when necessarily $H = K_{3,3}$, we also assume that passing to the complement will not give us a bigger choice of J). Now we investigate how the remainder of G can attach onto $L(H)$. The edges of J correspond to edge-disjoint paths of H, which in turn become vertex-disjoint paths of $L(H)$, which we call "rungs" (we will do the definitions properly later). One thing we find is that the remainder of G can contain alternative rungs — paths that could replace one of the rungs in $L(H)$ to give a new $L(H')$, for some other bipartite subdivision H' of the same graph J. We find it advantageous to assemble all these alternative rungs in one "strip", for each edge of J, and to maximize the union of these strips (being careful that there are no unexpected edges of G between strips). Each strip corresponds to an edge of J, and runs between two sets of vertices (called "potatoes" for now) that correspond to vertices of J. Let the union of the strips be Z say. Again we ask, how does the remainder of G attach onto this "generalized line graph" Z? This turns out to be quite pretty. There are only two kinds of vertices in the remainder of G, vertices with very few neighbours in Z, and vertices with a lot of neighbours. For the first kind, all their neighbours lie either in one of the strips, or in one of the potatoes; and we can show that for any connected set of these "minor" vertices, the union of their neighbours in Z has the same property (they all lie in one strip or in one potato). For the second kind of vertex, they have so many neighbours in Z that all their *non-neighbours* in any one potato lie inside one strip incident with the potato; and the same is true for the union of the nonneighbours of any anticonnected set of these "major" vertices. In other words, every anticonnected set of these major vertices has a great many common neighbours in Z, so many that they separate all the strips from one another, and that is where we find skew partitions. (If there are no major vertices then we need a different argument, but that case is basically easy.)

In this section and the next few, we have to pay for our convention that "path" means "induced path", because here we need paths in the conventional sense, and therefore need to use a different word for them. A *track* P is a non-null connected graph, not a cycle, in which every vertex has degree ≤ 2 ; and its *length* is the number of edges in it. (Its ends and internal vertices are defined in the natural way.) A *track* in a graph H means a subgraph of H (not necessarily induced) which is a track. Note that there is a correspondence between the tracks (with at least one edge) in a graph H and the paths in $L(H)$; the edge-set of a track becomes the vertex-set of a path, and vice versa. And two tracks are vertex-disjoint if and only if the corresponding paths are vertex-disjoint and there is no edge of $L(H)$ between them. However, *the parity changes*; a track in H and the corresponding path in $L(H)$ have lengths differing by one, and therefore of opposite parity.

A *branch-vertex* of a graph H means a vertex with degree ≥ 3; and a *branch* of H means a maximal track P in H such that no internal vertex of P is a branch-vertex. *Subdividing* an edge uv means deleting the edge uv, adding a new vertex w, and adding two new edges uw and wv. Starting with a graph J, the effect of repeatedly subdividing edges is to replace each edge of J by a track joining the same pair of vertices, where these tracks are disjoint except for their ends. We call the graph we obtain a *subdivision* of J. Note that $V(J) \subseteq V(H)$. Let J be a 3-connected graph. (We use the convention that a k-connected graph must have $> k$ vertices.) If H is a subdivision of J then $V(J)$ is the set of branch-vertices of H, and the branches of H are in 1-1 correspondence with the edges of J. We say H is *cyclically 3-connected* if it is a subdivision of some 3-connected graph J. (We remind the reader that in this paper, all graphs are simple by definition.)

In general, if F, K are induced subgraphs of G with $V(F \cap K) = \emptyset$, a vertex in $V(K)$ is said to be an *attachment* of F (or of $V(F)$) if it has a neighbour in $V(F)$. We need the following lemma:

5.3 *Let H be bipartite and cyclically 3-connected. Then either $H = K_{3,3}$, or H is a subdivision of K_4, or H has a subgraph H' such that H' is a subdivision of K_4 and $L(H')$ is nondegenerate.*

Proof. There is a subgraph of H which is a subdivision of K_4, and we may assume that it does not satisfy the theorem. Hence there are tracks $p_1\text{-}\cdots\text{-}p_m$ ($= P$ say) and $q_1\text{-}\cdots\text{-}q_n$ ($= Q$ say) of H, vertex-disjoint, such that $p_1q_1, p_1q_n, p_mq_1, p_mq_n$ are edges, and $m, n \geq 3$ are odd. Suppose every track in H between $\{p_1, \ldots, p_m\}$ and $\{q_1, \ldots, q_n\}$ uses one of the edges $p_1q_1, p_1q_n, p_mq_1, p_mq_n$. Then there are no edges between P and Q except the given four, and for every component F of $H \setminus (V(P) \cup V(Q))$, the set of attachments of F in $V(P) \cup V(Q)$ is a subset of one of $V(P), V(Q)$. Since H is cyclically 3-connected, it follows that H is a subdivision of K_4 and the theorem holds. So we may assume that there is a track R of H, say $r_1\text{-}\cdots\text{-}r_t$, from $V(P)$ to $V(Q)$, not using any of $p_1q_1, p_1q_n, p_mq_1, p_mq_n$. We may assume that $r_1 \in \{p_1, \ldots, p_{m-1}\}$, $r_t \in \{q_1, \ldots, q_{n-1}\}$, and none of $r_2, \ldots, r_{t-1}$ belong to $V(P) \cup V(Q)$. The subgraph H' formed by the edges $E(P) \cup E(Q) \cup E(R) \cup \{p_1q_n, p_mq_1, p_mq_n\}$ (and the vertices of H incident with them) is a subdivision of K_4, and we may assume it does not satisfy the theorem. There is therefore a cycle of H' with vertex set $\{r_1, r_t, p_m, q_n\}$. Since H is bipartite and p_mq_n is an edge, it follows that $t = 2$. Hence not both $r_1 = p_1$ and $r_2 = q_1$, and so $r_1 = p_{m-1}$ and $r_2 = q_{n-1}$. By the same argument with p_1, p_m exchanged, it follows that $r_1 = p_2$, and so $m = 3$, and similarly $n = 3$. Hence there is a subgraph J of H isomorphic to $K_{3,3}$.

It is helpful now to change the notation. Let J have vertex set $\{a_1, a_2, a_3, b_1, b_2, b_3\}$, where a_1, a_2, a_3 are adjacent to b_1, b_2, b_3. Suppose that there is a component F of $H \setminus V(J)$. Since H is cyclically 3-connected, at least two vertices of J are attachments of F. If say a_1, b_1 are attachments, choose a track P between a_1, b_1 with interior in F; then the union of P and $J \setminus \{a_1b_1, a_2b_2\}$ satisfies the theorem. If say a_1, a_2 are attachments of F, choose a track P between a_1, a_2 with interior in F; then the union of P and $J \setminus \{a_1b_1, a_2b_3\}$ satisfies the theorem. So we may assume there is no such F. Since H is bipartite, it follows that $H = J = K_{3,3}$, and so the theorem holds. This proves 5.3. ∎

If G,J are graphs, we say that J *appears* in G if there is a bipartite subdivision H of J so that $L(H)$ is isomorphic to an induced subgraph of G. We call $L(H)$ an *appearance* of J in G. Note that if $L(H)$ is isomorphic to some induced subgraph K of G, there is another subdivision H' isomorphic to H, made from H by replacing each edge of H by the corresponding vertex of K; and now $L(H') = K$ (rather than just being isomorphic to it). So whenever it is convenient we shall assume that the isomorphism between $L(H)$ and K is just equality, without further explanation. Note in particular that $E(H) = V(K)$, and so some vertices of G are edges of H.

When $J = K_4$, we have already defined what we mean by a degenerate appearance of J. When $J \neq K_4$, let us say that an appearance $L(H)$ of J in G is *degenerate* if $J = H = K_{3,3}$, and otherwise it is *nondegenerate*. So all appearances of any graph $J \neq K_4, K_{3,3}$ are nondegenerate. If J is 3-connected, we say a graph J' is a *J-enlargement* if J' is 3-connected, and has a proper subgraph which is isomorphic to a subdivision of J.

Our goal remains to prove 5.1 and 5.2. Before we start on the main argument, let us observe that it suffices to prove the following.

5.4 *Let G be Berge. Let J be a 3-connected graph, such that there is no J-enlargement with a nondegenerate appearance in G. Let $L(H_0)$ be an appearance of J in G, such that if $L(H_0)$ is degenerate, then $H_0 = J = K_{3,3}$ and no J-enlargement appears in $\overline{G}$. Then either $G = L(H_0)$, or $H_0 \neq K_{3,3}$ and G admits a proper 2-join, or G admits a balanced skew partition.*

The proof of this will take several sections, but let us see now that 5.4 implies 5.1 and 5.2.

Proof of 5.1, assuming 5.4.

Let G be Berge, and assume there is a nondegenerate appearance of K_4 in G. Choose a 3-connected graph J maximal (under J-enlargement) such that there is a nondegenerate appearance of J in G; then the hypotheses of 5.4 are satisfied, and the claim follows from 5.4. This proves 5.1. ▮

Proof of 5.2, assuming 5.4.

Let G be Berge, and let $L(H_0)$ be an appearance of $K_{3,3}$ in G, where $H_0 = K_{3,3}$. We may assume that both $G, \overline{G}$ contain no nondegenerate $L(H)$ where H is a bipartite subdivision of K_4. By 5.3, no $K_{3,3}$-enlargement appears in either $G, \overline{G}$. By 5.4, either $G = L(K_{3,3})$, or G admits a balanced skew partition. This proves 5.2. ▮

Now we start on the proof of 5.4. We assume that $L(H)$ is an appearance of J in G, and we need to study how the remaining vertices of G attach to $L(H)$. In the remainder of this section we examine how individual vertices attach to $L(H)$, and how connected sets of minor vertices attach. In the next section we think about anticonnected sets of major vertices.

A vertex in $V(G) \setminus V(L(H))$ has a set of neighbours in $V(L(H))$, that we want to investigate; but this set is more conveniently thought of as a subset of $E(H)$, and we begin with some lemmas about subsets of edges of a graph H.

5.5 *Let H be cyclically 3-connected, and let C, D be subgraphs with $C \cup D = H$, $|V(C \cap D)| \leq 2$, and $V(C), V(D) \neq V(H)$. Then one of C, D is contained in a branch of H.*

The proof is clear.

5.6 *Let c_1, c_2 be nonadjacent vertices of a graph H, such that $H \setminus \{c_1, c_2\}$ is connected. For $i = 1, 2$, let the edges incident with c_i be partitioned into two sets A_i, B_i, where A_1, A_2 are both nonempty and at least one of B_1, B_2 is nonempty. Assume that for every edge $uv \in A_1 \cup A_2$, $H \setminus \{u, v\}$ is connected, and that no vertex of $V(H)$ is incident with all edges in $A_1 \cup A_2$. Then one of the following holds:*

1. *there is a track in H with first edge in A_1, second edge in B_1 (and hence second vertex c_1), last vertex c_2 and last edge in A_2, or*
2. *there is a track in H with first edge in A_2, second edge in B_2 (and hence second vertex c_2), last vertex c_1 and last edge in A_1.*

Proof. For $i = 1, 2$ let X_i be the set of ends (different from c_i) of edges in A_i, and define Y_i similarly for B_i. So by hypothesis, X_1, X_2 are nonempty, $|X_1 \cup X_2| \geq 2$, and we may assume Y_1 is nonempty. Choose $x_1 \in X_1$ such that $X_2 \not\subseteq \{x_1\}$ (this is possible since $|X_1 \cup X_2| \geq 2$). Both Y_1 and X_2 meet the connected graph $H \setminus \{c_1, x_1\}$, and so there is a track in $H \setminus \{c_1, x_1\}$ from Y_1 to $X_2 \cup Y_2$, say P, with vertices $p_1, \ldots, p_n$ say. We may assume that $p_1 \in Y_1$, and no other p_i is in Y_1; and $p_n \in X_2 \cup Y_2$, and no other p_i is in $X_2 \cup Y_2$. In particular it follows that $c_2 \notin V(P)$. Since $x_1 \notin V(P)$ we may assume that $p_n \notin X_2$ (for otherwise the theorem holds), so $p_n \in Y_2$. If any vertex of X_1 is in P then again the theorem holds (since X_2 is nonempty and none of its vertices are in P), so we may assume that P is disjoint from $X_1 \cup X_2$. Since $H \setminus \{c_1, c_2\}$ is connected, there is a minimal track Q in $H \setminus \{c_1, c_2\}$ from $X_1 \cup X_2$ to $V(P)$, and we may assume that only its first vertex (q say) is in $X_1 \cup X_2$. If $q \in X_1 \setminus X_2$, choose $x \in X_2$; if $q \in X_2 \setminus X_1$ choose $x \in X_1$; and if $q \in X_1 \cap X_2$ choose $x \in X_1 \cup X_2$ different from q. Thus we may assume that $q \in X_1$ and there exists $x \in X_2$ different from q and hence not in Q. So $P \cup Q$ contains a path from q to B_2 not containing x, and hence the theorem holds. This proves 5.6. ▮

If v is a vertex of H, the set of edges of H incident with v is denoted by $\delta(v)$ or $\delta_H(v)$. Let H be bipartite and cyclically 3-connected, and let X be some set. We say that X *saturates* $L(H)$ if for every branch-vertex v of H, at most one edge of $\delta_H(v)$ is not in X (or equivalently, for every K_3 subgraph of $L(H)$, at least two of its vertices are in X). When H is connected and bipartite, we speak of vertices having the same or different *biparity* depending whether every track between them is even or odd respectively. Two edges of G are *disjoint* if they have no end in common, and otherwise they *meet*.

5.7 *Let H be bipartite and cyclically 3-connected. Let $X \subseteq E(H)$, such that there is no track in H of even length ≥ 4, with its end-edges in X and with no other edge in X. Then either:*

1. *X saturates $L(H)$, or*
2. *there is a branch-vertex b of H with $X \subseteq \delta(b)$, or*
3. *there is a branch B of H with $X \subseteq E(B)$, or*
4. *there is a branch B of H with ends b_1, b_2 say, such that $X \setminus E(B) = \delta(b_1) \setminus E(B)$, or*
5. *there is a branch B of H of odd length with ends b_1, b_2 say, such that*

$$X \setminus E(B) = (\delta(b_1) \cup \delta(b_2)) \setminus E(B),$$

or

6. *there are two vertices c_1, c_2 of H, of different biparity and not in the same branch of H, such that $X = \delta(c_1) \cup \delta(c_2)$.*

In particular, either statements 1 or 6 hold, or there are at most two branch-vertices of H incident with more than one edge in X; and exactly two only if statement 5 holds.

Proof. The second assertion (the final sentence) follows from the first, because if statements 2,3 or 4 hold then there is at most one branch-vertex incident with more than one edge in X; while if

B, b_1, b_2 are as in statement 5, then since B is odd, it follows that b_1, b_2 have no common neighbour, and so no branch-vertex different from b_1, b_2 is incident with more than one edge in X. So it remains to prove the first assertion.

(1) *We may assume that there are two disjoint edges in X.*

For if not, then by König's theorem, there is a vertex of H incident with every edge in X, and then one of statements 2,3 of the theorem hold. This proves (1).

(2) *If there is a branch B of H such that every edge in X has at least one end in $V(B)$ then the theorem holds.*

For suppose there is a such a branch B, and let $C \subseteq B$ be a track, minimal such that every edge in X has an end in $V(C)$. By (1) we may assume that C has length ≥ 1. Let c_1, c_2 be the ends of C. For $i = 1, 2$ let A_i be the set of edges in $\delta(c_i)$ that are in X and not in C; and let B_i be the set of edges in $\delta(c_i)$ that are not in X and not in C. From the minimality of C, it follows that A_1, A_2 are both nonempty.

Suppose first that c_1, c_2 have the same biparity. Choose $c_i a_i \in A_i$ for $i = 1, 2$, if possible such that $a_1 \neq a_2$. Since c_1, c_2 belong to the same branch of H and H is cyclically 3-connected, it follows that there is a track in $H \setminus \{c_1, c_2\}$ from a_1 to a_2; and therefore there is a track T in H from c_1 to c_2 with end-edges $c_1 a_1$ and $c_2 a_2$. Since c_1, c_2 have the same biparity, it follows that T is even; and since only its end-edges are in X (because every edge in X either belongs to C or is incident with one of c_1, c_2), it follows from the hypothesis of the theorem that T has length 2, that is, $a_1 = a_2$. We deduce that there is a vertex $a \in V(H)$ such that $A_i = \{c_i a\}$ for $i = 1, 2$. Now there is only one branch of H containing c_1 and c_2, since J is simple, so a is not in the interior of a branch, and therefore it is a branch-vertex. Moreover it does not belong to the branch B, for the same reason, and so $C = B$ and c_1, c_2 are branch-vertices. Choose a branch-vertex b of H different from c_1, c_2, a, and choose three paths P_1, P_2, P_3 between b and c_1, c_2, a respectively, pairwise disjoint except for b. So P_1 and P_2 have lengths of the same parity, and P_3 has length of different parity. By (1) we may assume there is an edge in X not incident with a, and any such edge belongs to C; so for $i = 1, 2$ there is a minimal subtrack Q_i of C containing c_i and an edge in X. If $Q_1 = C$ then (since C has even length) $P_1 \cup P_2$ is the interior of an even track with end-edges in X and no internal edges in X, contrary to the hypothesis. So c_2 is not a vertex of Q_1, and similarly c_1 is not in Q_2. From the track Q_1-c_1-P_1-b-P_2-c_2-a and the hypothesis it follows that Q_1 is even; and from the track Q_1-c_1-P_1-b-P_3-a-c_2 and the hypothesis it follows that Q_1 is odd, a contradiction.

We may assume therefore that c_1, c_2 have different biparity. It follows that no vertex of $V(H)$ is incident with all edges in $A_1 \cup A_2$. Let H' be the graph obtained from H by deleting the internal vertices and edges of C. There is no track T in H' with first edge in A_1, second edge in B_1 (and hence second vertex c_1), last vertex c_2 and last edge in A_2; for any such track would be even, since c_1, c_2 have opposite biparity, and have length ≥ 4, and have only its end-edges in X, contrary to the hypothesis. A similar statement holds with c_1, c_2 exchanged. By 5.6 applied to H', it follows that $B_1 \cup B_2 = \emptyset$, and so one of statements 3,4,5 of the theorem holds. This proves (2).

(3) *There do not exist three tracks of H with an end (b say) in common and otherwise vertex-*

disjoint, such that each contains an edge in X, and at least two of the three edges of the tracks incident with b do not belong to X.

For suppose that P_1, P_2, P_3 are three such tracks, where P_i is between a_i and b, for $1 \leq i \leq 3$. We may assume that for each i, the only edge of P_i in X is the edge incident with a_i. Now two of P_1, P_2, P_3 have lengths of the same parity, say P_1, P_2; and their union is an even track with end-edges in X and its other edges not in X. By hypothesis it has length 2, and so P_1, P_2 both have length 1. But then at most one edge of $P_1 \cup P_2 \cup P_3$ incident with b does not belong to X, a contradiction. This proves (3).

(4) *There do not exist a connected subgraph A of $H \setminus X$ and three mutually disjoint edges $x_1, x_2, x_3 \in X$ such that each x_i has at least one end in $V(A)$.*

For suppose such A, x_1, x_2, x_3 exist. We may assume A is a maximal connected subgraph of $H \setminus X$. For $1 \leq i \leq 3$ let x_i have ends a_i, b_i, where a_1, a_2, a_3 have the same biparity. Let K be the graph with vertex set $\{a_1, a_2, a_3, b_1, b_2, b_3\}$, in which two vertices of K are adjacent if there is a track in A joining them not using any other vertex of K. Since A is connected and meets all of x_1, x_2, x_3 it follows that there is a component of K containing an end of each of these three edges. If some two of a_1, a_2, a_3 are adjacent in K, then the corresponding track in A is even, contrary to the hypothesis of the theorem; so a_1, a_2, a_3 are pairwise nonadjacent in K, and similarly so are b_1, b_2, b_3, and therefore all the edges of K join some a_i to some b_j. Also, by (3) it follows that a_3 is not adjacent in K to both b_1 and b_2, and five similar statements. Since there is a component of K containing an end of each of x_1, x_2, x_3, we may assume that $a_1b_3, b_2a_3, a_3b_3 \in E(K)$, and the only other possible edges of K are a_1b_1, a_2b_2, a_2b_1. In particular, there are no more edges of K incident with a_3 or b_3. Let the tracks in A corresponding to $a_1b_3, b_2a_3, a_3b_3 \in E(K)$ be P_1, P_2, P_3 respectively. Since P_3 joins the adjacent vertices a_3, b_3 and does not use the edge x_3, it follows that P_3 has nonempty interior. Choose a maximal connected subgraph S of A including the interior of P_3 and not containing either of a_3, b_3. Since there are no more edges of K incident with a_3 or b_3, it follows that none of a_1, b_1, a_2, b_2 is in $V(S)$, and therefore S is vertex-disjoint from P_1 and P_2 as well. Consequently the only edges of A between $V(S) \cup \{a_3, b_3\}$ and the remainder of H are incident with a_3 or b_3. Since H is cyclically 3-connected and a_3, b_3 are adjacent, it follows that $H \setminus \{a_3, b_3\}$ is connected, and therefore there is an edge sv of H such that $s \in V(S)$ and $v \in V(H) \setminus (V(S) \cup \{a_3, b_3\})$. Since S is maximal it follows that $sv \notin E(A)$; and since A is maximal, it follows that $sv \in X$; and from the symmetry we may assume $v \notin \{a_1, b_1\}$. Choose a minimal track in S between s and the interior of P_3; then it can be extended via a subpath of P_3 and via sv to become a track P_4 in H, of length ≥ 2, from v to b_3, using none of a_1, b_1, a_3, and with only its first edge in X. But then the tracks b_1-a_1-P_1-b_3, P_4, and the one-edge track made by x_3, violate (3). This proves (4).

We may assume that statement 1 of the theorem does not hold, and so there is a branch-vertex of H incident with ≥ 2 edges not in X. Hence there is a connected subgraph A of $H \setminus X$, containing a branch-vertex and at least two edges incident with it. Choose such a subgraph A maximal. It follows that $V(A)$ is not contained in any branch of H. By (4), there is no 3-edge matching among the edges in X that meet $V(A)$; and since this set of edges forms a bipartite subgraph, it follows from König's theorem that there are two vertices $c_1, c_2 \in V(G)$ such that every edge in X with an end in $V(A)$ is incident with one of c_1, c_2. From the maximality of A, every edge of H between $V(A)$

and $V(H) \setminus V(A)$ belongs to X and therefore is incident with one of c_1, c_2; and so there are two subgraphs C, D of H with $V(C) = V(A) \cup \{c_1, c_2\}$, $V(D) = (V(H) \setminus V(A)) \cup \{c_1, c_2\}$ and $C \cup D = H$.

Now $V(C)$ is not contained in a branch of H, because it contains $V(A)$ and we already saw that $V(A)$ is not contained in a branch; and we may assume that $V(D)$ is not contained in a branch by (2), since every edge in X has an end in $V(D)$. But $V(D) \neq V(G)$ since $|V(C)| \geq |V(A)| \geq 3 > |V(C \cap D)|$; and since H is cyclically 3-connected, it follows that $V(C) = V(G)$. Hence every edge in X is incident with one of c_1, c_2. For $i = 1, 2$ let $A_i = \delta(c_i) \cap X$, and let $B_i = \delta(c_i) \setminus A_i$. By (2), we may assume that c_1, c_2 do not belong to the same branch. Consequently c_1, c_2 are nonadjacent, and $H \setminus \{c_1, c_2\}$ is connected, by 5.5. By (1) we may assume that there exist disjoint edges $a_1c_1 \in A_1$ and $a_2c_2 \in A_2$. Take a minimal track in $H \setminus \{c_1, c_2\}$ between a_1, a_2; then by the hypothesis of the theorem, this track has odd length, and so c_1, c_2 have opposite biparity. There is therefore no track T in H with first edge in A_1, second edge in B_1 (and hence second vertex c_1), last vertex c_2 and last edge in A_2; and a similar statement holds with c_1, c_2 exchanged. By 5.6, it follows that $B_1, B_2 = \emptyset$, and therefore statement 6 of the theorem holds. This proves 5.7. ∎

Suppose that $L(H)$ is an appearance of J in G. We recall that H is a subdivision of J, and $L(H)$ is an induced subgraph of G. If $X \subseteq V(L(H)$, we say that X is *local* if either $X \subseteq \delta_H(v)$ for some $v \in V(J)$, or X is a subset of the edge-set of some branch of H. We say a vertex $v \in V(G) \setminus V(L(H))$ is *major* (with respect to $L(H)$) if the set of its neighbours in $L(H)$ saturates $L(H)$.

5.8 *Let G be Berge, let J be a 3-connected graph, and let $L(H)$ be an appearance of J in G. For each vertex v of J, let N_v be the set of edges of H incident with v; and for each edge uv of J, let R_{uv} be the path of $L(H)$ with vertex set the set of edges of the branch of H between u and v. Let $F \subseteq V(G) \setminus V(L(H))$ be connected, such that the set of attachments of F in $L(H)$ is not local. Assume that no member of F is major. Then there is a path P of G with $V(P) \subseteq F$ and with ends p_1 and p_2, such that either:*

1. *there are vertices c_1, c_2 of H, not in the same branch of H, such that for $i = 1, 2$ p_i is complete in G to N_{c_i}, and there are no other edges between $V(P)$ and $V(L(H))$, or*
2. *there is an edge b_1b_2 of J such that one of the following holds (for $i = 1, 2$, r_i denotes the unique vertex in $N_{b_i} \cap V(R_{b_1b_2})$):*
   - *(a) p_1 is adjacent in G to all vertices in $N_{b_1} \setminus \{r_1\}$, and p_2 has a neighbour in $R_{b_1b_2} \setminus r_1$, and every edge from $V(P)$ to $V(L(H)) \setminus \{r_1\}$ is either from p_1 to $N_{b_1} \setminus \{r_1\}$, or from p_2 to $V(R_{b_1b_2}) \setminus \{r_1\}$, or*
   - *(b) for $i = 1, 2$, p_i is adjacent in G to all vertices in $N_{b_i} \setminus \{r_i\}$, and there are no other edges between $V(P)$ and $V(L(H))$ except possibly p_1r_1, p_2r_2, and P has the same parity as $R_{b_1b_2}$, or*
   - *(c) $p_1 = p_2$, and p_1 is adjacent to all vertices in $(N_{b_1} \cup N_{b_2}) \setminus \{r_1, r_2\}$, and all neighbours of p_1 in $V(L(H))$ belong to $N_{b_1} \cup N_{b_2} \cup V(R_{b_1b_2})$, and $R_{b_1b_2}$ is even, or*
   - *(d) $r_1 = r_2$, and for $i = 1, 2$, p_i is adjacent in G to all vertices in $N_{b_i} \setminus \{r_i\}$, and there are no other edges between $V(P)$ and $V(L(H)) \setminus \{r_1\}$, and P is even.*

Proof. We remark that the set N_v is just the set $\delta_H(v)$, but now we are going to think of it as a subset of the vertex set of $L(H)$ and it is convenient to change notation. We may assume F is minimal such that its set of attachments is not local. Let X be the set of attachments of F in $L(H)$. Suppose first that $|F| = 1$, $F = \{y\}$ say. Apply 5.7 to H, X. Now 5.7.1 is false since by hypothesis y is not major, and 5.7.2 and 5.7.3 are false since X is not local. So one of 5.7.4-6 holds, and the claim follows. Consequently we may assume that $|F| \geq 2$.

(1) *There exist two attachments x_1, x_2 of F such that $\{x_1, x_2\}$ is not local.*

For $X \subseteq E(H)$. If there exists $x_1 \in X$ not incident in H with a branch-vertex, and in some branch B, choose any $x_2 \in X$ not in B; then $\{x_1, x_2\}$ is not local. So we may assume that every edge in X is incident with a branch-vertex of H. Choose $x_1 \in X$, in some branch B_1 of H, and incident with a branch-vertex b_1. There exists $x_2 \in X$ not incident with b_1, and we may assume that $x_2 \in E(B_1)$, for otherwise $\{x_1, x_2\}$ is not local. Hence x_2 is incident with the other end b_2 say of B_1. There exists $x_3 \in X$ not belonging to $E(B)$, and it cannot share an end both with x_1 and with x_2, so we may assume x_3 is not incident with b_1. But then $\{x_1, x_3\}$ is not local, as required. This proves (1).

From the minimality of F, it follows that F is minimal such that x_1 and x_2 are both attachments of F, and so (since x_1 and x_2 are nonadjacent), F is the interior of a path with vertices $x_1, p_1, \ldots, p_n, x_2$ in order. Let X_1 be the set of attachments in $L(H)$ of $F \setminus \{p_n\}$, and let X_2 be the attachments of $F \setminus \{p_1\}$. From the minimality of F, X_1 and X_2 are both local.

(2) *If there is an edge uv of J such that $X_1 \subseteq N_u$ and $X_2 \subseteq V(R_{uv})$ then the theorem holds.*

For let the ends of R_{uv} be r_1, r_2 where $r_1 \in N_u$. Since X is not local, it follows that p_1 has a neighbour in $N_u \setminus \{r_1\}$ and p_n has a neighbour in $V(R_{uv}) \setminus \{r_1\}$. If p_1 is adjacent to every vertex in $N_u \setminus \{r_1\}$ then statement 2.a of the theorem holds, so we may assume p_1 has a neighbour s_1 and a nonneighbour s_2 in $N_u \setminus \{r_1\}$. Let Q be the path between r_2 and s_1 with interior in $F \cup V(R_{uv} \setminus \{r_1\})$. Choose $w \in V(J)$ such that $s_1 \in V(R_{uw})$. Now H is a subdivision of a 3-connected graph, so if we delete all edges of H incident with u except s_1, the graph we produce is still connected. Consequently there is a track of H from u to v with first edge s_1; and hence there is a path S_1 of $L(H)$ from s_1 to r_2, vertex-disjoint from $V(R_{uv}) \cup N_u$ except for its ends. Indeed, if we delete from H both the vertex w and all edges incident with u except s_2, the graph remains connected; so there is a path S_2 of $L(H)$ between s_2 and r_2, vertex-disjoint from $R_{uv} \cup N_u \cup V(R_{uw}) \cup N_w$ except for its ends. Now S_1 and S_2 have the same parity since H is bipartite. Yet S_1 can be completed via r_2-Q-s_1 and S_2 can be completed via r_2-Q-s_1-s_2, a contradiction. This proves (2).

(3) *If there are nonadjacent vertices $v_1, v_2 \in V(J)$ such that $X_i \subseteq N_{v_i}$ for $i = 1, 2$, then the theorem holds.*

Let A_1 be the set of vertices in N_{v_1} adjacent to p_1, and $B_1 = N_{v_1} \setminus A_1$; and let A_2 be the set of vertices in N_{v_2} adjacent to p_n, and $B_2 = N_{v_2} \setminus A_2$. So $X = A_1 \cup A_2$. If both B_1 and B_2 are empty then statement 1 of the theorem holds, so we may assume that at least one of B_1,B_2 is nonempty. Certainly A_1 and A_2 are both nonempty, so there is a track in H from v_1 to v_2 with end-edges in A_1 and A_2 respectively. Hence there is a path S_1 in $L(H)$ from A_1 to A_2, vertex-disjoint from $N_{v_1} \cup N_{v_2}$ except for its ends. Since $X = A_1 \cup A_2$ is not local, there is no $w \in V(J)$ with $A_1 \cup A_2 \subseteq N_w$.

Hence we can apply 5.6, and we deduce (possibly after exchanging v_1 and v_2) that there is a path S_2 in $L(H)$ with first vertex in A_1, second vertex in B_1, last vertex in A_2, and otherwise disjoint from $N_{v_1} \cup N_{v_2}$. Since H is bipartite, S_1 and S_2 have opposite parity; but they can both be completed via F, a contradiction. This proves (3).

(4) *If there are adjacent vertices $v_1, v_2 \in V(J)$ such that $X_i \subseteq N_{v_i}$ for $i = 1, 2$, then the theorem holds.*

For $i = 1, 2$ let r_i be the end of $R_{v_1v_2}$ in N_{v_i}. Let A_1 be the set of vertices in $N_{v_1} \setminus \{r_1\}$ adjacent to p_1, and $B_1 = N_{v_1} \setminus (A_1 \cup r_1)$; and define A_2, B_2 similarly. Then $X \subseteq A_1 \cup A_2 \cup \{r_1, r_2\}$. By (2) we may assume that A_1 and A_2 are both nonempty. Suppose that both B_1 and B_2 are empty. There is a cycle in J of length ≥ 4 using the edge v_1v_2, and so there is a path in $L(H)$ of length ≥ 2 from A_1 to A_2 with no internal vertex in $N_{v_1} \cup V(R_{v_1v_2}) \cup N_{v_2}$. The union of this path with $R_{v_1v_2}$ induces a hole, and so does its union with F, and therefore these two paths have lengths of the same parity. Consequently either statement 2.b or 2.d of the theorem holds. So we may assume that at least one of B_1, B_2 is nonempty. There is a path S_1 from A_1 to A_2 with no vertex in $N_{v_1} \cup N_{v_2} \cup V(R_{v_1v_2})$ except for its ends. Suppose that there is no vertex $w \in V(J)$ with $A_1 \cup A_2 \subseteq N_w$. Then we can apply 5.6 to the graph obtained from H by deleting the edges and internal vertices of the branch between v_1 and v_2. We deduce (possibly after exchanging v_1 and v_2) that there is a path S_2 of $L(H)$ with first vertex in A_1, second vertex in B_1, last vertex in A_2, and otherwise disjoint from $N_{v_1} \cup N_{v_2} \cup V(R_{v_1v_2})$. Since H is bipartite, S_1 and S_2 have opposite parity; but they can both be completed via F, a contradiction. Consequently there is a vertex $w \in V(J)$ with $A_1 \cup A_2 \subseteq N_w$. Since H is bipartite, and there is a 2-edge track of H between v_1, v_2 (via w), it follows that the branch of H with ends v_1, v_2 has even length, and therefore $R_{v_1v_2}$ has odd length, and in particular $r_1 \neq r_2$. Since $|N_{v_i} \cap N_w| \leq 1$ (since J is simple) it follows that $|A_i| = 1$, $A_i = \{a_i\}$ say, for $i = 1, 2$. Since X is not local it is not a subset of N_w and so there is a vertex of $R_{v_1v_2}$ in X. Since $X_i \subseteq N_{v_i}$ for $i = 1, 2$, no internal vertex of $R_{v_1v_2}$ is in X, so we may assume that $r_1 \in X$. Since $r_1 \notin N_{v_2}$ it follows that $r_1 \notin X_2$, and hence p_1 is the only vertex in F adjacent to r_1. Now the hole p_1-$\cdots$-p_n-a_2-a_1-p_1 is even, and so n is even. If we delete the vertex v_2 and the edge a_1 from H, what remains is still connected, and so contains a track from w to v_1. Hence there is a path T in $L(H)$ from some $a_3 \in N(w)$ to r_1, disjoint from $N_{v_2} \cup a_1$. But T can be completed to a hole via r_1-$R_{v_1v_2}$-r_2-a_2-a_3 and via r_1-p_1-$\cdots$-p_n-a_2-a_3, and these two completions have different parity, a contradiction. This proves (4).

(5) *If $X_1 \cap X_2$ is nonempty, and in particular if one of $p_2, \ldots, p_{n-1}$ has a neighbour in $L(H)$, then the theorem holds.*

For any neighbour in $L(H)$ of one of $p_2, \ldots, p_{n-1}$ belongs to $X_1 \cap X_2$, so assume $x \in X_1 \cap X_2$. Then $x \in V(R_{v_1v_2})$ for a unique edge v_1v_2 of J, and $x \in N_v$ for at most two $v \in V(J)$, namely v_1 and v_2. Since both X_1 and X_2 are local, each is a subset of one of $N_{v_1}, N_{v_2}, V(R_{v_1v_2})$, and they are not both subsets of the same one. So we may assume that $X_1 \subseteq N_{v_1}$. Hence either $X_2 \subseteq N_{v_2}$ or $X_2 \subseteq V(R_{v_1v_2})$, and therefore the theorem holds by (5) or (2). This proves (5).

(6) *If there is a vertex u and an edge v_1v_2 of J such that $X_1 \subseteq N_u$ and $X_2 \subseteq V(R_{v_1v_2})$ then the theorem holds.*

For by (2) we may assume u is different from v_1 and v_2. Choose a cycle C_1 of H using the branch between v_1 and v_2 and not using u, and choose a minimal track S in $H \setminus \{v_1, v_2\}$ between u and $V(C_1)$. Let the ends of S be u and w say. Hence in $L(H)$ there are three vertex-disjoint paths, from N_{v_1}, N_{v_2},N_u respectively to N_w, and there are no edges between them except in the triangle T formed by their ends in N_w. If p_n has a unique neighbour (say r) in $R_{v_1v_2}$, then r can be linked onto the triangle T, contrary to 2.4. If p_n has two nonadjacent neighbours in $R_{v_1v_2}$, then p_n can be linked onto the triangle T, contrary to 2.4. So p_n has exactly two neighbours in $R_{v_1v_2}$, and they are adjacent. If p_1 is adjacent to all of N_u, then statement 1 of the theorem holds, so we may assume that p_1 has a neighbour and a non-neighbour in N_u. Let A be the neighbours of p_1 in N_u and $B = N_u \setminus A$. In H there is a cycle C_2 using the branch between v_1 and v_2, and using an edge in A and an edge in B. (To see this, divide u into two adjacent vertices, one incident with the edges in A and the other with those in B, and use Menger's theorem to deduce that there are two vertex-disjoint paths between these two vertices and $\{v_1, v_2\}$.) Hence in G, there is a path between N_{v_1} and N_{v_2} using a unique edge of $N(u)$, and that edge is between a vertex $a \in A$ say and some vertex in B. Hence a can be linked onto the triangle formed by p_n and its two neighbours in $R_{v_1v_2}$, a contradiction. This proves (6).

(7) *If there are edges u_1v_1 and u_2v_2 of J with $X_i \subseteq V(R_{u_iv_i})$ for $i = 1, 2$, then the theorem holds.*

For in this case it follows that the edges u_1v_1 and u_2v_2 are different, and hence we may assume that v_2 is different from u_1 and v_1, and v_1 is different from u_2 and v_2; possibly $u_1 = u_2$. If p_1 has exactly two neighbours in $R_{u_1v_1}$ and they are adjacent, and also p_n has exactly two neighbours in $R_{u_2v_2}$ and they are adjacent, then statement 1 of the theorem holds; so we may assume that p_1 has either only one neighbour, or two nonadjacent neighbours, in $R_{u_1v_1}$. There is a cycle in H using the branch between u_1 and v_1, and using u_2 and not v_2 (since $J \setminus v_2$ is 2-connected). There correspond two paths in $L(H)$, say P and Q, from N_{u_1} and N_{v_1} respectively to N_{u_2}, disjoint from each other, and there is a third path R say from p_1 to N_{u_2} via F and a subpath of $R_{u_2v_2}$. There are no edges between these paths except within the triangle T formed by their ends in N_{u_2}. If p_1 has only one neighbour $r \in R_{u_1v_1}$, then we may assume that r is in the interior of $R_{u_1v_1}$, by (6), and so r can be linked onto T, contrary to 2.4. If p_1 has two nonadjacent neighbours in $R_{u_1v_1}$, then p_1 can be linked onto T, again a contradiction. This proves (7).

But (2)-(7) cover all the possibilities for the local sets X_1 and X_2, and so this proves 5.8. ∎

6 Major attachments to a line graph

We continue to study appearances $L(H)$ of a 3-connected graph J in a Berge graph G. In this section we study anticonnected sets of major vertices, and their common neighbours in $L(H)$.

An appearance $L(H)$ of J in G is *overshadowed* if there is a branch B of H with odd length ≥ 3, with ends b_1, b_2, such that some vertex of G is nonadjacent in G to at most one vertex in $\delta_H(b_1)$ and at most one in $\delta_H(b_2)$. Thus for instance an appearance is overshadowed if there is a major vertex and some branch has odd length at least 3. This section is devoted to proving the following.

6.1 *Let G be Berge, let $L(H)$ be an appearance in G of a 3-connected graph J, and let Y be an anticonnected set of major vertices. Assume that the set of all Y-complete vertices in $L(H)$ does not saturate $L(H)$. Then either*

- *$J = K_{3,3}$ or K_4, and there is an overshadowed appearance of J in G, or*
- *$J = K_{3,3}$ or K_4, $L(H)$ is degenerate, and there is an overshadowed appearance of J in $\overline{G}$, or*
- *$J = K_{3,3}$, $L(H)$ is degenerate, and there is a J-enlargement that appears in $\overline{G}$, or*
- *$J = K_4$ and $|V(H)| = 6$, or*
- *$J = K_4$ and $L(H)$ is degenerate, and there exist nonadjacent $y, y' \in Y$ with the following property. Let the 4-cycle in H formed by the branch-vertices of H have edges a-b-c-d in order. Let p be the third edge of H such that a, b, p have a common end, and similarly let b, c, q have a common end, and c, d, r and d, a, s. Then (up to symmetry) the neighbours of y in $L(H)$ are a, b, d, q, r and possibly c, and the neighbours of y' in $L(H)$ are b, c, d, p, s and possibly a.*

Proof. We may assume that Y is minimal such that it is anticonnected and its common neighbours do not saturate $L(H)$. Let X be the set of all Y-complete vertices in $L(H)$. Choose two vertices of $L(H)$, both incident in H with the same branch-vertex of H, and both not in X. Then there is an antipath joining them with interior in Y, and the common neighbours of the interior of this antipath do not saturate $L(H)$. From the minimality of Y it follows that this antipath contains all vertices in Y. Consequently, Y is the vertex set of an antipath with ends y_1,y_2, say. From the hypothesis, $|Y| \geq 2$, since the neighbours of any vertex in Y saturate $L(H)$, so y_1, y_2 are distinct. Now for $i = 1, 2$, $Y \setminus \{y_i\}$ is anticonnected; let X_i be the set of $Y \setminus \{y_i\}$-complete vertices in $L(H)$ that are not in X. So $X \cup X_i$ is the set of all $Y \setminus \{y_i\}$-complete vertices in $L(H)$. From the minimality of Y, both $X \cup X_1$ and $X \cup X_2$ saturate $L(H)$. In terms of H, we see that X, X_1, X_2 are mutually disjoint subsets of $E(H)$, and for every branch-vertex b of H and for $i = 1, 2$, at most one edge of H incident with b does not belong to $X \cup X_i$.

(1) *If the branch-vertices of H form a 4-cycle C and X consists of at most three edges of C, then the theorem holds.*

For in this case H has only four branch-vertices and $J = K_4$. Let the edges of C be a, b, c, d in order, and let p, q, r, s be edges of $H \setminus \{a, b, c, d\}$ such that the sets of edges incident with branch-vertices of H are $\{a, b, p\}, \{b, c, q\}, \{c, d, r\}$ and $\{d, a, s\}$. Since every branch-vertex is incident with at least one edge in X, we may assume that $X = \{b, d\}$ or $\{b, c, d\}$. Since $a, p \notin X$, it follows that one is in X_1 and the other in X_2, say $a \in X_1$ and $p \in X_2$. Similarly, since $a, s \notin X$ it follows that $s \in X_2$. Let P be the path in $L(H)$ between p, r whose vertex set is the edge-set of the branch of H containing p, r, and choose Q containing q, s similarly. Thus P is odd, and so is Q. If they both have length 1 then H has 6 vertices and the fourth outcome of the theorem holds. We may therefore assume that P has length ≥ 3. The path b-p-P-r-d is odd and has length ≥ 5; its ends are Y-complete and its internal vertices are not, so by 2.1, Y contains a leap. Hence there exist nonadjacent $y, y' \in Y$ such that y-r-P-p-y' is a path in G. Since $p \in X_2$ and y is nonadjacent to p it follows that $y = y_2$; and since $s \in X_2$ and $y \neq y'$, it follows that y' is adjacent to s. Now y-r-P-p-y' is an odd path, and it cannot be completed to an odd hole, so y, y' have no common neighbour in

Q. But b-q-Q-s-d is an odd path; its ends are $\{y,y'\}$-complete, and its internal vertices are not, so by 2.1, y,y' form a leap for this path, that is, y-q-Q-s-y' is a path of G. (Note that this holds even if b-q-Q-s-d has length 3, since the anticonnected set in question has cardinality 2.) Since y' is major and nonadjacent to q it follows that y' is adjacent to c, and similarly y is adjacent to a. But then the fifth outcome of the theorem holds. This proves (1).

In the arguments to come there is a certain amount of moving from H to $L(H)$ and back, and to facilitate this, for every subgraph H' of H we denote by $L(H')$ the induced subgraph of $L(H)$ formed by the edges of H'. So for any track P of H, $L(P)$ is a path of $L(H)$. We say a branch-vertex b of H is a *triad* if b is incident with at most one edge in X. It follows that every triad has degree 3 in H, and is incident with exactly one edge in each of X, X_1, X_2.

We recall that Y is the vertex set of an antipath between y_1, y_2; let Q be this antipath. There are two cases, depending whether Q is odd or even.

(2) *If Q is odd then there is no cycle of H with edge-set $\{h_1,h_2,h_3,h_4\}$ in order, such that the common end of h_1 and h_2 is a branch-vertex, $h_1 \in X_1$, $h_2 \in X_2$, and $h_3, h_4 \in X$.*

For if there is such a cycle, then Q can be completed to an odd antihole via y_2-h_2-h_4-f-h_3-h_1-y_1 (where f is a third edge of H such that h_1, h_2, f have a common end), a contradiction. This proves (2).

(3) *If Q is odd and $h_1 \in X_1$ meets $h_2 \in X_2$, then every edge in X meets at least one of h_1, h_2.*

For if $h_1 \in X_1$ meets $h_2 \in X_2$, and $f \in X$ meets neither of h_1, h_2, then Q can be completed to an odd antihole via y_2-h_2-f-h_1-y_1, a contradiction. This proves (3).

There is a branch-vertex b of H incident with at least two edges not in X. For $i = 1,2$ let $e_i \in X_i$ be incident with b, and let e_3 be some third edge incident with b. For $i = 1,2,3$, let B_i be the branch of H containing e_i, and let b_i be its other end. If Q is odd, let $f_i \in X$ be incident with b_i, chosen in addition such that $f_i \notin E(B_i)$ if possible ($1 \leq i \leq 3$). (If Q is even we choose the f_i's a little differently, described later.)

(4) *If Q is odd then b_3 is a triad.*

For suppose not; then $f_3 \notin E(B_3)$, and there is a second edge $f_3' \in X$ incident with b_3. By (3), the edge f_3 meets one of e_1, e_2, and from the symmetry we may assume that it meets e_1. Thus $f_3 = b_1b_3$ and $E(B_1) = \{e_1\}$. Since H is bipartite, it follows that B_3 is even. Thus f_3' is not incident with b, and by (3) applied to f_3', e_1 and e_2 we deduce that $f_3' = b_2b_3$ and $E(B_2) = \{e_2\}$. But the edges e_1, e_2, f_3', f_3 contradict (2). This proves (4).

(5) *If Q is odd and either B_3 has length > 1 or b is not a triad, then the theorem holds.*

For assume that B_3 has length ≥ 2. By (3) applied to e_3 and the two edges of $E(H) \setminus X$ incident with b_3 it follows that B_3 has length two and $f_3 \notin E(B_3)$. (Later we will use the shorthand "by (3) applied to e_3 and b_3".) By (3) applied to f_3, e_1 and e_2 we deduce that f_3 is incident with

e_1 or e_2, and so from the symmetry we may assume that $f_3 = b_1b_3$ and $E(B_1) = \{e_1\}$. Suppose that B_2 has length at least two. By (3) applied to f_2, e_1 and e_2, it follows that either $f_2 = b_1b_2$ or $E(B_2) = \{e_2, f_2\}$; and therefore in both cases b_2, b_3 are nonadjacent, since H is bipartite. But this contradicts (3) applied to f_2 and b_3. It follows that B_2 has length 1, and $E(B_2) = \{e_2\}$. From (3) applied to f_2 and b_3 we deduce that b_2 is adjacent to b_3 and $b_2b_3 \notin X$. The vertex b has degree 3, for a fourth edge incident with b would violate (3) applied to that edge and b_3. Since H is cyclically 3-connected, it follows that H is the union of B_1, B_2, B_3, the edges b_1b_3, b_2b_3 and a branch B with ends b_1 and b_2. The branch B includes f_2, and its edge incident with b_1, say e, is not in X by (3) applied to e and b_3. But e meets f_2, by (3) applied to f_2 and b_1. Thus B has length two, and hence the fourth outcome of the theorem holds. We may therefore assume that $E(B_3) = \{e_3\}$. In this case e_3 is the only member of X incident with b_3, and from (4) with b, b_3 exchanged it follows that b is a triad. This proves (5).

(6) *If Q is odd and one of B_1, B_2 has length > 1 then the theorem holds.*

For suppose first that they both have length at least two. Then, for $i = 1, 2$, by (3) applied to b and f_i we deduce that $E(B_i) = \{e_i, f_i\}$ and therefore f_i is the unique edge of X incident with b_i. This contradicts (3) applied to f_1 and b_2. So at least one of B_1, B_2 has length 1, and from the symmetry we may assume that $E(B_1) = \{e_1\}$ and B_2 has length at least two. If $f_2 \in E(B_2)$ then b_2 is a triad, and the theorem holds by (5) with b, b_2 exchanged, so we may assume that $f_2 \notin E(B_2)$. Let e_2' be the edge of B_2 incident with b_2. By (3) applied to f_2 and b we deduce that $f_2 = b_1b_2$, and that no edge incident with b_2 belongs to X except f_2 and possibly e_2'. By (3) and (4) applied to f_2 and b_3, it follows that b_3 is adjacent to b_2 and $b_2b_3 \notin X$. Suppose for a contradiction that b_1 is not a triad, and choose $e_1' \in X \setminus \{b_1b_2\}$ incident with b_1. By (3) applied to e_1' and b_2, it follows that $e_2' \in X$, and from (3) applied to e_2' and b we deduce that $E(B_2) = \{e_2, e_2'\}$. But now the edges e_1, e_2, e_2', f_2 contradict (2). This proves that b_1 is a triad, and from (4) with b, b_1 exchanged, we deduce that b_2 is a triad. Since H is cyclically 3-connected, it follows that H is the union of B_1, B_2, B_3, the edges b_1b_2 and b_2b_3 and a branch B with ends b_1 and b_3. From (3) applied to b_1, we deduce that no edge of B_2 belongs to X, and by (3) applied to b_2 it follows that no edge of B belongs to X. But then the theorem holds by (1). This proves (6).

(7) *If Q is odd then the theorem holds.*

For by (5) and (6) we may assume that $E(B_i) = \{e_i\}$ for $i = 1, 2, 3$. For $i = 1, 2$ let $f_i = b_ix_i$. Then $x_1 \neq x_2$, for otherwise the edges e_1, e_2, f_2, f_1 violate (2). By (3) applied to f_i and b_3 we deduce that x_i is adjacent to b_3 and $x_ib_3 \notin X$, and therefore f_i is the unique edge in X incident with b_i, and b_i is a triad ($i = 1, 2$). By (3) applied to f_2 and b_1 we deduce that b_1 is adjacent to x_2, and, similarly, x_1 is adjacent to b_2. Since H is a subdivision of a 3-connected graph, $J = K_{3,3}$, and $L(H)$ is a degenerate appearance of J, and there is a J-enlargement that appears in $\overline{G}$, so the third outcome of the theorem holds. This proves (7).

In view of (7) we may henceforth assume that Q is even.

(8) *Every edge in X_1 meets every edge in X_2.*

For if $h_1 \in X_1$ does not meet some $h_2 \in X_2$, then Q can be completed to an odd antihole via y_2-h_2-h_1-y_1, a contradiction. This proves (8).

A vertex of a track P is *penultimate* if it is adjacent in P to an end of P.

(9) *For all $W \in \{X, X \cup X_1, X \cup X_2\}$ and for every even track P in H of length ≥ 4 and with both end-edges and no internal edges in W, every edge in W is incident with a penultimate vertex of P.*

For let $f \in W$. If $W = X$ let $Y' = Y$, and if $W = X \cup X_i$ where $i \in \{1, 2\}$, let $Y' = Y \setminus \{y_i\}$. So W is the set of Y'-complete vertices of $L(H)$. The path $L(P)$ of G is odd and has length ≥ 3; its ends are Y'-complete, and its internal vertices are not. By 2.2, f is adjacent (in G) to vertices in the interior of $L(P)$; that is, f is incident in H with an internal vertex of P. We must show that f is incident with a penultimate vertex. Let P have vertices p_1-$\cdots$-p_n and edges $h_1, \ldots, h_{n-1}$, where h_i is incident with p_i, p_{i+1} for $1 \leq i < n$; so n is odd and $n \geq 5$. Suppose first that both ends of f belong to P, say $f = p_i p_j$ where $i < j$. Since H is bipartite, $j - i$ is odd, and so either $i - 1$ or $n - j$ is odd, and from the symmetry we may assume the former, that is, i is even. Hence the track T with edge-set $\{h_1, \ldots, h_{i-1}, f\}$ has even length, at least 4 (since we may assume that $i \neq 2$); and yet in G the Y'-complete vertex h_{n-1} has no neighbour in the interior of the odd path $L(T)$, contrary to 2.2. So not both ends of f belong to P. Hence f is incident with a unique vertex p_i of P, and again we may assume that $3 \leq i \leq n - 2$. In G, h_1-$\cdots$-h_{i-1}-f is a path; its ends are Y'-complete, and its internal vertices are not, and the Y'-complete vertex h_{n-1} has no neighbour in its interior; so by 2.2, this path is even, that is, i is odd. Since p_i is a branch-vertex of H, and at least two of the edges incident with it do not belong to W, it follows that $W = X$ and $Y' = Y$; and we may assume that $h_{i-1} \in X_1$ and $h_i \in X_2$. Since every edge in X_1 meets every edge in X_2, it follows that $h_1, \ldots, h_{i-2} \notin X_1$. In G, the path h_1-$\cdots$-h_{i-1} is odd; its ends are Y-complete, its internal vertices are not, and the Y-complete vertex h_{n-1} has no neighbour in its interior, so it has length 1, that is, $i = 3$. Similarly $n - i = 2$, that is, $n = 5$. But then Q can be completed to an odd antihole via y_2-h_3-h_1-h_4-h_2-y_1, a contradiction. This proves (9).

(10) *If P_1, P_2, P_3 are tracks in H with a common end v, say, and otherwise vertex-disjoint, each with an edge in X, then at least two of the three edges of $P_1 \cup P_2 \cup P_3$ incident with v belong to X.*

For we may assume that for $i = 1, 2, 3$, P_i is between v and v_i say, and the only edge of P_i in X is the edge incident with v_i. Some two of P_1, P_2, P_3 have lengths of the same parity, say P_1, P_2, and so $P_1 \cup P_2$ is a track of even length. If it has length 2 then P_1, P_2 both have length 1 and the claim holds, so we assume it has length ≥ 4. The edge of P_3 incident with v_3 is incident with a penultimate vertex of this track, by (9), and so P_3 and one of P_1, P_2 have length 1, and again the claim holds. This proves (10).

Earlier (preceding (4)) we chose b such that at least two edges of H incident with b did not belong to X. Let us refine this choice; now in addition we choose b such that B_3 is as long as possible.

(11) *For $i = 1, 2$ there is an edge $f_i \in X$ incident with b_i that does not meet e_3.*

For it suffices to prove this for $i = 1$, and it clearly holds if there are at least two members of X incident with b_1. So we may assume that there is a unique member of X incident with b_1, and that this edge meets e_3, and therefore is the edge b_1b_3. But then b_1 is a triad, and $E(B_3) = \{e_3\}$, and $|E(B_1)| > 1$, because H is bipartite. The unique edge of X_1 incident with b_1 meets e_2 by (8); and hence this edge is b_1b_2, and $e_2 = bb_2$. Suppose for a contradiction that there is a fourth edge bv incident with b, and let f be an edge incident with b_3 different from bb_2, b_1b_2; then $v \neq b, b_1, b_2, b_3$, and there is a track of length 4 with vertices b_3-b_1-b_2-b-v in order; its end-edges belong to X and its internal edges do not; and $f \in X$ is not incident with any penultimate vertex of this track, contrary to (9). This proves that b has degree three. Since H is cycically 3-connected, it follows that H consists of B_1, B_2, B_3, the edges b_1b_2, b_1b_3, and a branch B with ends b_2 and b_3 that includes a member of X incident with b_2. Since H is bipartite, it follows that $|E(B)| > 1$, and hence b_2 and B contradict the choice of b and B_3. This proves (11).

(12) *If there exist f_1, f_2 as in (11) with $f_1, f_2 \neq b_1b_2$ then the theorem holds.*

For it follows from (10) applied to subtracks of the tracks with edge-sets $E(B_1) \cup \{f_1\}$, $E(B_2) \cup \{f_2\}$ and $\{e_3\}$ that B_1, B_2 include no member of X, and that f_1 meets f_2. Thus b_1 is not adjacent to b_2. We claim that for $i = 1, 2$ the edge f_i is the only edge of X incident with b_i. For suppose that say $f_1' \in X$ is incident with b_1. By (10) applied to the vertex b and the tracks with edge-sets $E(B_1) \cup \{f_1'\}$, $E(B_2) \cup \{f_2\}$ and $\{e_3\}$, we deduce that f_1' meets e_3. Thus B_1 is even. Let P be the track obtained from B_1 by adding e_3 and f_1; then P and the edge f_2 violate (9). This proves our claim that f_i is the only edge of X incident with b_i for $i = 1, 2$. Consequently, b_1 and b_2 are triads. From (8) we deduce that B_1 and B_2 have length one. For $i = 1, 2$ let d_i be the edge incident with b_i different from e_i, f_i; so $d_1 \in X_2$ and $d_2 \in X_1$. By (8) the edges d_1, d_2 meet; let v denote their common end. Every edge g incident with v other than d_1 and d_2 belongs to X. If some such g does not meet e_3 then the edges g, d_2, e_2, e_3 form a track with end-edges in X and internal edges not in X, and f_1 is not incident with a penultimate vertex of this track, contrary to (9). So every such edge g meets e_3 and hence is incident with b_3 (since H is bipartite). Thus v has degree two or three. If $v = b_3$, then B_3 has length 2 and both its edges belong to X, and the fourth outcome of the theorem holds. If $v \neq b_3$ and v has degree 3, then the third edge incident with v is vb_3, and b is a triad, and H consists of the vertices b, b_1, b_2, b_3, v and a branch B with ends b_3 and u, where u is the common end of f_1 and f_2; but then $J = K_{3,3}$, and if B has length 1 then the second outcome of the theorem holds, and otherwise the first outcome holds. Finally, if $v \neq b_3$ and v has degree two, then b_3 is the common end of f_1, f_2, and $J = K_4$ and the second outcome of the theorem holds. This proves (12).

From (11) and (12) we may therefore assume that b_1, b_2 are adjacent, and the edge $b_1b_2 \in X$. From the symmetry we may assume that B_1 is even and B_2 is odd. Let T be the track formed by B_1 and the edges e_3, b_1b_2. So T is even. Suppose that there is an edge (say f) in X incident with b_2 and different from b_1b_2. By (10) no edge of B_1 belongs to X, and yet f is not incident with a penultimate vertex of T, contrary to (9). So there is no such edge f, and therefore b_2 is a triad. Let e_4 be the edge incident with b_2 different from b_1b_2 and not in B_2. So $e_4 \in X_1 \cup X_2$, and therefore by (8), e_4 meets one of e_1, e_2. Since it is not incident with e_1, it follows that $E(B_2) = \{e_2\}$, and $e_4 \in X_1$. Let B_4 be the branch of H containing e_4, and let b_4 be the other end of B_4.

(13) *$b_4 = b_3$, and B_3 has length 1, and H is a subdivision of K_4, and B_4 is even.*

For b_4 is different from b, b_1, b_2. Since B_1 is even, and e_2 is the unique edge in X_2 incident with b, it follows that no edge in X_2 incident with b_4 meets e_1, and therefore by (8), no edge in X_2 is incident with b_4. Consequently b_4 is not a triad, and so there are at least two edges (say g_1, g_2) in X incident with b_4. By (10) (applied to three tracks with common end b_2), each of them meets either b_1b_2 or e_3. But no edge in X is incident with both b_2 and b_4, since $e_4 \in X_1$; so g_1, g_2 are either incident with b_1 or meet e_3.

Suppose that b_4 is not incident with e_3. Then at most one of g_1, g_2 is incident with b_1, and at most one meets e_3 (since H is bipartite), so there is exactly one of each. Hence b_1 is adjacent to b_4, and $b_1b_4 \in X$; and (since H is bipartite and B_1 is even) b is adjacent to b_4 and $bb_4 \in X$, and b_4 has degree 3. Since b_4 is not incident with e_3, and b_4 is adjacent to b, it follows that $b_4 \neq b_3$; and since H is cyclically 3-connected and b_2 is a triad, this is impossible. So b_4 is incident with e_3, that is, $b_4 = b_3$ and B_3 has length 1. Since this holds for every choice of e_3, we deduce that b has degree 3, and therefore H is a subdivision of K_4. It follows that B_4 is even. This proves (13).

Let B_5 be the branch of H between b_1, b_3. Since no edge incident with b_3 meets e_1 except e_3, it follows that b_3 is not a triad. Suppose that no edge of B_1 is in X. Then by (9) applied to T, every edge in X is incident with one of b, b_1. In particular, no edge of B_4 is in Xi; and since b_3 is not a triad, it follows that B_5 has length 1 and its edge is in X. Thus b_3 is adjacent to both b, b_1, and the edges bb_3, b_1b_3 both belong to X; but then the theorem holds by (1).

So we may assume that some edge of B_1 is in X. This edge is not incident with a penultimate vertex of the track formed by B_4 and the edges b_1b_2, e_3, so by (9), some edge of B_4 belongs to X. By (10) applied to B_1, a subtrack of $B_2 \cup B_4$ and the track consisting of the edge e_3, we deduce that the only edge of B_4 in X is the edge incident with b_3. By (10) applied to the track with edge-set $E(B_2) \cup \{b_1b_2\}$, a subtrack of B_1 and the track consisting of the edge e_3, we deduce that the only edge of B_1 in X is the edge incident with b_1. But B_5 is odd, and if it has length > 1 then the first outcome of the theorem holds. So we may assume that b_1b_3 is an edge. Now the tracks B_1, B_4 are even; their end-edges belong to $X \cup X_1$, and their other edges do not (by (8)), and e_3 is not incident with a penultimate vertex of these tracks; so by (9), B_1 and B_4 both have length 2. But then the fourth outcome of the theorem holds. This proves 6.1. ∎

7 Rung replacement

Before we apply 6.1, let us simplify it a little. We can effectively eliminate the cases of $L(H)$ being overshadowed. We need a few lemmas.

7.1 *Let c_1, c_2 be adjacent vertices of a 3-connected graph J, and let e, f be edges of J incident with c_1 and different from c_1c_2. There are three tracks of J from c_1 to c_2, pairwise vertex-disjoint except for their ends, and with first edges c_1c_2, e, f respectively.*

Proof. Since J is 3-connected, if we delete from J all edges incident with c_1 except e and f, the graph we make is still 2-connected, and so it has a cycle containing c_1 and c_2. This proves 7.1. ∎

A *prism* means a graph consisting of two vertex-disjoint triangles $\{a_1, a_2, a_3\}$, $\{b_1, b_2, b_3\}$, and three paths P_1, P_2, P_3, where each P_i has ends a_i, b_i, and for $1 \leq i < j \leq 3$ the only edges between

$V(P_i)$ and $V(P_j)$ are a_ia_j and b_ib_j. The three paths P_1, P_2, P_3 are said to *form* the prism. The prism is *long* if at least one of the three paths has length > 1.

7.2 *Let R_1, R_2, R_3 form a prism in a Berge graph G; then R_1, R_2, R_3 all have the same parity.*

The proof is clear.

7.3 *Let G be Berge, let $Y \subseteq V(G)$ be anticonnected, and for $i = 1, 2, 3$ let a_i-P_i-b_i be a path in $G \setminus Y$, forming a prism with triangles $\{a_1, a_2, a_3\}, \{b_1, b_2, b_3\}$. Assume P_1, P_2, P_3 all have length > 1, and that every vertex in Y is adjacent to at least two of a_1, a_2, a_3 and to at least two of b_1, b_2, b_3. Then at least two of a_1, a_2, a_3 and at least two of b_1, b_2, b_3 are Y-complete.*

Proof. Suppose not; then there is an antipath with interior in Y, joining two vertices either both in $\{a_1, a_2, a_3\}$ or both in $\{b_1, b_2, b_3\}$. Let Q be the shortest such antipath. We may assume Q joins a_1 and a_2 say. Since every vertex in Y is adjacent to either a_1 or a_2 it follows that Q has length ≥ 3. From the minimality of Q, a_3 is Q^*-complete, and so is at least one of b_1, b_2, b_3, say b_i. Since Q can be completed to an antihole via a_1-b_i-a_2 it follows that Q is even. From 3.3 applied to the hole formed by $P_1 \cup P_2$ and hat a_3, neither of b_1, b_2 is Q^*-complete, and so there is an antipath between b_1 and b_2 with interior in Q^*. By the minimality of Q, the two antipaths have the same interior; but this again contradicts 3.3. This proves 7.3. ∎

In fact it is easy to find strengthenings of 7.3 in which some of the paths P_i have length 1, but for the moment 7.3 will suffice.

7.4 *Let G be Berge, and for $1 \leq i \leq 3$ let P_i be a path of even length ≥ 2, from a_i to b_i, such that these three paths form a prism with triangles $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$. Let P_1' be a path from a_1' to b_1, such that P_1', P_2, P_3 also form a prism. Let $y \in V(G)$ have at least two neighbours in A and in B. Then it also has at least two neighbours in $\{a_1', a_2, a_3\}$.*

Proof. Suppose not. By 7.2 P_1' has even length. Let X be the set of neighbours of Y in G. Then $a_1' \notin X$, and $a_1 \in X$, and exactly one of $a_2, a_3 \in X$, say $a_2 \in X$. Also, y cannot be linked onto the triangle $A' = \{a_1', a_2, a_3\}$, by 2.4, and since one of $b_2, b_3 \in X$ it follows that no internal vertex of P_1' is in X. Hence $b_1 \notin X$, for otherwise y-a_2-a_1'-P_1'-b_1- would be an odd hole. So $b_2, b_3 \in X$. Since y-a_1-a_3-P_3-b_3-y is not an odd hole, there is a member of X in $P_3 \setminus b_3$. But then y can be linked onto A', via b_2-b_1-P_1'-a_1, the path a_2, and the path between y and a_3 with interior in $V(P_3) \setminus \{b_3\}$, contrary to 2.4. This proves 7.4. ∎

We shall only need the following when $J = K_4$ or $K_{3,3}$, but we might as well prove it in general.

7.5 *Let G be Berge, and let $L(H)$ be an overshadowed appearance of J in G, where J is 3-connected. Then either:*

- *there is a J-enlargement with a nondegenerate appearance in G, or*
- *G admits a balanced skew partition.*

Proof. For each edge uv of J, let B_{uv} be the branch of H with ends u, v, and let R_{uv} be the path $L(B_{uv})$ of $L(H)$. For each $v \in V(J)$ let N_v be the clique of $L(H)$ with vertex set $\delta_H(v)$. There is an edge c_1c_2 of J such that $B_{c_1c_2}$ has odd length ≥ 3, and some vertex of G is nonadjacent in G to at most one vertex of N_{c_1} and to at most one vertex of N_{c_2}. We say such a vertex v is $B_{c_1c_2}$*-dominant with respect to L(H)*. Let the ends of $R_{c_1c_2}$ (that is, the end-edges of $B_{c_1c_2}$) be r_1, r_2, where $r_i \in N_{c_i}$. Let Y be a maximal anticonnected set of vertices each with at most one non-neighbour in N_{c_1} and at most one non-neighbour in N_{c_2}. We shall prove that Y and some of its common neighbours separate the interior of $R_{c_1c_2}$ from the remainder of $L(H)$ in G, so that will be the skew partition we are looking for. Let X be the set of all Y-complete vertices in G.

(1) *For $i = 1, 2$, at most one vertex of N_{c_i} is not in X.*

For let a_1, a_2 be any two distinct vertices in $N_{c_1} \setminus \{r_1\}$; we shall show that at most one of a_1, a_2, r_1 is not in X. By 7.1, there are two paths Q_1,Q_2 of H between c_1 and c_2, such that Q_1,Q_2,$B_{c_1c_2}$ are vertex-disjoint except for their ends, and for $i = 1, 2$, a_i is the first edge of Q_i. Let b_i be the other end-edge of Q_i. Both Q_1 and Q_2 have odd length, since $B_{c_1c_2}$ is odd and H is bipartite; and they have length ≥ 3 since c_1, c_2 are nonadjacent (for they are the ends of a branch of length > 1.) Hence there are two paths P_1,P_2 of $L(H)$ from N_{c_1} to N_{c_2}, such that P_1,P_2,$R_{c_1c_2}$ are vertex-disjoint and form a prism, and P_i is from a_i to b_i. Now $B_{c_1c_2}$ is odd and therefore $R_{c_1c_2}$ is even, and similarly P_1 and P_2 are even. By hypothesis, each member of Y is adjacent to at least two vertices of the triangle $\{a_1, a_2, r_1\}$ and to two vertices of the triangle $\{b_1, b_2, r_2\}$. By 7.3 it follows that X contains at least two members of $\{a_1, a_2, r_1\}$. This proves (1).

Let

$$\begin{aligned}
X_1 &= X \cap (N_{c_1} \cup N_{c_2}) \\
X_2 &= X \cap (V(L(H)) \setminus (N_{c_1} \cup N_{c_2})) \\
X_0 &= X \setminus V(L(H)) \\
S &= V(R_{c_1c_2}) \setminus X_1 \\
T &= (V(L(H)) \setminus V(R_{c_1c_2})) \setminus X_1.
\end{aligned}$$

We observe first that no vertex of S is adjacent to any vertex in T; for such an edge would join two vertices both in N_{c_i} for some i, and therefore both not in X, contradicting (1).

(2) *If $F \subseteq V(G)$ is connected and some vertex of S has a neighbour in F, and so does some vertex of T, and $F \cap (X_0 \cup X_1 \cup Y) = \emptyset$, then the theorem holds.*

We shall prove this by induction on $|F|$; so, we assume it holds for all smaller choices of F (even for different choices of $L(H)$). Hence we may assume that $G|F$ is a path with vertices $f_1, \ldots, f_n$ say, where f_1 is the only vertex of F with a neighbour in S, and f_n is the only vertex with a neighbour in T. From the minimality of F it also follows that F is disjoint from $L(H)$; for any vertex of F in $L(H)$ would be in S or T, since it is not in X_1, and then we could make F shorter by omitting this vertex. Consequently $F \cap X = \emptyset$. Suppose some vertex in $v \in F$ is major with respect to $L(H)$. Then since $v \notin X$ it follows that v has a nonneighbour in Y, and so $Y \cup v$ is anticonnected; the maximality of Y therefore implies that $v \in Y$, and hence $F \cap Y \neq \emptyset$, a contradiction. So we may assume that no vertex in F is major. On the other hand, the set of attachments of F in $L(H)$ is not local, because it has an attachment in $R_{c_1c_2}$, and its attachments are not all contained in any of $V(R_{c_1c_2})$, N_{c_1},N_{c_2}. Let us apply 5.8. Suppose first that 5.8.1 holds. Then we obtain an appearance $L(H')$ in G of some J-enlargement, with $L(H)$ an induced subgraph of $L(H')$. Since $R_{c_1c_2}$ has even nonzero length, it follows that $L(H)$ is not degenerate, and therefore neither is $L(H')$, and hence the theorem holds. So we may assume that 5.8.2 holds, and there is an edge b_1b_2 of J, (for $i = 1, 2$, s_i denotes the unique vertex in $N_{b_i} \cap R_{b_1b_2}$) and a path P of G with $V(P) \subseteq F$ and with ends p_1 and p_2, such that one of the following holds:

1. p_1 is adjacent in G to all vertices in $N_{b_1} \setminus \{s_1\}$, and p_2 has a neighbour in $R_{b_1b_2} \setminus s_1$, and every edge from $V(P)$ to $V(L(H)) \setminus \{s_1\}$ is either from p_1 to $N_{b_1} \setminus \{s_1\}$, or from p_2 to $R_{b_1b_2} \setminus s_1$, or
2. for $i = 1, 2$, p_i is adjacent in G to all vertices in $N_{b_i} \setminus \{s_i\}$, and there are no other edges between $V(P)$ and $V(L(H))$ except possibly p_1s_1, p_2s_2, and P has the same parity as $R_{b_1b_2}$, or
3. $p_1 = p_2$, and p_1 is adjacent to all vertices in $(N_{b_1} \cup N_{b_2}) \setminus \{s_1, s_2\}$, and all neighbours of p_1 in $V(L(H))$ belong to $N_{b_1} \cup N_{b_2} \cup R_{b_1b_2}$, and $R_{b_1b_2}$ is even, or
4. $s_1 = s_2$, and for $i = 1, 2$, p_i is adjacent in G to all vertices in $N_{b_i} \setminus \{s_i\}$, and there are no other edges between $V(P)$ and $V(L(H)) \setminus \{s_1\}$, and P is even.

In case 1, let R' be the (unique) path from p_1 to s_2 in $(V(P) \cup V(R_{b_1b_2})) \setminus \{s_1\}$, and in the other cases let R' be P. So if in $L(H)$ we replace $R_{b_1b_2}$ by R' we obtain another appearance of J in G, say $L(H')$, where H' is obtained from H by replacing the branch $B_{b_1b_2}$ by some new branch B' joining the same two vertices. For each $v \in V(J)$ let N'_v be the clique in $L(H')$ formed by the edges in $\delta_{H'}(v)$. So $N'_v = N_v$ for all vertices v of J except for b_1 and b_2. Let R' be between r'_1 and r'_2, where $r'_i \in N'_{b_i}$ for $i = 1, 2$.

Now suppose that b_1b_2 and c_1c_2 are different edges of J. Then $B_{c_1c_2}$ is still a branch of H', and we claim that every $y \in Y$ is $B_{c_1c_2}$-dominant with respect to $L(H')$. For let e, f be two edges of J incident with c_1 and different from c_1c_2. By 7.1 there are three tracks of J from c_1 to c_2, vertex-disjoint except for their ends, and one of them is the edge c_1c_2, and the first edges of the other two are e and f. There are three tracks corresponding to these in H, and their line graph is a prism in $L(H)$. There also correspond three tracks in H', yielding a prism in $L(H')$. Since $R_{b_1b_2} \neq R_{c_1c_2}$, it follows that $R_{b_1b_2}$ is incident with at most one of c_1, c_2, so these two prisms are related as in 7.4. Hence by 7.4, since y has two neighbours in both triangles of the first prism, it also has two neighbours in the triangles of the second. This proves that y is $B_{c_1c_2}$-dominant with respect to $L(H')$. The same argument in the reverse direction shows that Y remains a maximal anticonnected set of $B_{c_1c_2}$-dominant vertices. Since there is a proper subset F' of F with attachments in S and in the new set T' in $V(H')$ corresponding to T (for T' contains all the vertices of R' that are in F, and there is at least one such vertex), it follows that we may apply the inductive hypothesis. So F', and hence F, contains a vertex of X. This completes the argument when b_1b_2 and c_1c_2 are distinct edges.

Now we assume that $b_i = c_i$ for $i = 1, 2$. There were four cases in the definition of P, listed above. Case 3 is impossible, since then the vertex p_1 would be $B_{c_1c_2}$-dominant with respect to $L(H)$, and therefore would be in either X or Y, a contradiction. Also, case 1 is impossible, by applying 7.4 as before to show that Y remains a maximal anticonnected set of B'-dominant vertices, and applying the inductive hypothesis. Case 4 is impossible since $B_{c_1c_2}$ has length ≥ 3. So case 2 applies; that is, p_2 is adjacent to all vertices in $N_{c_2} \setminus \{r_2\}$, and to no vertex of $R_{c_1c_2}$ except possibly r_2. So $N'_{c_i} = (N_{c_i} \setminus \{r_i\}) \cup \{r'_i\}$ for $i = 1, 2$. We recall that in this case $R' = P$, and P is a subpath of the path with vertices $f_1, \ldots, f_n$. Choose h with $1 \leq h \leq n$ minimum such that f_h is a vertex of R'. Since both R' and $G|F$ are paths it follows that f_h is one end of R', say r'_1. (This is without loss of generality, because in this case 2, there is symmetry between $b_1 = c_1$ and $b_2 = c_2$.) From the minimality of F, r'_1 has no neighbour in T, and in particular every vertex in $N_{c_1} \setminus \{r_1\}$ is in X. We claim also that every vertex of $N_{c_2} \setminus \{r_2\}$ is in X. For if not, then $r_2 \in X$, and by 7.1 there is a prism $R_{c_1c_2}, P_1, P_2$ say, in $L(H)$, where each P_i has an end $a_i \in N_{c_1}$ and an end $b_i \in N_{c_2}$, and $b_2 \notin X$. (Consequently $r_2, b_1 \in X$.) Hence at most one vertex of the triangle $\{r'_2, b_1, b_2\}$ is in X, and some vertex in X (namely a_1) has no neighbour in this triangle, so by 2.8, Y cannot be linked onto this triangle. In particular, no vertex of P_2 is in X except a_2. But then a_2-P_2-b_2-r_2 is an odd path between members of X, and none of its internal vertices are in X, and a_1 has no neighbour in its interior, contrary to 2.2. This proves that every vertex of $N_{c_2} \setminus \{r_2\}$ is in X. Consequently all vertices of Y are B'-dominant with respect to $L(H')$. We claim also that Y is still maximal. For suppose not, and let $Y \subset Y'$ for some larger anticonnected set Y' of B'-dominant vertices. Since r'_1, r'_2 are not in X, they are certainly not Y'-complete, and since by (1) applied to Y', at most one vertex of N'_{c_i} is not Y'-complete for $i = 1, 2$, it follows that every vertex of $N'_{c_1} \setminus \{r'_1\}$ and $N'_{c_2} \setminus \{r'_2\}$ are Y'-complete. But then all the members of Y' are $B_{c_1c_2}$-dominant with respect to $L(H)$, contrary to the maximality of Y. This proves that Y is a maximal anticonnected set of B'-dominant vertices with respect to $L(H')$. Hence we can apply induction on F, and the result follows. This proves (2).

It follows from (2) that there is a partition of $V(G)\setminus(X_0 \cup X_1 \cup Y)$ into two sets L and M say, where there is no edge between L and M, and $S \subseteq L$ and $T \subseteq M$. So $(L \cup M, X_0 \cup X_1 \cup Y)$ is a skew partition of G. By 4.2 we may assume it is not loose, and so X_2 is empty; and we shall show it is balanced.

(3) *For $i = 1, 2$, all vertices of $N_{c_i} \setminus \{r_i\}$ belong to X_1.*

For suppose there is a vertex n_1 of $N_{c_1} \setminus \{r_1\}$ not in X. Therefore all other vertices of N_{c_1} belong to X, and in particular, $r_1 \in X$. Suppose no other vertex of $R_{c_1c_2}$ is in X; then $r_2 \notin X$, so X includes $N_{c_2} \setminus \{r_2 i\}$. Choose any $n_2 \in N_{c_2} \setminus \{r_2\}$, and any $n'_1 \in N_{c_1} \setminus \{r_1\}$ different from n_1. Then r_1-$R_{c_1c_2}$-r_2-n_2 is an odd path between Y-complete vertices, and none of its internal vertices are Y-complete, and yet n'_1 has no neighbour in its interior, contrary to 2.2. This proves that some vertex of $R_{c_1c_2}$ different from r_1 is in X; yet X_2 is empty, so the interior of $R_{c_1c_2}$ contains no vertex in X. Consequently $r_2 \in X$. Choose $n_2 \in N_{c_2} \setminus \{r_2\}$ such that $N_{c_2} \setminus \{n_2\} \subseteq X$. Since J is 3-connected, there is a track of H from c_1 to c_2 with first edge n_1 and last edge different from n_2. This track is odd since c_1 and c_2 have opposite biparity; and so in G there is an even path, P say, from n_1 to some $n'_2 \in N_{c_2} \setminus \{n_2\}$, with no vertex in $N_{c_1} \cup N_{c_2}$ except its ends. But then r_1-n_1-P-n'_2 is an odd path between Y-complete vertices, no vertex in its interior is Y-complete, and the Y-complete vertex r_2 has no neighbour in its interior, contrary to 2.2. This proves (3).

Let $W = (N_{c_1} \setminus \{r_1\}) \cup (N_{c_2} \setminus \{r_2\})$. Then $W \subseteq X_1$ by (3), and since there are no edges between N_{c_1} and N_{c_2}, it follows that W has exactly two components, both cliques. In particular, W is anticonnected. Now every W-complete vertex is $B_{c_1c_2}$-dominant, and so belongs to $X \cup Y$; and hence there are no W-complete vertices in $L \cup M$. Consequently W is a kernel for the skew partition.

Let us verify the hypotheses of 4.6. Suppose $u_1, u_2 \in W$ are nonadjacent. Then one of u_1, u_2 is in $N_{c_1} \setminus \{r_1\}$ and the other in $N_{c_2} \setminus \{r_2\}$, and therefore they are joined by a path in $L(H)$ using no more vertices in $N_{c_1} \cup N_{c_2}$, which is even (since H is bipartite). In particular, by 4.3, u_1, u_2 are not joined by any odd path with interior in L. Finally, suppose there is a pair of vertices of L joined by an odd antipath with interior in W, necessarily of length ≥ 5 (since we already did the odd path case). Then $G|W$ contains an antipath of length 3, which is impossible since its components are cliques. From 4.6 it follows that G admits a balanced skew partition. This proves 7.5. ∎

8 Generalized line graphs

In this section we complete the proofs of 5.1 and 5.2. As we said earlier, our strategy is to find the biggest line graph in G that we can, and then assemble all the alternative rungs for a given edge of J into a "strip". Let us make that precise.

Let J be 3-connected, and let G be Berge. A *J-strip system* (S, N) in G means:

- for each edge uv of J, a subset $S_{uv} = S_{vu} \subseteq V(G)$
- for each vertex v of J, a subset $N_v \subseteq V(G)$

satisfying the following conditions (for $uv \in E(J)$, a *uv-rung* means a path R of G with ends s, t say, where $V(R) \subseteq S_{uv}$, and s is the unique vertex of R in N_u, and t is the unique vertex of R in N_v):

- The sets S_{uv} ($uv \in E(J)$) are pairwise disjoint
- For each $u \in V(J)$, $N_u \subseteq \bigcup(S_{uv} : v \in V(J)$ adjacent to $u)$
- For each $uv \in E(J)$, every vertex of S_{uv} is in a uv-rung
- If $uv, wx \in E(J)$ with u, v, w, x all distinct, then there are no edges between S_{uv} and S_{wx}
- If $uv, uw \in E(J)$ with $v \neq w$, then $N_u \cap S_{uv}$ is complete to $N_u \cap S_{uw}$, and there are no other edges between S_{uv} and S_{uw}
- For each $uv \in E(J)$ there is a special uv-rung such that for every cycle C of J, the sum of the lengths of the special uv-rungs for $uv \in E(C)$ has the same parity as $|V(C)|$.

It follows that for distinct $u, v \in V(J)$, $N_u \cap N_v$ is empty if u,v are nonadjacent, and otherwise $N_u \cap N_v \subseteq S_{uv}$; and for $uv \in E(J)$ and $w \in V(J)$, if $w \neq u, v$ then $S_{uv} \cap N_w = \emptyset$. The final axiom looks strange, but we shall show immediately that the same property holds for *every* choice of uv-rungs.

8.1 *Let (S, N) be a J-strip system in a Berge graph G, where J is 3-connected. Then for every $uv \in E(J)$, all uv-rungs have lengths of the same parity.*

Proof. Since J is 3-connected, there is a cycle C of J with $|V(C)| \geq 4$ and with $uv \in E(C)$. For each $xy \in E(C)$ different from uv, choose an xy-rung R_{xy}. For every uv-rung R, the union of $V(R)$ and all the $V(R_{xy})$'s induces a cycle in G. This has length ≥ 4 since C has length ≥ 4, so it is a hole and therefore even. Hence all choices of R have lengths of the same parity. This proves 8.1. ∎

For each edge uv of J, choose a uv-rung R_{uv}. It follows from 8.1 and the final axiom above that the subgraph of G induced on the union of the vertex sets of these rungs is a line graph of a bipartite subdivision H of J. For brevity we say that this choice of rungs *forms* $L(H)$.

We need two easy observations:

8.2 *Let (S, N) be a J-strip system in a Berge graph G, where J is 3-connected. If there is an edge uv of J such that some uv-rung has length 0 and another uv-rung has length ≥ 1, then there is an overshadowed appearance of J in G.*

Proof. For each edge ij of J choose an ij-rung R_{ij}, arbitrarily for every edge of J different from uv, and such that R_{uv} has length ≥ 1; and let this choice of rungs form $L(H)$. Let y be the vertex of some uv-rung of length 0. By 8.1, R_{uv} has even length. Let B be the branch of H between u and v, so $E(B) = V(R_{uv})$. Then B is odd and has length ≥ 3 and y is nonadjacent in G to at most one vertex of G in $\delta_H(u)$ and at most one in $\delta_H(v)$. Hence $L(H)$ is overshadowed. This proves 8.2. ∎

A J-strip system is *nondegenerate* if there is some choice of rungs such that the line graph $L(H)$ they form is a nondegenerate appearance of J. 8.2 has the following corollary:

8.3 *Let (S, N) be a nondegenerate J-strip system in a Berge graph G, where J is 3-connected. If there is no overshadowed appearance of J in G, then for every choice of rungs, the line graph they form is a nondegenerate appearance of J in G.*

Proof. Take a choice of rungs $R_{ij} (ij \in E(J))$, forming $L(H)$ say, where $L(H)$ is nondegenerate; and suppose there is another choice, $R'_{ij} (ij \in E(J))$, forming $L(H')$ say, where $L(H')$ is degenerate. Then for some $ij \in E(J)$, R_{ij} has nonzero length and R'_{ij} has length 0. By 8.2 there is an overshadowed appearance of J in G. This proves 8.3. ∎

Given a J-strip system (S, N), we define $V(S, N) = \bigcup(S_{uv} : uv \in E(J))$. Hence every $N_v \subseteq V(S, N)$. If $u, v \in V(J)$ are adjacent, we define $N_{uv} = N_u \cap S_{uv}$. So every vertex of N_u belongs to N_{uv} for exactly one v. Note that N_{uv} is in general different from N_{vu}, but S_{uv} and S_{vu} mean the same thing. We say $X \subseteq V(S, N)$ *saturates* the strip system if for every $u \in V(J)$, there is at most one neighbour v of u in J such that $N_{uv} \not\subseteq X$; and a vertex $y \in V(G) \setminus V(S, N)$ is *major* (with respect to the strip system) if the set of its neighbours in $V(S, N)$ saturates (S, N). We say $X \subseteq V(S, N)$ is *local* (with respect to the strip system) if either $X \subseteq N_v$ for some $v \in V(J)$, or $X \subseteq S_{uv}$ for some edge $uv \in E(J)$.

8.4 *Let G be Berge, and let J be a 3-connected graph. Let (S, N) be a J-strip system in G, nondegenerate if $J = K_4$. Let $y \in V(G) \setminus V(S, N)$, and let X be the set of neighbours of y. If there is a choice of rungs forming a line graph that is saturated by X, then either:*

- *X saturates the strip system, or*
- *there is a J-enlargement with a nondegenerate appearance in G, or*
- *$J = K_4$ and there is an overshadowed appearance of J in G.*

Proof. We define the *fork number* of a choice of rungs to be the number of branch-vertices of H incident in H with ≥ 2 edges in $X \cap E(H)$, where $L(H)$ is the line graph formed by this choice of rungs. Let us say that a choice of rungs R_{ij} forming a line graph $L(H)$ is *saturated* if X saturates $L(H)$, and in this case its fork number is $|V(J)|$. If every choice of rungs is saturated, then X saturates the strip system as required, so we may therefore assume that there is some choice of rungs that is not saturated. Hence there are two choices of rungs R_{ij} ($ij \in E(J)$) and R'_{ij} ($ij \in E(J)$), so that the first is saturated and the second is not, differing only on one edge of J. Let the line graphs made by R_{ij} and R'_{ij} be $L(H)$ and $L(H')$ respectively.

Let us apply 5.7 to H' and $X \cap E(H')$. Then 5.7.1 is false; suppose that 5.7.6 holds. Then $G|(V(L(H')) \cup \{y\}) = L(H'')$ say, and $L(H'')$ is an appearance in G of a J-enlargement. We may assume that $L(H'')$ is degenerate, for otherwise the theorem holds. Hence $J = K_4$ and $L(H')$ is degenerate. Since the strip system is nondegenerate, the result follows from 8.3. So we may assume that one of 5.7.2-5 holds. Hence the choice of rungs R'_{ij} ($ij \in E(J)$) has fork number is ≤ 2. Since the two choices of rungs R_{ij} ($ij \in E(J)$) and R'_{ij} ($ij \in E(J)$) differ only on one edge of J, their fork numbers differ by at most 2; and so $|V(J)| = 4$, and $J = K_4$.

Let $V(J) = \{1, 2, 3, 4\}$, and $R_{ij} \neq R'_{ij}$ only for the edge 1-2. Let the ends of each R_{ij} be r_{ij} and r_{ji}, where $\{r_{ij} : j \in \{1, \ldots, 4\} \setminus \{i\}\}$ is a triangle T_i for each i. Similarly each R'_{ij} is between r'_{ij} and r'_{ji}, where for each i, $\{r'_{ij} : j \in \{1, \ldots, 4\} \setminus \{i\}\}$ is a triangle T'_i. Since X saturates $L(H)$, it has at least two members in each of $T_1, \ldots, T_4$; and since X does not saturate $L(H')$, there is some T'_i containing at most one member of X. Since $T_3 = T'_3$ and $T_4 = T'_4$, we may assume that $|X \cap T_1| = 2$ and $|X \cap T'_1| = 1$; and so $r_{1,2} \in X$, $r'_{1,2} \notin X$, and exactly one of $r_{1,3}, r_{1,4} \in X$, say $r_{1,3} \in X$ and $r_{1,4} \notin X$.

Also, at least two vertices of T_3 and T_4 are in X, so there are at least two branch-vertices of H' incident in H' with more than one edge in X. By 5.7 applied to H', we deduce that 5.7.5 holds, and so there is an edge ij of J such that R'_{ij} is even and

$$(X \cap V(L(H'))) \setminus V(R'_{ij}) = (T'_i \cup T'_j) \setminus V(R'_{ij}).$$

In particular, T'_i and T'_j both contain at least two vertices in X, and so $i, j \geq 2$. Since $r_{1,3} \in X$ it follows that one of $i, j = 3$, say $j = 3$, and $r_{1,3} \in T_3$; so $R_{1,3}$ has length 0. Now there are two cases, $i = 2$ and $i = 4$. Suppose first that $i = 2$. Then

$$(X \cap V(L(H'))) \setminus V(R_{2,3}) = \{r_{1,3}, r_{3,4}, r_{2,4}, r'_{2,1}\},$$

and since at least two vertices of T_4 are in X it follows that $R_{2,4}, R_{3,4}$ both have length 0, a contradiction since $R'_{ij} = R_{2,3}$ is even. So $i = 4$, and hence $R_{3,4}$ is even and

$$(X \cap V(L(H'))) \setminus V(R_{3,4}) = \{r_{3,1}, r_{4,1}, r_{3,2}, r_{4,2}\}.$$

Since the path $r_{3,2}$-$R_{2,3}$-$r_{2,3}$-$r_{2,4}$-$R_{2,4}$-$r_{4,2}$ can be completed to a hole via $r_{4,2}$-$r_{4,3}$-$R_{3,4}$-$r_{3,4}$-$r_{3,2}$, it follows that the first path is even, and so exactly one of $R_{2,3}, R_{2,4}$ is odd; and since the same path can be completed to a hole via $r_{4,2}$-$r_{4,1}$-$R_{1,4}$-$r_{1,4}$-$r_{1,3}$-$r_{3,2}$ it follows that $R_{1,4}$ is odd. Since one of $R_{2,3}, R_{2,4}$ is odd, they do not both have length 0, and hence at most one of $r_{2,3}, r_{2,4} \in X$. Since X saturates $L(H)$, it follows that exactly one of $r_{2,3}, r_{2,4} \in X$ (and hence one of $R_{2,3}, R_{2,4}$ has length 0), and also that $r_{2,1} \in X$. Since no vertex of $R'_{1,2}$ is in X, this restores the symmetry between T'_1 and T'_2.

Suppose that $R_{2,3}$ has length 0. Then $R_{2,4}$ and $R_{1,2}$ are odd, and in particular $r_{2,1} \neq r_{1,2}$. If $r_{2,1}$ has no neighbour in $R'_{1,2}$, then y-$r_{2,1}$-$r_{2,4}$-$r'_{2,1}$-$R'_{1,2}$-$r'_{1,2}$-$r_{1,4}$-$R_{1,4}$-$r_{4,1}$-y is an odd hole, a contradiction. So $r_{2,1}$ has a neighbour in $R'_{1,2}$; but then y can be linked onto the triangle T'_1 via $R'_{1,2}$ and $R_{1,4}$, contrary to 2.4. This proves that $R_{2,3}$ has length ≥ 1. Hence $R_{2,3}$ has odd length and $R_{2,4}$ has length 0, and consequently $R_{1,2}$, $R_{3,4}$ have even length and $R_{1,4}$ is odd. If $R_{3,4}$ has positive length then $L(H)$ is overshadowed (because of the vertex y), and so the theorem holds. We may therefore assume that $R_{3,4}$ has length 0. If $r_{2,1} \neq r_{1,2}$ and $r_{2,1}$ has no neighbour in $R'_{1,2}$, then y-$r_{2,1}$-$r_{2,4}$-$r'_{2,1}$-$R'_{1,2}$-$r'_{1,2}$-$r_{1,3}$-y is an odd hole, a contradiction; while if $r_{2,1} \neq r_{1,2}$ and $r_{2,1}$ has a neighbour in $R'_{1,2}$, then then y can be linked onto the triangle T'_1 via $R'_{1,2}$ and $R_{1,4}$, contrary to 2.4. So $r_{2,1} = r_{1,2}$. But then $L(H)$ is degenerate. Since the strip system is nondegenerate, it follows from 8.3 that there is an overshadowed appearance of K_4 in G. This proves 8.4. ∎

A J-strip system (S, N) in G is *maximal* if there is no J-strip system (S', N') in G such that $V(S, N) \subset V(S', N')$, and $S'_{uv} \cap V(S, N) = S_{uv}$ for every $uv \in E(J)$, and $N_v \subseteq N'_v$ for every $v \in V(J)$. We need to analyze maximal strip systems. For an edge $uv \in E(J)$, we call the set S_{uv} a *strip* of the strip system.

8.5 *Let G be Berge, let J be a 3-connected graph, and let (S, N) be a maximal J-strip system in G. Assume that there is no J-enlargement with a nondegenerate appearance in G. Assume moreover that if $J = K_4$ then (S, N) is nondegenerate and there is no overshadowed appearance of J in G. Let $F \subseteq V(G) \setminus V(S, N)$ be connected, so that no member of F is major with respect to (S, N). Then the set of attachments of F in $V(S, N)$ is local.*

Proof. Let X be the set of attachments of F in $V(S, N)$, and suppose for a contradiction that X is not local. We may assume that F is minimal (connected) with this property.

(1) *For every choice of rungs, forming $L(H)$ say:*

- *for each $y \in F$, the set of neighbours of y does not saturate $L(H)$, and*
- *if $J = K_4$ then $L(H)$ is not degenerate.*

For no $y \in F$ is major with respect to the strip system, and no J-enlargement has a nondegenerate appearance in G, and if $J = K_4$ then there is no overshadowed appearance of J in G, so the first claim follows from 8.4. For the second claim, assume $J = K_4$; then by hypothesis, the strip system is not degenerate, and the claim follows from 8.3. This proves (1).

(2) *There is no $v \in V(J)$ such that $X \subseteq \bigcup(S_{uv} : uv \in E(J))$.*

For assume that v is such a vertex. Consequently, for every vertex $w \in V(J)$ except at most one, only one strip meets both N_w and X. Since X is not local, there exists $x \in X \cap S_{uv} \setminus N_v$ for some edge uv of J. Since $X \not\subseteq S_{uv}$, there exists $x' \in X \cap S_{u'v}$ for some edge $u'v$ of J with $u' \neq u$. For $w \in V(J)$, x belongs to N_w only if $w = u$, and x' belongs to N_w only if $w \in \{v, u'\}$; and since x, x' do not belong to the same strip it follows that $\{x, x'\}$ is not local with respect to the strip system. Make a choice of rungs R_{ij} $ij \in E(J)$ such that $x \in V(R_{uv})$ and $x' \in V(R_{u'v})$, forming $L(H)$. Then $\{x, x'\}$ is not local with respect to $L(H)$, so by (1) we can apply 5.8. Suppose that 5.8.1 holds. Then there is an appearance $L(H')$ in G of some J-enlargement J', with $L(H)$ an induced subgraph of $L(H')$. Moreover, if $J' = K_{3,3}$ then $J = K_4$, and so $L(H)$ is nondegenerate and therefore so is $L(H')$. Since $J' \neq K_4$ it follows that $L(H')$ is nondegenerate, contrary to hypothesis. So 5.8.1 does not hold, and therefore 5.8.2 holds. Since for every vertex $w \in V(J)$ except at most one, only one strip meets both N_w and X, it follows that 5.8.2.a holds, and there is a branch D of H with an end d such that $\delta_H(d) \setminus E(D) = (X \cap E(H)) \setminus E(D)$. Since x and x' are disjoint edges in $X \cap E(H)$, they are not both incident with d, and so one of them is in $E(D \setminus d)$. The branch containing x' does not meet x, so D is the branch between u and v, and $d = v$. Hence x' is incident with v in H, and $\delta_H(v) \subseteq X \cup E(D)$. Consequently, for all neighbours $w \neq u$ of v in J, X contains the vertex of R_{vw} that belongs to N_v, and contains no other vertex of R_{vw}. This restores the symmetry between u' and the other neighbours of v different from u; and since it holds for all choices of the rungs R_{vw}, we deduce that $X \setminus S_{uv} = N_v \setminus S_{uv}$. The minimality of F implies that there is a path P with $V(P) = F$, with ends p_1, p_2 such that p_1 is complete to $N_v \setminus N_{vu}$, and no other vertex of P has any neighbours in $N_v \setminus N_{vu}$, and p_2 is adjacent to x, and no other vertex of P has any neighbours in $S_{uv} \setminus N_v$. But then we can add p_1 to N_v and F to S_{uv}, contradicting the maximality of (S, N). This proves (2).

Let $K = \{uv \in E(J) : X \cap S_{uv} \neq \emptyset\}$.

(3) *There are two disjoint edges in K.*

For make a choice of rungs R_{uv} ($uv \in E(J)$) such that $X \cap V(R_{uv}) \neq \emptyset$ for each $uv \in K$, forming $L(H)$. If there are no two disjoint edges in K, then by (1) and 5.8, it follows that either $X \cap V(L(H))$ is local (with respect to $L(H)$) or 5.8.2.a holds, and in either case there is a branch D of H with an end d such that every edge of $X \cap E(H)$ either is in $E(D)$ or is incident with d. In particular, every branch containing an edge of X is incident with d, and so d meets all edges of J in K, contrary to (2). This proves (3).

From (3) it follows that there exists a 2-element subset of X that is not local, and so from the minimality of F it follows that F is the vertex set of a path, say $f_1, \dots, f_n$. Let us say a choice R_{uv} ($uv \in E(J)$) of rungs is *broad* if there are two disjoint edges ij and hk of J such that X meets both R_{ij} and R_{hk}. From (3) there is a broad choice. We denote the ends of R_{uv} by r_{uv} and r_{vu}, where $r_{uv} \in N_u$ and $r_{vu} \in N_v$.

(4) *For every broad choice of rungs R_{uv} ($uv \in E(J)$), there is a unique pair (i, j) of adjacent vertices of J such that:*

- *for every $w \in V(J)$ different from j and adjacent to i in J, $r_{iw}f_1$ is the unique edge of G between $V(R_{iw})$ and F,*
- *for every $w \in V(J)$ different from i and adjacent to j in J, $r_{jw}f_n$ is the unique edge of G between $V(R_{jw})$ and F,*
- *for every edge uv of J disjoint from ij, there are no edges of G between $V(R_{uv})$ and F.*

For by (1) we can apply 5.8, and since the choice of rungs is broad, the minimality of F implies that one of 5.8.2.b, 5.8.2.c, 5.8.2.d holds. Hence there is an edge ij as in (4). Suppose there is another, say $i'j'$. Since $i'j'$ meets all edges of J that share exactly one end with ij, and J is 3-connected, it follows that $J = K_4$ and the two edges $ij, i'j'$ are disjoint. Moreover, the unique vertex of $R_{ii'}$ in X is both $r_{ii'}$ and $r_{i'i}$, so $R_{ii'}$ has length 0. Similarly $R_{ij'}, R_{ji'}, R_{jj'}$ all have length 0, and so $L(H)$ is degenerate, contrary to (1). This proves (4).

(5) *Every choice of rungs is broad.*

For from (3), there is a broad choice, and from (4) in any broad choice R_{uv} ($uv \in E(J)$) there are four different edges $a_1b_1, \ldots, a_4b_4$ of J, such that a_1b_1 is disjoint from a_2, and a_3b_3 is disjoint from a_4b_4, and X meets $R_{a_ib_i}$ for $1 \leq i \leq 4$. Consequently, if we take another choice of rungs, differing from this one on only one edge, then it too is broad. It follows that every choice is broad. This proves (5).

For a given choice of rungs, let us call the edge ij as in (4) the *traversal* for the choice.

(6) *There are two choices of rungs with different traversals.*

Take a choice of rungs, and let ij be its traversal; and suppose that all other choices of rungs have the same traversal. Let $A_1 = N_i \setminus S_{ij}$, and $A_2 = N_j \setminus S_{ij}$. From (4),(5), and the uniqueness of ij it follows that $X \cap (V(S, N) \setminus S_{ij}) = A_1 \cup A_2$. Hence $n \geq 2$, for if $n = 1$ then we can add f_1 to N_i, N_j and S_{ij}, contrary to the maximality of the strip system. Choose $x_1 \in A_1$ and $x_2 \in A_2$ in disjoint strips. From (4), x_1 is adjacent to exactly one of f_1, f_n, say f_1. For any other vertex $x_3 \in A_2$, let R_{uv} ($uv \in E(J)$) be a choice of rungs forming $L(H)$ say, such that $x_1, x_3 \in V(H)$. From (4) and (5) it follows that f_n is adjacent to x_3; and so f_n is complete to A_2, and similarly f_1 is complete to A_1. From the minimality of F, there are no other edges between F and $A_1 \cup A_2$; but then we can add f_1 to N_i, f_n to N_j, and F to S_{ij}, contrary to the maximality of the strip system. This proves (6).

Let us say a choice R_{uv} ($uv \in E(J)$) is *optimal* if R_{uv} has a vertex in X for all edges uv in K. For any choice of rungs, there is an optimal choice with the same traversal (just replace rungs that miss X by rungs that meet X wherever possible); so (6) implies that there are two optimal choices of rungs with different traversals. Now for any optimal choice of rungs, if hi is its traversal, then by (4) and the optimality of the choice, it follows that K consists precisely of the edges of J with exactly one end in common with hi, together possibly with hi itself. In particular hi meets all edges in K. We may assume that some other edge jk is the traversal for some other optimal choice; and hence (since J is 3-connected) it follows that $J = K_4$ and jk is disjoint from hi, and neither edge is in K. Hence $V(J) = \{h, i, j, k\}$. Now since the strip system is not degenerate, there is one of the four edges hj, hk, ij, ik whose strip contains a rung of nonzero length; some hj-rung R has length > 0 say. From (4) it follows that exactly one vertex of R is in X, one of its ends; say the end in N_h. Let R_{uv} ($uv \in E(J)$) be any choice of rungs such that $R_{hj} = R$. Since the end of R in N_j does not belong to X, it follows from (4) that for each of R_{hk}, R_{ij}, R_{ik}, its unique vertex in X is its end in $N_h \cup N_i$. Since the choice of these rungs was arbitrary, it follows that $X \cap S_{hk} = N_{hk}$, $X \cap S_{ij} = N_{ij}$, and $X \cap S_{ik} = N_{ik}$. If also $X \cap S_{hj} = N_{hj}$ then hi is the traversal for every choice of rungs, contrary to (6), so $X \cap S_{hj} \neq N_{hj}$. It follows that every ij-rung has length 0; for if one, R' say, has length > 0, then its unique vertex in X is its end in N_i, and by exchanging h and i it follows that $X \cap S_{hj} = N_{hj}$, a contradiction. Similarly all hk and ik-rungs have length 0, and therefore all hj-rungs have even length, since G is Berge. From (1), we may assume that f_1 is adjacent to r_{hj}

and complete to S_{hk}, and f_n is complete to $S_{ij} \cup S_{ik}$, and there are no other edges between F and $S_{hk} \cup S_{ij} \cup S_{ik} \cup \{r_{hj}\}$. Let R' be an hj-rung such that its vertex in N_h (r'_{hj}, say) is not its unique vertex in X. Consequently, its other end (r'_{jh}) is its unique vertex in X. By the same argument with hi and jk exchanged, it follows that one of f_1, f_n is complete to $S_{ij} \cup \{r'_{jh}\}$ and the other to $S_{hk} \cup S_{ik}$; and hence $n = 1$. But then the path f_1-r_{hj}-R_{hj}-r_{jh}-r_{ji}-f_1 is an odd hole, a contradiction. This proves 8.5. ∎

We are now ready to prove 5.4, which we restate:

8.6 *Let G be Berge. Let J be a 3-connected graph, such that there is no J-enlargement with a nondegenerate appearance in G. Let $L(H_0)$ be an appearance of J in G, such that if $L(H_0)$ is degenerate, then $H_0 = J = K_{3,3}$ and no J-enlargement appears in $\overline{G}$. Then either $G = L(H_0)$, or $H_0 \neq K_{3,3}$ and G admits a proper 2-join, or G admits a balanced skew partition.*

Proof. By 7.5, we may assume that if $J = K_4$ or $K_{3,3}$ then no appearance of J in G is overshadowed. Regard $L(H_0)$ as a J-strip system in the natural way, and enlarge it to a maximal J-strip system (S, N). If $L(H_0)$ is nondegenerate then so is the strip system. Let Y be the set of vertices in $V(G) \setminus V(S,N)$ that are major with respect to the strip system, and let $Z = V(G) \setminus (V(S,N) \cup Y)$. By 8.5, for each component of Z, its set of attachments in $V(S,N)$ is local.

(1) *If $Y \neq \emptyset$ then G admits a balanced skew partition.*

For suppose not. Let Y' be an anticomponent of Y, and let X be the set of all Y'-complete vertices in $V(G)$. For every choice of rungs, forming $L(H)$ say, every member of Y' is major with respect to $L(H)$. We claim that X saturates $L(H)$; for suppose not. By 6.1, one of the five outcomes of 6.1 holds. The first we have already assumed is false. Thus 6.1 implies that $L(H)$ is degenerate, and consequently 8.3 implies that $L(H_0)$ is degenerate. By hypothesis, $J = K_{3,3}$, and no J-enlargement appears in $\overline{G}$. By 6.1, there is an overshadowed appearance of J in $\overline{G}$, contrary to 7.5 applied in $\overline{G}$. This proves that X saturates $L(H)$. Since this holds for every choice of rungs, it follows that X saturates the strip system. Let b_1b_2 be an edge of J, chosen if possible such that $S_{b_1b_2} \not\subseteq X$. Now the sets ($N_{b_1v}$: $b_1v \in E(J)$) form a partition of N_{b_1} into say m sets, and at least $m-1$ of them are subsets of X. Choose $m-1$ of them that are subsets of X, not using $N_{b_1b_2}$ if possible (that is, if the other $m-1$ sets are all subsets of X), and let their union be X_1. Define $X_2 \subseteq N_{b_2}$ similarly. We note that $S_{b_1b_2} \not\subseteq X_1 \cup X_2$; for if some vertex of $S_{b_1b_2}$ is not in X then this is clear, while if $S_{b_1b_2} \subseteq X$ then $V(S,N) \subseteq X$ from our choice of b_1b_2, and then from the way we chose X_1 it follows that $X_1 \cap S_{b_1b_2} = \emptyset$, and similarly $X_2 \cap S_{b_1b_2} = \emptyset$, and again our claim holds. This proves that $S_{b_1b_2} \not\subseteq X_1 \cup X_2$. Define X_3 to be the set of vertices in $X \cap V(S,N)$ that are not in $X_1 \cup X_2$, and let X_0 be the set of vertices of X that are not in $V(S,N)$. So X_0, X_1, X_2, X_3 are four disjoint subsets of X, with union X. Note that $Y \setminus Y' \subseteq X_0$. Let B be the union of all components of $G \setminus (Y' \cup X_0 \cup X_1 \cup X_2)$ that have nonempty intersection with $V(S,N) \setminus S_{b_1b_2}$, and let A be the union of all the other components. We claim that B is nonempty; for there is an edge c_1c_2 of J disjoint from b_1b_2, and no vertex of $S_{c_1c_2}$ is in $N_{b_1} \cup N_{b_2} \cup S_{b_1b_2}$, and therefore no vertex of $S_{c_1c_2}$ is in $Y' \cup X_0 \cup X_1 \cup X_2$. Suppose that A is also nonempty. Then $(A \cup B, Y' \cup X_0 \cup X_1 \cup X_2)$ is a skew partition of G. By 4.2 it is not loose; and so X_3 is empty (since any vertex of X_3 is in $A \cup B$ and yet is complete to Y'). In particular, $X \cap V(L(H_0)) \subseteq N_{b_1} \cup N_{b_2}$. Since $X \cap V(L(H_0))$ saturates $L(H_0)$, it follows that for every vertex w of J different from b_1, b_2, w has at most one neighbour in

J different from b_1, b_2, and w is adjacent in J to both b_1 and b_2, and all wb_1 and wb_2-rungs have length 0. Since J is 3-connected it follows that $J = K_4$, and $L(H_0)$ is degenerate, a contradiction. Thus A is empty. Now we already saw that $S_{b_1b_2} \not\subseteq X_1 \cup X_2$. Since A is empty, it follows that there is a path of G between $S_{b_1b_2}$ and $V(S,N) \setminus S_{b_1b_2}$, disjoint from $Y' \cup X_0 \cup X_1 \cup X_2$. Choose such a path, minimal. From the choice of X_1 and X_2 this path has a nonempty interior; from its minimality, none of its internal vertices belong to $V(S,N)$; since all major vertices are in $Y' \cup X_0$, its interior contains no major vertices; by 8.5, the set of attachments of its interior is local; yet its ends are both attachments of its interior, so there exist $u \in S_{b_1b_2}$ and $v \in V(S,N) \setminus S_{b_1b_2}$, such that $u, v \notin X_1 \cup X_2$, and yet $\{u, v\}$ is local. Now u, v do not lie in the same strip, and therefore there is some N_w containing them both; and the only $w \in V(J)$ with $u \in N_w$ are b_1, b_2, so we may assume that $u, v \in N_{b_1}$. Since they are not in X_1, and not in the same strip, this is impossible. This proves (1).

We may therefore assume that Y is empty.

(2) *If there is a component F of Z such that for some $v \in V(J)$, all attachments of F in $V(S,N)$ belong to N_v, then G admits a balanced skew partition.*

For let $F' = V(G) \setminus (F \cup N_v)$; then $F' \neq \emptyset$, and every path in G from F to F' meets N_v. Since N_v is not anticonnected, it follows that $(F \cup F', N_v)$ is a skew partition. By 4.2 we may assume it is not loose, and we will prove that it is balanced. Let the neighbours of v in J be $u_1, \ldots, u_k$; then every anticomponent of N_v is a subset of one of $N_{vu_1}, \ldots, N_{vu_k}$. Choose a neighbour w of u_1 in J different from v, u_2, choose $n_1 \in N_{u_1w}$, and choose $n_2 \in N_{vu_2}$. Then n_1, n_2 belong to strips S_{u_1w}, S_{vu_2}, where u_1w, vu_2 are disjoint edges of J; and so n_1, n_2 are not adjacent in G. Let $K = \{n_1\} \cup S_{vu_1} \setminus N_{vu_1}$. Then K is connected (since every vertex of S_{vu_1} is in a vu_1-rung and n_1 is complete to N_{u_1v}), every vertex in N_{vu_1} has a neighbour in K (for the same reason), and n_2 is not in K and has no neighbour in K. (For the last claim, n_2 is not in K since it is in only one strip; and it has no neighbour in $S_{vu_1} \setminus N_{vu_1}$ from the definition of a strip system; and it is not adjacent to n_1 as we already saw.) By 2.6, (K, N_{vu_1}) is balanced, and therefore by 2.7.1, so is (F, N_{vu_1}). By 4.5, G admits a balanced skew partition. This proves (2).

We assume therefore that there are no such components F of Z. Consequently, for every component F of Z, there is an edge b_1b_2 of J such that all the attachments of F in $V(S,N)$ are in $S_{b_1b_2}$. If Z is empty and for all b_1b_2 there is only one b_1b_2-rung, then $G = L(H_0)$ and the theorem holds. So we may assume that there is an edge b_1b_2 of J such that either there is more than one b_1b_2-rung in $S_{b_1b_2}$ or there is a component F of Z with all its attachments in $S_{b_1b_2}$. Let A be the union of $S_{b_1b_2}$ and any components of Z that have an attachment in $S_{b_1b_2}$ (and which therefore have attachments only in $S_{b_1b_2}$), and let $B = V(G) \setminus A$. Let $A_1 = N_{b_1b_2}$, $A_2 = N_{b_2b_1}$, $B_1 = N_{b_1} \setminus N_{b_1b_2}$, and $B_2 = N_{b_2} \setminus N_{b_2b_1}$. Then $A_1, A_2 \subseteq A$, and B_1, B_2 are disjoint subsets of B, and for $i = 1, 2$ A_i is complete to B_i, and there are no other edges between A and B. Also $|B_1| \geq 2$, and we chose b_1b_2 such that if A_1, A_2 both have only one vertex then A is not the vertex set of a path joining them. If $A_1 \cap A_2 = \emptyset$ then $H_0 \neq K_{3,3}$ and G admits a proper 2-join, and the theorem holds. Thus we may assume that there exists $a \in A_1 \cap A_2 \neq \emptyset$. Then a is complete to $B_1 \cup B_2$, and since $|A| \geq 2$, it follows that $((B \setminus (B_1 \cup B_2)) \cup (A \setminus \{a\}), B_1 \cup B_2 \cup \{a\})$ is a skew partition of G. Since $\{a\}$ is an anticomponent of $B_1 \cup B_2 \cup \{a\}$, 4.1 implies that G admits a balanced skew partition. This proves

5.4. ∎

9 Double split graphs

In this section we handle degenerate appearances of K_4. There is another way to view them, not as line graphs but as sets of paths and antipaths with certain properties, as we shall see.

Let P_1, P_2 be paths in a graph G, and let Q_1, Q_2 be antipaths. Suppose that P_1, P_2, Q_1, Q_2 are pairwise disjoint, and we can label the ends of each P_i as a_i, b_i, and label the ends of each Q_j as x_j, y_j, such that:

- P_1, P_2, Q_1, Q_2 all have length ≥ 1
- there are no edges between P_1 and P_2, and Q_1 is complete to Q_2
- for $(i, j) = (1, 1), (1, 2)$ or $(2, 1)$, the only edges between $V(P_i)$ and $\{x_j, y_j\}$ are $a_i x_j$ and $b_i y_j$, and the only edges between $V(P_2)$ and $\{x_2, y_2\}$ are $a_2 y_2$ and $b_2 x_2$,
- for $(i, j) = (1, 1), (1, 2)$ or $(2, 1)$, the only nonedges between $V(Q_j)$ and $\{a_i, b_i\}$ are $a_i y_j$ and $b_i x_j$, and the only nonedges between $V(Q_2)$ and $\{a_2, b_2\}$ are $a_2 x_2$ and $b_2 y_2$.

In these circumstances we call the quadruple (P_1, P_2, Q_1, Q_2) a *knot* in G. Note that if (P_1, P_2, Q_1, Q_2) is a knot then so is (P_2, P_1, Q_1, Q_2), with a suitable relabelling of the ends of the paths and antipaths.

If $L(H)$ is a degenerate appearance of K_4 in G, it can be viewed as a knot. For, in our usual notation, let $R_{1,3}, R_{1,4}, R_{2,3}, R_{2,4}$ have length 0; let $P_1 = R_{1,2}$, $P_2 = R_{3,4}$, let Q_1 be the antipath $r_{1,3}$-$r_{2,4}$, and Q_2 the antipath $r_{1,4}$-$r_{2,3}$. It is easy to check that this is a knot. In fact, this and its complement are the only knots in Berge graphs, as the next theorem shows.

9.1 *Let (P_1, P_2, Q_1, Q_2) be a knot in a Berge graph G. Then all four of P_1, P_2, Q_1, Q_2 have odd length; and either both P_1, P_2 have length 1, or both Q_1, Q_2 have length 1.*

Proof. Define a_i, b_i, x_i, y_i $(i = 1, 2)$ as usual. Certainly P_1 is odd since x_1-a_1-P_1-b_1-y_2-x_1 is a hole, and similarly the other three are odd. Suppose one of P_1, P_2 has length > 1 and one of Q_1, Q_2 has length > 1. By exchanging P_1, P_2 or Q_1, Q_2 we may therefore assume that P_1, Q_1 both have length > 1. Let Y be the interior of Q_1. Then a_1, b_1, a_2, b_2 are all Y-complete, from the last condition in the definition of a knot, and since a_2 has no neighbours in the interior of P_1 it follows from 2.2 that there is a Y-complete vertex (v say) in the interior of P_1. But x_1, y_1 are not Y-complete, and they are adjacent, so a_1-x_1-y_1-b_1 is an odd path between Y-complete vertices and v has no neighbour in its interior, contrary to 2.2. This proves 9.1. ∎

Nevertheless, it turns out to be advantageous to make only limited use of 9.1; it is better to preserve the symmetry between the paths and the antipaths.

Let (P_1, P_2, Q_1, Q_2) be a knot in a Berge graph G; we define K to be the subgraph of G induced on $V(P_1) \cup V(P_2) \cup V(Q_1) \cup V(Q_2)$. (For brevity we say that the knot *induces* K.) We say a subset $X \subseteq V(K)$ is *local* (with respect to the knot) if X is disjoint from one of $V(P_1), V(P_2)$, and X includes neither of $V(Q_1), V(Q_2)$, and $X \cap (V(P_1) \cup V(P_2))$ is complete to $X \cap (V(Q_1) \cup V(Q_2))$. We say X *resolves* the knot if $V(K) \setminus X$ is local with respect to the knot (Q_1, Q_2, P_1, P_2) in $\overline{G}$; that is, if

X includes one of $V(Q_1), V(Q_2)$, and X meets both P_1 and P_2, and X contains at least one end of every edge between $V(P_1) \cup V(P_2)$ and $V(Q_1) \cup V(Q_2)$. Conveniently, these definitions almost agree with what we did for line graphs, because of the following.

9.2 *Let (P_1, P_2, Q_1, Q_2) be a knot in a graph G, inducing K, where Q_1, Q_2 both have length 1, and so $K = L(H)$ is an appearance of K_4. Let $X \subseteq V(K)$. Then:*

- *X is local with respect to the knot if and only if it is local with respect to $L(H)$*
- *X resolves the knot if and only if X saturates $L(H)$ and X meets both $V(P_1)$ and $V(P_2)$.*

The proof is obvious and we omit it. This allows us to unify some portions of 5.8 and 6.1, as follows. (The expression "up to symmetry" means here "possibly after exchanging P_1 and P_2 and exchanging Q_1 and Q_2, and renaming the ends of P_1, P_2, Q_1, Q_2 accordingly.")

9.3 *Let (P_1, P_2, Q_1, Q_2) be a knot in a Berge graph G, inducing K. Assume that there is no appearance in G or in $\overline{G}$ of any K_4-enlargement, and there is no overshadowed appearance of K_4 in G or in $\overline{G}$. Let F be a connected subset of $V(G) \setminus V(K)$, such that its set of attachments in K is not local. Then either:*

1. *there is a vertex in F such that its neighbour set in K resolves the knot, or*
2. *(up to symmetry) there is a path R in F with ends r_1, r_2 such that r_1, a_1 have the same neighbours in $V(P_2) \cup V(Q_1) \cup V(Q_2)$, and there are no edges between $R \setminus r_1$ and $V(P_2) \cup V(Q_1) \cup V(Q_2)$, and r_2 has a neighbour in $P_1 \setminus a_1$, and there are no edges between $R \setminus r_2$ and $P_1 \setminus a_1$, or*
3. *(up to symmetry) there is an odd path R in F with ends r_1, r_2 such that r_1, a_1 have the same neighbours in $V(P_2) \cup V(Q_1) \cup V(Q_2)$, and r_2, b_1 have the same neighbours in $V(P_2) \cup V(Q_1) \cup V(Q_2)$, and there are no edges between $V(R^*)$ and $V(P_2) \cup V(Q_1) \cup V(Q_2)$, and no edges between R and P_1 except possibly r_1a_1 and r_2b_1, or*
4. *there is a vertex $f \in F$ such that (up to symmetry) f, x_1 have the same neighbours in $V(P_1) \cup V(P_2) \cup V(Q_2)$ and f is not adjacent to y_1.*

Proof. Define a_i, b_i, x_i, y_i $(i = 1, 2)$ as usual. By 9.1 there are two cases, depending whether Q_1 and Q_2 have length 1 or P_1, P_2 have length 1.

(1) *If Q_1, Q_2 have length 1 then the theorem holds.*

For assume Q_1, Q_2 have length 1. Then K is a degenerate appearance of K_4 in G, say $K = L(H)$. Suppose that the neighbour set of some $f \in F$ saturates $L(H)$. If f has a neighbour in both $V(P_1)$ and $V(P_2)$ then statement 1 of the theorem holds, so we assume it has no neighbour in $V(P_1)$. But then f is adjacent to all four of x_1, x_2, y_1, y_2, since it has two neighbours in every triangle of K, and then f-x_1-a_1-P_1-b_1-y_1-f is an odd hole, a contradiction. So we assume there is no such f, and hence we may apply 5.8. If 5.8.1 holds then there is an appearance in G of some K_4-enlargement, a contradiction. So 5.8.2 holds. In the notation of 5.8.2, the edge b_1b_2 of J is of one of two types; either N_{b_1} meets N_{b_2} or it does not. In the first case, we may assume from the symmetry that those two sets are $\{x_1, x_2, a_1\}$ and $\{x_1, y_2, a_2\}$, and there is a path R of G with $V(R) \subseteq F$ and with ends r_1 and r_2, such that r_1 is adjacent to a_1, x_2, and r_2 is adjacent to a_2, y_2, and there are no other edges between $V(P)$ and $K \setminus x_1$. If R has length 0 then statement 4 of the theorem holds, while if R has length > 0 then it is even and there is an overshadowed appearance of K_4 in G, a contradiction. In the second case, when the sets called $N(b_1), N(b_2)$ in the notation of 5.8.2 are disjoint, we may assume that these sets are $\{x_1, x_2, a_1\}$ and $\{y_1, y_2, b_1\}$ respectively, and one of 5.8.2(a), 5.8.2(b), 5.8.2(c) holds. In the first two cases statements 2,3 of the theorem hold, respectively, and the last case is impossible since P_1 is odd. This proves (1).

Henceforth we may therefore assume that one of Q_1, Q_2 has length > 1, and therefore by 9.1, both P_1 and P_2 have length 1. Hence $\overline{K} = L(H)$, where $L(H)$ is a degenerate appearance of K_4 in $\overline{G}$.

(2) *If there exists $f \in F$ such that f is not major with respect to $L(H)$ in $\overline{G}$, then the theorem holds.*

For let $f \in F$ have this property. If the set of neighbours of f in K resolves the knot (P_1, P_2, Q_1, Q_2), then statement 1 of the theorem holds, so we assume not. Therefore, in $\overline{G}$, the set of neighbours of f in $\overline{K}$ is not local with respect to the knot $(\overline{Q_1}, \overline{Q_2}, \overline{P_1}, \overline{P_2})$. But this set does not saturate $L(H)$; so we can apply 5.8 (or, indeed, 5.7) in $\overline{G}$, and deduce, as before, that either there is a K_4-enlargement that appears in $\overline{G}$ (a contradiction), or (up to symmetry) f, a_1 have the same neighbours in $K \setminus a_1$ (but then statement 2 of the theorem holds), or (up to symmetry) f, x_1 have the same neighbours in $V(P_1) \cup V(P_2) \cup V(Q_2)$ (but then either statement 1 or statement 4 of the theorem holds). This proves (2).

We may therefore assume that every $f \in F$ is major with respect to $L(H)$ in $\overline{G}$. Let X be the set of vertices of K which, in G, have no neighbours in F. By hypothesis, $V(K) \setminus X$ is not local with respect to the knot (P_1, P_2, Q_1, Q_2) in G, and hence X does not resolve the knot $(\overline{Q_1}, \overline{Q_2}, \overline{P_1}, \overline{P_2})$ in $\overline{G}$. If X does not saturate $L(H)$ in $\overline{G}$, then by (2) we may apply 6.1. Since Q_1 has length > 1 it follows that the last outcome of 6.1 holds, and hence statement 3 of the theorem holds. We may therefore assume that X saturates $L(H)$ in $\overline{G}$. By 9.2, X is disjoint from one of $V(Q_1), V(Q_2)$, say $X \cap V(Q_1) = \emptyset$. Hence $a_1, a_2, b_1, b_2 \in X$. Since a_1-y_1-Q_1-x_1-b_1 is an odd antipath in G, and its internal vertices all have neighbours in F, and its ends do not, it follows from 2.2 applied in $\overline{G}$ that every vertex in X has a non-neighbour in $V(Q_1)$; and hence no vertex of Q_2 belongs to X. This restores the symmetry between Q_1, Q_2. Now one of Q_1, Q_2 has length > 1, say Q_1 without loss of generality. Hence, in $\overline{G}$, the path a_1-y_1-Q_1-x_1-b_1 is odd and has length ≥ 5; its ends are complete to F, and its internal vertices are not. By 2.1, F contains a leap; so there exist nonadjacent $f_1, f_2 \in F$ such that Q_1 is the interior of a path R between them. (All this is in $\overline{G}$ - we will tell the reader when we switch back to G.) Now f_1, f_2 have no common neighbour in Q_2 (because R could be completed to an odd hole through any such common neighbour), so by 2.1, f_1, f_2 is also a leap for the path a_1-y_2-Q_2-x_2-b_1 (this path might have length 3, but still we get a leap by 2.1.3, since $\{f_1, f_2\}$ cannot include the interior of any longer antipath between x_2 and y_2). Hence from the symmetry we may assume that f_1 is adjacent to y_1, y_2, and f_2 to x_1, x_2, and there are no other edges between $\{f_1, f_2\}$ and $V(Q_1) \cup V(Q_2)$. Therefore, back in G, we see that a_1, f_1 have the same neighbours in $V(P_2) \cup V(Q_1) \cup V(Q_2)$, and so do b_1, f_2, and therefore statement 3 of the theorem holds. This proves 9.3. ∎

9.3 suggests that we should attempt to combine paths into strips, as in the section on "Generalized line graphs", and combine antipaths into "antistrips". Let us make that precise.

Let A, B, C be disjoint subsets of $V(G)$. We call $S = (A, C, B)$ a *strip* if A, B are nonempty, and every vertex of $A \cup B \cup C$ belongs to a path between A and B with only its first vertex in A, only its last vertex in B, and interior in C. Such a path is called a *rung* of the strip S, or an S-rung. When $S = (A, C, B)$ is a strip, $V(S)$ means $A \cup B \cup C$. The *reverse* of a strip (A, C, B) is the strip (B, C, A). An *antistrip* is a triple that is a strip in $\overline{G}$, and the corresponding antipaths are called *antirungs*. If P is a rung with ends $a \in A$ and $b \in B$, we speak of the "rung a-P-b" for brevity; the reader can deduce which end is in which set from the names of the ends, because we shall always use a, a', a_1 etc. for ends in a set called something like A, and so on.

Let $S = (A, C, B)$ be a strip and $T = (X, Z, Y)$ an antistrip, with $V(S) \cap V(T) = \emptyset$. We say S, T are *parallel* if:

- A is complete to $X \cup Z$, and B is complete to $Y \cup Z$, and
- X is anticomplete to $B \cup C$, and Y is anticomplete to $A \cup C$.

We say S, T are *co-parallel* if S, T' are parallel, where T' is the reverse of T.

Now let S_1, S_2 be strips and T an antistrip, where S_1, S_2, T are pairwise disjoint. We say that S_1, S_2 *agree on* T if either S_1, T are parallel and S_2, T are parallel, or both pairs are co-parallel; and they *disagree* if one pair is parallel and the other pair is co-parallel. If S is a strip and T_1, T_2 are antistrips, pairwise disjoint, we define whether T_1, T_2 agree or disagree on S similarly.

Now let S_1, S_2 be strips, and let T_1, T_2 be antistrips, all pairwise disjoint. We call the quadruple (S_1, S_2, T_1, T_2) a *twist* if S_1, S_2 agree on one of T_1, T_2 and disagree on the other. (Equivalently, if T_1, T_2 agree on one of S_1, S_2, and disagree on the other.) Note that if (S_1, S_2, T_1, T_2) is a twist, then so is (S_1', S_2, T_1, T_2), where S_1' is the reverse of S_1.

A *striation* in a graph G is a family of strips $S_i = (A_i, C_i, B_i)$ $(1 \le i \le m)$ together with a family of antistrips $T_j = (X_j, Z_j, Y_j)$ $(1 \le j \le n)$, satisfying the following conditions:

- all the strips and antistrips are pairwise disjoint, and all their rungs and antirungs have odd length
- $m, n \ge 2$
- for $1 \le i < i' \le m$, S_i is anticomplete to $S_{i'}$, and for $1 \le j < j' \le n$, T_j is complete to $T_{j'}$
- for $1 \le i \le m$ and $1 \le j \le n$, S_i and T_j are either parallel or co-parallel
- for $1 \le i < i' \le m$ there exist distinct j, j' with $1 \le j, j' \le n$ such that $(S_i, S_{i'}, T_j, T_{j'})$ is a twist
- for $1 \le j < j' \le n$ there exist distinct i, i' with $1 \le i, i' \le m$ such that $(S_i, S_{i'}, T_j, T_{j'})$ is a twist.

(Note that if we replace some (A_i, C_i, B_i) by its reverse, we obtain another striation.) We denote the striation by L, and the union of the vertex sets of all its strips and antistrips by $V(L)$. By analogy with what we did for knots, let us say that a subset $X \subseteq V(L)$ is *local* with respect to L if

- at most one of $X \cap V(S_1), \dots, X \cap V(S_m)$ is nonempty,

- for $1 \leq j \leq n$, every T_j-antirung has a vertex not in X, and
- $X \cap (V(S_1) \cup \cdots \cup V(S_m))$ is complete to $X \cap (V(T_1) \cup \cdots \cup V(T_n))$.

We say X *resolves* L if $V(L) \setminus X$ is local with respect to the striation in $\overline{G}$ obtained from L by exchanging the strips and antistrips; that is, if

- there is at most one of $T_1, \ldots, T_n$ that is not a subset of X,
- for $1 \leq i \leq m$, every S_i-rung meets X, and
- X contains at least one end of every edge between $V(S_1) \cup \cdots \cup V(S_m)$ and $V(T_1) \cup \cdots \cup V(T_n)$.

A striation L in G is *maximal* if there is no striation L' in G with $V(L) \subset V(L')$.

9.4 *Let G be Berge, such that there is no appearance in G or in $\overline{G}$ of any K_4-enlargement, and there is no overshadowed appearance of K_4 in G or in $\overline{G}$. Let L be a maximal striation in G. Let $f \in V(G) \setminus V(L)$, and let X be the set of neighbours of f in $V(L)$. Then either X is local with respect to L, or X resolves L.*

Proof. Let L have strips $S_i = (A_i, C_i, B_i)$ $(1 \leq i \leq m)$ and antistrips $T_j = (X_j, Z_j, Y_j)$ $(1 \leq j \leq n)$.

(1) *Let $1 \leq i \leq m$, and $1 \leq j \leq n$; let a_i-P_i-b_i be an S_i-rung, and x_j-Q_j-y_j a T_j-antirung. Then either $X \cap V(P_i) \neq \emptyset$, or $V(Q_j) \not\subseteq X$.*

For suppose that X includes $V(Q_1)$ and is disjoint from $V(P_1)$ say. By reversing S_2 we may assume that S_1 and S_2 agree on T_1; and we may assume they disagree on T_2. Let a_2-P_2-b_2 be any S_2-rung, and x_2-Q_2-y_2 any T_2-antirung. Then (P_1, P_2, Q_1, Q_2) is a knot, so by 9.1, we may assume (taking complements if necessary) that Q_1 has length 1. But then f-x_1-a_1-P_1-b_1-y_1-f is an odd hole, a contradiction. This proves (1).

From (1), taking complements if necessary, we may assume that for all $1 \leq j \leq n$, and for all T_j-antirungs Q_j, $V(Q_j) \not\subseteq X$.

(2) *X meets at most one of $V(S_1), \ldots, V(S_m)$.*

For suppose that X meets both S_1 and S_2 say. We may assume that (S_1, S_2, T_1, T_2) is a twist. For $i = 1, 2$ choose an S_i-rung P_i such that $X \cap V(P_i) \neq \emptyset$, and for $j = 1, 2$ choose any T_j-antirung Q_j. By our assumption above, f has nonneighbours in both Q_1, Q_2. But then (P_1, P_2, Q_1, Q_2) is a knot, and setting $F = \{f\}$ violates 9.3, a contradiction. This proves (2).

We may assume that X is not local with respect to L, and so we may assume that there is an S_1-rung a_1-P_1-b_1 and a T_1-antirung x_1-Q_1-y_1 containing nonadjacent members of X. By reversing each T_j if necessary, we may assume that S_1 is parallel to each T_j. In particular, a_1x_1 is an edge, and so is b_1y_1. Since one of P_1, Q_1 has length 1 by 9.1, the interior of Q_1 is complete to $V(P_1)$, we may assume that $x_1 \in X$, and $X \cap V(P_1 \setminus a_1) \neq \emptyset$. Let $2 \leq j \leq n$, and let x_j-Q_j-y_j be any T_j-antirung. For definiteness we assume $j = 2$. Now T_1, T_2 agree on S_1, and so there is some S_i on which they disagree, say S_2. Let a_2-P_2-b_2 be any S_2-rung. Then (P_1, P_2, Q_1, Q_2) is a knot, with union K say, and $X \cap V(K)$ is not local with respect to K (since $x_1 \in X$, and $X \cap V(P_1 \setminus a_1) \neq \emptyset$). By 9.3, it follows that 9.3.2 holds, and hence f, a_1 have the same neighbours in $V(Q_1) \cup V(Q_2)$. In particular, $V(Q_2) \setminus \{y_2\} \subseteq X$. Since $V(Q_2) \not\subseteq X$, it follows that $y_2 \notin X$; since this holds for all Q_2, we deduce that $X \cap V(T_2) = X_2 \cup Z_2$; and since the same holds for all antistrips of L except T_1, we deduce that $X \cap V(T_j) = X_j \cup Z_j$ for $2 \leq j \leq n$. Since our only assumption about T_1 was that $X \cap X_1 \neq \emptyset$, and since we have shown that the same is true for all T_j, we can replace T_1 by T_2 say, and deduce similarly that $X \cap V(T_1) = X_1 \cup Z_1$. But then we can add f to A_1, contrary to the maximality of the striation. This proves 9.4. ∎

9.5 *Let G be Berge, such that there is no appearance in G or in $\overline{G}$ of any K_4-enlargement, and there is no overshadowed appearance of K_4 in G or in $\overline{G}$. Let L be a maximal striation in G. Let $F \subseteq V(G) \setminus V(L)$ be connected, such that for each $f \in F$, the set of its neighbours in $V(L)$ is local with respect to L. Then the set of attachments of F in $V(L)$ is local with respect to L.*

Proof. Let L have strips $S_i = (A_i, C_i, B_i)$ $(1 \leq i \leq m)$ and antistrips $T_j = (X_j, Z_j, Y_j)$ $(1 \leq j \leq n)$. Suppose not, and choose a counterexample F with F minimal. Let X be its set of attachments in $V(L)$.

(1) $X \not\subseteq V(T_1) \cup \cdots \cup V(T_n)$.

For suppose it is. Since X is not local, we may assume that X includes $V(Q_1)$ for some T_1-antirung x_1-Q_1-y_1. Let $2 \leq j \leq n$, and let x_j-Q_j-y_j be a T_j-antirung. Then we can choose some $S_i, S_{i'}$ to make a twist, and if we choose an S_i-rung and $S_{i'}$-rung and apply 9.3 to the resultant knot, we deduce (since no vertices of S_i and $S_{i'}$ are in X) that 9.3.3 holds. This has several consequences. First, it implies that there is an odd path in F with vertices $f_1, \ldots, f_k$ say, which is either parallel or co-parallel to Q_1, and either parallel or co-parallel to Q_j; and there are no edges between $\{f_2, \ldots, f_{k-1}\}$ and $Q_1 \cup Q_j$. Hence the set of attachments of $\{f_1, \ldots, f_k\}$ is not local with respect to L, and so $F = \{f_1, \ldots, f_k\}$ from the minimality of F. Second, every vertex of Q_j is in X, and since this holds for all Q_j it follows that $V(T_j) \subseteq X$. By exchanging T_1 and T_j it follows that $V(T_1) \subseteq X$. Moreover, since this holds for all j we deduce that $X = V(T_1) \cup \cdots \cup V(T_n)$. This restores the symmetry between T_1 and $T_2, \ldots, T_n$. Third, this shows that there are no edges between $\{f_2, \ldots, f_{k-1}\}$ and $V(T_1) \cup \cdots \cup V(T_n)$. Fourth, for $1 \leq j \leq n$ every vertex in Z_j is adjacent to both f_1, f_k. Since k is even, this proves that either $k = 2$ or $Z_1 \cup \cdots \cup Z_n = \emptyset$. Fifth, every vertex in $X_1 \cup Y_1 \cdots \cup X_n \cup Y_n$ is adjacent to exactly one of f_1, f_n; let U be the set of those adjacent to f_1, and V those adjacent to f_n. For the moment fix j with $1 \leq j \leq n$. Every T_j-antirung has one end in U and the other in V; let M_j be the union of the vertex sets of all T_j-antirungs x_j-Q_j-y_j such that $x_j \in U$, and N_j the union of all those with $x_j \in V$. Since there is no T_j-antirung with both ends in M_j or both ends in N_j, it follows that $M_j \cap N_j = \emptyset$, and there are no nonedges betwen M_j and N_j except possibly between $M_j \cap X_j$ and $N_j \cap X_j$, or between $M_j \cap Y_j$ and $N_j \cap Y_j$. Suppose there is such a nonedge; and choose T_j-antirungs x_j-Q_j-y_j,x'_j-Q'_j-y'_j where $x_j \in U$ is nonadjacent to $x'_j \in V$, say. Now x_j, x'_j have a common neighbour $d_1 \in A_1 \cup B_1$, and then d_1-x_j-f_1-$\cdots$-f_k-x'_j-d_1 is an odd hole. This proves that M_j is complete to N_j. Now if M_j is nonempty, then $(M_j \cap X_j, M_j \cap Z_j, M_j \cap Y_j)$ is an antistrip, and similarly if N_j is nonempty it also induces an antistrip. We call these the *offspring* of T_j. (If one of M_j, N_j is empty, then the other equals $V(T_j)$, and so the only offspring of T_j is T_j itself; and otherwise it has two.) Also, there is a new strip $S_0 = (\{f_1\}, \{f_2, \dots, f_{k-1}\}, \{f_k\})$. Note that

- for all j with $1 \le j \le n$, S_0 is parallel or antiparallel with the offspring of T_j
- for all i with $1 \le i \le m$, there exists j with $1 \le j \le n$ such that S_0, S_i disagree on one of the offspring of T_j, and there exists j such that S_0, S_i agree on one of the offspring of T_j. For if the first were false, say, then each of the T_j's has only one offspring, and we could add f_1 to A_i, $\{f_2, \dots, f_{k-1}\}$ to C_i, and f_k to B_i, contradicting the maximality of the striation; while if the second were false we could do the same with f_1, f_k exchanged.
- if T_1', T_2' are each offspring of one of $T_1, \dots, T_n$, then there exists i with $0 \le i \le m$ such that T_1', T_2' agree on S_i; and there exists i such that they disagree. For this is clear if they are offspring of different parents, since their parents were in a twist together; while if they are both offspring of the same T_j, then they disagree on S_0 and agree on all of $S_1, \dots, S_m$.

It follows from these observations that the set of strips $S_0, \dots, S_m$, together with the set of offspring of $T_1, \dots, T_n$, forms a new striation, contrary to the maximality of L. This proves (1).

(2) *X meets exactly one of $S_1, \dots, S_m$.*

For by (1) it meets at least one of these sets; suppose it meets two, say S_1 and S_2. We may assume that (S_1, S_2, T_1, T_2) is a twist. For $i = 1, 2$ choose an S_i-rung a_i-P_i-b_i such that X meets P_i, and for $j = 1, 2$ let x_j-Q_j-y_j be a Q_j-antirung. Then (P_1, P_2, Q_1, Q_2) is a knot K say, and $X \cap V(K)$ is not local with respect to K. From the minimality of F, F is minimal such that $X \cap V(K)$ is not local with respect to K. It follows from 9.3 that one of 9.3.1, 9.3.4 holds; and in either case there is a vertex $f \in F$ with neighbours in P_1 and in P_2. Hence the set of neighbours of f in $V(L)$ is not local with respect to L. But this contradicts a hypothesis of the theorem, and hence proves (2).

(3) *$V(Q_j) \not\subseteq X$, for $1 \le j \le n$, and for every T_j-antirung Q_j.*

For suppose that $V(Q_1) \subseteq X$ for some T_1-antirung x_1-Q_1-y_1. By (2) we may assume that X meets S_1 and none of $S_2, \dots, S_m$. Let $2 \le j \le n$, and choose i with $2 \le i \le m$ such that (S_1, S_i, T_1, T_j) is a twist. Let Q_j be an x_j-T_j-y_j-antirung, let a_1-P_1-b_1 be an S_1-rung such that X meets P_1, and let a_i-P_i-b_i be an S_i-rung. Hence (P_1, P_i, Q_1, Q_j) is a knot. Let us apply 9.3. By (2) and the minimality of F it follows that 9.3.3 holds. This has several consequences. First, from the minimality of F, $G|F$ is an odd path f_1-$\cdots$-f_k such that f_1, a_1 have the same neighbours in $V(Q_1 \cup Q_j)$, and so do f_k, b_1, and there are no edges between F and $V(P_1)$ except possibly f_1a_1 and f_kb_1. Since X meets P_1, it follows that at least one of these two edges is present; and therefore they both are, since f_1-$\cdots$-f_k is an odd path and so is P_1 (for otherwise the union of these two paths, with one of x_1, y_1, would induce an odd hole). So f_1 is adjacent to a_1 and to no other vertex of P_1, and f_n to b_1 and to no other vertex of P_1. Second, $V(Q_j) \subseteq X$. Since this holds for all Q_j it follows that $V(T_j) \subseteq X$; and by exchanging T_1 and T_j we deduce that $V(T_1) \cup \cdots \cup V(T_n) \subseteq X$. Moreover $\{f_2, \dots, f_{k-1}\}$ is anticomplete to $V(T_1) \cup \cdots \cup V(T_n)$. Third, let x_j'-Q_j'-y_j' be some other T_j-antirung. By the same argument applied to the knot (P_1, P_i, Q_1, Q_j'), we deduce that again 9.3.3 holds, and so one of f_1, f_k is adjacent to x_j' and the other to y_j'. Furthermore, the one adjacent to x_j' is also adjacent to a_1;

and so in fact f_1 is adjacent to x'_j. Since this holds for all choices of Q_j and of j, it follows that f_1, a_1 have the same neighbours in $V(T_1) \cup \cdots \cup V(T_n)$, and so do f_k, b_1. Hence we can add f_1 to A_1, $\{f_2, \ldots, f_{k-1}\}$ to C_1 and f_k to B_1, contrary to the maximality of the striation. This proves (3).

Since X is not local with respect to L, we may assume from (2) and (3) that there exist a vertex of $X \cap V(S_1)$ and a vertex of $X \cap V(T_1)$ that are nonadjacent. By reversing $T_1, \ldots, T_n$ we may assume that S_1 is parallel to each T_j. Since by 9.1 every vertex of Z_1 is complete to $V(S_1)$, we may assume that there is an S_1-rung a_1-P_1-b_1 and a T_1-antirung x_1-Q_1-y_1 such that $x_1 \in X$ and $X \cap V(P_1 \setminus a_1) \neq \emptyset$. Let $2 \leq j \leq n$, and choose i with $2 \leq i \leq m$ such that (S_1, S_i, T_1, T_j) is a twist. Let P_i be an S_i-rung, and let Q_j be a T_j-antirung. So (P_1, P_i, Q_1, Q_j) is a knot K say, and $X \cap V(K)$ is not local with respect to K. Let us apply 9.3; we deduce that one of the outcomes of 9.3 holds. The first and fourth outcomes contradict (2), and the third contradicts (3), so there is a path with vertex set in F satisfying 9.3.2. From the minimality of F, it follows that this path has vertex set F, and so F is a path with vertices f_1-$\cdots$-f_k say. Since $x_1 \in X$, it follows that one of f_1, f_k is adjacent to x_1, and we may assume that f_1 is adjacent to x_1. By 9.3.2, f_1 is also adjacent to x_j and to all internal vertices of Q_1, Q_j, and to neither of y_1, y_j, and none of $f_2, \ldots, f_{k-1}$ have neighbours in $V(Q_1 \cup Q_j)$, and f_k has a neighbour in $P_1 \setminus a_1$, and f_k has no neighbours in $V(Q_1 \cup Q_j)$. For any other choice of Q_j the same happens, and f_1, f_k cannot become exchanged since f_1 has neighbours in Q_1 and f_k has none. We deduce that f_1 is complete to $X_j \cup Z_j$ and anticomplete to Y_j; and $\{f_2, \ldots, f_k\}$ is anticomplete to $V(T_j)$. In particular there is a vertex of $X \cap V(S_1)$ and a vertex of $X \cap V(T_j)$ that are nonadjacent, and so by exchanging T_1 and T_j in the above argument, we deduce that f_1 is complete to $X_1 \cup Z_1$ and anticomplete to Y_1; and $\{f_2, \ldots, f_k\}$ is anticomplete to $V(T_1)$. Since this holds for all j, it follows that a_1, f_1 have the same neighbours in $V(T_1) \cup \cdots \cup V(T_n)$, and there are no edges between $\{f_2, \ldots, f_k\}$ and $V(T_1) \cup \cdots \cup V(T_n)$. But then we can add f_1 to A_1 and $\{f_2, \ldots, f_k\}$ to C_1, contrary to the maximality of the striation. This proves 9.5. ∎

Now we can prove 1.8.3, which we restate.

9.6 *Let G be a Berge graph, such that every appearance of K_4 in G and in $\overline{G}$ is degenerate, and there is no induced subgraph of G isomorphic to $L(K_{3,3})$. Then either G is a double split graph, or G admits a balanced skew partition, or one of $G, \overline{G}$ admits a proper 2-join, or there is no appearance of K_4 in either G or $\overline{G}$.*

Proof. If there is an appearance in G of some K_4-enlargement, say $L(H')$, then by 5.3, either $H' = K_{3,3}$, which is impossible by hypothesis, or there is a subgraph H'' of H' which is a bipartite subdivision of K_4, such that $L(H'')$ is nondegenerate, and again this is impossible by hypothesis. So there is no appearance in G of a K_4-enlargement, and similarly there is none in $\overline{G}$. Moreover, by 7.5, we may assume that there is no overshadowed appearance of K_4 in G or in $\overline{G}$. We may assume that there is an appearance of K_4 in one of $G, \overline{G}$, and consequently $|V(G)| \geq 8$; and by taking complements if necessary we may assume that $L(H)$ is an appearance of K_4 in G. By hypothesis it is degenerate, and hence there is a striation in G; choose a maximal striation L. Let L have strips $S_i = (A_i, C_i, B_i)$ $(1 \leq i \leq m)$ and antistrips $T_j = (X_j, Z_j, Y_j)$ $(1 \leq j \leq n)$. By 9.4 we can partition $V(G) \setminus V(L)$ into two sets M, N, where for every vertex in M its set of neighbours in $V(L)$ is local with respect to L, and for every vertex in N, its set of neighbours in $V(L)$ resolves L.

(1) *If there exists $f \in N$ with a nonneighbour in $V(S_1) \cup \cdots \cup V(S_m)$ then the theorem holds.*

For let f have a nonneighbour in S_1 say. Let N_1 be the anticomponent of N containing f, and let X be the set of all N_1-complete vertices in $V(G)$. From 9.5 applied in the complement, it follows that X resolves L. Since f has a nonneighbour in $V(S_1)$, there is a vertex u of S_1 not in X. Let U be the component of $V(G) \setminus (X \cup N)$ containing u. We claim that U is disjoint from $V(L) \setminus V(S_1)$, and no vertex in $V(S_2) \cup \cdots \cup V(S_m)$ has a neighbour in U. For suppose not; then there is a path P say in G, from $V(S_1)$ to $V(L) \setminus V(S_1)$, with $(X \cup N) \cap V(P) \subseteq V(S_2) \cup \cdots \cup V(S_m)$; choose such a path minimal. It follows that no internal vertex of P is in $V(L)$ or in $X \cup N$; and since X meets every edge between $V(S_1)$ and $V(L) \setminus V(S_1)$, and there are no edges between $V(S_1)$ and $V(S_2) \cup \cdots \cup V(S_m)$, it follows that P^* is nonempty. Now no vertex of P^* is in N, since $N \subseteq N_1 \cup X$; and so there is a component M_1 of M including P^*. From 9.5, the set of attachments of M_1 in $V(L)$ is local with respect to L. Since it has an attachment in $V(S_1)$ it therefore has none in $V(S_2) \cup \cdots \cup V(S_m)$. But the ends of P are attachments of M_1, they are nonadjacent, and one is in $V(S_1)$ and the other is not, a contradiction. This proves that U is disjoint from $V(L) \setminus V(S_1)$. Let X' be the set of vertices in X with neighbours in U, and let $V = V(G) \setminus (U \cup N_1 \cup X')$. Then V is nonempty because $V(S_2) \subseteq V$; and so $U \cup V, N_1 \cup X'$ is a skew partition of G. Since there is a vertex of S_2 in X (because X resolves L), and this vertex is in V, we deduce that the skew partition is loose, and hence by 4.2 G admits a balanced skew partition. This proves (1).

From (1) we may assume that N is complete to $V(S_1) \cup \cdots \cup V(S_m)$, and by taking complements, that M is anticomplete to $V(T_1) \cup \cdots \cup V(T_n)$.

(2) *If M, N are both nonempty then the theorem holds.*

For let M_1 be a component of M, and N_1 an anticomponent of N. By taking complements we may assume that there is a nonedge between M_1 and N_1. Since the set of attachments of M_1 in $V(L)$ is local by 9.5, and since it has no attachments in $V(T_1) \cup \cdots \cup V(T_n)$, we may assume that all its attachments are in $V(S_1)$. Let $V = V(G) \setminus (M_1 \cup N_1 \cup V(S_1))$. Since every vertex of S_1 is N_1-complete, it follows that $(M_1 \cup V, N_1 \cup V(S_1))$ is a skew partition of G, and since there are N_1-complete vertices with no neighbours in M_1 (for instance, any vertex of $V(S_2)$), the skew partition is loose, and by 4.2 G admits a balanced skew partition. This proves (2).

(3) *If M, N are both empty then the theorem holds.*

For then by 9.1, we may assume that for $1 \leq j \leq n$ all Q_j-antirungs have length 1. If $|V(S_1)| > 2$, then $(V(S_1), V(L) \setminus V(S_1))$ is a proper 2-join of G; for every vertex in $V(T_1) \cup \cdots \cup V(T_n)$ is either complete to A_1 and anticomplete to $B_1 \cup C_1$, or complete to B_1 and anticomplete to $A_1 \cup C_1$ (since all the antirungs have length 1). So we may assume that each S_i has only two vertices. In particular, every S_i-rung has length 1, so by taking complements the same argument shows that we may assume every $V(T_j)$ has only two vertices. But then G is a double split graph and the theorem holds. This proves (3).

From (2) and (3), and taking complements if necessary, we may assume that N is empty and M is nonempty. For $1 \leq i \leq m$ let M_i be the union of the components of M that have an attachment in $V(S_i)$, and let M_0 be the union of the components of M that have no attachments in $V(L)$.

Then $M_0, M_1, \ldots, M_n$ are pairwise disjoint and have union M. If M_0 is nonempty then G is not connected, and since $|V(G)| \geq 8$ it therefore admits a balanced skew partition, so we may assume that M_0 is empty. Since M is nonempty we may assume that M_1 is nonempty. We recall that $T_1 = (X_1, Z_1, Y_1)$; suppose that $z \in Z_1$. Then z is complete to $V(S_1)$ by 9.1, and hence if we define $V = V(G) \setminus M_1 \cup V(S_1) \cup \{z\}$, then $(M_1 \cup V, V(S_1) \cup \{z\})$ is a skew partition of G, and by 4.1 G admits a balanced skew partition. So we may assume that Z_1 is empty, and similarly every Z_j is empty. Then $(M_1 \cup V(S_1), V(G) \setminus (M_1 \cup V(S_1))$ is a proper 2-join of G. This proves 9.6. ∎

It is convenient to combine three earlier results as follows.

9.7 *Let G be a Berge graph, such that there is an appearance of K_4 in G. Then either one of $G, \overline{G}$ is a line graph, or G is a double split graph, or one of $G, \overline{G}$ admits a proper 2-join, or G admits a balanced skew partition.*

Proof. This is immediate from 9.6, 5.1 and 5.2. ∎

10 The even prism

We have completed the first of the main steps of the proof, handling graphs that contain an appearance of K_4. The next big step is to handle graphs that do not contain an appearance of K_4, but do contain a long prism. For our purposes, "even" prisms are easier than odd ones, and we treat them in this section. (Odd prisms are treated in sections 11-13.) Incidentally, in this section all we need is that there is no nondegenerate appearance of K_4 in G, and so the results of this section are independent of those in the previous one; these two sections could be in either order. We begin with some results about prisms in general.

For $i = 1, 2, 3$ let a_i-R_i-b_i be a path in G, such that these three paths form a prism K with triangles $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$. A subset $X \subseteq V(G)$ *saturates* the prism if at least two vertices of each triangle belong to X; and a vertex is *major* with respect to the prism if its neighbour set saturates it. A subset $X \subseteq V(K)$ is *local* with respect to the prism if either $X \subseteq V(R_i)$ for some i, or X is a subset of one of the triangles. By 7.2, the three paths R_1, R_2, R_3 all have lengths of the same parity. A prism is *even* if the three paths R_1, R_2, R_3 have even length, and *odd* otherwise.

10.1 *Let R_1, R_2, R_3 form a prism K in a Berge graph G, with triangles $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$, where each R_i has ends a_i and b_i. Let $F \subseteq V(G) \setminus V(K)$ be connected, such that its set of attachments in K is not local. Assume no vertex in F is major with respect to K. Then there is a path f_1-$\cdots$-f_n in F with $n \geq 1$, such that (up to symmetry) either:*

1. *f_1 has two adjacent neighbours in R_1, and f_n has two adjacent neighbours in R_2, and there are no other edges between $\{f_1, \ldots, f_n\}$ and $V(K)$, and (therefore) G has an induced subgraph which is the line graph of a bipartite subdivision of K_4, or*

2. *$n \geq 2$, f_1 is adjacent to a_1, a_2, a_3, and f_n is adjacent to b_1, b_2, b_3, and there are no other edges between $\{f_1, \ldots, f_n\}$ and $V(K)$, or*

3. *$n \geq 2$, f_1 is adjacent to a_1, a_2, and f_n is adjacent to b_1, b_2, and there are no other edges between $\{f_1, \ldots, f_n\}$ and $V(K)$, or*

4. *f_1 is adjacent to a_1, a_2, and there is at least one edge between f_n and $V(R_3) \setminus \{a_3\}$, and there are no other edges between $\{f_1, \ldots, f_n\}$ and $V(K) \setminus \{a_3\}$.*

Proof. We may assume that F is minimal such that it is connected and its set of attachments in K is not local. Let X be the set of attachments of F in K. For $1 \leq i \leq 3$, if $X \cap V(R_i) \neq \emptyset$, let c_i and d_i be the vertices of R_i in X closest (in R_i) to a_i and to b_i respectively, and let C_i, D_i be the subpaths of R_i between a_i and c_i, and between d_i and b_i respectively. Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$.

We claim that some two-element subset of X is not local. For since $X \not\subseteq B$ we may assume that c_1 exists and $c_1 \neq b_1$. Since $X \not\subseteq V(R_1)$, we may assume d_2 exists. If $d_2 \neq a_2$ then $\{c_1, d_2\}$ is the desired subset; so we may assume $d_2 = a_2$, and similarly $d_3 = a_3$ if d_3 exists. Since $X \not\subseteq A$, it follws that $d_1 \neq a_1$, and then $\{a_2, d_1\}$ is the desired subset. So some two-element subset $\{x_1, x_2\}$ of X is not local. Consequently x_1, x_2 are not adjacent. From the minimality of F, there is a path with vertices $x_1, f_1, \ldots, f_n, x_2$ such that $F = \{f_1, \ldots, f_n\}$.

(1) *If $n = 1$ then the theorem holds.*

For assume $n = 1$; then $F = \{f_1\}$. Since X is not local it meets at least two of the paths; suppose it only meets R_1 and R_2. Suppose that $c_1 = d_1$. Then we may assume that $c_1 \notin A$ and $c_2 \neq b_2$, by exchanging A and B if necessary; but then c_1 can be linked onto the triangle A, via the paths c_1-C_1-a_1, c_1-f_1-c_2-C_2-a_2, and c_1-D_1-b_1-b_3-R_3-a_3, contrary to 2.4, since f has at most one neighbour in A. So c_1 is different from d_1, and similarly c_2 is different from d_2 (and in particular, $c_2 \neq b_2$). Suppose that c_1 is nonadjacent to d_1. Then since f_1 is not major, we may assume it has at most one neighbour in A, by exchanging A and B if necessary; but it can be linked onto A, via f_1-c_1-C_1-a_1, f_1-c_2-C_2-a_2 and f_1-d_1-D_1-b_1-b_3-R_3-a_3, contrary to 2.4. So c_1, d_1 are adjacent, and similarly so are c_2, d_2, but then statement 1 of the theorem holds. So we may assume that X meets all three of R_1, R_2, R_3. Since f_1 is not major, we may assume that it has at most one neighbour in A, by exchanging A and B if necessary, and therefore cannot be linked onto A. Since it has neighbours in all three of R_1, R_2, R_3, it follows that for at least two of these paths, the only neighbour of f_1 in this path is in B. We may assume therefore that $c_1 = b_1$ and $c_2 = b_2$. Since X is not local, $c_3 \neq b_3$; but then statement 4 of the theorem holds. This proves (1).

We may therefore assume that $n \geq 2$. Let X_1 be the set of attachments of $F \setminus \{f_1\}$, and X_2 the set of attachments of $F \setminus \{f_n\}$. From the minimality of F, both X_1 and X_2 are local. Moreover, $X = X_1 \cup X_2$, and for $2 \leq i \leq n-1$, every neighbour of f_i in K belongs to $X_1 \cap X_2$.

(2) *If $X_1 \subseteq A$ and $X_2 \subseteq V(R_1)$ then the theorem holds.*

For then f_1 has at least one neighbour in $R_1 \setminus a_1$, and f_n is adjacent to at least one of a_2, a_3, and there are no other edges between F and $V(K) \setminus \{a_1\}$. If f_n is adjacent to both a_2, a_3 then statement 4 of the theorem holds, so we assume it is not adjacent to a_3. But then a_2 can be linked onto the triangle B, via a_2-f_n-f_{n-1}-$\cdots$-f_1-d_1-D_1-b_1, a_2-R_2-b_2, a_2-a_3-R_3-b_3, contrary to 2.4. This proves (2).

From (2), since both X_1 and X_2 are local, we may assume that either $X_1 \subseteq A$ and $X_2 \subseteq B$, or $X_1 \subseteq V(R_2)$ and $X_2 \subseteq V(R_1)$. In either case $X_1 \cap X_2 = \emptyset$, so none of $f_2, \ldots, f_{n-1}$ has any neighbours in $V(K)$. Therefore X_1 is the set of neighbours of f_n in $V(K)$, and X_2 is the set of neighbours of f_1 in $V(K)$.

(3) *If $X_1 \subseteq A$ and $X_2 \subseteq B$ then the theorem holds.*

For then we may assume that f_n is adjacent to a_1 and f_1 to b_2. Suppose first that n has the same parity as the length of R_1. Since a_2-R_2-b_2-f_1-$\cdots$-f_n-a_2 is not an odd hole, it follows that f_n is not adjacent to a_2, and similarly f_1 is not adjacent to b_1. Since a_3-R_3-b_3-b_2-f_1-$\cdots$-f_n-a_1-a_3 is not an odd hole, either f_n is adjacent to a_3 or f_1 to b_3, and not both, as we saw before. But then statement 4 of the theorem holds. Now suppose that n has different parity from the length of R_1. Since a_1-a_2-R_2-b_2-f_1-$\cdots$-f_n-a_1 is not an odd hole, f_n is adjacent to a_2, and similarly f_1 to b_1. If there are no more edges between F and $V(K)$ then statement 3 of the theorem holds, so we may assume that f_n is adjacent to a_3. By the same argument as before it follows that f_1 is adjacent to b_3, and then statement 2 of the theorem holds. This proves (3).

From (2) and (3) we may assume that $X_1 \subseteq V(R_2)$ and $X_2 \subseteq V(R_1)$. So f_1 is adjacent to the vertices of R_1 that are in X, and f_n to those of R_2 in X. If $c_1 = d_1$, then from the symmetry we may assume that $c_1 \neq a_1$, and $c_2 \neq b_2$; but then c_1 can be linked onto A, via c_1-C_1-a_1, c_1-f_1-$\cdots$-f_n-c_2-C_2-a_2, c_1-D_1-b_1-b_3-R_3-a_3, contrary to 2.4. So $c_1 \neq d_1$ and similarly $c_2 \neq d_2$; and in particular $c_2 \neq b_2$. If c_1, d_1 are nonadjacent, then f_1 can be linked onto A via f_1-c_1-C_1-a_1, f_1-$\cdots$-f_n-c_2-C_2-a_2, f_1-d_1-D_1-b_1-b_3-R_3-a_3; but f_1 has at most one neighbour in A (because $n \geq 2$), contrary to 2.4. So c_1, d_1 are adjacent, and similarly so are c_2, d_2; but then statement 1 of the theorem holds. This proves 10.1. ∎

10.2 *Let R_1, R_2, R_3, K, F be as in 10.1, and suppose that 10.1.1 holds. Then either R_1 and R_2 both have length 1, or there is a nondegenerate appearance of K_4 in G.*

Proof. For let f_1-$\cdots$-f_n be a path in F such that f_1 has two adjacent neighbours in R_1, and f_n has two adjacent neighbours in P_2, and there are no other edges between $\{f_1, \ldots, f_n\}$ and $V(K)$. Then $G|(V(K) \cup \{f_1, \ldots, f_n\}$ is a line graph of a bipartite subdivision of K_4. We may assume it is degenerate. Hence the prism is odd, for all prisms contained in a degenerate appearance of K_4 are odd. So R_3 is odd, and therefore so is the path f_1-$\cdots$-f_n, and the other four "rungs" of this line graph have length 0. In particular, R_1 and R_2 both have length 1. This proves 10.2. ∎

There is also a tighter version of 10.1, the following.

10.3 *Let G be a Berge graph, such that there is no nondegenerate appearance of K_4 in G. Let R_1, R_2, R_3 form a prism K in G, with triangles $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$, where each R_i has ends a_i and b_i. Let $F \subseteq V(G) \setminus V(K)$ be connected, such that no vertex in F is major with respect to K. Let x_1 be an attachment of F in the interior of R_1, and assume that there is another attachment x_2 of F not in R_1. Then there is a path f_1-$\cdots$-f_n in F such that (up to the symmetry between A and B) f_1 is adjacent to a_2, a_3, and f_n has at least one neighbour in $R_1 \setminus a_1$, and there are no other edges between $\{f_1, \ldots, f_n\}$ and $V(K) \setminus \{a_1\}$.*

Proof. We may assume F is minimal such that it is connected, x_1 is one of its attachments, and it has some attachment x_2 in $R_2 \cup R_3$. Hence there is a path x_2-v_1-$\cdots$-v_m-x_1 where $F = \{v_1, \ldots, v_m\}$. By

10.1, there is a subpath f_1-$\cdots$-f_n of v_1-$\cdots$-v_m such that one of 10.1.1-4 holds. From the minimality of F, v_1 is the only vertex of F with a neighbour in $V(R_2) \cup V(R_3)$, and in particular, at most one vertex of f_1-$\cdots$-f_n has a neighbour in $V(R_2) \cup V(R_3)$. We deduce that f_1-$\cdots$-f_n does not satisfy 10.1.2 or 10.1.3. Suppose it satisfies 10.1.1. By 10.2 the path f_1-$\cdots$-f_n joins two of R_1, R_2, R_3 that are both of length 1, and therefore n is even. Since R_1 has length ≥ 2 (because x_1 is in its interior) it follows that f_1, f_n are distinct vertices of F both with neighbours in $V(R_2) \cup V(R_3)$, a contradiction. So f_1-$\cdots$-f_n satisfies 10.1.4, and therefore we may assume that for some i with $1 \leq i \leq 3$, f_1 is adjacent to the two vertices in $A \setminus \{a_i\}$, and f_n has at least one neighbour in $R_i \setminus a_i$, and there are no other edges between $\{f_1, \ldots, f_n\}$ and $V(K) \setminus \{a_i\}$. Suppose first that $i > 1$, $i = 2$ say. Then both f_1, f_n have neighbours in $V(R_2) \cup V(R_3)$, and so from the minimality of F it follows that $n = 1$ and $f_1 = v_1$. But then f_1 can be linked onto the triangle B, via the path between f_1 and b_1 with interior in $\{v_2, \ldots, v_m\} \cup V(R_1 \setminus a_1)$, the path between f_1 and b_2 with interior in $V(R_2 \setminus a_2)$, and the path f_1-a_3-R_3-b_3, contrary to 2.4. Hence $i = 1$, and the theorem is satisfied. This proves 10.3. ∎

Another useful corollary of 10.1 is the following.

10.4 *Let G be Berge, such that there is no nondegenerate appearance of K_4 in G. Let R_1, R_2, R_3 form a prism K in a Berge graph G, with triangles $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$, where each R_i has ends a_i and b_i. Let $F \subseteq V(G) \setminus V(K)$ be connected, such that if the prism is even then no vertex in F is major with respect to K. Assume that the set of attachments of F in K is not local, but none are in $V(R_3)$. Then $|F| \geq 2$, and the set of attachments of F in K is precisely $\{a_1, b_1, a_2, b_2\}$.*

Proof. If there is a major vertex $v \in F$, then since it has no neighbours in R_3, it is adjacent to a_1 and b_2, and since v-a_1-a_3-R_3-b_3-b_2-v is a hole, it follows that the prism is even, contrary to the hypothesis. So there is no major vertex in F. By 10.3 no internal vertex of R_1 or R_2 is an attachment of F. By 10.1, there is a path in F satisfying one of 10.1.1-4; and since it has no attachments in R_3, it must satisfy 10.1.1 or 10.1.3, and in either case a_1, b_1, a_2, b_2 are all attachments of F. Since no vertex in F is major it follows that $|F| \geq 2$. This proves 10.4. ∎

The next result is a close relative of 7.5.

10.5 *Let G be Berge, such that there is no nondegenerate appearance of K_4 in G. If there is an even prism K in G, such that some vertex of G is major with respect to K, then G admits a balanced skew partition.*

Proof. Any prism has six vertices of degree 3, called *triangle-vertices*; choose a prism K and a nonempty anticonnected set $Y \subseteq V(G) \setminus V(K)$, such that every vertex in Y is major with respect to the prism, and as few triangle-vertices of K are Y-complete as possible. Let the paths a_i-R_i-$b_i (i = 1, 2, 3)$ form K, where $\{a_1, a_2, a_3\}, \{b_1, b_2, b_3\}$ are its triangles. We may assume that Y is maximal with the given property. Let X be the set of all Y-complete vertices in G. By 7.3, X saturates K. Consequently there is one of R_1, R_2, R_3 with both ends in X, say R_1. Let $X_0 = X \setminus V(K)$ and $X_1 = \{a_1, b_1\}$.

(1) *If $F \subseteq V(G)$ is connected and some vertex of $V(R_1^*)$ has a neighbour in F, and so does some vertex of $V(R_2) \cup V(R_3)$, then $F \cap (X_0 \cup X_1 \cup Y)$ is nonempty.*

Suppose for a contradiction that some F exists not satisfying (1), and choose it minimal. Hence $G|F$ is a path, disjoint from K. Consequently $F \cap X = \emptyset$. Suppose some vertex in $v \in F$ is major with respect to K. Then since $v \notin X$ it follows that v has a nonneighbour in Y, and so $Y \cup \{v\}$ is anticonnected; the maximality of Y therefore implies that $v \in Y$, and hence $F \cap Y \neq \emptyset$ and the claim holds. So we may assume that no vertex in F is major. Let x_1 be an attachment of F in R_1^*. By 10.3 we may assume that there is a path f_1-$\cdots$-f_n in F such that f_1 is adjacent to a_2, a_3, and f_n has neighbours in $R_1 \setminus a_1$, and f_1a_2, f_1a_3 are the only edges between $\{f_1, \ldots, f_n\}$ and $V(R_2) \cup V(R_3)$ Now there is a path R from f_1 to b_1 with interior in $\{f_2, \ldots, f_n\} \cup V(R_1 \setminus a_1)$, and hence R, R_2, R_3 form a prism K' say. By 7.4, every vertex in Y is major with respect to K', and since a_1 is Y-complete and f_1 is not, it follows that the number of Y-complete triangle-vertices in K' is smaller than the number in K, a contradiction. This proves (1).

It follows from (1) that there is a partition of $V(G) \setminus (X_0 \cup X_1 \cup Y)$ into two sets L and M say, where there is no edge between L and M, and $V(R_1^*) \subseteq L$ and $V(R_2) \cup V(R_3) \subseteq M$. So $(L \cup M, X_0 \cup X_1 \cup Y)$ is a skew partition of G. Since at least two vertices of A are in X and only one is in X_1, there is a vertex of X in M, and so the skew partition is loose. By 4.2 the result follows. This proves 10.5. ∎

The main result of this section is 1.8.4, which we restate.

10.6 *Let G be a Berge graph, such that there is no nondegenerate appearance of K_4 in G. If G contains an even prism, then either G is an even prism with $|V(G)| = 9$, or G admits a proper 2-join or a balanced skew partition.*

Proof. Since G contains an even prism, we can choose in G a collection of nine sets

$$\begin{matrix} A_1 & C_1 & B_1 \\ A_2 & C_2 & B_2 \\ A_3 & C_3 & B_3 \end{matrix}$$

with the following properties:

- all these sets are nonempty and pairwise disjoint
- for $1 \leq i < j \leq 3$, A_i is complete to A_j and B_i is complete to B_j, and there are no other edges between $A_i \cup B_i \cup C_i$ and $A_j \cup B_j \cup C_j$
- for $1 \leq i \leq 3$, every vertex of $A_i \cup B_i \cup C_i$ belongs to a path between A_i and B_i with interior in C_i
- some path between A_1 and B_1 with interior in C_1 is even.

We call this collection of nine sets a *hyperprism*. Let H be the subgraph of G induced on the union of the nine sets. Choose the hyperprism with $V(H)$ maximal. For $1 \leq i \leq 3$, a path from A_i to B_i with interior in C_i is called an i-rung. Let us write $S_i = A_i \cup B_i \cup C_i$ for $1 \leq i \leq 3$, and $A = A_1 \cup A_2 \cup A_3$, and $B = B_1 \cup B_2 \cup B_3$.

(1) *For $1 \leq i \leq 3$, all i-rungs have even length.*

For we are given that some 1-rung R_1 say has even length. Let R_2 be an 2-rung; then the union of R_1 and R_2 induces a hole, and so R_2 is even. Hence every 2- or 3-rung is even, and hence so is every 1-rung. This proves (1).

A subset $X \subseteq V(H)$ is *local* (with respect to the hyperprism) if X is a subset of one of S_1, S_2, S_3, A or B.

(2) *We may assume that for every connected subset F of $V(G) \setminus V(H)$, its set of attachments in H is local.*

For suppose not. Choose F minimal, and let X be the set of attachments of F in H. Suppose first that there exists $x_1 \in X \cap C_1$. Since X is not local, we may assume that there exists $x_2 \in X \cap S_2$. For $i = 1, 2, 3$ choose an i-rung R_i with ends $a_i \in A_i$ and $b_i \in B_i$, such that $x_i \in V(R_i)$ for $i = 1, 2$. Then R_1, R_2, R_3 form an even prism K say. By 10.5 we may assume no vertex in F is major with respect to K; so by 10.3, we may assume that there is a path f_1-$\cdots$-f_n in F such that f_1 is adjacent to a_2, a_3, and f_n has at least one neighbour in $R_1 \setminus a_1$, and there are no other edges between $\{f_1, \ldots, f_n\}$ and $V(K) \setminus \{a_1\}$. From the minimality of F it follows that $F = \{f_1, \ldots, f_n\}$. Since this holds for all choices of R_3 it follows that f_1 is complete to A_3 and there are no edges between $\{f_1, \ldots, f_n\}$ and $B_3 \cup C_3$. Since $a_3 \in X$ the same conclusion follows for all choices of R_2, and so f_1 is complete to A_2 and there are no edges between $\{f_1, \ldots, f_n\}$ and $B_2 \cup C_2$. But then we can add f_1 to A_1 and $\{f_2, \ldots, f_n\}$ to C_1, contradicting the maximality of the hyperprism.

It follows that $X \cap C_1 = \emptyset$, and similarly $X \cap C_2, X \cap C_3 = \emptyset$. We claim there is a 2-element subset of X which is also not local. For we may assume $X \cap A_1 \neq \emptyset$; and hence if X meets B_2 or B_3 our claim holds. If not, then it meets B_1 (since it is not a subset of A) and meets $A_2 \cup A_3$ (since it is not a subset of S_1), and again the claim holds. So there is a subset $\{x_1, x_2\}$ of X which is not local. We may assume that $x_1 \in A_1$ and $x_2 \in B_2$. From the minimality of F, there is a path x_1-f_1-$\cdots$-f_n-x_2 with $F = \{f_1, \ldots, f_n\}$.

Suppose first that n is even. For any 3-rung R_3 with ends $a_3 \in A_3$ and $b_3 \in B_3$,

$$x_1\text{-}f_1\text{-}\cdots\text{-}f_n\text{-}x_2\text{-}b_3\text{-}R_3\text{-}a_3\text{-}x_1$$

is not an odd hole, and so some vertex of R_3 is in X. Since $X \cap C_3 = \emptyset$, and a_3 has no neighbour in $\{f_2, \ldots, f_n\}$ from the minimality of F, and similarly b_3 has no neighbour in $\{f_1, \ldots, f_{n-1}\}$, it follows that either f_1 is adjacent to a_3, or f_n to b_3 (and not both, since otherwise f_1-$\cdots$-f_n-b_3-R_3-a_3 is an odd hole). From the symmetry we may assume that f_n is adjacent to b_3. By exchanging S_2 and S_3 it follows that for every 2-rung with ends $a_2 \in A_2$ and $b_2 \in B_2$, either f_1 is adjacent to a_2 or f_n to b_2, and not both. Suppose that f_n is complete to $B_2 \cup B_3$; then f_1 has no neighbours in $S_2 \cup S_3$, and we can add f_n to B_1 and $f_1, \ldots, f_{n-1}$ to C_1, contrary to the maximality of the hyperprism. So f_n is not complete to $B_2 \cup B_3$, and hence f_1 has a neighbour in one of A_2, A_3, say A_3; and by exchanging S_1 and S_2 it follows that for every 1-rung with ends $a_1 \in A_1$ and $b_1 \in B_1$, either f_1 is adjacent to a_1 or f_n to b_1 and not both. In particular, f_1 has no neighbours in B and f_n has none in A. For $i = 1, 2, 3$ let A'_i be the set of neighbours of f_1 in A_i, and let $A''_i = A_i \setminus A'_i$; let B''_i be the set of neighbours of f_n in B_i, and let $B'_i = B_i \setminus B''_i$. We have shown so far that every i-rung is either between A'_i and B'_i or between A''_i and B''_i. Let C'_i be the union of the interiors of the i-rungs between A'_i and B'_i, and C''_i the union of the interiors of the i-rungs between A''_i and B''_i. We observe that $C_i = C'_i \cup C''_i$. Moreover, $C'_i \cap C''_i = \emptyset$, for otherwise there would be an i-rung between A'_i and B''_i. For the same reason there are no edges between $A'_i \cup C'_i$ and $C''_i \cup B''_i$, and no edges between $A''_i \cup C''_i$ and $C'_i \cup B'_i$. We claim that A'_i is complete to A''_i. For if not, let R'' be an i-rung with ends $a'' \in A''_i$ and $b'' \in B''_i$, and let $a' \in A'_i$ be nonadjacent to a''. Since we have seen that f_n has neighbours in at least two of B_1, B_2, B_3, it follows that at least two of A''_1, A''_2, A''_3 are nonempty, and therefore we may choose $a \in A''_j$ for some $j \neq i$. Then

$$a\text{-}a'\text{-}f_1\text{-}\cdots\text{-}f_n\text{-}b''\text{-}R''\text{-}a''\text{-}a$$

is an odd hole, a contradiction. So A'_i is complete to A''_i for each i, and similarly B'_i is complete to B''_i for each i. We showed already that we may assume that A'_1, A''_2, A'_3, A''_3 are all nonempty. But then the nine sets

$$\begin{array}{ccc} A'_1 & C'_1 & B'_1 \\ A'_2 \cup A'_3 & C'_2 \cup C'_3 & B'_2 \cup B'_3 \\ A''_1 \cup A''_2 \cup A''_3 \cup \{f_1\} & C''_1 \cup C''_2 \cup C''_3 \cup \{f_2, \dots, f_n\} & B''_1 \cup B''_2 \cup B''_3 \end{array}$$

form a hyperprism, contrary to the maximality of $V(H)$. This completes the argument when n is even.

Now assume n is odd. f_1 has a neighbour a_1 say in A_1; let R_1 be a 1-rung with ends a_1 and b_1 say. Similarly let R_2 be a 2-rung with ends a_2 and b_2, where $b_2 \in B_2$ is adjacent to f_n. Since $a_1\text{-}f_1\text{-}\cdots\text{-}f_n\text{-}b_2\text{-}b_1\text{-}R_1\text{-}a_1$ is not an odd hole, it follows that $b_1 \in X$, and similarly $a_2 \in X$. From the minimality of F, one of b_1, a_2 is adjacent to f_1 and the other to f_n, and neither has any more neighbours in F. Suppose that f_n is not adjacent to b_1; so f_1 is adjacent to b_1, and $n \geq 2$, and f_n is adjacent to a_2. But then $b_1\text{-}f_1\text{-}\cdots\text{-}f_n\text{-}b_2\text{-}b_1$ is an odd hole, a contradiction. This proves that f_n is adjacent to b_1 and f_1 to a_2. Hence for all $1 \leq i \leq 3$, and for every i-rung with ends $a \in A$ and $b \in B$, $a \in X$ if and only if $b \in X$, and if so then f_1 is adjacent to a and f_n to b. Consequently, for every vertex in $X \cap A$, f_1 is its unique neighbour in F, and for every vertex in $X \cap B$, f_n is its unique neighbour in F. For $1 \leq i \leq 3$, let

$$\begin{aligned} A'_i &= A_i \cap X \\ B'_i &= B_i \cap X \\ A''_i &= A_i \setminus X \\ B''_i &= B_i \setminus X. \end{aligned}$$

Let C'_i be the union of the interior of the i-rungs between A'_i and B'_i, and C''_i the union of the interior of the i-rungs between A''_i and B''_i. We have seen that every i-rung is of one of these two types, and so $C_i = C'_i \cup C''_i$. Moreover, since there is no rung between A'_i and B''_i, it follows that $C'_i \cap C''_i = \emptyset$, and there are no edges between $A'_i \cup C'_i$ and $C''_i \cup B''_i$, and similarly no edges between $A''_i \cup C''_i$ and $C'_i \cup B'_i$. We have seen that f_1 has neighbours in at least two of A_1, A_2, A_3, and f_n has neighbours in at least two of B_1, B_2, B_3. We claim that also f_1 has nonneighbours in at least two of A_1, A_2, A_3, and the same for f_n. For suppose not, and f_1 is complete to $A_1 \cup A_2$ say. Then f_n is complete to $B_1 \cup B_2$; by 10.5 we may assume that $n > 1$, and so we can add f_1 to A_3, f_n to B_3 and $f_2, \dots, f_{n-1}$ to C_3, contrary to the maximality of $V(H)$. This proves that f_1 has nonneighbours in at least two of A_1, A_2, A_3, and similarly f_n has nonneighbours in at least two of B_1, B_2, B_3. Let $1 \leq i \leq 3$; we claim that A'_i is complete to A''_i. For we may assume that $i = 1$; suppose that $a' \in A'_1$ and $a'' \in A''_1$

are nonadjacent, and let R'' be a 1-rung with ends a'', b''. Choose $a \in A_2'' \cup A_3''$ and $b \in B_2' \cup B_3'$; then a, b are not adjacent since all rungs have even length, and so a-a'-f_1-$\cdots$-f_n-b-b''-R''-a''-a is an odd hole, a contradiction. This proves that A_i' is complete to A_i'' for $i = 1, 2, 3$, and similarly B_i' is complete to B_i''. We have seen that we may assume that A_1', A_2' are nonempty. But then

$$\begin{array}{ccc} A_1' & C_1' & B_1' \\ A_2' \cup A_3' & C_2' \cup C_3' & B_2' \cup B_3' \\ A_1'' \cup A_2'' \cup A_3'' \cup \{f_1\} & C_1'' \cup C_2'' \cup C_3'' \cup \{f_2, \ldots, f_{n-1}\} & B_1'' \cup B_2'' \cup B_3'' \cup \{f_n\} \end{array}$$

is a hyperprism, contrary to the maximality of $V(H)$. This proves (2).

Suppose F is a component of $V(G) \setminus V(H)$, and all its attachments are in A. Then $(V(G) \setminus A, A)$ is a skew partition of G. We must show that G admits a balanced skew partition. Choose $b_2 \in B_2$ and $a_3 \in A_3$. Then $B_1 \cup C_1 \cup \{b_2\}$ is connected, and all vertices in A_1 have neighbours in it. By 2.6, $(B_1 \cup C_1 \cup \{b_2\}, A_1)$ is balanced, and so by 2.7.1, so is (A_1, F). By 4.5, G admits a balanced skew partition. So we may assume there is no such F, and the same for B.

From (2) it follows that for every component of $V(G) \setminus V(H)$, all its attachments in H are a subset of one of S_1, S_2, S_3. Let X be the union of S_1 and all components of $V(G) \setminus V(H)$ whose attachment set is a subset of S_1, and let $Y = V(G) \setminus X$. Then $|Y| \geq 4$, and so either (X, Y) is a proper 2-join in G, or both A_1, B_1 have one element and X is the vertex set of a path between these two vertices. We may assume the latter, and the same for S_2 and S_3; and so G is an even prism. Then either it admits a proper 2-join, or $|V(G)| = 9$. This proves 10.6. ∎

11 Step-connected strips

Our next target is the statement analogous to 10.6 for long odd prisms, but we need to creep up on it in stages. (A warning: we shall not prove the exact analogue, and we don't know if it is true. We need to permit more types of decomposition, namely proper 2-joins in $\overline{G}$, and proper homogeneous pairs.) The key idea is to start with a prism of three paths, R_0, R_1, R_2, where R_0 has length ≥ 3, and to grow the union of the other two paths into a kind of strip (*one* strip, not two) with a richer internal structure than we have seen hitherto, that we call being "step-connected". If we expand the union of these two paths into a maximal step-connected strip, then the remainder of the graph attaches to this structure in ways that we can exploit. In this section we introduce step-connected strips, and prove some preliminary lemmas about them.

Let (A, C, B) be a strip in G. A *step* is a pair a_1-R_1-b_1, a_2-R_2-b_2 of rungs such that

- $V(R_1) \cap V(R_2) = \emptyset$
- a_1 is adjacent to a_2, and b_1 to b_2, and there are no other edges between $V(R_1)$ and $V(R_2)$.

The edges a_1a_2 and b_1b_2 such that there exists a step as above are called *stepped* edges. We say that the strip is *step-connected* if every vertex of $A \cup B \cup C$ is in a step, and for every partition (X, Y) of A or of B into two nonempty sets, there is a step R_1, R_2 such that R_1 has an end in X and R_2 has an end in Y. (This second condition is equivalent to requiring that the subgraph of G with vertex set A and edges the stepped edges within A be connected, and the same for B.)

Let (A, C, B) be a step-connected strip in a Berge graph G. A vertex $v \in V(G) \setminus (A \cup B \cup C)$ is a *left-star* for the strip if it is complete to A and anticomplete to $B \cup C$, and it is a *right-star* if it is complete to B and anticomplete to $A \cup C$. A *banister* (with respect to the strip) is a path a-R-b of $G \setminus (A \cup B \cup C)$, such that a is a left-star, b is a right-star, and there are no edges between the interior of R and $V(S)$. (Here we distinguish between a-R-b and b-R-a; we follow the convention that when describing a banister relative to a strip, the end which is the left-star is listed first.) A banister can have length 1.

11.1 *Let G be a Berge graph, such that there is no nondegenerate appearance of K_4 in G. Let $S = (A, C, B)$ be a step-connected strip in G, and let a_0-R_0-b_0 be a banister. Suppose that $v \in V(G) \setminus V(S)$ has a neighbour in $A \cup C$, and has no neighbour in B; and that P is a path in $G \setminus (V(S) \cup \{a_0\})$ from v to b_0, such that there are no edges between P^* and $V(S)$. Then v is a left-star.*

Proof. Let F be a connected subset of $V(P)$, containing v and disjoint from $V(R_0)$, and with an attachment in $R_0 \setminus a_0$.

(1) *For every step a_1-R_1-b_1, a_2-R_2-b_2 , if v has a neighbour in $R_1 \cup R_2$ then v is adjacent to a_1, a_2 and to no other vertices of $R_1 \cup R_2$.*

For assume v has a neighbour in R_1 say, and hence in $R_1 \setminus b_1$. Now R_0, R_1, R_2 form a prism K say, and no vertex in F is major with respect to K since no vertex in F is adjacent to b_1 or b_2. Yet F has an attachment in $R_0 \setminus a_0$ and one in $R_1 \setminus b_1$, so its set of attachments is not local. Since b_1 is not an attachment of F, it follows from 10.4 that F has an attachment in R_2; and therefore v has a neighbour in $R_2 \setminus b_2$. If v has any neighbours in $R_1 \cup R_2$ different from a_1, a_2, say a neighbour in the interior of R_1, then v can be linked onto the triangle b_0, b_1, b_2, via the paths v-P-b_0, from v to b_1 with interior in $R_1 \setminus a_1$, and from v to b_2 with interior in R_2; but this contradicts 2.4. This proves (1).

From (1) it follows that v has no neighbour in C (since every vertex is in a step), and therefore v has at least one neighbour in A; and from (1) again, v has no nonneighbour in A (for otherwise we could choose the step in (1) with v adjacent to a_1 and not to a_2, since the strip is step-connected.) This proves 11.1. ∎

11.2 *Let G be Berge, such that there is no appearance of K_4 in G. Let $S = (A, C, B)$ be a step-connected strip in G, and let a_0-R_0-b_0 be a banister. Let $v \in V(G) \setminus V(S)$ have a neighbour in $V(S)$, and be nonadjacent to b_0. Let P be a path in $G \setminus (V(S) \cup \{a_0\})$ from v to b_0, and let Q be a path in $G \setminus (V(S) \cup \{b_0\})$ from v to a_0, such there are no edges from $P^* \cup Q^*$ to $V(S)$. Then either v is B-complete, or v is a left-star.*

Proof. If v has no neighbours in B, then by 11.1 v is a left-star, so we may assume v has a neighbour in B. Since we may assume it is not B-complete, there is a step a_1-R_1-b_1, a_2-R_2-b_2 such that v is adjacent to b_1 and not to b_2. Let $F \subseteq V(Q)$ be connected, containing v and disjoint from $V(R_0)$, with an attachment in $R_0 \setminus b_0$. Now R_0, R_1, R_2 form a prism K say, and no vertex of F is major with respect to K since none of them has two neighbours in $\{b_0, b_1, b_2\}$. But there is an attachment of F in $R_0 \setminus b_0$, and b_1 is also an attachment of F, so its set of attachments is not local with respect to the prism. By 10.1, one of 10.1.1-4 holds. Since there is no appearance of K_4 in G, 10.1.1 does not hold. Also 10.1.2, 10.1.3 do not hold, since v is the only vertex in F with neighbours in $A \cup B$. So 10.1.4 holds, and therefore F has an attachment in R_2, and so v has a neighbour in R_2. But then v can be linked onto the triangle $\{b_0, b_1, b_2\}$, via v-P-b_0, v-b_1, and the path from v to b_2 with interior in R_2, contrary to 2.4. This proves 11.2. ∎

We remark:

11.3 *Let G be Berge, containing no even prism, let $S = (A, C, B)$ be a step-connected strip in G, and let a_0-R_0-b_0 be a banister. Then every rung of the strip has odd length, and so does R_0.*

Proof. Let a_1-R_1-b_1, a_2-R_2-b_2 be a step. Then these three paths form a prism, and it is not an even prism by hypothesis. In particular R_0 has odd length, by 7.2. For any rung a-R-b, the hole a_0-R_0-b_0-b-R-a-a_0 has even length, and so R is odd. This proves 11.3. ∎

11.4 *Let G be a Berge graph, such that there is no appearance of K_4 in G and no even prism in G. Let $S = (A, C, B)$ be a step-connected strip in G. Let $F \subseteq V(G) \setminus (A \cup B \cup C)$ be connected, such that there are no edges between F and $A \cup B \cup C$. There is no anticonnected set $Q \subseteq V(G) \setminus (A \cup B \cup C \cup F)$ such that:*

- *some right-star has a neighbour in F and a nonneighbour in Q,*
- *some vertex in B has a nonneighbour in Q,*
- *some left-star with a neighbour in F is Q-complete,*
- *every vertex in Q has a neighbour in F,*
- *every vertex in Q has a neighbour in $A \cup B \cup C$, and*
- *no vertex in Q is a left-star.*

Proof. Suppose that such a set Q exists. Let a_0 be a left-star with a neighbour in F complete to Q, and let b_0 be a right-star with a neighbour in F and a nonneighbour in Q. Let R_0 be a path between a_0 and b_0 with interior in F. Hence a_0-R_0-b_0 is a banister. By 11.3 R_0 and every rung has odd length. Since some vertex in B has a nonneighbour in F, there is an antipath q_1-$\cdots$-q_n in Q such that q_1 is not adjacent to b_0 and q_n is not adjacent to some vertex in B. Choose such an antipath with n minimum. Let B_1 be the set of neighbours of q_n in B, and $B_2 = B \setminus B_1$. So $B_2 \neq \emptyset$. Since q_n is not a left-star, and there is a path from q_n to b_0 with interior in F, it follows from 11.1 that $B_1 \neq \emptyset$. Choose a step a_1-R_1-b_1, a_2-R_2-b_2 with $b_1 \in B_1$ and $b_2 \in B_2$.

(1) $n \geq 2$.

For suppose $n = 1$. Then q_1 is adjacent to a_0 and to b_1, and not to b_0, so by 10.4, q_1 has a neighbour in $R_2 \setminus b_2$. Since q_1 also has a neighbour in F, it can be linked onto the triangle $\{b_0, b_1, b_2\}$, via a path from q_1 to b_0 with interior in F, the path q_1-b_1, and the path from q_1 to b_2 with interior in R_2, contrary to 2.4. This proves (1).

(2) $(A \cup B \cup C, \{b_0, q_1, \dots, q_n\})$ *is balanced.*

For $b_1 \in B_1$ is complete to $\{b_0, q_1, \dots, q_n\}$ from the minimality of n. But b_1 has no neighbour in F, so by 2.6, $(F, \{b_0, q_1, \dots, q_n\})$ is balanced. Since F is connected and every vertex in $\{b_0, q_1, \dots, q_n\}$ has a neighbour in F, the claim follows from 2.7.1. This proves (2).

Now the path a_0-a_2-R_2-b_2-b_1 is odd, and its ends are complete to $\{q_1, \dots, q_n\}$; so by (2) and 2.1, there are two adjacent vertices u, v in this path, both complete to $\{q_1, \dots, q_n\}$. Since b_2 is not adjacent to q_n, it follows that $u, v \in \{a_0\} \cup V(R_2 \setminus b_2)$. Suppose that the hole a_0-R_0-b_0-b_2-R_2-a_2-a_0 has length ≥ 6. Then one of u, v is nonadjacent to both b_0, b_2, say v, and hence n is odd, since v-b_0-q_1-$\cdots$-q_n-b_2-v is an antihole; but b_1 is adjacent to b_0 and b_2, and has no other neighbours in this hole, and is complete to $\{q_1, \dots, q_n\}$, contrary to 3.3. So the hole has length 4, and in particular a_2 is adjacent to b_2 and is complete to $\{q_1, \dots, q_n\}$, and a_0 is adjacent to b_0. Hence n is odd, because b_1-a_2-b_0-q_1-$\cdots$-q_n-b_2-a_0-b_1 is an antihole, and so a_2-b_0-q_1-$\cdots$-q_n-b_2 is an odd antipath, contrary to (2). This proves 11.4. ∎

A triple (S, F, Q) is called a *1-breaker* in G if it satisfies the following.

- $S = (A, C, B)$ is a step-connected strip in G,
- $F \subseteq V(G) \setminus V(S)$ is connected, such that there are no edges between F and $V(S)$, and there is a left- and right-star, both with neighbours in F,
- $Q \subseteq V(G) \setminus (V(S) \cup F)$ is anticonnected,
- some vertex in A has a nonneighbour in Q, and so does some vertex in B,
- every vertex in Q has a neighbour in F and a neighbour in $A \cup B \cup C$,
- some left-star with a neighbour in F is Q-complete,
- no vertex in Q is a left-star.

11.5 *Let G be a Berge graph, such that there is no appearance of K_4 in G and no even prism in G. If there is a 1-breaker in G then G admits a balanced skew partition.*

Proof. Suppose that some 1-breaker (S, F, Q) exists, and for fixed G and S, choose F and Q with $|F| + |Q|$ maximum such that all the hypotheses of the theorem remain satisfied (possibly exchanging "left" and "right"). Let N be the set of vertices of G not in F but with a neighbour in F. Hence $Q \subseteq N$, and every left- or right-star with a neighbour in F is in N. Let $S = (A, C, B)$.

(1) *Every vertex in N has a neighbour in $A \cup B \cup C$.*

For suppose $v \in V(G) \setminus F$ has a neighbour in F and has none in $A \cup B \cup C$. Let $F' = F \cup \{v\}$. Certainly F' is connected and disjoint from $A \cup B \cup C$, and there are no edges between F' and $A \cup B \cup C$; and F' is disjoint from Q since every vertex in Q has a neighbour in $A \cup B \cup C$. It follows that the hypotheses of the theorem remain true, contrary to the maximality of $|F| + |Q|$. This proves (1).

(2) *There is no left- or right-star in Q, and every left- and right-star with a neighbour in F is Q-complete.*

For we are given that there is no left-star in Q. Suppose there is a right-star with a neighbour in F, either in Q or with a nonneighbour in Q. Then there is an antipath with interior in Q, between B and some right-star with a neighbour in F; but the set of vertices in such an antipath contradicts 11.4. So there is no right-star in Q, and every right-star with a neighbour in F is Q-complete. We are given that there is a right-star with a neighbour in F, and so all hypotheses of the theorem are true with "left" and "right" exchanged. It follows by the same argument, therefore, that every left-star with a neighbour in F is Q-complete. This proves (2).

Since $Q \subseteq N$ is anticonnected, it is contained in some anticonnected component of N, say N_1. We may assume that G admits no balanced skew partition, for otherwise the theorem holds.

(3) *There is a left- or right-star in N_1.*

For let N_2 be the union of all the anticomponents of N different from N_1. Assume that no left- and right-star is in N_1. Let $Y = V(G) \setminus (F \cup N)$; then there are no edges between F and Y, from definition of N. Also, $A \cup B \cup C \subseteq Y$, so in particular $Y \neq \emptyset$, and also $N_2 \neq \emptyset$ since by hypothesis there is a left-star in N. Hence $(F \cup Y, N)$ is a skew partition of G. By (1), every vertex in N has a neighbour in $A \cup B \cup C$ and in F, and so every vertex in N_1 has a neighbour in B (since otherwise it would be a left-star by 11.1 and therefore belong to N_2). Now $(B \cup C, N_1)$ is balanced, by 2.6, since any left-star is complete to N_1 and anticomplete to $B \cup C$. Since $B \cup C$ is connected (because every vertex of $B \cup C$ is in a step and the strip is step-connected), it follows from 2.7.1 that (F, N_1) is balanced. From 4.5, G admits a balanced skew partition, a contradiction. This proves (3).

From (3), $N_1 \neq Q$; and hence there is a vertex $v \in N \setminus Q$ with a nonneighbour in Q. From the maximality of $|F| + |Q|$, replacing Q by $Q \cup \{v\}$ violates one of the hypotheses of the theorem. But v has a neighbour in $A \cup B \cup C$ by (1); $v \notin F$ since it belongs to N; v is not a left-star since all left-stars in N are Q-complete by (2); and so no left-star in N is $Q \cup \{v\}$-complete. Since they are all Q-complete, it follows that v is nonadjacent to every left-star in N . Similarly v is nonadjacent to every right-star in N.

(4) *v is complete to $A \cup B$.*

For suppose not; then from the symmetry we may assume that v has a nonneighbour in B. By 11.2, v is a left-star, a contradiction. This proves (4).

Choose an antipath v-q_1-$\cdots$-q_k in Q, such that q_k has a nonneighbour in $A \cup B$, with k minimum. From (4), $k \geq 1$. From the minimality of k, $\{v, q_1, \ldots, q_{k-1}\}$ is complete to $A \cup B$. Let A_1 be the set of neighbours of q_k in A, and $A_2 = A \setminus A_1$, and define $B_1, B_2 \subseteq B$ similarly. So $A_2 \cup B_2$ is nonempty.

(5) *k is odd.*

For $A_2 \cup B_2$ is nonempty. If there exists $a_2 \in A_2$, let $b_0 \in N$ be a right-star; then

$$b_0\text{-}v\text{-}q_1\text{-}\cdots\text{-}q_k\text{-}a_2\text{-}b_0$$

is an antihole, so it follows that k is odd. The result follows similarly if B_2 is nonempty.

(6) *A_1 is complete to B_2, and A_2 is complete to B_1.*

For suppose that $a_1 \in A_1$ and $b_2 \in B_2$ are nonadjacent. Let $b_0 \in N$ be a right-star; then by (5),

$$b_0\text{-}v\text{-}q_1\text{-}\cdots\text{-}q_k\text{-}b_2\text{-}a_1\text{-}b_0$$

is an odd antihole, a contradiction. So A_1 is complete to B_2 and similarly A_2 is complete to B_1. This proves (6).

(7) *A_1, B_1, A_2, B_2 are all nonempty.*

For we may assume that A_2 is nonempty. Since the strip is step-connected, every vertex in A has a nonneighbour in B, and so by (6), $B_1 \neq B$. Hence B_2 is also nonempty. Since q_k has a neighbour in $A \cup B \cup C$ it follows that it has a neighbour in B, by 11.1, and similarly it has a neighbour in A. This proves (7).

Now the strip is step-connected, and so there is a step $a_1\text{-}R\text{-}b_2, a_2\text{-}R'\text{-}b_1$ with $a_1 \in A_1$ and $a_2 \in A_2$. Since a_1 is not adjacent to b_1 it follows that $b_1 \in B_1$ by (6), and similarly $b_2 \in B_2$. Also by (6), R and R' both have length 1. Let $a_0 \in N$ be a left-star and $b_0 \in N$ a right-star. Since $v\text{-}a_1\text{-}a_0\text{-}b_0\text{-}b_2\text{-}v$ is not an odd hole, it follows that a_0 is not adjacent to b_0.

For every vertex $u \in V(G) \setminus F$, let F_u be the set of vertices in F adjacent to u.

(8) *$F_{a_0} \cap F_{b_0} = \emptyset$, and every path in F between F_{a_0} and F_{b_0} meets both F_v and F_{q_k}.*

For if $f \in F_{a_0} \cap F_{b_0}$, then $f\text{-}a_0\text{-}a_1\text{-}b_2\text{-}b_0\text{-}f$ is an odd hole, so $F_{a_0} \cap F_{b_0} = \emptyset$. Let $p_1\text{-}P\text{-}p_2$ be a path in F between F_{a_0} and F_{b_0}, with $V(P)$ minimal, where $p_1 \in F_{a_0}$ and $p_2 \in F_{b_0}$. Hence

$$a_0\text{-}p_1\text{-}P\text{-}p_2\text{-}b_0\text{-}b_1\text{-}a_2\text{-}a_0$$

is a hole, and so P is odd. If P does not meet F_v then

$$v\text{-}a_1\text{-}a_0\text{-}p_1\text{-}P\text{-}p_2\text{-}b_0\text{-}b_1\text{-}v$$

is an odd hole, while if P does not meet F_{q_k} then

$$q_k\text{-}a_0\text{-}p_1\text{-}P\text{-}p_2\text{-}b_0\text{-}q_k$$

is an odd hole, in both cases a contradiction. This proves (8).

(9) *Every path in F between F_v and F_{q_k} meets both F_{a_0} and F_{b_0}.*

For suppose not; then since F is connected and $F_{a_0} \cap F_{b_0} = \emptyset$, there is a connected subset F' of F meeting both F_v, F_{q_k} and meeting exactly one of F_{a_0}, F_{b_0}. From the symmetry we may assume F' meets F_{a_0} and not F_{b_0}. Define $q_{k+1} = a_2$; then q_{k+1} has no neighbour in F', so we may choose i with $1 \leq i \leq k+1$ minimum such that q_i has no neighbour in F'. Note that v has a neighbour in F' (because F' meets F_v). If i is even, then b_0-v-q_1-$\cdots$-q_i is an odd antipath; its internal vertices have neighbours in F', and its ends do not, and a_1 is complete to its interior and has no neighbours in F', contrary to 2.2 in the complement. If i is odd, then b_1-a_0-v-q_1-$\cdots$-q_i is an odd antipath, and its internal vertices have neighbours in F' and its ends do not, and again a_1 is complete to its interior and has no neighbours in F', contrary to 2.2 in the complement. This proves (9).

Let f_1-f_2-$\cdots$-f_n be a minimal path in F between F_{a_0} and F_{b_0}, where $f_1 \in F_{a_0}$ and $f_n \in F_{b_0}$. Then $n \geq 2$ by (8), and by (8) and (9) it follows that f_1-f_2-$\cdots$-f_n is also a minimal path between F_v and F_{q_k}, so we may assume that $f_1 \in F_v$, $f_n \in F_{q_k}$, and no other vertex of the path is in either set. Then f_1-f_2-$\cdots$-f_n-q_k-a_0-f_1 and f_1-f_2-$\cdots$-f_n-b_0-b_1-v-f_1 are both holes, of different parity, a contradiction. This proves 11.5. ∎

12 Attachments in a staircase

For the next step of our approach towards the long odd prism, let us fix a little more than just the strip. Let $S = (A, C, B)$ be a step-connected strip in G, and let a_0-R_0-b_0 be a banister of length ≥ 3. We call the pair $K = (S, R_0)$ a *staircase*, and define $V(K) = V(R_0) \cup V(S)$. (For brevity we often speak of the staircase $K = (S = (A, C, B), a_0\text{-}R_0\text{-}b_0)$, meaning that $K = (S, R_0)$ is a staircase, and $S = (A, C, B)$, and R_0 has ends a_0, b_0, where a_0 is a left-star and b_0 is a right-star.) The staircase is *maximal* if there is no staircase $(S' = (A', C', B'), a'_0\text{-}R'_0\text{-}b'_0)$ such that $A \subseteq A', B \subseteq B', C \subseteq C'$ and $V(S) \subset V(S')$.

Let $K = (S = (A, C, B), a_0\text{-}R_0\text{-}b_0)$ be a staircase in G. Some definitions (all with respect to K):

- A subset $X \subseteq V(K)$ is *local* if X is a subset of one of $V(S), V(R_0), A \cup \{a_0\}, B \cup \{b_0\}$
- $v \in V(G) \setminus V(K)$ is *minor* if its set of neighbours in $V(K)$ is local
- $v \in V(G) \setminus V(K)$ is *major* if it has neighbours in all of A, B and $V(R_0)$
- $v \in V(G) \setminus V(K)$ is *left-diagonal* if v is $(A \cup \{b_0\})$-complete, and *right-diagonal* if it is $(B \cup \{a_0\})$-complete
- $v \in V(G) \setminus V(K)$ is *central* if it is $(A \cup B)$-complete, and is nonadjacent to both a_0 and b_0.

First let us examine the possible types of vertices outside the staircase.

12.1 *Let G be a Berge graph, such that there is no appearance of K_4 in G, no even prism in G, and no 1-breaker in G. Let $K = (S = (A, C, B), a_0\text{-}R_0\text{-}b_0)$ be a maximal staircase in G, and let $v \in V(G) \setminus V(K)$. Then exactly one of the following holds:*

1. *v is minor; and in that case, either v is a left-star or v is not A-complete, and either v is a right-star or v is not B-complete.*

2. v is major; and in that case, it is either left- or right-diagonal or central.

3. v is a left-star with a neighbour in $R_0 \setminus a_0$, or a right-star with a neighbour in $R_0 \setminus b_0$.

Proof.

(1) *If v is left- or right-diagonal then the theorem holds.*

For assume v is right-diagonal say. If it has no neighbours in $A \cup C$ then statement 3 of the theorem holds, so we assume there is a step a_1-R_1-b_1, a_2-R_2-b_2 such that v has a neighbour in $R_1 \setminus b_1$. Hence it can be linked onto the triangle $\{a_0, a_1, a_2\}$, via v-a_0, the path from v to a_1 with interior in $R_1 \setminus b_1$, and the path from v to a_2 with interior in R_2, and so by 2.4, v has a neighbour in A. So it is major, and therefore statement 2 holds. This proves (1).

(2) *If v is adjacent to both a_0, b_0 then the theorem holds.*

For then it has a neighbour in R_0^*, since R_0 is odd and has length ≥ 3 and v is adjacent to both its ends; and we may assume that v has a neighbour in $V(S)$, for otherwise statement 1 of the theorem holds. If v has no neighbour in B then it is a left-star by 11.1, and statement 3 of the theorem holds, so we may assume it has neighbours in B and similarly in A. Hence it is major. Since $(S, V(R_0^*), \{v\})$ is not a 1-breaker, v does not have nonneighbours in both A and B, so it is either left- or right-diagonal and the claim follows from (1). This proves (2).

(3) *If v is adjacent to a_0 and not to b_0 then the theorem holds.*

For we may assume v has a neighbour in $V(S)$. If v has a neighbour in R_0^*, then by 11.2 it is either B-complete (when it is right-diagonal and the claim follows from (1)) or a left-star (when statement 3 holds). So we may assume it has no neighbour in R_0^*. We may assume it has a neighbour in $B \cup C$, for otherwise it is minor and statement 1 of the theorem holds; let a_1-R_1-b_1, a_2-R_2-b_2 be a step such that v has a neighbour in $R_1 \setminus a_1$, and in addition such that v is not adjacent to b_2 if possible. By 10.4, v has a neighbour in R_2. If a_2 is its only neighbour in R_2, then the strip $S' = (A \cup \{v\}, C, B)$ is step-connected, since v-R-b_1, a_2-R_2-b_2 is an S'-step where R is the path from v to b_1 with interior in $R_1 \setminus a_1$; and since v is adjacent to a_0 and has no other neighbours in R_0, this is contrary to the maximality of the staircase. So v has a neighbour in $R_2 \setminus a_2$; and hence v can be linked onto the triangle $\{b_0, b_1, b_2\}$ via v-a_0-R_0-b_0, and for $i = 1, 2$, the path from v to b_i with interior in $R_i \setminus a_i$. By 2.4 it follows that v is adjacent to both b_1, b_2; and hence from our choice of the step R_1, R_2, and since the strip is step-connected, it follows that v is right-diagonal, and the claim follows from (1). This proves (3).

(4) *If v is nonadjacent to both a_0, b_0 then the theorem holds.*

For then we may assume that v has a neighbour in $V(S)$, since otherwise it is minor, and statement 3 of the theorem holds. Suppose first that v also has a neighbour in R_0^*. If v is a left-star then statement 3 holds, so we assume not; and then by 11.2, v is B-complete. Similarly v is A-complete and therefore central, and statement 2 holds. Thus we may assume that v has no neighbour in

$V(R_0)$, and therefore v is minor. We claim that statement 1 holds, and to show this we may assume that v is A-complete. Let a_1-R_1-b_1, a_2-R_2-b_2 be a step; then by 10.4, v has no neighbour in $R_1 \setminus a_1$ or in $R_2 \setminus a_2$, and therefore v is a left-star, and statement 1 holds. This proves (4).

But (2)-(4) cover all the possibilities, up to symmetry, and this completes the proof of 12.1. ∎

Now let us do the same thing for connected sets.

12.2 *Let G be a Berge graph, such that there is no appearance of K_4 in G, no even prism in G, and no 1-breaker in G. Let $K = (S = (A, C, B), a_0$-R_0-$b_0)$ be a maximal staircase in G, and let $F \subseteq V(G) \setminus V(K)$ be connected, such that its set of attachments in $V(K)$ is not local with respect to K. Then F contains either:*

1. *a major vertex, or*
2. *a banister u-R-v, such that there are no edges between $V(R)$ and $V(R_0)$, or*
3. *(up to symmetry) a path u-R-v, where u is a left-star, v has a neighbour in $R_0 \setminus a_0$, and there are no edges between $V(R \setminus u)$ and $V(S)$.*

Proof. Let X be the set of attachments of F in $V(K)$. We may assume that F is minimal (connected) such that X is not local. Now a subset of $V(K)$ is local if and only if it does not meet both $A \cup C$ and $V(R_0 \setminus a_0)$ and does not meet both $B \cup C$ and $V(R_0 \setminus b_0)$; so we may assume that X meets both $A \cup C$ and $V(R_0 \setminus a_0)$, and therefore from the minimality of F, there is a path f_1-$\cdots$-f_k where $F = \{f_1, \ldots, f_k\}$ and f_1 is the unique vertex of F with a neighbour in $A \cup C$, and f_k is the unique vertex of F with a neighbour in $V(R_0 \setminus a_0)$. If $k = 1$ then the claim follows from 12.1, so we may assume that $k \geq 2$.

(1) *If f_1 is A-complete then the theorem holds.*

For assume f_1 is A-complete. If there is no edge between F and $B \cup C$, then statement 3 of the theorem holds, so we assume that there is such an edge. Choose i with $1 \leq i \leq k$ minimum such that f_i has a neighbour in $B \cup C$. Suppose first that there is no edge between $\{f_1, \ldots, f_i\}$ and $V(R_0)$. Let a_1-R_1-b_1, a_2-R_2-b_2 be a step such that f_i has a neighbour in $R_1 \setminus a_1$, and in addition such that f_i is nonadjacent to b_2 if possible. With respect to the prism formed by R_0, R_1, R_2, the set of attachments of $\{f_1, \ldots, f_i\}$ is not local, and so by 10.4, $i \geq 2$ and its attachments in the prism are a_1, a_2, b_1, b_2. Hence the only edges between $\{f_1, \ldots, f_i\}$ and $V(R_1 \cup R_2)$ are $f_1a_1, f_1a_2, f_ib_1, f_ib_2$. From our choice of the step it follows that f_i is B-complete. Consequently any step satisfies the condition we imposed on R_1, R_2, and so the same conclusion follows for every step; that is, statement 2 of the theorem holds. Now assume that there is an edge between $\{f_1, \ldots, f_i\}$ and $V(R_0)$. Suppose that $i < k$; then there is no edge between $\{f_1, \ldots, f_i\}$ and $R_0 \setminus a_0$, from the minimality of F, and so a_0 is an attachment of $\{f_1, \ldots, f_i\}$. But this set also has an attachment in $B \cup C$, so its set of attachments is not local, contrary to the minimality of F. This proves that $i = k$. Since $k \geq 2$, the minimality of F implies that there are no edges between $\{f_2, \ldots, f_k\}$ and $V(R_0 \setminus b_0)$; and so b_0 is the unique neighbour of f_k in R_0. Hence there are no edges between $\{f_2, \ldots, f_k\}$ and $A \cup C$, from the minimality of F. Also, there are no edges between $\{f_1, \ldots, f_{k-1}\}$ and $B \cup C$, from the minimality of i. Choose a step a_1-R_1-b_1, a_2-R_2-b_2 such that f_k is adjacent to b_1, and in addition such that f_k is nonadjacent to b_2 if possible. Since R_1 is odd and a_1-f_1-$\cdots$-f_k-b_1-R_1-a_1 is a hole, it follows that k is even. Since a_2-f_1-$\ldots$-f_k-b_0-b_2-R_2-a_2 is not an odd hole, f_k is adjacent to b_2, and therefore to all B from our choice of the step. Since a_1-f_1-$\cdots$-f_k-b_0-R_0-a_0-a_1 is not an odd hole and R_0 is odd, it follows that f_1 is adjacent to a_0. But then we can add f_1 to A, f_k to B, and $\{f_2, \ldots, f_{k-1}\}$ to C, contrary to the maximality of the staircase. This proves (1).

By (1), we may assume there is a step a_1-R_1-b_1, a_2-R_2-b_2 such that f_1 has a neighbour in $R_1 \setminus b_1$, and a_2 is not adjacent to f_1. (To see this, first choose a step a_1-R_1-b_1, a_2-R_2-b_2 such that f_1 has a neighbour in $R_1 \setminus b_1$; this satisfies our requirements unless a_2 is adjacent to f_1. We may therefore assume that f_1 has a neighbour and a non-neighbour in A; but then since the strip is step-connected, we may choose a step a_1-R_1-b_1, a_2-R_2-b_2 so that f_1 is adjacent to a_1 and not to a_2, and again our requirements are satisfied.) Then R_0, R_1, R_2 form a prism K' say, and the set of attachments of F in $V(K')$ is not local with respect to K'. Suppose that some vertex v in F is major with respect to K'. Then we claim v is major with respect to K. For it has a neighbour in A and in B, and if it has none in R_0 then it is adjacent to all of a_1, a_2, b_1, b_2, in which case v-a_1-a_0-R_0-b_0-b_2-v is an odd hole. So v is major with respect to K, and hence the theorem holds. Hence we may assume that no vertex in F is major with respect to K', and so we may apply 10.1. By hypothesis, 10.1.1 does not hold. Since no vertex of F is adjacent to a_2, 10.1.2 does not hold.

Suppose that 10.1.3 holds. Since f_1 is not adjacent to a_2, it follows that f_1 is adjacent to a_0, a_1, and there exists i with $2 \leq i \leq k$ such that f_i is adjacent to b_0, b_1, and there are no other edges between $\{f_1, \ldots, f_i\}$ and $V(K')$. Then we can add f_1 to A, f_i to B and $\{f_2, \ldots, f_{i-1}\}$ to C, contrary to the maximality of the staircase. So 10.1.3 does not hold.

Hence 10.1.4 holds, that is, there is a path p_1-P-p_2 in F, such that for some j with $0 \leq j \leq 2$, either:

- p_1 is adjacent to the two vertices in $\{a_0, a_1, a_2\} \setminus \{a_j\}$, and p_2 has neighbours in $R_j \setminus a_j$, and there are no other edges between $V(P)$ and $V(K') \setminus \{a_j\}$, or
- p_1 is adjacent to the two vertices in $\{b_0, b_1, b_2\} \setminus \{b_j\}$, and p_2 has neighbours in $R_j \setminus b_j$, and there are no other edges between $V(P)$ and $V(K') \setminus \{b_j\}$

From the minimality of F, $F = V(P)$. If $j > 0$ then in the first case we can add p_1 to A and $V(P \setminus p_1)$ to C, contrary to the maximality of the staircase; and in the second case we do the same with A and B exchanged. So $j = 0$. The first case is impossible since no vertex in F is adjacent to a_2; and the second case is impossible since $f_1 \in F = V(P)$ and f_1 has a neighbour in $R_1 \setminus b_1$. This proves 12.2. ∎

The previous result can be strengthened as follows.

12.3 *Let G be a Berge graph, such that there is no appearance of K_4 in G, no even prism in G, and no 1-breaker in G. Let $K = (S = (A, C, B), a_0$-R_0-$b_0)$ be a maximal staircase in G, and let $F \subseteq V(G) \setminus V(S)$ be connected, containing a left-star and with an attachment in $B \cup C$. (Note that F may intersect $V(R_0)$.) Then F contains either a major vertex or a banister.*

Proof. We may assume F is minimal (possibly exchanging A and B); so F is the vertex set of a path f_1-$\cdots$-f_k, where f_1 is the unique left-star in F, and f_k is the only vertex in F with a neighbour in $B \cup C$. Since f_1 is a left-star and f_k has a neighbour in $B \cup C$ it follows that $k \geq 2$. We may assume there is no major vertex in F.

(1) *We may assume that none of $f_1, \dots, f_k$ is a right-star, and that f_k is not B-complete.*

For if there is a right-star in F, then it must be f_k; and then from the minimality of F (exchanging A and B), no vertex of F different from f_1 has a neighbour in $A \cup C$, and so f_1-$\cdots$-f_k is a banister. So we may assume that there is no right-star in F. Since f_k is neither major nor a right-star, by 12.1 it is not B-complete. This proves (1).

(2) *$F \cap V(R_0) = \emptyset$, and there are no edges between $\{f_2, \dots, f_k\}$ and $V(R_0 \setminus b_0)$.*

For by (1), $b_0 \notin F$. Suppose that either $\{f_2, \dots, f_k\}$ intersects $V(R_0 \setminus b_0)$, or there is an edge joining these two sets. Choose i with $2 \le i \le k$ maximum such that either $f_i \in V(R_0 \setminus b_0)$ or f_i has a neighbour in $V(R_0 \setminus b_0)$. We claim that $f_i \notin V(R_0)$. For if $i = k$ this is true, since f_k has neighbours in $B \cup C$; and if $i < k$ then f_{i+1} has no neighbour in $V(R_0 \setminus b_0)$ from the maximality of i, and therefore again $f_i \notin V(R_0)$. So none of $f_i, \dots, f_k$ belong to $V(R_0)$. Since $\{f_i, \dots, f_k\}$ has attachments in $V(R_0 \setminus b_0)$ and in $B \cup C$, and contains no major vertex or left- or right-star, this contradicts 12.2. So $\{f_2, \dots, f_k\}$ is disjoint from $V(R_0 \setminus b_0)$ and hence from $V(R_0)$, and there are no edges between $\{f_2, \dots, f_k\}$ and $V(R_0 \setminus b_0)$. Since there is an edge between $\{f_2, \dots, f_k\}$ and f_1 it follows that $f_1 \notin V(R_0)$, and so $F \cap V(R_0) = \emptyset$. This proves (2).

Let a_1-R_1-b_1, a_2-R_2-b_2 be a step such that f_k has a neighbour in $R_1 \setminus a_1$ and f_k is nonadjacent to b_2. (To see that such a step exists, we argue as follows: since f_k has a neighbour in $B \cup C$, there is a step a_1-R_1-b_1, a_2-R_2-b_2 such that f_k has a neighbour in $R_1 \setminus a_1$, and so we may assume that f_k is adjacent to b_2. Hence f_k has a neighbour and a nonneighbour in B, and the required step exists since the strip is step-connected.)

(3) *$f_1 a_2$ is the only edge between F and R_2.*

For if f_k has a neighbour in R_2, then its neighbour set in the prism formed by R_0, R_1, R_2 is not local with respect to that prism, and therefore by 10.4, f_k has a neighbour in R_0; and then by 12.1 it is major, a contradiction. So f_k has no neighbours in R_2. From the minimality of F, there are no edges between F and $R_2 \setminus a_2$. Suppose that a_2 has a neighbour in $\{f_2, \dots, f_k\}$, and choose i maximum such that a_2 is adjacent to f_i. Since f_k has a neighbour in $V(R_1 \setminus a_1)$, the set of attachments of $\{f_i, \dots, f_k\}$ is not local with respect to the prism formed by R_0, R_1, R_2; and since b_2 is not an attachment, it follows from 10.4 that there is an attachment of $\{f_i, \dots, f_k\}$ in $V(R_0)$. By (2), b_0 has a neighbour in $\{f_i, \dots, f_k\}$; but then $\{f_i, \dots, f_k\}$ violates 12.2. This proves (3).

(4) *b_0 has neighbours in $\{f_1, \dots, f_{k-1}\}$.*

For first suppose that b_0 has no neighbour in F. Since b_2 is not an attachment of F, it follows from 10.4 (applied to F and the prism formed by R_0, R_1, R_2) that there is an edge between F and $V(R_0)$, and so f_1 has a neighbour in R_0. But then f_1 can be linked onto the triangle $\{b_0, b_1, b_2\}$, via the path between f_1 and b_0 with interior in $V(R_0)$, the path between f_1 and b_1 with interior in $\{f_2, \dots, f_k\} \cup (V(R_1) \setminus \{a_1, b_1\})$, and the path f_1-a_2-R_2-b_2. This contradicts 2.4, and therefore proves that b_0 has a neighbour in F. Suppose that f_k is the only neighbour of b_0 in F. Then since f_k is not major, its unique neighbour in R_1 is b_1. From 11.3, R_1, R_2 are odd, and from the hole f_1-$\cdots$-f_k-b_0-b_2-R_2-a_2-f_1 it follows that k is odd. If a_1 has no neighbour in $\{f_2, \ldots, f_k\}$ then f_1-$\cdots$-f_k-b_1-R_1-a_1-f_1 is an odd hole, and if a_1 has a neighbour in $\{f_2, \ldots, f_k\}$ then $\{f_2, \ldots, f_k\}$ violates 12.2. So f_k is not the unique neighbour of b_0 in F. This proves (4).

Choose i with $1 \leq i < k$ minimum such that b_0 is adjacent to f_i, and let R_0' be the path f_1-$\cdots$-f_i-b_0. There are no edges between $\{f_1, \ldots, f_i\}$ and $B \cup C$ from the minimality of F, and from 12.2 there are no edges between $\{f_2, \ldots, f_i, b_0\}$ and $A \cup C$. Hence f_1-R_0'-b_0 is a banister, and in particular the three paths R_0', R_1, R_2 form a prism, K' say. Let $F' = \{f_{i+1}, \ldots, f_k\}$. Then F' is connected and disjoint from $V(K')$, and F' has attachments in $R_1 \setminus a_1$, and in $R_0' \setminus b_0$, and by (3) it has no attachments in R_2. By 10.4 applied to K', it follows that F' contains a path with one end adjacent to a_1, f_1, the other end adjacent to b_0, b_1, and with no more edges between this path and $V(R_0') \cup V(R_1)$. Since the only vertex of F' adjacent to f_1 is f_2, and that only if $i = 1$, and the only vertex in F' adjacent to b_1 is f_k, it follows that $i = 1$, and the only edges between $\{f_2, \ldots, f_k\}$ and $V(R_0') \cup V(R_1)$ are $f_kb_1, f_kb_0, f_2a_1, f_2f_1$. But then by (2), a_1 can be linked onto the triangle $\{b_0, b_1, f_k\}$, via a_1-a_0-R_0-b_0, a_1-R_1-b_1, a_1-f_2-$\cdots$-f_k, contrary to 2.4. This proves 12.3. ∎

Now we turn to anticonnected sets of major vertices. We have already defined what it is for a staircase to be maximal in G. We say a staircase $K = (S = (A, C, B), a_0\text{-}R_0\text{-}b_0)$ is *strongly maximal* if it is maximal, and in addition, either $C \neq \emptyset$, or there is no staircase (S', R') in $\overline{G}$ with $V(S) \subset V(S')$. A *2-breaker* in G is a pair (K, Q) such that

- $K = (S = (A, C, B), a_0\text{-}R_0\text{-}b_0)$ is a strongly maximal staircase in G,
- $Q \subseteq V(G) \setminus V(K)$ is anticonnected,
- some vertex of A is Q-complete, and some vertex of B is Q-complete
- a_0, b_0 are not Q-complete, and
- some vertex of R_0 is Q-complete.

We observe that if q is a central vertex with respect to a strongly maximal staircase K, then $(K, \{q\})$ is a 2-breaker, so it follows from the next result that we no longer have to worry about central vertices.

12.4 *Let G be a Berge graph, containing no appearance of K_4, no even prism, and no 1-breaker. If there is a 2-breaker in G then G admits a balanced skew partition.*

Proof. Choose a 2-breaker (K, Q) in G, with notation as above, such that for fixed K the set Q is maximal. Let a_0-S-s and b_0-T-t be the subpaths of R_0 such that s is the unique Q-complete vertex of S, and t is the unique Q-complete vertex of T.

(1) *S, T both have odd length, and therefore s, t are different.*

For choose $a \in A$ and $b \in B$, both Q-complete; then a-a_0-S-s has length > 1, and its ends are Q-complete and its internal vertices are not, and b is also Q-complete and has no neighbours in the interior of a-a_0-S-s. By 2.2, this path is even, and so S is odd, and similarly T is odd. Since R_0 is odd it follows that s, t are different. This proves (1).

(2) *Every vertex in $A \cup B$ is Q-complete.*

For suppose some vertex in A say is not Q-complete. Choose a step a_1-R_1-b_1, a_2-R_2-b_2 such that a_1 is Q-complete and a_2 is not. Since s, t are different it follows that t is nonadjacent to both a_0, a_2; and so by 2.8, Q cannot be linked onto the triangle $\{a_0, a_1, a_2\}$. Hence there is no Q-complete vertex in R_2. Assume s, t are nonadjacent; then the subpath of R_0 between them is odd, and a_1 has no neighbour in its interior, so by 2.2 it contains another Q-complete vertex u say; and then s-S-a_0-a_2-R_2-b_2-b_0-T-t is an odd path, its ends are Q-complete and its internal vertices are not, and u has no neighbour in its interior, contrary to 2.2. So s, t are adjacent. Hence the hole a_0-R_0-b_0-b_2-R_2-a_2-a_0 has length ≥ 6, and the only Q-complete vertices in it are the adjacent vertices s, t. By 2.10 Q contains a hat or a leap; and in either case there is a vertex $q \in Q$ with no neighbours in R_2. But q is adjacent to s and a_1, contrary to 10.4 applied to the prism formed by R_0, R_1, R_2. This proves (2).

(3) *Every major vertex is either in Q or complete to Q.*

For let v be a major vertex, and suppose $v \notin Q$, and Q' is anticonnected, where $Q' = Q \cup \{v\}$. From 12.1, v is either left- or right-diagonal, or central; and in either case it has neighbours $a_1 \in A$ and $b_1 \in B$ that are nonadjacent. It follows that a_1-a_0-R_0-b_0-b_1 is an odd path of length ≥ 5, and its ends are Q'-complete. From the maximality of Q, none of its internal vertices are Q'-complete, and so by 2.1, Q' contains a leap q_1, q_2 say. So neither of q_1, q_2 has neighbours in the interior of R_0; but this is impossible since one of them is in Q and is therefore adjacent to s. This proves (3).

(4) *There is no edge uv of $G \setminus V(S)$ such that u is a left-star, v is a right-star, and u, v are not Q-complete.*

For suppose uv is such an edge. Since u, v have neighbours in $A \cup B$, they do not belong to R_0^*. Since u, v have nonneighbours in Q and Q is anticonnected, there is an antipath u-q_1-$\cdots$-q_k-v with $q_1, \ldots, q_k \in Q$. Choose a step a_1-R_1-b_1, a_2-R_2-b_2. Then a_1-b_2-u-q_1-$\cdots$-q_k-v-a_1 is an antihole, so k is even. Hence every Q-complete vertex w say is adjacent to one of u, v, for otherwise w-u-q_1-$\cdots$-q_k-v-w would be an odd antihole. In particular, there are no Q-complete vertices in C; and therefore a_1-R_1-b_1 is an odd path with both ends Q-complete and no internal vertex Q-complete. Since a_2 is Q-complete and has no neighbour in the interior of R_1, it follows from 2.2 that R_1 has length 1, and similarly R_2 has length 1. Since this step was arbitrary, and every vertex is in a step, it follows that $C = \emptyset$. Suppose that u has no neighbour in R_0^*. Then all Q-complete vertices in R_0^* are adjacent to v. In particular, v is adjacent to s, t and hence does not belong to R_0 (because v is a right-star); and s-S-a_0-a_1-b_1 is an odd path, its ends are $(Q \cup \{v\})$-complete, its internal vertices are not, and the $(Q \cup \{v\})$-complete t has no neighbour in its interior, contrary to 2.2. So u has a neighbour in R_0^*, and similarly so does v. Now b_1-u-Q-v-a_1 is an odd antipath, all its internal vertices have neighbours in the connected set R_0^*, and its ends do not. By 2.1 applied in $\overline{G}$, there is a leap; that is, there exist adjacent $a, b \in R_0^*$, both Q-complete, such that b-u-Q-v-a is an antipath. Define $A' = A \cup \{a\}$ and $B' = B \cup \{b\}$; then $(A', \emptyset, B')$ is a strip (S' say) in $\overline{G}$. For every edge a_1b_1 of G with $a_1 \in A$ and

$b_1 \in B$, the pair a-b_1, a_1-b is a step of S' (in $\overline{G}$), and every vertex of $A \cup B$ is in such an edge, and so S' is step-connected. Hence $((A', \emptyset, B'), v$-q_k-$\cdots$-q_1-$u)$ is a staircase in $\overline{G}$, contrary to the hypothesis that K is strongly maximal. This proves (4).

(5) *Every path in G from an A-complete vertex to a vertex with a neighbour in $B \cup C$ contains either a vertex in Q or a Q-complete vertex.*

For suppose not, and choose a path p_1-$\cdots$-p_k say, with k minimum such that p_1 is A-complete and p_k has a neighbour in $B \cup C$, and none of $p_1, \ldots, p_k$ is in Q or Q-complete. Since $A \cup B$ is complete to Q it follows that none of $p_1, \ldots, p_k$ is in $A \cup B$. Now p_1 is not in C since no vertex in C is A-complete (because they are all in steps), and if some $p_i \in C$ for $i > 1$, then p_1-$\cdots$-p_{i-1} is a shorter path with the same properties, contrary to the minimality of k. So none of $p_1, \ldots, p_k$ is in $V(S)$. (Some may be in R_0, however.) Since none of $p_1, \ldots, p_k$ is major by (3), it follows from 12.3 and the minimality of k that p_1-$\cdots$-p_k is a banister. From (4), since none of $p_1, \ldots, p_k$ is Q-complete, it follows that $k > 2$. Let a_1-R_1-b_1, a_2-R_2-b_2 be a step. From the hole a_1-p_1-$\cdots$-p_k-b_1-R_1-a_1 it follows that k is even; and so a_1-p_1-$\cdots$-p_k-b_2 is an odd path of length ≥ 5; its ends are Q-complete, and its internal vertices are not. By 2.1, Q contains a leap a, b; so a-p_1-$\cdots$-p_k-b is a path. But then $(A \cup \{a\}, C, B \cup \{b\})$ is a step-connected strip S' say (since for every nonadjacent $a' \in A$ and $b' \in B$, the two paths a-b', a'-b make a step in this strip), and so $(S', p_1$-$\cdots$-$p_k)$ is a staircase, contrary to the maximality of (S, R_0). This proves (5).

Let X be the set of all Q-complete vertices in G; let M be the component of $G \setminus (Q \cup X)$ that contains a_0, and N the union of all the other components. By (5), $b_0 \in N$, so N is nonempty, and hence $(M \cup N, Q \cup X)$ is a skew partition of G. Choose $b \in B$; then $b \in X$, and it has no neighbour in M by (5). Hence the skew partition is loose, and so G admits a balanced skew partition, by 4.2. This proves 12.4. ∎

12.5 *Let G be a Berge graph, containing no appearance of K_4, no even prism, no 1-breaker and no 2-breaker. Let $K = (S = (A, C, B), a_0$-R_0-$b_0)$ be a strongly maximal staircase in G. Let q_1-$\cdots$-q_k be an antipath such that $q_2, \ldots, q_{k-1}$ are both left- and right-diagonal, and q_1 is left- and not right-diagonal, and q_k is right- and not left-diagonal. Then q_1 is a left-star and q_k is a right-star.*

Proof. First, obviously $k \geq 2$. Let $Q = \{q_1, \ldots, q_k\}$.

(1) *If q_1 is adjacent to a_0 and q_k to b_0 then the theorem holds.*

For then both a_0, b_0 are Q-complete, and q_1 has a nonneighbour in B (for otherwise it would be right-diagonal), and q_k has a nonneighbour in A. Since R_0 has odd length ≥ 3, it follows that each of $q_1, \ldots, q_k$ has a neighbour in R_0^*. Since (S, R_0^*, Q) is not a 1-breaker, it follows that Q contains a left-star, which must be q_1; and similarly q_k is a right-star. Then the theorem holds. This proves (1).

(2) *If q_1 is adjacent to a_0 and q_k is nonadjacent to b_0 then the theorem holds.*

For in this case, q_1 has a nonneighbour in B, say b. From the antihole a_0-b-q_1-$\cdots$-q_k-b_0-a_0 we deduce that k is odd. Now R_0 is odd, of length ≥ 3, and its ends are complete to $Q \setminus \{q_k\}$, and so is every $a \in A$, and a has no neighbour in the interior of R_0, so by 2.2, there is a $(Q \setminus \{q_k\})$-complete vertex in the interior of R_0, say t. Let T be the subpath of t to b_0, and let us choose t with T of minimum length, that is, such that t is the unique $(Q \setminus \{q_k\})$-complete vertex of T. If t is nonadjacent to q_k then t-b-q_1-$\cdots$-q_k-t is an odd antihole (since $k \geq 2$) , a contradiction. Hence t is Q-complete, and in particular, all of $q_1, \ldots, q_k$ have neighbours in the interior of R_0. By 11.4 it follows that Q contains a left-star, which must be q_1. We may assume that q_k is not a right-star, for otherwise the theorem holds. Since q_k is right-diagonal, from 12.1 it follows that q_k is major and therefore has a neighbour in A. Choose a step a_1-R_1-b_1, a_2-R_2-b_2 such that q_k is adjacent to a_1, and if possible nonadjacent to a_2. Then t-T-b_0-b_1-R_1-a_1 is a path, and both its ends are Q-complete, and none of its internal vertices are Q-complete (since q_1 is a left-star). By 3.2 applied to t-T-b_0-b_1-R_1-a_1 and the antipath b_1-q_1-$\cdots$-q_k-b_0, it follows that t-T-b_0-b_1-R_1-a_1 has length 4, and so R_1 has length 1 and T has length 2; let its middle vertex be u say. Also from 3.2, u is $Q \setminus \{q_1\}$-complete, and nonadjacent to q_1. Suppose that q_k is nonadjacent to a_2. Then there is no Q-complete vertex in R_2. If t is nonadjacent to a_0 then a_0-a_2-R_2-b_2-b_0-u-t is an odd path of length ≥ 5; its ends are Q-complete and its internal vertices are not, so by 2.1, Q contains a leap, which is impossible since every vertex in Q is adjacent to one of b_0, b_2. If t is adjacent to a_0, then a_0-a_2-R_2-b_2-b_0-R_0-a_0 is a hole of length ≥ 6, and the only Q-complete vertices in it are a_0, t, and these are adjacent; so by 2.10 there is a hat or a leap in Q; and again this is impossible since every vertex in Q is adjacent to one of b_0, b_2. This proves that $q_k a_2$ is an edge. From our choice of the step, it follows that q_k is A-complete. But therefore any step satisfies the condition we imposed on the step R_1, R_2; and therefore every path in every step has length 1, that is $C = \emptyset$. Then $S = (A \cup \{t\}, \emptyset, B \cup \{u\})$ is a step-connected strip in $\overline{G}$, and $(S', b_0$-q_k-$\cdots$-$q_1)$ is a staircase in $\overline{G}$, contradicting that (S, R_0) is strongly maximal. This proves (2).

(3) *If q_1 is nonadjacent to a_0 and q_k is nonadjacent to b_0 then the theorem holds.*

For then a_0-q_1-$\cdots$-q_k-b_0-a_0 is an antihole, so k is even. Let A_1 be the set of vertices in A adjacent to q_k, and $A_2 = A \setminus A_1$; and let B_1 be the set of vertices in B adjacent to q_1, and $B_2 = B \setminus B_1$. If $a_1 \in A_1$ and $b_2 \in B_2$, then a_1-b_2-q_1-$\cdots$-q_k-b_0-a_1 is not an odd antihole, and so a_1 is adjacent to b_2; and hence A_1 is complete to B_2, and similarly A_2 is complete to B_1. If A_1, B_1 are both empty then by 12.1, the theorem holds; so we may assume that A_1 is nonempty. Choose $a_1 \in A_1$. Since a_1 is in a step, it has a nonneighbour in B, say b_1. Since a_1 is B_2-complete it follows that $b_1 \in B_1$. Then a_1, b_1 are both Q-complete, and since (K, Q) is not a 2-breaker, no internal vertex of R_0 is Q-complete. So a_1-a_0-R_0-b_0-b_1 is an odd path of length ≥ 5, and its ends are Q-complete, and its internal vertices are not. By 2.1, Q contains a leap. Since every vertex of Q except q_1, q_k has ≥ 2 neighbours in R_0, it follows that $k = 2$ and q_1, q_2 both have no neighbours in the interior of R_0. Then $S' = (A \cup \{q_2\}, C, B \cup \{q_1\})$ is a step-connected strip (since a_1-q_1, q_2-b_1 is a step of it), and (S', R_0) is a staircase, contrary to the maximality of (S, R_0). This proves (3).

From (1),(2),(3), the theorem follows. This proves 12.5. ∎

13 The long odd prism

In this section we apply the results of the previous section to prove that a Berge graph containing a long odd prism has a decomposition unless it is a line graph.

Let $K = ((A, C, B), a_0\text{-}R_0\text{-}b_0)$ be a strongly maximal staircase in a Berge graph G. From 12.1 there are three possible kinds of B-complete vertices; right-stars, vertices complete to both A and B, and B-complete vertices adjacent to some but not all of A. The most difficult step in handling the long odd prism is when there is a vertex of the third kind. In that case, we shall construct a subset of B-complete vertices, including all these "mixed" vertices and some of the others, such that they and their common neighbours form a cutset of the graph, and thereby give us a skew partition. We define the set recursively as follows: initially let X be the set of all B-complete vertices adjacent to some but not all of A. Then enlarge X by repeatedly applying the following two rules, in any order:

1. if there is an $A \cup B$-complete vertex v that is not in X and not X-complete, add v to X
2. if there is a banister a-R-b such that a is not X-complete and b is not in X, add b to X.

The process eventually stops with some set X. We shall prove that X and its common neighbours (say Y) separate A (or at least the part of A that is not X-complete) from b_0, and this will provide a balanced skew partition. To prove that $X \cup Y$ separates G as described, we have to show that every path from A to b_0 meets $X \cup Y$, and it turns out that there are only two kinds of paths to worry about; banisters, and 1-vertex paths consisting of a major vertex. Any banister a-R-b is automatically hit, because of the rule above; if $a \notin Y$ then $b \in X$. The 1-vertex paths are trickier. Let v be a major vertex. If it is B-complete, then it is in either Y or X by the rule above, so assume it is not B-complete. By 12.1, it is left- and not right-diagonal, and now we have to show it belongs to Y. If only we knew that every vertex in X was adjacent to a_0, then it follows easily that $v \in Y$, because of 12.5. So that is what we need to do — to prove that every vertex in X is adjacent to a_0.

Let us start again, more formally. Let $K = ((A, C, B), a_0\text{-}R_0\text{-}b_0)$ be a staircase in a Berge graph G. We define a *right-sequence* to be a sequence $x_1, \dots, x_t$, with the following properties (which we refer to as the *right-sequence axioms*):

1. $x_1, \dots, x_t$ are distinct and B-complete
2. for $1 \leq i \leq t$, if x_i is A-complete then there exists h with $1 \leq h < i$ such that x_h is nonadjacent to x_i
3. for $1 \leq i \leq t$, if x_i is A-anticomplete then there is a banister r-R-x_i such that r has a nonneighbour in $\{x_1, \dots, x_{i-1}\}$.

Any initial subsequence of a right-sequence is therefore another right-sequence. We say x_i is *earlier* than x_j if $i < j$. Let $X = \{x_1, \dots, x_t\}$. For each $x_i \in X$ that has an earlier nonneighbour, we define its *predecessor* to be x_h, where h is minimum such that $1 \leq h < i$ and x_h is nonadjacent to x_i. From the second axiom, every x_i either has a nonneighbour in A or a predecessor, so we can follow the sequence of predecessors until we get to some vertex that is not A-complete. For each x_i we therefore define the *trajectory* of x_i to be the sequence $w_1\text{-}\cdots\text{-}w_n$ with the following properties:

- $n \geq 1$, and $w_1 = x_i$

- w_n has a nonneighbour in A
- for $1 \le j < n$, w_j is A-complete, and w_{j+1} is the predecessor of w_j.

Clearly the trajectory is unique, and is an antipath. If $v \in V(G)$ is A-complete, not in X and not X-complete, we define the *trajectory* of v to be the antipath v-w_1-$\cdots$-w_n, where w_1 is the earliest nonneighbour of v in X, and w_1-$\cdots$-w_n is the trajectory of w_1.

Let a be a left-star. If it is not X-complete, we define the *birth* of a to be the earliest nonneighbour of a in X. Now let b be a right-star. A banister a-R-b is said to be *b-optimal* if a is not X-complete, and there is no banister a'-R'-b such that a' is not X-complete and the birth of a' is earlier than the birth of a.

13.1 *Let G be Berge, containing no appearance of K_4, no even prism, no 1-breaker and no 2-breaker. Let $K = (S = (A, C, B), a_0$-R_0-$b_0)$ be a strongly maximal staircase in G, and let $x_1, \dots, x_t$ be a right-sequence. Let b be a right-star, and let a-R-b be a b-optimal banister. Let a-w_1-$\cdots$-w_n be the trajectory of a. Then n is odd, and either:*

- *b is the unique vertex of R which is $\{w_1, \dots, w_n\}$-complete, or*
- *R has length 1, and there exists some even m with $1 \le m < n$ such that a-w_1-$\cdots$-w_m-b is an antipath.*

Proof. We proceed by induction on t, and assume the result holds for all smaller values of t. Hence we may assume that $w_1 = x_t$, for otherwise the result follows from the inductive hypothesis. Let $W = \{w_1, \dots, w_n\}$; then every vertex in B is W-complete.

(1) *n is odd.*

For choose $a_2 \in A$ nonadjacent to w_n, and $b_1 \in B$ nonadjacent to a_2; then b_1-a-w_1-$\cdots$-w_n-a_2-b_1 is an antihole, so n is odd. This proves (1).

(2) *If w_n has a neighbour in A then the theorem holds.*

For choose a step a_1-R_1-b_1, a_2-R_2-b_2 such that w_n is adjacent to a_1 and not to a_2. Then a_1,b_2 are W-complete. Suppose first that there are no W-complete vertices in R. Then a_1-a-R-b-b_2 is an odd path between W-complete vertices. If R has length 1 then there is an antipath Q joining a, b with interior in W, and since it can be completed to an antihole via b-a_1-b_2-a, it has odd length and the theorem holds. So we may assume R has length > 1, and hence by 2.1 W contains a leap. Since all vertices of W except w_1 are adjacent to a, the leap is w_1, w_2; and hence the only edges between w_1, w_2 and R are $w_1 b$ and $w_2 a$. Since n is odd it follows that $n > 2$ and so w_1, w_2 are both $A \cup B$-complete. But then $S' = (A \cup \{w_2\}, C, B \cup \{w_1\})$ is a step-connected strip, and $(S', a$-R-$b)$ is a staircase, contrary to the maximality of (S, R_0). So we may assume there are W-complete vertices in R. If b is the only one then the theorem holds, so assume there is another. But then W can be linked onto the triangle $\{a, a_1, a_2\}$, via a subpath of $R \setminus b$, the 1-vertex path a_1, and a subpath of R_2. Since b_1 is W-complete and nonadjacent to both a, a_2, this contradicts 2.8. This proves (2).

From (2) we may assume that w_n has no neighbour in A. Let $w_n = x_s$ say. From the third axiom, there is a banister r'-R'-w_n, such that r' has a nonneighbour in $\{x_1, \dots, x_{s-1}\}$, and therefore we may choose this banister to be w_n-optimal.

(3) *R' is disjoint from R, and there are no edges between $V(R \setminus a)$ and $V(R' \setminus w_n)$.*

Suppose that $(R \setminus a) \cup (R' \setminus w_n)$ is connected. Then it contains a path between r' and b, with interior in the union of the interiors of R and R', and therefore this path is a banister. But R is b-optimal, and the birth of r' is earlier than the birth of a, a contradiction. So $R \setminus a$ is disjoint from $R' \setminus w_n$, and there are no edges between them. Since $a \neq r'$ (because their births are different), and $b \neq w_n$ (because R is optimal for b) it follows that R is disjoint from R'. This proves (3).

Let r'-v_1-$\cdots$-v_m be the trajectory of r', and let $V = \{v_1, \ldots, v_m\}$. By the inductive hypothesis, it follows that either w_n is the unique V-complete vertex in R', or R' has length 1 and there is an odd antipath between r' and w_n with interior in V. Since each of $v_1, \ldots, v_m$ is earlier than w_n, it follows from the definition of trajectory that $v_1, \ldots, v_m$ are all $\{a, w_1, \ldots, w_{n-1}\}$-complete.

(4) *If $n = 1$ then the theorem holds.*

For let $n = 1$, and choose a step a_1-R_1-b_1, a_2-R_2-b_2 with a_1 nonadjacent to v_m. Suppose first that a has no neighbour in R'. Now a is V-complete, and either w_1 is the unique V-complete vertex in R', or R' has length 1 and there is an odd antipath Q between r' and w_1 with interior in V. In the first case, a-a_1-r'-R'-w_1 is an odd path, its ends are V-complete, its internal vertices are not, and the V-complete vertex b_2 has no neighbour in its interior, contrary to 2.2. In the second case, a-r'-Q-w_1-a is an odd antihole. This proves that a has a neighbour in R'. Now suppose it has a neighbour different from r'; then R' has length > 1, and so w_1 is the unique V-complete vertex in R'; and there is a path P' say from a to w_1 with interior in $R' \setminus r'$. Since the ends of this path are V-complete and its internal vertices are not, and the V-complete vertex b_1 has no neighbour in its interior, it is even by 2.2. But it can be completed to an odd hole via w_1-b_1-R_1-a_1-a, a contradiction. This proves that r' is the unique neighbour of a in R'. Since a-r'-R'-w_1-b_1-b-R-a is not an odd hole, it follows from (3) that w_1 has a neighbour in R. If b is its unique neighbour in R then the theorem holds, so we assume not. Then there is a path P say from w_1 to a with interior in $R \setminus b$. Since w_1-P-a-r'-R'-w_1 is a hole it follows that P is even; but P can be completed via a-a_1-R_1-b_1-w_1, a contradiction. This proves (4).

We may therefore assume that $n \geq 3$ (since it is odd.)

(5) *$C = \emptyset$.*

For suppose not, and choose a step a_1-R_1-b_1, a_2-R_2-b_2 where R_1 has length > 1. Since R_1 is odd, and its ends are $(W \setminus \{w_n\})$-complete, and the $(W \setminus \{w_n\})$-complete vertex b_2 has no neighbour in its interior, there is a $(W \setminus \{w_n\})$-complete vertex v in the interior of R_1, by 2.2. But then v is nonadjacent to both a and w_n, since they are left- and right-stars respectively, and so v-a-w_1-$\cdots$-w_n-v is an odd antihole, a contradiction. This proves (5).

(6) *If b is not $(W \setminus \{w_n\})$-complete and no edge of R is $(W \setminus \{w_n\})$-complete then the theorem holds.*

For choose a step a_1-R_1-b_1, a_2-R_2-b_2. Then a_1-a-R-b-b_2 is an odd path, its ends are $(W \setminus \{w_n\})$-complete, and none of its edges are $(W \setminus \{w_n\})$-complete. Suppose first that R has length ≥ 3. Then by 2.1 there is a leap in $W \setminus \{w_n\}$; and so there are nonadjacent vertices $x, y \in W \setminus \{w_n\}$ such that x-a-R-b-y is a path. But then $((A \cup \{x\}, \emptyset, B \cup \{y\}), a\text{-}R\text{-}b)$ is a staircase, contrary to the maximality of (S, R_0). So R has length 1, and there exists i with $1 \leq i < n$ such that a-w_1-$\cdots$-w_i-b is an odd antipath. But then the theorem holds. This proves (6).

(7) *If no vertex in R is W-complete then the theorem holds.*

For by (6) we may assume that there is a vertex v of R which is $(W \setminus \{w_n\})$-complete. Hence v is nonadjacent to w_n. Since $n \geq 3$ and is odd, and a-w_1-$\cdots$-w_n-v-a is not an odd antihole, it follows that v is adjacent to a. Consequently v is the unique $(W \setminus \{w_n\})$-complete vertex in R. From (6) we may assume that $v = b$, and R has length 1. Choose a step a_1-R_1-b_1, a_2-R_2-b_2. Then b_1-a-w_1-$\cdots$-w_n-b is an odd antipath, of length ≥ 5. All its internal vertices have neighbours in the connected set $V(R' \setminus w_n) \cup \{a_2\}$, and its ends do not. By 2.1 applied in $\overline{G}$, there are adjacent vertices x, y in $V(R' \setminus w_n) \cup \{a_2\}$, such that x-a-w_1-$\cdots$-w_n-y is an odd antipath. Since x is adjacent to w_n, it follows that x is the neighbour of w_n in R', and therefore either y is the second neighbour of x in R', or R' has length 1 and $y = a_2$. Assume first that R' has length > 1, and so both x, y belong to the interior of R'. Hence x, y are both anticomplete to $A \cup B$, and so $((B \cup \{x\}, \emptyset, A \cup \{y\}), a\text{-}w_1\text{-}\cdots\text{-}w_n)$ is a staircase in $\overline{G}$, contradicting that (S, R_0) is strongly maximal. Now assume that R' has length 1. Then $x = r'$ and $y = a_2$, and $((B \cup \{r'\}, \emptyset, A \cup \{b\}), a\text{-}w_1\text{-}\cdots\text{-}w_n)$ is a staircase in $\overline{G}$, a contradiction as before. This proves (7).

We may therefore assume that some vertex of $R \setminus b$ is W-complete, for otherwise the theorem holds by (7). Let a-P-p be the subpath of $R \setminus b$ such that p is the unique W-complete vertex of P. Choose $a_1 \in A$ and $b_1 \in B$, adjacent (this is possible by (5)). Let us apply 3.2 to the path p-P-a-a_1-b_1, and the even antipath a-w_1-$\cdots$-w_n-a_1. Both ends of the path are complete to the interior of the antipath, so by 3.2 it follows that P has length 2, and if q denotes its middle vertex then q is nonadjacent to w_n and adjacent to $w_1, \ldots, w_{n-1}$. But then $((B \cup \{p\}, \emptyset, A \cup \{q\}), a\text{-}w_1\text{-}\cdots\text{-}w_n)$ is a staircase in $\overline{G}$, a contradiction. This completes the proof of 13.1. ∎

13.2 *Let G be Berge, containing no appearance of K_4, no even prism, no 1-breaker and no 2-breaker. Let $K = (S = (A, C, B), a_0\text{-}R_0\text{-}b_0)$ be a strongly maximal staircase in G, and let $x_1, \ldots, x_t$ be a right-sequence. Then $x_1, \ldots, x_t$ are all adjacent to a_0.*

Proof. Suppose the theorem is false, and choose t is small as possible such that the statement of the theorem does not hold. So $t \geq 1$, and $x_1, \ldots, x_{t-1}$ are all adjacent to a_0, and x_t is not.

(1) *a_0-R_0-b_0 is not an optimal banister for b_0.*

For suppose it is, and let a_0-w_1-$\cdots$-w_n be the trajectory of a_0. Since R_0 has length > 1 it follows from 13.1 that b_0 is the unique W-complete vertex of R_0, where $W = \{w_1, \ldots, w_n\}$. Suppose first that $n = 1$. Then b_0 is the unique neighbour of w_1 in R_0, and w_1 has a nonneighbour in A, and so by 12.1 it is a right-star. By axiom 3 there is a banister r-R-w_1 such that the birth of r

is earlier than w_1. Since a_0-R_0-b_0 is optimal for b_0, it follows as in the proof of 13.1 that R is disjoint from R_0, and there are no edges between $R_0 \setminus a_0$ and $R \setminus w_1$. Choose an S-rung a_1-R_1-b_1. Since a_1-a_0-R_0-b_0-w_1-R-r-a_1 is not an odd hole it follows that a_0 has neighbours in R. If it has a neighbour different from r, then the path from a_0 to w_1 with interior in $R \setminus r$ can be completed via w_1-b_0-R_0-a_0 and via w_1-b_1-R_1-a_1-a_0, and one of the resulting holes is odd, a contradiction. So the unique neighbour of a_0 in R is r. But then we can add r to A, w_1 to B and the interior of R to C, contradicting the maximality of (S, R_0). So $n \geq 2$. Now all of $w_1, \dots, w_{n-1}$ are left-diagonals, and all of $w_2, \dots, w_n$ are right-diagonals. But w_1 is not a right-diagonal, and w_n is not a left-diagonal, and w_1 is not a right-star, contrary to 12.5. This proves (1).

Now since a_0 has a nonneighbour in $\{x_1, \dots, x_t\}$, it follows that there is an optimal banister r-R-b_0 for b_0. From (1), r has a nonneighbour in $\{x_1, \dots, x_{t-1}\}$. From the minimality of t (replacing R_0 by R) it follows that R has length 1, and so rb_0 is an edge. Let r-w_1-$\cdots$-w_n be the trajectory of r; so w_1 is earlier than x_t. Let $W = \{w_1, \dots, w_n\}$; hence a_0 is W-complete. By 13.1, n is odd.

(2) *b_0 is W-complete.*

For suppose not. Then by 13.1, there exists i with $1 \leq i < n$ such that r-w_1-$\cdots$-w_i-b_0 is an odd antipath. Now $r, w_1, \dots, w_{i-1}$ are all left-diagonals; $w_1, \dots, w_i$ are all right-diagonals; r is not a right-diagonal (since it is a left-star); and w_i is not a left-diagonal (since it is nonadjacent to b_0) and not a right- or left-star (since it is $A \cup B$-complete, because $i < n$). This contradicts 12.5, and so proves (2).

(3) *a_0 is adjacent to r, and w_n is a right-star.*

Let a_1-R_1-b_1 be an S-rung with w_n nonadjacent to a_1. Since a_0-r-w_1-$\cdots$-w_n-a_1-b_0-a_0 is not an odd antihole it follows that a_0 is adjacent to r. So each of $r, w_1, \dots, w_{n-1}$ is left-diagonal, each of $w_1, \dots, w_n$ is right-diagonal, r is not right-diagonal, w_n is not left-diagonal, and the claim follows from 12.5. This proves (3).

(4) *There is no $(W \cup \{r\})$-complete vertex in the interior of R_0.*

For suppose there is, v say. Let a_1-R_1-b_1 be an S-rung. Then a_0-a_1-R_1-b_1-b_0 is an odd path; both its ends are $(W \cup \{r\})$-complete; and the $(W \cup \{r\})$-complete vertex v has no neighbour in its interior, so by 2.2 there is a $(W \cup \{r\})$-complete vertex in R_1. But r is a left-star and by (3), w_n is a right-star, so they have no common neighbour in R_1, a contradiction. This proves (4).

(5) $n = 1$.

For assume $n > 1$. Now R_0 is odd, and both its ends are $(W \cup \{r\})$-complete. Suppose first that R_0 has length ≥ 5. By 2.1 and (4) there is a leap; that is, there are two nonadjacent vertices $x, y \in W \cup \{r\}$ joined by an odd path P whose interior is the interior of R_0. Choose $b_1 \in B$; then b_1-x-P-y-b_1 is not an odd hole, and so one of x, y is nonadjacent to b_1. Since b_1 is W-complete, we may assume $y = r$; and hence $x = w_1$ since that is the only vertex in W nonadjacent to r. Choose $a_1 \in A$; then since a_1-r-P-w_1-a_1 is not an odd hole it follows that a_1 is not adjacent to w_1 and so $n = 1$. Now assume that R_0 has length 3, and let its internal vertices be x, y (in some order). By 2.1 there exists an odd antipath Q joining x, y with interior in $W \cup \{r\}$. If $r \notin V(Q)$ then b_1-x-Q-y-b_1 is an odd antihole, where $b_1 \in B$; and if $w_n \notin V(Q)$ then a_1-x-Q-y-a_1 is an odd antihole, where $a_1 \in A$. Hence we may assume that x-r-w_1-$\cdots$-w_n-y is an antipath. We claim that $C = \emptyset$. For suppose there is an S-rung a_1-R_1-b_1 say of length > 1. Then a_1-R_1-b_1-b_0-r-a_1 is a hole of length ≥ 6; and r-w_1-$\cdots$-w_n-a_1 is an even antipath of length ≥ 4; and a_0 is complete to the antipath, and has no other neighbours on the hole; and at least two vertices of the hole are complete to the interior of the antipath, namely b_0 and b_1. This contradicts 3.3. So $C = \emptyset$. Hence $((B \cup \{x\}, \emptyset, A \cup \{y\}), r$-$w_1$-$\cdots$-$w_n)$ is a staircase in $\overline{G}$, a contradiction. This proves (5).

From (4), (5) we may apply 2.1 to R_0 and the anticonnected set $\{r, w_1\}$, and since the latter has only two members, 2.1 implies that there is an odd path P joining r and w_1 with interior equal to the interior of R_0. From (3), w_1 is a right-star, and from axiom 3 there is a banister r'-R'-w_1 (and we may choose it optimal for w_1) such that the birth of r' is earlier than w_1. Now R' is disjoint from R_0, and there are no edges between $R_0 \setminus a_0$ and $R' \setminus w_1$; for otherwise there would be a banister from r' to b_0, contradicting that r-b_0 is optimal for b_0. Suppose that r has a neighbour in R'; then the path between r and w_1 with interior in R' can be completed to holes via w_1-b_0-r and via w_1-P-r, a contradiction since one of these holes is odd. So r has no neighbour in R'. Let r'-v_1-$\cdots$-v_m be the trajectory of r'. Since $v_1, \ldots, v_m$ are earlier than w_1, and w_1 is the earliest nonneighbour of r, it follows that r is adjacent to all of $v_1, \ldots, v_m$. Now by 13.1, either

- w_1 is the unique $\{v_1, \ldots, v_m\}$-complete vertex in R'; but then w_1-R'-r'-a_1-r (where $a_1 \in A$ is nonadjacent to v_m) is an odd path; its ends are $\{v_1, \ldots, v_m\}$-complete and its internal vertices are not; and the $\{v_1, \ldots, v_m\}$-complete vertex b_1 (for any $b_1 \in B$ nonadjacent to a_1) has no neighbour in its interior, contrary to 2.2.
- R' has length 1, and there is an odd antipath Q between r' and w_1 with interior in $\{v_1, \ldots, v_m\}$; but then r-r'-Q-w_1-r is an odd antihole, a contradiction.

This completes the proof of 13.2. ∎

Now we are ready to apply 13.2 to produce a skew partition. Let us say a *3-breaker* in G is a pair (K, x) such that $K = (S = (A, C, B), a_0$-R_0-$b_0)$ is a strongly maximal staircase in G, and $x \in V(G) \setminus V(K)$ is B-complete, and not A-complete, and not A-anticomplete.

13.3 *Let G be Berge, containing no appearance of K_4, no even prism, no 1-breaker and no 2-breaker. Suppose that there is a 3-breaker in G; then G admits a balanced skew partition.*

Proof. Let (K, x_1) be a 3-breaker, where $K = (S = (A, C, B), a_0$-R_0-$b_0)$. The 1-vertex sequence x_1 is a right-sequence; so there exists a right-sequence $x_1, \ldots, x_t$ of maximum length, with $t \geq 1$. Let $X = \{x_1, \ldots, x_t\}$, and let Y be the set of all $A \cup X$-complete vertices in $V(G) \setminus V(S)$. So $a_0 \in Y$ by 13.2.

(1) *$X \cup Y \cup B$ meets the interior of every path in G from $A \cup C$ to b_0.*

For suppose P is a path from $A \cup C$ to b_0 with no internal vertex in $X \cup Y \cup B$. Note that $b_0 \notin X$ by 13.2, and so $b_0 \notin X \cup Y \cup B$ (since it is not A-complete). We may assume P is minimal, and therefore no internal vertex of P is in $V(S)$. Let P be from $p \in A \cup C$ to b_0. By 12.3, $P \setminus p$

contains either a major vertex or a banister. Suppose first that it contains a banister a-R-b say. Hence $a, b \notin X \cup Y \cup B$. Since a is A-complete it is therefore not X-complete (because it is not in Y), and then we can set $x_{t+1} = b$, contradicting the maximality of the right-sequence. So $P \setminus p$ contains no banister. Now assume it contains a major vertex v say. Since $v \notin X \cup Y \cup B$, it follows that v is not $X \cup A$-complete. Suppose v is B-complete. Since it is major it has a neighbour in A. If it is not A-complete we can set $x_{t+1} = v$ and obtain a longer right-sequence, a contradiction; and if v is A-complete then since it is not $X \cup A$-complete, it is not X-complete and so again we can set $x_{t+1} = v$ and obtain a longer right-sequence, a contradiction. So v is not B-complete. By 12.1 and since there is no 2-breaker in G and therefore no central vertex, v is left-diagonal, and not right-diagonal; and since it is not $X \cup A$-complete, it is not X-complete. Let v-w_1-$\cdots$-w_n be the trajectory of v. Then each of $w_1, \ldots, w_n$ is right-diagonal, since they are all $B \cup \{a_0\}$-complete. Since w_n has a nonneighbour in A, it is not left-diagonal; and so there is a minimum i with $1 \leq i \leq n$ such that w_i is not left-diagonal. By 12.5 applied to the sequence $v, w_1, \ldots, w_i$, we deduce that v is a left-star, contradicting that v is major. This proves (1).

Now since S is step-connected, it follows that $A \cup C$ is connected; and therefore belongs to a component A_1 of $G \setminus (X \cup Y \cup B)$. Let A_2 be the union of all the other components. So by (1), $b_0 \in A_2$, and $(A_1 \cup A_2, X \cup Y \cup B)$ is a skew partition of G (since $Y \cup B$ is complete to X, and X is nonempty). We need to find a balanced skew partition. By 4.2 we may assume this skew partition is not loose; so every X-complete vertex in G either belongs to B or is also A-complete. Every vertex in $Y \cup B$ has a neighbour in $A \cup C$, so $A \cup C$ is a kernel for this skew partition, in $\overline{G}$. By 4.6 it suffices to show that in G, any two nonadjacent vertices in $Y \cup B$ are joined by an even path with interior in $A_1 \cup A_2$, and any two adjacent vertices of $A \cup C$ are joined by an even antipath with interior in $X \cup Y \cup B$. Now let $u, v \in Y \cup B$ be nonadjacent. If they are both adjacent to b_0, then any path joining them with interior in $A \cup C$ (and there is one) is even, since it can be completed to a hole via v-b_0-u. So we may assume that u is nonadjacent to b_0, and hence $u \notin B$, so $u \in Y$. If they are both in Y, then they are joined by an even path u-a_1-v for any $a_1 \in A$. So we may assume that $v \in B$. Since u is nonadjacent to b_0 and to v, it is neither left- nor right-diagonal, and it is not central since there is no 2-breaker; so from 12.1 it is a left-star. Let a_1-R_1-v be an S-rung; then u-a_1-R_1-v is the desired even path between u and v. Now for antipaths, let uv be an edge with $u, v \in A \cup C$. They both therefore have nonneighbours in B, and since $B \cup \{a_0\}$ is anticonnected, they are joined by an antipath Q with interior in $B \cup \{a_0\}$. It suffices to show that Q is even, since $Q^* \subseteq Y \cup B$. If $a_0 \notin Q^*$, then Q is even since b_0-u-Q-v-b_0 is an antihole. So a_0 is in Q^*. But there are no edges between a_0 and B, and so a_0 is nonadjacent to every other vertex in the interior of Q; and since Q is an antipath, it therefore has at most 3 internal vertices, so its length is ≤ 4. If it is odd, then it has length 3, that is, there are nonadjacent vertices $u' \in Y$ and $v' \in B$, joined by an odd path with interior in $A \cup C$. But we have already shown that they are joined by an even path, and the result follows from 4.3. This proves 13.3. ∎

Now we can prove 1.8.5, the main result of this section. We restate it (proper homogeneous pairs were defined in section 1.)

13.4 *Let G be Berge, such that there is no appearance of K_4 in either G or $\overline{G}$. Suppose that G contains a long odd prism as an induced subgraph. Then either one of $G, \overline{G}$ admits a proper 2-join, or G admits a balanced skew partition, or G admits a proper homogeneous pair.*

Proof. We assume that G does not admit a balanced skew partition, and $G, \overline{G}$ do not admit proper 2-joins. Since G contains a long odd prism, and therefore $G, \overline{G}$ are not even prisms, it follows from 10.6 that $G, \overline{G}$ contain no even prism. By 11.5, 12.4 and 13.3, $G, \overline{G}$ contain no 1-, 2- or 3-breaker.

Since G contains a long odd prism, it contains a staircase; and therefore (possibly by replacing G by its complement) there is a strongly maximal staircase $K = (S = (A, C, B), a_0\text{-}R_0\text{-}b_0)$ say in G. Let A_0 be the set of all left-stars, B_0 the set of all right-stars, and N the set of all vertices that are $A \cup B$-complete. By 12.1, every non-major A-complete vertex is in A_0, and since there is no 3-breaker, every major A-complete vertex is in N, so every A-complete vertex is in $A_0 \cup N$; and similarly every B-complete vertex is in $B_0 \cup N$. Let $H = G \setminus (V(S) \cup A_0 \cup B_0 \cup N)$.

(1) *Let F be a component of H, and let X be the set of attachments of F in $V(S) \cup A_0 \cup B_0$. Then either $X \cap V(S) = \emptyset$, or $X \subseteq V(S)$ and X meets both $A \cup C$ and $B \cup C$.*

We may assume that X meets $V(S)$, and therefore from the symmetry we may assume that X meets $A \cup C$. Since no vertex in F is A- or B-complete, and therefore no vertex in F is major or a left- or right-star, it follows from 12.3 that X is disjoint from B_0. If X meets $B \cup C$ then similarly X is disjoint from A_0, and so $X \subseteq V(S)$ and the claim holds. We assume therefore that $X \subseteq A \cup A_0$. Now if $v \in V(G) \setminus F$ has a neighbour in F, then $v \notin V(H)$, and so $v \in V(S) \cup A_0 \cup B_0 \cup N$, and therefore $v \in X \cup N \subseteq A \cup A_0 \cup N$. Hence $(V(G) \setminus (A \cup A_0 \cup N), A \cup A_0 \cup N)$ is a skew partition of G, since F is a component of $V(G) \setminus (A \cup A_0 \cup N)$ and b_0 is in a different component, and $A, A_0 \cup N$ are both nonempty and complete to each other. Now by 2.6, $(B \cup C, A)$ is balanced, since a_0 is complete to A and anticomplete to $B \cup C$; and therefore from 2.7, (F, A) is balanced (since $B \cup C$ is connected and all vertices in A have neighbours in it). Hence from 4.5, G admits a balanced skew partition, a contradiction. This proves (1).

Let M be the union of all components of H with no attachment in $V(S)$. Then M is nonempty, since by (1) the component of H containing the interior of R_0 has no attachments in $V(S)$. Let D be the union of all the components of H that have an attachment in $V(S)$. Hence $V(G)$ is partitioned into $A, B, C, D, A_0, B_0, N, M$, where possibly C, D or N may be empty.

(2) *$N \neq \emptyset$.*

For assume that $N = \emptyset$. Then the only edges between $V(S) \cup D$ and $A_0 \cup B_0 \cup M$ are the edges from A to A_0 and those from B to B_0; and since R_0 is an odd path from A_0 to B_0 of length ≥ 3 and with $V(R_0) \subseteq A_0 \cup B_0 \cup M$, and both A and B contain at least two vertices, it follows that G admits a proper 2-join, a contradiction. This proves (2).

(3) *$C \cup D = \emptyset$.*

For assume that $C \cup D$ is nonempty. By (1) there are no edges between $C \cup D$ and $A_0 \cup B_0 \cup M$. Since N is complete to $A \cup B$, it follows that $(C \cup D \cup A_0 \cup B_0 \cup M, N \cup A \cup B)$ is a skew partition of G. By 4.2, it is not loose, and so there is no N'-complete vertex in R_0, where N' is an anticomponent of N. Let $a_1\text{-}R_1\text{-}b_1, a_2\text{-}R_2\text{-}b_2$ be a step; then $a_1\text{-}a_0\text{-}R_0\text{-}b_0\text{-}b_2$ is an odd path of length ≥ 5; its ends are N'-complete, and its internal vertices are not. By 2.1, there is a leap in N', and so there exist nonadjacent x, y in N such that $x\text{-}a_0\text{-}R_0\text{-}b_0\text{-}y$ is a path. But then $((A \cup \{x\}, C, B \cup \{y\}), a_0\text{-}R_0\text{-}b_0)$

is a staircase, contradicting the maximality of (S, R_0). This proves (3).

But then (A, B) is a proper homogeneous pair in G. (This is the only place in the entire paper where we use such pairs.) This proves 13.4. ∎

Let us say a graph G is *recalcitrant* if:

- G is Berge
- G and $\overline{G}$ are not line graphs, and G is not a double split graph
- G and $\overline{G}$ do not admit proper 2-joins, and
- G does not admit a proper homogeneous pair or balanced skew partition.

The remainder of the paper is basically a proof of the following.

13.5 *If G is recalcitrant then either G or $\overline{G}$ is bipartite.*

Clearly any counterexample to 1.3 is recalcitrant, so 13.5 will imply 1.3.

On the other hand, for some future applications, it is desirable to keep closer track of which results hold under which hypotheses, instead of just using the blanket "recalcitrant" hypothesis. But at least, for the remainder of the paper we shall only be concerned with Berge graphs G such that in both $G, \overline{G}$ there is no appearance of K_4 and no long prism; that is, with the members of the class $\mathcal{F}_5$ introduced in section 1. Certainly every recalcitrant graph belongs to $\mathcal{F}_5$, by 10.6 and 9.7.

It turns out that for such graphs, there is a useful strengthening of 2.1 — the second alternative of that theorem can no longer hold.

13.6 *Let $G \in \mathcal{F}_5$, and let P be a path in G with odd length. Let $X \subseteq V(G) \setminus V(P)$ be anticonnected, such that both ends of P are X-complete. Then either:*

1. *some edge of P is X-complete, or*
2. *P has length 3 and there is an odd antipath joining the internal vertices of P with interior in X.*

Proof. Let P be $p_1\text{-}\cdots\text{-}p_n$. By 2.1, we may assume that P has length ≥ 5 and X contains a leap u, v say; so $u\text{-}p_2\text{-}\cdots\text{-}p_{n-1}\text{-}v$ is a path. But then the three paths $p_1\text{-}v, u\text{-}p_n, p_2\text{-}\cdots\text{-}p_{n-1}$ form a long prism, contrary to $G \in \mathcal{F}_5$. This proves 13.6. ∎

There is an analogous strengthening of 2.9, as follows.

13.7 *Let $G \in \mathcal{F}_5$, and let X, Y be disjoint nonempty anticonnected subsets of $V(G)$, complete to each other. Let P be a path in G with even length > 0, with vertices $p_1, \ldots, p_n$ in order, such that p_1 is the unique X-complete vertex of P and p_n is the unique Y-complete vertex of P. Then P has length 2 and there is an antipath Q between p_2 and p_3 with interior in X, and an antipath R between p_1 and p_2 with interior in Y, and exactly one of Q,R has odd length.*

Proof. Let us apply 2.9. We may therefore assume that P has length ≥ 4 and there are nonadjacent $x_1, x_2 \in X$ such that $x_1\text{-}p_2\text{-}\cdots\text{-}p_n\text{-}x_2$ is a path P' say, of odd length ≥ 5. But the ends of P' are $Y \cup \{p_1\}$-complete, and its internal vertices are not, contrary to 13.6. This proves 13.7. ∎

14 The double diamond

We are finished with prisms — we cannot dispose of the prism where all three paths have length 1 (yet), and we have disposed of all others. Now we turn to a different type of subgraph, the double diamond. A *double diamond* means the graph with eight vertices $a_1, \ldots, a_4, b_1, \ldots, b_4$ and with the following adjacencies: every two a_i's are adjacent except a_3a_4, every two b_i's are adjacent except b_3b_4, and a_ib_i is an edge for $1 \leq i \leq 4$.

Let G be Berge. If A, B are disjoint subsets of $V(G)$, we say a *square* in (A, B) is a hole a_1-b_1-b_2-a_2-a_1 of length 4, where $a_1, a_2 \in A$ and $b_1, b_2 \in B$. The pair (A, B) is *square-connected* if:

- $|A|, |B| \geq 2$,
- for every partition (X, Y) of A with X, Y nonempty, there is a square a_1-b_1-b_2-a_2-a_1 with $a_1 \in X$ and $a_2 \in Y$
- for every partition (X, Y) of B with X, Y nonempty, there is a square a_1-b_1-b_2-a_2-a_1 with $b_1 \in X$ and $b_2 \in Y$.

It follows that if (A, B) is square-connected then every vertex of $A \cup B$ is in a square. An *antisquare* is a square in $\overline{G}$; that is, an antihole a_1-b_1-b_2-a_2-a_1 with $a_1, a_2 \in A$ and $b_1, b_2 \in B$; and (A, B) is *antisquare-connected* if it is square-connected in $\overline{G}$. For strips in which every rung has length 1 (and from now on, those are the only kind of strips we shall need), being square-connected is the same as being step-connected. We have renamed the concepts because we wanted to improve our notation for a step.

We say a quadruple (A, B, C, D) of subsets of $V(G)$ is a *cube* in G if it satisfies the following conditions:

- A, B, C, D are pairwise disjoint and nonempty
- A is complete to C, and B to D, and A is anticomplete to D, and B to C
- (A, B) is square-connected, and (C, D) is antisquare-connected.

If G contains a double diamond, then it contains a cube in which A, B, C, D all have two elements, and that turns out to be the right approach to the double diamond — grow the cube until it is maximal, and analyze how the remainder of G attaches to it. That is our goal in this section. A cube (A, B, C, D) is *maximal* if there is no cube (A', B', C', D') with $A \subseteq A', B \subseteq B', C \subseteq C'$, and $D \subseteq D'$ such that $(A, B, C, D) \neq (A', B', C', D')$. The subgraph $G|(A \cup B \cup C \cup D)$ is called the graph *formed* by the cube. Note that if (A, B, C, D) is a cube in G, then (C, D, B, A) is a cube in $\overline{G}$. (This is very convenient, because it reduces our work by half — we are going to have the usual minor vertices and major vertices, and they switch when we take complements, so whatever we can prove about minor vertices will also give us information about major vertices by going to the complement.)

14.1 *Let $G \in \mathcal{F}_5$. Let (A, B, C, D) be a maximal cube in G, forming K, let $v \in V(G) \setminus V(K)$, and let X be the set of neighbours of v in $V(K)$. Then either*

- *X is a subset of one of $A \cup B, C \cup D, A \cup C, B \cup D$, and $X \cap (A \cup C)$ is complete to $X \cap (B \cup D)$, or*

- *X includes one of $A\cup B, C\cup D, A\cup D, B\cup C$, and $(A\cup D)\setminus X$ is anticomplete to $(B\cup C)\setminus X$.*

Proof. Note that under taking complements the two outcomes become exchanged. If $X\subseteq A\cup B$, and there exists $a\in X\cap A$ and $b\in X\cap B$, nonadjacent, then choose $c\in C$ and $d\in D$, adjacent, and v-a-c-d-b-v is an odd hole. So if $X\subseteq A\cup B$ then the theorem holds. Similarly it holds if $X\subseteq C\cup D$; and trivially it holds if X is a subset of one of $A\cup C, B\cup D$. So we may assume that X meets both A and D. From the same argument in $\overline{G}$, we may also assume that none of $A\cup B, C\cup D, A\cup D, B\cup C$ is a subset of X, that is, either X includes neither of A,C or it includes neither of B,D. These two cases are exchanged when we pass to the complement; so we may assume by taking complements that X includes neither of B, D. Let $A_1 = A\cap X$, and $A_2 = A\setminus A_1$; and define B_1, B_2 etc. similarly. We have shown so far that A_1, B_2, D_1, D_2 are nonempty. Choose an antisquare c_2-d_1-d_2-c_1-c_2 such that $d_1\in D_1$ and $d_2\in D_2$, and choose $b_2\in B_2$. Since v-c_2-d_2-b_2-d_1-v is not an odd hole, it follows that $c_2\in C_2$. Hence A_1 is complete to B_1; for if $a_1\in A_1$ and $b_1\in B_1$ are nonadjacent then v-a_1-c_2-d_2-b_1-v is an odd hole. If $A_1 = A$, then since (A,B) is square-connected and A_1 is complete to B_1 it follows that B_1 is empty; but then we can add v to C (because v-d_2-d_1-c_2-v becomes a new antisquare), contrary to the maximality of the cube. So A_2 is nonempty. Hence there is a square a_1-b_1-b_2-a_2-a_1 with $a_1\in A_1$ and $a_2\in A_2$. Since a_1 is nonadjacent to b_2 and complete to B_1, it follows that $b_2\in B_2$; but then v-a_1-a_2-b_2-d_1-v is an odd hole, a contradiction. This proves 14.1. ▮

Say a vertex $v\in V(G)\setminus V(K)$ is *minor* if the first case of 14.1 applies to it, and *major* if the second case applies. Then every such vertex is either minor or major and not both; and by taking complements, the minor and major vertices are exchanged.

14.2 *Let $G\in\mathcal{F}_5$. Let (A,B,C,D) be a maximal cube in G, forming K, let $F\subseteq V(G)\setminus V(K)$ be a connected set of minor vertices, and let X be the set of attachments of F in $V(K)$. Then X is a subset of one of $A\cup B, C\cup D, A\cup C, B\cup D$. Moreover, $X\cap(A\cup C)$ is complete to $X\cap(B\cup D)$.*

Proof. Suppose the first assertion is false, and choose F minimal with this property. We may assume that X meets both of A, D. Since all vertices in F are minor, it follows that F is a path f_1-f_2-$\cdots$-f_k of length ≥ 1. We may assume f_1 is the unique vertex of F with a neighbour in A, and f_k is the unique vertex of F with a neighbour in D. Let X_1, X_2 be the sets of attachments in $V(K)$ of $F\setminus\{f_k\}$, $F\setminus\{f_1\}$ respectively. From the minimality of F it follows that X_1 is a subset of one of $A\cup B, A\cup C$, and X_2 is a subset of one of $B\cup D, C\cup D$.

(1) *Not both $X_1\subseteq A\cup B$ and $X_2\subseteq B\cup D$.*

For suppose that both these hold. If k is even, choose $a\in A$ is adjacent to f_1, and $d\in D$ is adjacent to f_k, and $c\in C$ is adjacent to d; then a-f_1-$\cdots$-f_k-d-c-a is an odd hole , a contradiction. So k is odd. Suppose first that f_1 is complete to A. Since it is minor, it has no neighbours in B (for no vertex in B is A-complete). If there are no edges between B and F, let a_1-b_1-b_2-a_2-a_1 be a square, and let $d\in D$ be adjacent to f_k; then a_1-b_1, a_2-b_2, f_1-$\cdots$-f_k-d form a long prism, a contradiction. So there are edges between B and F. Choose i with $1\leq i\leq k$ minimum such that f_i has a neighbour in B. If f_i is not complete to B, choose a square a_1-b_1-b_2-a_2-a_1 such that f_i is adjacent to b_1 and not to b_2; then b_1 can be linked onto the triangle $\{f_1, a_1, a_2\}$, via b_1-f_i-$\cdots$-f_1, b_1-a_1, b_1-b_2-a_2, contrary to 2.4. So f_i is complete to B. Let a_1-b_1-b_2-a_2-a_1 be a square; then since a_1-b_1, a_2-b_2, f_1-$\cdots$-f_i do not form a long prism (because $G\in\mathcal{F}_5$), it follows that $i=2$. But $k>2$ since k is odd; so we can add f_1

to C and f_2 to D, contrary to the maximality of the cube. This proves (1) if f_1 is A-complete. Now assume f_1 is not A-complete, and choose a square a_1-b_1-b_2-a_2-a_1 such that f_1 is adjacent to a_1 and not to a_2. Since a_1-f_1-$\cdots$-f_k-d-b_2-a_2-a_1 is not an odd hole (where $d \in D$ is adjacent to f_k), it follows that b_2 has a neighbour in F. Choose i minimum such that b_2 is adjacent to f_i. Let $c \in C$ and $d \in D$ be any adjacent pair of vertices. Then the three paths a_1-b_1, a_2-b_2, c-d form a prism, and since the set of attachments of $\{f_1, \ldots, f_i\}$ in this prism is not local, and does not include a_2, it has an attachment in the third path c-d, by 10.4; and hence $i = k$, and f_k is D-complete. Again, let $c \in C$ and $d \in D$ be adjacent. Then the prism formed by a_1-f_1-$\cdots$-f_k,a_2-b_2,c-d is long, contrary to $G \in \mathcal{F}_5$. This proves (1).

(2) *Not both* $X_1 \subseteq A \cup C$ *and* $X_2 \subseteq C \cup D$.

For assume these both hold. Choose a square a_1-b_1-b_2-a_2-a_1 such that f_1 is adjacent to a_1, and choose $d \in D$ adjacent to f_k. If a_2 is adjacent to f_1 then a_1-b_1, a_2-b_2, f_1-$\cdots$-f_k-d form a long odd prism, a contradiction. If a_2 is not adjacent to f_1 then a_1 can be linked onto the triangle $\{b_1, b_2, d\}$, via a_1-b_1,a_1-a_2-b_2, a_1-f_1-$\cdots$-f_k-d, a contradiction. This proves (2).

(3) *Not both* $X_1 \subseteq A \cup B$ *and* $X_2 \subseteq C \cup D$.

For assume these both hold. Then $X_1 \cap X_2 = \emptyset$, and so f_1 is the unique neighbour in F of the vertices in X_1, and f_k is the unique neighbour of those in X_2. From (1), $X_2 \not\subseteq B \cup D$ and so $X_2 \cap C \neq \emptyset$; and similarly from (2), $X_1 \cap B \neq \emptyset$. Also we are given that $X_1 \cap A, X_2 \cap D \neq \emptyset$. Since a_1-f_1-$\cdots$-f_k-c_1-a_1 is a hole (where $a_1 \in A \cap X_1$ and $c_1 \in C \cap X_2$) it follows that k is even. Since f_1 is minor, $X_1 \cap A$ is complete to $X_1 \cap B$, and so A, B are not subsets of X_1; and similarly $C \cap X_2$ is complete to $D \cap X_2$ and therefore C, D are not subsets of X_2. So all the eight sets $A \cap X_1, A \setminus X_1$ etc. are nonempty. Choose a square a_1-b_1-b_2-a_2-a_1 such that f_1 is adjacent to a_1 and not to a_2; and choose an antisquare c_1-d_1-d_2-c_2-c_1 such that f_k is adjacent to d_1 and not to d_2. It follows that f_1 is nonadjacent to b_2, since $X_1 \cap A$ is complete to $X_1 \cap B$, and f_k is not adjacent to c_1 since $X_2 \cap C$ is complete to $X_2 \cap D$. But then a_1-f_1-$\cdots$-f_k-d_1-b_2-d_2-c_1-a_1 is an odd hole, a contradiction. This proves (3).

(4) *Not both* $X_1 \subseteq A \cup C$ *and* $X_2 \subseteq B \cup D$.

For assume both these hold. Then again, the only edges between $V(K)$ and F are between X_1 and f_1 and between X_2 and f_k. By (1) and (2), again all four of the sets $A \cap X_1, B \cap X_2, C \cap X_1, D \cap X_2$ are nonempty. There are two cases, depending on the parity of k. First assume k is odd. Then $A \cap X_1$ is anticomplete to $B \cap X_2$ (for if ab were an edge there, then a-f_1-$\cdots$-f_k-b-a would be an odd hole), and so $A \setminus X_1, B \setminus X_2$ are nonempty; and similarly $C \cap X_1$ is anticomplete to $D \cap X_2$, and therefore $C \setminus X_1, D \setminus X_2$ are nonempty. Choose a square a_1-b_1-b_2-a_2-a_1 such that f_1 is adjacent to a_1 and not to a_2, and choose an antisquare c_1-d_1-d_2-c_2-c_1 such that f_k is adjacent to d_1 and not to d_2. Since $A \cap X_1$ is anticomplete to $B \cap X_2$ it follows that $b_1 \notin X_2$, and $c_2 \notin X_1$ similarly; and since a_1-f_1-$\cdots$-f_k-d_1-b_2-a_2-a_1 is not an odd hole it follows that $b_2 \in X_2$. But then the three paths a_2-b_2, c_2-d_1, a_1-f_1-$\cdots$-f_k form a long prism, contrary to $G \in \mathcal{F}_5$. Now assume k is even. Then $A \cap X_1$ is anticomplete to $B \setminus X_2$ (for if $a \in A \cap X_1$ is adjacent to $b \in B \setminus X_2$ then a-f_1-$\cdots$-f_k-d-b-a is an odd hole, where $d \in X_2 \cap D$) . Similarly $A \setminus X_1$ is anticomplete to $B \cap X_2$, $C \cap X_1$ is anticomplete to $D \setminus X_2$, and $C \setminus X_1$ is anticomplete to $D \cap X_2$. Choose $a \in A \cap X_1$ and a neighbour b of a in B; then $b \in X_2$.

Similarly choose $c \in C \cap X_1$ and $d \in D \cap X_2$, adjacent. Then the three paths a-b, c-d, f_1-$\cdots$-f_k form a prism, and so $k = 2$. If f_1 is C-complete then since $C \cap X_1 = C$ is anticomplete to $D \setminus X_2$, it follows that f_2 is D-complete; and then we can add f_1 to A and f_2 to B, contrary to the maximality of the cube. So $C \not\subseteq X_1$. Choose an antisquare c_1-d_1-d_2-c_2-c_1 such that f_1 is adjacent to c_1 and not to c_2. It follows that f_2 is adjacent to d_2 and not to d_1. If f_1 is A-complete, then as before f_2 is B-complete, and we can add f_1 to C and f_2 to D (because f_1-d_1-f_2-c_2-f_1 is a new antisquare), a contradiction. So f_1 has a nonneighbour in A, and we can choose a square a_1-b_1-b_2-a_2-a_1 such that f_1 is adjacent to a_1 and not to a_2. It follows that f_2 is adjacent to b_1 and not to b_2. But then a_1-f_1-f_2-d_2-b_2-d_1-c_2-a_1 is an odd hole, a contradiction. This proves (4).

From (1)-(4), the first assertion of the theorem follows. Now let us prove the second assertion. We may assume X meets both $A \cup C$ and $B \cup D$, and so from what we just proved, either $X \subseteq C \cup D$ or $X \subseteq A \cup B$. Suppose first that $X \subseteq C \cup D$. If possible, choose $c \in C \cap X$ and $d \in D \cap X$, nonadjacent, and choose a path P joining them with interior in F. Let a_1-b_1-b_2-a_1-a_1 be a square; then the three paths a_1-b_1, a_2-b_2, c-P-d form a long prism, a contradiction. So there are no such c, d, and the theorem holds.

Now assume that $X \subseteq A \cup B$. Assume $X \cap A$ is not complete to $X \cap B$, and choose a path a-f_1-$\cdots$-f_k-b, where $a \in A, b \in B$ are nonadjacent and $f_1, \ldots, f_k \in F$, with k minimum. Since f_1 is minor, its neighbours in A are complete to its neighbours in B, and so $k \geq 2$. Let A' be the set of all vertices $a \in A$ such that a is adjacent to f_1 and there is a nonneighbour b of a in B adjacent to f_k. By assumption $A' \neq \emptyset$. Define B' similarly in B. If $A' = A$ and $B' = B$, then f_1 is A-complete, and so there are no edges between $\{f_1, \ldots, f_{k-1}\}$ and B, from the minimality of k; and similarly f_k is B-complete and there are no edges between $\{f_2, \ldots, f_k\}$ and A. Choose a square a_1-b_1-b_2-a_2-a_1; then a_1-b_1, a_2-b_2, f_1-$\cdots$-f_k form a prism, so $k = 2$, and we can add f_1 to C and f_2 to D, contrary to the maximality of the cube. So we may assume that $A' \neq A$. Choose a square a_1-b_1-b_2-a_2-a_1 such that $a_1 \in A'$ and $a_2 \notin A'$. Choose $c \in C$ and $d \in D$, adjacent. Choose $b \in B'$ nonadjacent to a_1 (this exists from the definition of A'). From the minimality of k, a_1-f_1-$\cdots$-f_k-b is a path. From the hole a_1-f_1-$\cdots$-f_k-b-d-c-a_1 we deduce that k is even. Since b is not adjacent to a_1, b is different from b_1. Suppose that f_k is adjacent to b_2. Then the set of attachments of $\{f_1, \ldots, f_k\}$ with respect to the prism formed by a_1-b_1, a_2-b_2, c-d is not local, and yet it has no attachment in c-d, so by 10.4, both a_2 and b_1 are attachments. Since a_2, b_1 are nonadjacent, it follows from the minimality of k and 10.1 that a_2 is adjacent to f_1 and b_1 to f_k, contradicting that $a_2 \notin A'$.

So f_k is not adjacent to b_2. Then b is different from b_2. Since c has no neighbour in the connected set $F' = \{f_1, \ldots, f_k, b\}$, and the set of attachments of F' is not local with respect to the prism formed by a_1-b_1, a_2-b_2, c-d, it follows from 10.4 that F' has an attachment in a_2-b_2. If a_2 is not an attachment then b_2 is, and from the minimality of k it follows that b is the unique neighbour of b_2 in F'; but then a_2-b_2, c-d, a_1-f_1-$\cdots$-f_k-b form a long prism, a contradiction. So a_2 is an attachment of F'. Since a_2-a_1-f_1-$\cdots$-f_k-b-a_2 is not an odd hole, a_2 has a neighbour in $\{f_1, \ldots, f_k\}$. If b_1 also has a neighbour in $\{f_1, \ldots, f_k\}$, then (since a_2, b_1 are nonadjacent) from the minimality of k and 10.1 it follows that a_2 is adjacent to f_1 and b_1 to f_k, and hence $a_2 \in A'$, a contradiction. So b_1 has no neighbour in $\{f_1, \ldots, f_k\}$. Since a_1-f_1-$\cdots$-f_k-b-b_1-a_1 is not an odd hole it follows that b_1 is not adjacent to b, and therefore has no neighbours in F'. Let P be the path between a_2 and b with interior in F'. From 10.4, a_1 has a neighbour in $P \setminus a_2$. But the only neighbour of a_1 in F' is f_1, so f_1 is in $P \setminus a_2$, and therefore f_1 is adjacent to a_2, and there are no other edges between a_2 and F'. Since $a_2 \notin A'$ it follows that a_2 is adjacent to b. But then the set of neighbours of b in the prism formed by a_1-b_1, a_2-b_2, c-d is not local, and yet none are in the path a_1-b_1, contrary to 10.4. This proves 14.2. ∎

The main result of this section is 1.8.6, which we restate, the following:

14.3 *Let $G \in \mathcal{F}_5$. If G contains a double diamond as an induced subgraph, then either one of $G, \overline{G}$ admits a proper 2-join, or G admits a balanced skew partition. In particular, every recalcitrant graph belongs to $\mathcal{F}_6$.*

Proof. We may assume that $G, \overline{G}$ do not admit proper 2-joins, and G does not admit a balanced skew partition. Suppose for a contradiction that G contains a double diamond; then it contains a cube, and so there is a maximal cube (A, B, C, D) in G, forming K. Let F be the set of all minor vertices in $V(G) \setminus V(K)$, and Y the set of all major ones.

(1) *Every anticomponent Y_1 of Y is complete to one of $A \cup B, C \cup D, A \cup D, B \cup C$, and every edge from $A \cup D$ to $B \cup C$ has a Y_1-complete end.*

This is immediate from 14.2 by taking complements.

(2) *There is no anticomponent of Y that is complete to $A \cup D$ or $B \cup C$.*

For suppose such a component exists, say Y_1. From the symmetry we may assume it is complete to $A \cup D$. Define L to be the union of C and all components of F with an attachment in C, and M to be the union of B and all other components of F; and define X to be the set of all Y_1-complete vertices of G not in $L \cup M$. So all major vertices belong to $Y_1 \cup X$, and the four sets $L, M, X \cup A \cup D, Y_1$ are nonempty and partition $V(G)$; and since Y_1 is complete to $X \cup A \cup D$, and there are no edges between L, M by 14.2, it follows that $(L \cup M, X \cup A \cup D \cup Y_1)$ is a skew partition of G. By 4.2 it is not loose. We claim it is balanced. For by 2.6, (L, D) is balanced, since any vertex in B is D-complete and L-anticomplete. Let $u, v \in L$ be adjacent, and suppose they are joined by an odd antipath Q_1 with interior in Y_1. If they both have nonneighbours in D, then since D is anticonnected they are also joined by an antipath Q_2 with interior in D, which is also odd since its union with Q_1 is an antihole, contradicting that (L, D) is balanced. So we may assume that u is D-complete. Hence $u \notin C$, and so u belongs to some component F_1 of F with an attachment in C. Since u is minor, all its neighbours in C are adjacent to all its neighbours in D, and hence it has no neighbours in C; so $v \in F_1$. Since F_1 has an attachment in C and in D (because u has neighbours in D) it follows that F has no attachments in A, and so u, v have no neighbours in A. But then a-u-Q_1-v-a is an odd antihole (where $a \in A$), a contradiction. Next suppose there exist nonadjacent $u, v \in Y_1$, joined by an odd path P with interior in L. By what we just proved about odd antipaths, it follows that P has length ≥ 5. Now $A \cup D$ is anticonnected, and there is no $A \cup D$-complete vertex in L, since every vertex in L is minor or belongs to C. Hence the ends of P are $A \cup D$-complete and its internal vertices are not. But this contradicts 13.6. By 4.5, G admits a balanced skew partition, a contradiction. This proves (2).

(3) *There is no component of F such that its set of attachments in K is a subset of one of $A \cup C, B \cup D$.*

This follows from (2) by taking complements.

(4) *There do not exist both a component F_1 of F with set of attachments contained in $A \cup B$ and an anticomponent Y_1 of Y complete to $A \cup B$; and the same holds with $A \cup B$ replaced by $C \cup D$.*

For the first assertion, assume that such F_1, Y_1 exist. Define $M = C \cup D \cup (F \setminus F_1)$, and X to be the set of all Y_1-complete vertices in $V(G) \setminus (M \cup F_1)$. So $A \cup B \subseteq X$, and the four sets F_1, M, Y_1, X are all nonempty and form a partition of $V(G)$. Since Y_1 is complete to X and there are no edges between F_1 and M, it follows that $(F_1 \cup M, Y_1 \cup X)$ is a skew partition of G. Choose $a \in A$ and $b \in B$, nonadjacent. By 14.2, not both a, b are attachments of F_1, and therefore the skew partition is loose, and so by 4.5 G admits a balanced skew partition, a contradiction. This proves the first assertion and the second is proved similarly. This proves (4).

Now if $Y = \emptyset$, then by (3) it follows that G admits a proper 2-join, a contradiction. So Y is nonempty, and by taking complements, F is nonempty. By (4), passing to the complement if necessary, we may assume that there is no anticomponent of Y that is complete to $A \cup B$. Hence Y is complete to $C \cup D$, by (1) and (2). Since Y is nonempty, it follows from (4) that there is no component F_1 of F with set of attachments contained in $C \cup D$; so by (3), all attachments of F belong to $A \cup B$. Choose an anticomponent Y_1 of Y. By (3) and 14.2, Y_1 is not A-complete or B-complete. Let X be the set of Y_1-complete vertices in $A \cup B \cup C \cup D$. Let L be the union of $A \setminus X$ and all components of F that have an attachment in $A \setminus X$; and let M be the union of $B \setminus X$ and all other components of F. By (1) there are no edges between $A \setminus X$ and $B \setminus X$; and therefore by 14.2, no component of F has attachments in both $A \setminus X$ and $B \setminus X$. Hence there is no edge between L and M. Since $L, M, X \cup (Y \setminus Y_1), Y_1$ is a partition of $V(G)$, and Y_1 is complete to $X \cup (Y \setminus Y_1)$, it follows that $(L \cup M, X \cup (Y \setminus Y_1) \cup Y_1)$ is a skew partition of G. No vertex of D has a neighbour in L, and so it is loose, contrary to 4.2. Hence there is no such graph G. This proves 14.3.

15 Consequences

Disposal of the long prism and double diamond has a number of consequences that we develop in this section. First, since we have shown that every minimum imperfect graph is recalcitrant and therefore belongs to $\mathcal{F}_6$, the next result (together with 1.5) implies that that no minimum imperfect graph G admits a skew partition. This is essentially Chvátal's skew partition conjecture [6]. (Chvátal actually conjectured that no *minimal* imperfect graph admits a skew partition, which is slightly stronger.)

15.1 *If $G \in \mathcal{F}_6$ admits a skew partition, then G admits a balanced skew partition.*

Proof. Let (A, B) be a skew partition in G, which by 4.2 we may assume is not loose. We may assume that there is an odd path P of length ≥ 3 with ends in B and with interior in A. Let P have ends b_1, b_1', and let their neighbours in P be a_1, a_1' respectively. Let A_1 be the component of A including the interior of P, and let B_1 be the anticomponent of B containing b_1, b_1'. Let A_2 be a second component of A, and B_2 a second anticomponent of B. Now the ends of P are B_2-complete, and its internal vertices are not, since the skew partition is not loose; suppose that P has length at least 5. Then by 2.1, B_2 contains a leap x, y for P, and then the subgraph induced on $V(P) \cup \{x, y\}$ is a long prism, a contradiction since $G \in \mathcal{F}_6$. So no such path has length ≥ 5; and similarly no odd antipath with ends in A and interior in B has length ≥ 5. Hence P has vertices b_1-a_1-a_1'-b_1' in order.

Now a_1, a'_1 both have non-neighbours in B_2, and hence are joined by an antipath with interior in B_2; this antipath is odd, since its union with b_1, b'_1 induces an antihole, and since all such antipaths have length 3 it follows that there exist nonadjacent $b_2, b'_2 \in B_2$ such that b_2-a_1-a'_1-b'_2 is a path. Now b_1, b'_1 both have neighbours in A_2, since the skew partition is not loose, and hence are joined by a path with interior in A_2, and it is odd as usual, and hence has length 3; so there exist adjacent $a_2, a'_2 \in A_2$ such that b_1-a_2-a'_2-b'_1 is a path. Since b_2-b_1-a_2-a'_2-b'_1-b_2 is not an odd hole, b_2 is adjacent to one of a_2, a'_2, and similarly so is b'_2. But b_2, b'_2 have no common neighbour in A_2, for if $v \in A_2$ were adjacent to them both then v-b_2-a_1-a'_1-b'_2-v would be an odd hole. So there are exactly two edges between $\{a_2, a'_2\}$ and $\{b_2, b'_2\}$, forming a 2-edge matching. There are two possible pairings; in one case the subgraph induced on these eight vertices is a double diamond, and in the other it is $L(K_{3,3} \setminus e)$. In both cases this contradicts that $G \in \mathcal{F}_6$. This proves 15.1. ∎

Consequently we have the following:

15.2 *Let $G \in \mathcal{F}_6$, and assume that G admits no balanced skew partition. Let $X, Y \subseteq V(G)$ be nonempty, disjoint, and complete to each other.*

- *If $X \cup Y = V(G)$, then either G is complete, or $\overline{G}$ has exactly two components, both with ≤ 2 vertices (and hence $|V(G)| \leq 4$).*
- *If $X \cup Y \neq V(G)$, then $V(G) \setminus (X \cup Y)$ is connected, and if in addition $|X| > 1$, then every vertex in X has a neighbour in $V(G) \setminus (X \cup Y)$.*

Proof. By 15.1, G admits no skew partition. Assume first that $X \cup Y = V(G)$. Then $\overline{G}$ is not connected; let the anticomponents of G be $B_1, \ldots, B_k$ say, where $k \geq 2$. We may assume that G is not complete, and therefore we may assume that some B_i, say B_1, has cardinality > 1. Choose $x, y \in B_1$, nonadjacent. Then $(\{x, y\}, V(G) \setminus \{x, y\})$ is not a skew partition, and so $G \setminus \{x, y\}$ is anticonnected. Hence $k = 2$ and $B_1 = \{x, y\}$. Similarly B_2 has cardinality ≤ 2, and so $|V(G)| \leq 4$ and the theorem holds. Now assume that $G \setminus (X \cup Y)$ is nonnull. Suppose that $V(G) \setminus (X \cup Y)$ is not connected; then $(V(G) \setminus (X \cup Y), X \cup Y)$ is a skew partition, a contradiction. So $V(G) \setminus (X \cup Y)$ is connected. Now suppose some $x \in X$ has no neighbour in $V(G) \setminus (X \cup Y)$. Hence $V(G) \setminus ((X \setminus \{x\}) \cup Y)$ is not connected, and since G admits no skew partition it follows that $X = \{x\}$. This proves 15.2. ∎

Here is another consequence:

15.3 *Let $G \in \mathcal{F}_6$. Let C be a cycle in G of length ≥ 6, with vertices $p_1, \ldots, p_n$ in order, and let $1 < h < i$ and $i+1 < j < n$. Let C be induced except possibly for an edge $p_h p_j$. Let $Y \subseteq V(G) \setminus V(C)$ be anticonnected, such that the only Y-complete vertices in C are p_n, p_1, p_i, p_{i+1}. Suppose there is a path F of $G \setminus Y$ from p_h to p_j (possibly of length 1), such that there are no edges between its interior and $V(C) \setminus \{p_h, p_j\}$. Then some vertex of F is Y-complete.*

Proof. Assume no vertex of F is Y-complete. Since the hole

$$p_1\text{-}\cdots\text{-}p_h\text{-}F\text{-}p_j\text{-}\cdots\text{-}p_n\text{-}p_1$$

is even, and the path p_1-$\cdots$-p_h-$\cdots$-p_i is even (by 2.2), it follows that the path

$$p_i\text{-}p_{i-1}\text{-}\cdots\text{-}p_h\text{-}F\text{-}p_j\text{-}\cdots\text{-}p_n$$

is odd, and therefore has length 3 by 13.6. So F has length 1, and $i = h+1$ and $n = j+1$. Similarly $h = 2$ and $j = i+2$, and so $n = 6$. Then p_2, p_5 are adjacent, so there is an antipath Q joining them with interior in Y. But then in $\overline{G}$, the three paths p_1-p_4,p_5-p_2, p_3-Q-p_6 form a long prism, a contradiction. This proves 15.3. ∎

There is a variant of 3.2, the following.

15.4 *Let $G \in \mathcal{F}_6$, and let p_1-$\cdots$-p_m be a path in G. Let $2 \leq s \leq m-2$, and let p_s-q_1-$\cdots$-q_n-p_{s+1} be an antipath, where $n \geq 2$. Assume that p_1, p_m are both adjacent to all of $q_1, \ldots, q_n$. Then n is even and $m = 4$.*

Proof. If n is even then p_s-q_1-$\cdots$-q_n-p_{s+1} is an odd antipath, and p_1, p_m are complete to its interior; and hence p_1, p_m are both adjacent to one of p_s, p_{s+1}. So $s = 2$ and $m = s+2$, and therefore $m = 4$. Now assume n is odd; then p_s-q_1-$\cdots$-q_n-p_{s+1} is an even antipath of length ≥ 4, contrary to 13.7 applied in $\overline{G}$ to this antipath and the sets $\{p_1, \ldots, p_{s-1}\}$,$\{p_{s+2}, \ldots, p_n\}$. This proves 15.4. ∎

There is a strengthening of 2.3:

15.5 *Let $G \in \mathcal{F}_6$, let C be a hole in G, and let $X \subseteq V(G) \setminus V(C)$ be anticonnected. Let P be a path in C of length > 1 such that its ends are X-complete and its internal vertices are not. Then P has even length.*

Proof. The claim is trivial if C has length 4, so we assume it has length ≥ 6. Let the vertices of C be $p_1, \ldots, p_n$ in order, and let P be p_1-$\cdots$-p_k say, where $3 \leq k < n$. Assume k is even. Then by 13.6 applied to P we deduce that P has length 3, so $k = 4$. By 2.2 every X-complete vertex is adjacent to one of p_2, p_3, so there are none in the interior of the odd path p_4-p_5-$\cdots$-p_n-p_1. By 13.6 this path also has length 3, so $n = 6$. Let Q be the shortest antipath with interior in X, joining either p_2, p_3 or p_5, p_6. From the symmetry we may assume its vertices are p_2-q_1-$\cdots$-q_m-p_3 say. Then Q is odd since it can be completed to an antihole via p_3-p_1-p_4-p_2; and since p_5-p_2-Q-p_3-p_5 is therefore not an antihole, it follows that p_5 (and similarly p_6) has a nonneighbour in the interior of Q. From the choice of Q it follows that p_5, p_6 both have exactly one nonneighbour in the interior of Q; one is nonadjacent to q_1 and the other to q_m. Suppose that $m > 2$. If p_5 is nonadjacent to q_1 then the three antipaths q_1-$\cdots$-q_m, p_5-p_3, p_2-p_6 for a long prism in $\overline{G}$, contrary to $G \in \mathcal{F}_6$; while if p_5 is nonadjacent to q_m then q_1-$\cdots$-q_m, p_6-p_3, p_2-p_5 form a long prism, again a contradiction. So $m = 2$. But then $G|\{p_1$-$\cdots$-$p_6, q_1, q_2\}$ is $L(K_{3,3} \setminus e)$ if p_5 is nonadjacent to q_1, and a double diamond if p_5 is nonadjacent to q_2, again contrary to $G \in \mathcal{F}_6$. This proves 15.5. ∎

There is also a strengthening of 3.3; we no longer need the vertex z.

15.6 *Let $G \in \mathcal{F}_6$, let C be a hole in G of length ≥ 6, with vertices $p_1, \ldots, p_m$ in order, and let Q be an antipath with vertices $p_1, q_1, \ldots, q_n, p_2$, with length ≥ 4 and even. There is at most one vertex in $\{p_3, \ldots, p_m\}$ complete to either $\{q_1, \ldots, q_{n-1}\}$ or $\{q_2, \ldots, q_n\}$, and any such vertex is one of p_3, p_m.*

Proof. Suppose first that one of $q_1, \ldots, q_n$ belongs to the hole. Since it is adjacent to at least one of p_1, p_2 (since Q is an antipath), we may assume that it is p_m; and since it is nonadjacent to p_2, it follows that $p_m = q_n$. So $p_3 \neq q_1$ (since q_1 is adjacent to q_n), and therefore no more of $q_1, \ldots, q_n$ belong to C. Suppose that there exists i with $3 \leq i < m$ such that p_i is complete to either $\{q_1, \ldots, q_{n-1}\}$ or $\{q_2, \ldots, q_n\}$. If $i < m-1$ then p_i is not adjacent to $p_m = q_n$, so p_i is complete to $\{q_1, \dots, q_{n-1}\}$; but then p_i-p_1-q_1-$\cdots$-q_n-p_i is an odd antihole. So $i = m - 1$. By 15.5 applied to the path p_{m-1}-p_m-p_1-p_2 it follows that p_{m-1} is not complete to $\{q_1, \dots, q_{n-1}\}$, and therefore it is complete to $\{q_2, \dots, q_n\}$ and nonadjacent to q_1. But then p_2-p_{m-1}-q_1-$\cdots$-q_n-p_2 is an odd antihole, a contradiction. So there is no such i, and therefore the theorem holds in this case.

So we may assume that none of $q_1, \dots, q_n$ belong to C. Let $X = \{q_1, \dots, q_n\}$, and let Y_1, Y_2 be the sets of vertices in $\{p_3, \dots, p_m\}$ complete to $X \setminus \{q_n\}$, $X \setminus \{q_1\}$ respectively.

(1) $Y_1 \subseteq Y_2 \cup \{p_m\}$, *and* $Y_2 \subseteq Y_1 \cup \{p_3\}$.

This is proved as in the proof of 3.3.

(2) *If* $Y_1 \not\subseteq \{p_m\}$ *then* $p_3 \in Y_1 \cap Y_2$, *and if* $Y_2 \not\subseteq \{p_3\}$ *then* $p_m \in Y_1 \cap Y_2$.

For assume $Y_1 \not\subseteq \{p_m\}$, and choose i with $3 \leq i \leq m - 1$ minimum so that $p_i \in Y_1$. By (1), $p_i \in Y_2$, so we may assume $i > 3$, for otherwise the claim holds. By 15.5 applied to the anticonnected set $X \setminus \{q_n\}$, i is even. The path p_1-$\cdots$-p_i is odd, and between $X \setminus \{q_1\}$-complete vertices, so by 15.5 it contains another in its interior, say p_h. From the minimality of i, $p_h \notin Y_1$, so by (1) $h = 3$, and 15.5 applied to the path p_3-$\cdots$-p_i implies that $i = 4$. Choose j with $4 \leq j \leq m$ maximum such that $p_j \in Y_2$. By (1), p_j is X-complete. By 15.4 applied to p_j-$\cdots$-p_m-p_1-$\cdots$-p_4 we deduce that $j \leq 5$, and so $j \neq m$. By 15.5 applied to the path p_j-$\cdots$-p_m-p_1 and anticonnected set $X \setminus \{q_1\}$, it follows that j is odd, and so $j = 5$. From 15.5 applied to the path p_5-$\cdots$-p_m-p_1-p_2 and anticonnected set $X \setminus \{q_n\}$, we deduce that there exists k with $6 \leq k \leq m$ such that $p_k \in Y_1$. Since it is not in Y_2, it follows from (1) that $k = m$, and so $p_m \in Y_1 \setminus Y_2$. But then p_3-q_1-$\cdots$-q_n-p_m-p_3 is an odd antihole, a contradiction. This proves (2).

Now not both p_3, p_m are in $Y_1 \cap Y_2$, for otherwise Q could be completed to an odd antihole via p_2-p_m-p_3-p_1. Hence we may assume $p_3 \notin Y_1 \cap Y_2$, and so from (2), $Y_1 \subseteq \{p_m\}$. By (1), $Y_2 \subseteq \{p_3\} \cup Y_1$, and so $Y_1 \cup Y_2 \subseteq \{p_3, p_m\}$. We may therefore assume that $Y_1 \cup Y_2 = \{p_3, p_m\}$, for otherwise the theorem holds. In particular, $p_3 \in Y_2$. If also $p_m \in Y_2$, then p_3-p_4-$\cdots$-p_m is an odd path between $X \setminus \{q_1\}$-complete vertices, and none of its internal vertices are $X \setminus \{q_1\}$-complete, contrary to 15.5. So $p_m \notin Y_2$, and so $p_m \in Y_1$; but then p_3-q_1-q_2-$\cdots$-q_n-p_m-p_3 is an odd antihole, a contradiction. This proves 15.6. ∎

This implies a strengthening of 3.1:

15.7 *Let* $G \in \mathcal{F}_6$. *Let* C *be a hole of length* > 4 *and* D *an antihole of length* > 4. *Then* $|V(C) \cap V(D)| \leq 2$.

Proof. Assume that $|V(C) \cap V(D)| \geq 3$; then by taking complements if necessary, we may assume that there are three vertices in $V(C) \cap V(D)$ such that exactly one pair of them is adjacent. Hence we can number the vertices of C as $p_1, \dots, p_m$ in order, and the vertices of D as $p_1, q_1, \dots, q_n, p_2, p_k$ for some k with $4 \leq k \leq m-1$. (Possibly the hole and antihole also share some fourth vertex.) Hence the antipath p_1-q_1-$\cdots$-q_n-p_2 has length ≥ 4 and even. The vertex p_k is complete to $\{q_1, \dots, q_n\}$, and different from p_3, p_m, contrary to 15.6. This proves 15.7. ∎

16 Odd wheels

Now we begin the third of the major parts of the proof, handling Berge graphs that do not contain appearances of K_4, long prisms or double diamonds, but do contain wheels. A *wheel* in a graph G is a pair (C, Y), satisfying:

- C is a hole of length ≥ 6
- Y is a non-null anticonnected set disjoint from C
- there are two disjoint Y-complete edges of C.

We need to study how the remainder of a recalcitrant graph can attach onto a wheel. (Conforti, Cornuéjols, Vušković and Zambelli [11] also made such a study, and their paper contains results related to ours.) We call C the *rim* and Y the *hub* of the wheel. A maximal path in a path or hole H whose vertices are all Y-complete is called a *segment* or *Y-segment* of H. A wheel (C, Y) is *odd* if some segment has odd length. Odd wheels are much easier to handle than general wheels, and in this section we prove that there are no odd wheels in a recalcitrant graph. (Gérard Cornuéjols informs us that he and his co-workers proved the same result, independently, but, like us, assuming the truth of 13.4 — see [11].)

Let us say that distinct vertices u, v of the rim of a wheel (C, Y) have the *same wheel-parity* if there is a path of the rim joining them containing an even number of Y-complete edges (and hence by 2.3, so does the second path, if u, v are nonadjacent); and *opposite wheel-parity* otherwise. In any odd wheel (C, Y), there are vertices u, v in C of opposite wheel-parity that are not Y-complete, and we shall show that, if the odd wheel has been chosen optimally, then Y and its common neighbours separate u, v and thereby give us a balanced skew partition.

16.1 *Let $G \in \mathcal{F}_6$, and let (C, Y) be a wheel in G. Let $v \in V(G) \setminus (V(C) \cup Y)$, such that v is not Y-complete. Suppose that there exist neighbours of v in C with opposite wheel-parity. Then in every path of C between them there is a $Y \cup \{v\}$-complete edge. Moreover, either:*

- *v has only two neighbours in C, and they are adjacent and both Y-complete, or*
- *there is a 3-vertex path p_1-p_2-p_3 in C, such that p_1, p_2, p_3 are all $Y \cup \{v\}$-complete, and every other neighbour of v in C has the same wheel-parity as p_1, or*
- *$(C, Y \cup \{v\})$ is a wheel.*

Proof.

(1) *Let P be a path in C of length ≥ 1, such that its ends are adjacent to v and have opposite wheel-parity. Then either some internal vertex of P is a neighbour of v, or P has length 1.*

For let C have vertices $p_1, \ldots, p_n$ in order, and let P be the path p_1-$\cdots$-p_j say, where $j < n$. We assume no internal vertex of P is a neighbour of v, and that $j \geq 3$. From the hole v-p_1-$\cdots$-p_j-v it follows that j is odd. Since p_1, p_j have opposite wheel-parity with respect to (C, Y), there are an odd number of Y-complete edges in P. Choose $Y' \subseteq Y$ minimal such that Y' is anticonnected and there are an odd number of Y'-complete edges in P. From 2.3 applied to the hole v-p_1-$\cdots$-p_j-v, it contains just one Y'-complete edge and only two Y'-complete vertices. Hence there exists i with $1 \le i < j$ such that p_i, p_{i+1} are the only Y'-complete vertices in P. Since j is odd, it follows that exactly one of $i-1, j-i$ is even; so (by replacing P by its reverse if necessary) we may assume that i is odd. So p_j is different from p_{i+1}, and hence p_j is not Y'-complete. There are two disjoint Y'-complete edges in C, so one of them does not use p_i; and therefore it does not use p_1 either (for p_1 is not Y'-complete unless $i = 1$). Hence both its ends are in $\{p_{j+1}, \dots, p_n\}$. Consequently $n \ge j+2$, and since n is even and j is odd it follows that $n \ge j+3$. Therefore there is a Y'-complete vertex in $\{p_{j+2}, \dots, p_{n-1}\}$.

Suppose that v has a neighbour in $\{p_{j+2}, \dots, p_{n-1}\}$. Then there is a path Q from v to a Y'-complete vertex u say, with $V(Q) \subseteq \{v, p_{j+2}, \dots, p_{n-1}\}$, such that no internal vertex of Q is Y'-complete. The path p_i-$\cdots$-p_1-v-Q-u has both ends Y'-complete, and no internal vertex Y'-complete, and the Y'-complete vertex p_{i+1} has no neighbour in its interior; so this path is even, that is, Q is odd. Hence the path p_{i+1}-$\cdots$-p_j-v-Q-u is odd, and so by 13.6 has length 3; and hence $j = i+2$ and Q has length 1. Also, every Y'-complete vertex is adjacent to one of p_j, v, by 2.2; and so p_i is adjacent to v, and so $i = 1, j = 3$; and v is adjacent to every Y'-complete vertex in C except p_2 and possibly p_4 (for no others are adjacent to p_3). In particular, there are two nonadjacent $Y' \cup \{v\}$-complete vertices in C, and so by 2.3 there are an even number of $Y' \cup \{v\}$-complete edges in C. But all Y'-complete edges of C are $Y' \cup \{v\}$-complete except p_1p_2 and possibly p_4p_5; and since there are also an even number of Y'-complete edges in C, it follows that p_4, p_5 are Y'-complete, and v is adjacent to p_5 and not to p_4. But then the vertices $v, p_1, p_2, p_3, p_4, p_5$ violate 15.3.

This proves that v has no neighbour in $\{p_{j+2}, \dots, p_{n-1}\}$. Choose k with $j \le k \le n$ minimum such that p_k is Y'-complete. Since there is a Y'-complete vertex in $\{p_{j+2}, \dots, p_{n-1}\}$, it follows that $k < n$. From 2.3 it follows that the path p_{i+1}-$\cdots$-p_k is even, and so k is even. Suppose that v is not adjacent to p_{j+1}. Since v-p_j-$\cdots$-p_n-v is not an odd hole, it follows that v is not adjacent to p_n, so p_1, p_j are its only neighbours in C. But p_i-$\cdots$-p_1-v-p_j-$\cdots$-p_k is odd, and therefore has length 3 by 13.6; and by 2.2, every Y'-complete vertex in C is adjacent to v except possibly p_{j-1}, p_{j+1}, a contradiction since there is a Y'-complete vertex in $\{p_{j+2}, \dots, p_{n-1}\}$. So v is adjacent to p_{j+1}. Since v-p_{j+1}-$\cdots$-p_n-p_1-v is not an odd hole, it follows that v is also adjacent to p_n, so it has exactly four neighbours in C. Choose m with $k \le m \le n$ maximum such that p_m is Y'-complete. It follows that $m \ge j+2$. If $m = n$ then a Y'-complete vertex in $\{p_{j+2}, \dots, p_{n-1}\}$ has no neighbours in the interior of the odd path p_{i+1}-$\cdots$-p_j-v-p_n, and the ends of this path are Y'-complete and its internal vertices are not, contrary to 2.2. So $m < n$. Then 2.3 applied to the path p_m-$\cdots$-p_n-p_1-$\cdots$-p_i implies that m is odd, and therefore $m > k$. Suppose that $m > k+1$. Then p_m-$\cdots$-p_n-v-p_{j+1}-$\cdots$-p_k is an odd path, and p_{i+1} has no neighbour in its interior, contrary to 2.2. So $m = k+1$, and there is symmetry between the paths p_1-$\cdots$-p_j and p_{j+1}-$\cdots$-p_n. Both these paths have length ≥ 2; suppose they both have length 2. Then $n = 6$, and the only $Y' \cup \{v\}$-complete vertices in C are p_1, p_4, contrary to 15.5. So one of the paths has length > 2, and from the symmetry we may assume that $j \ge 4$. Hence the hole $H = v$-p_1-$\cdots$-p_j-v has length ≥ 6, and the only Y'-complete vertices in it are p_i, p_{i+1}. By 2.10, Y' contains a hat or a leap. But p_{k+1} has no neighbour in this hole, so the pair $(V(H), Y')$ is balanced by 2.6, and hence there is no leap. So there is a hat; that is, there exists $y \in Y'$ with no neighbours in H except p_i, p_{i+1}. From the minimality of Y' it follows that $Y' = \{y\}$. But then $G|(V(C) \cup \{v, y\})$ is the line graph of a bipartite subdivision of K_4, a contradiction. This proves (1).

From (1) the first assertion of the theorem follows. Now we prove the second assertion. Suppose that v has at least four neighbours in C, two with the same wheel-parity, and two others with the opposite wheel-parity. Then there are two disjoint paths as in (1), and therefore from (1) there are two disjoint $Y \cup \{v\}$-complete edges in C, and so $(C, Y \cup \{v\})$ is a wheel and the theorem holds. So we may assume that C has vertices $p_1, \ldots, p_n$ in order, and v is adjacent to p_1, and v has no other neighbour in C with the same wheel-parity as p_1. Since v has at least one other neighbour, we may assume it has a neighbour in $V(C) \setminus \{p_1, p_n\}$. Choose $i > 1$ minimum such that v is adjacent to p_i; then $i < n$, so by (1), $i = 2$. So p_2 is $Y \cup \{v\}$-complete. If v has a third neighbour in C then similarly p_n is $Y \cup \{v\}$-complete and the theorem holds; and if not then again the theorem holds. This proves 16.1. ∎

16.2 *Let $G \in \mathcal{F}_6$, and let (C, Y) be a wheel in G. Let $F \subseteq V(G) \setminus (V(C) \cup Y)$ be connected, such that no vertex in F is Y-complete. Let $X \subseteq V(C)$ be the set of attachments of F in C. Suppose that there exist vertices in X with opposite wheel-parity, and there are two vertices in X that are nonadjacent. Then either:*

- *there is a vertex $v \in F$ such that $(C, Y \cup \{v\})$ is a wheel, or*
- *there is a vertex $v \in F$ with at least four neighbours in C, and a 3-vertex path p_1-p_2-p_3 in C, such that p_1, p_2, p_3 are all $Y \cup \{v\}$-complete, and every other neighbour of v in C has the same wheel-parity as p_1, or*
- *we can number the vertices of C as $p_1, \ldots, p_n$ in order, such that p_1, p_2, p_3 are all Y-complete, and there is a path p_1-f_1-$\cdots$-f_k-p_3 with interior in F, such that there are no edges between $\{f_1, \ldots, f_k\}$ and $\{p_4, \ldots, p_n\}$.*

Proof. We may assume that F is minimal. If $|F| = 1$ then the result follows from 16.1, so we assume $|F| \geq 2$.

(1) *If there do not exist nonadjacent vertices in X with different wheel-parity, then the theorem holds.*

For there exist vertices in X with different wheel-parity, which are therefore adjacent; say p_1, p_2, where C has vertices $p_1, \ldots, p_n$ in order. So p_1, p_2 are both Y-complete, since they have different wheel-parity. There is a third attachment of F, since there are two that are nonadjacent, say p_i where $3 \leq i \leq n$. Since p_1, p_2 have different wheel-parity, we may assume that p_2, p_i have different wheel-parity; and therefore p_2, p_i are adjacent, that is, $i = 3$, and p_3 is Y-complete. Suppose F has a fourth attachment p_j say, where $4 \leq j \leq n$. From the symmetry we may assume $j \neq n$; and so p_j is nonadjacent to both p_1, p_2, and one of these has different wheel-parity from p_j, a contradiction. So p_1, p_2, p_3 are the only attachments of F, and then the theorem holds. This proves (1).

From (1) we may assume there are nonadjacent vertices in X with opposite wheel-parity, say x_1, x_2, and therefore F is the interior of a path between x_1, x_2, from the minimality of F. Let C have vertices $p_1, \ldots, p_n$ in order; then we may assume that there exists m with $3 \leq m \leq n-1$ such that p_1, p_m have opposite wheel-parity, and there is a path p_1-f_1-$\cdots$-f_k-p_m where $F = \{f_1, \ldots, f_k\}$. Let X_1 be the set of attachments in C of $F \setminus \{f_k\}$, and X_2 the set of attachments of $F \setminus \{f_1\}$. From the minimality of F, for $i = 1, 2$ either all members of X_i have the same wheel-parity, or there are at most two members of X_i, adjacent if there are two. Since $k \geq 2$ it follows that $X_1 \cup X_2 = X$.

(2) *X_1 and X_2 do not both have members of opposite wheel-parity.*

For suppose they do; then X_1, X_2 both consist of exactly two adjacent vertices of opposite wheel-parity, say $X_1 = \{p_1, p_2\}$ and $X_2 = \{p_{m'}, p_{m'+1}\}$. So $p_1, p_2, p_{m'}, p_{m'+1}$ are all Y-complete, and all distinct since two of them are nonadjacent and of opposite wheel-parity. So the only edges between F and $\{p_1, p_2\}$ are incident with f_1, and similarly for f_k. But then G contains a long prism since $n \geq 6$, a contradiction. This proves (2).

(3) *If X_1 has members of opposite wheel-parity then the theorem holds.*

For assume X_1 has members of opposite wheel-parity. Then we may assume its only members are p_1, p_2, and they are both Y-complete. From (2) we may assume that all members of X_2 have the same wheel-parity as p_2. In particular, p_1 has no neighbour in $F \setminus \{f_1\}$. So the only edges between F and C are f_1p_1, edges incident with p_2, and edges incident with f_k. Suppose that p_2 also has no neighbour in $F \setminus \{f_1\}$, and therefore p_2 is adjacent to f_1. If f_k has a unique neighbour x in C, then x can be linked onto the triangle $\{p_1, p_2, f_1\}$; if f_k has two nonadjacent neighbours in C then f_k can be linked onto the same triangle; and if it has exactly two neighbours and they are adjacent, then G contains a long prism, in each case a contradiction. So p_2 has a neighbour in $F \setminus \{f_1\}$. Let R_1 be the path p_1-f_1-$\cdots$-f_k, and let R_2 be the path from p_2 to f_k with interior in $F \setminus \{f_1\}$. Then p_1 has no neighbours in $R_2 \setminus p_2$. Let Q_1 be the path from f_k to p_n with interior in $C \setminus p_1$. Now p_1-R_1-f_k-Q_1-p_n-p_1 is a hole, so R_1 and Q_1 have lengths of opposite parity; and since this hole contains an odd number of Y-complete edges (since all neighbours of f_k have wheel-parity opposite from that of p_1) it follows from 2.3 that it contains exactly one such edge and only two Y-complete vertices. Since p_1 is Y-complete, the other is therefore p_n. The path p_2-R_2-f_k-Q_1-p_n is between Y-complete vertices, and no internal vertex is Y-complete, and the Y-complete vertex p_1 has no neighbour in its interior; so it is even by 2.2, that is, R_1, R_2 have opposite parity. Now there is a Y-complete vertex in $\{p_4, \ldots, p_{n-1}\}$; for there are two disjoint Y-complete edges in C, and an even number of Y-complete edges in C. Let p_s be such a vertex, where $4 \leq s \leq n-1$. We claim that f_k has a neighbour in $\{p_4, \ldots, p_{n-1}\}$. For if not, then since $X \neq \{p_n, p_1, p_2\}$ (because there are nonadjacent vertices in X of opposite wheel-parity), it follows that f_k is adjacent to p_3. Since p_s is not in Q_1, it follows that p_3 is not in Q_1, and so f_k has another neighbour, which must be p_n; but then f_k-p_3-p_4-$\cdots$-p_n-f_k is an odd hole. So f_k has a neighbour in $\{p_4, \ldots, p_{n-1}\}$; and therefore there is a path Q_2 from f_k to some x, such that x is the unique Y-complete vertex in Q_2, and $V(Q_2 \setminus \{f_k\}) \subseteq \{p_4, \ldots, p_{n-1}\}$. Now the path p_2-R_2-f_k-Q_2 has both ends Y-complete, and no internal vertex Y-complete, and the Y-complete vertex p_1 has no neighbour in its interior, so it is even by 2.2. Therefore the path p_1-R_1-f_k-Q_2 is odd, since R_1, R_2 have opposite parity; and again its ends are Y-complete and its internal vertices are not. So it has length 3, by 13.6, and so $k = 2$; and every Y-complete vertex is adjacent to one of f_1, f_2. Consequently there is no Y-complete vertex in C different from p_1 with the same wheel-parity as p_1, a contradiction. This proves (3).

From (3) we may assume that all members of X_1 have the same wheel-parity, and all members of X_2 have the opposite wheel-parity. It follows that $X_1 \cap X_2 = \emptyset$, and so there are no edges between the interior of F and C. So X_1 is the set of neighbours of f_1 in C, and X_2 is the set of neighbours of f_k in C.

(4) *At least one of f_1, f_k has only one neighbour in C.*

For assume they both have at least two. Then there are disjoint paths Q, R of C, both containing neighbours of both f_1, f_k. Choose Q, R minimal, and let Q have ends q_1, q_2; then from the minimality of Q, q_1 is the unique neighbour of one of f_1, f_k in Q, and q_2 is the unique neighbour of the other. Let f_1q_1 and f_kq_2 be edges say. Similarly let R have ends r_1, r_2, where f_1r_1, f_kr_2 are edges. Since q_1, q_2 have opposite wheel-parity, it follows that there are an odd number of Y-complete edges in the hole $f_1\text{-}\cdots\text{-}f_k\text{-}q_2\text{-}Q\text{-}q_1\text{-}f_1$; so by 2.3 there is exactly one, and just two Y-complete vertices. If there are no edges between Q and R this contradicts 15.3, applied to the cycle $f_1\text{-}q_1\text{-}Q\text{-}q_2\text{-}f_k\text{-}r_2\text{-}R\text{-}r_1\text{-}f_1$. Since Q, R are disjoint subpaths of C, all the edges between them join their ends; so we may assume that q_1 is adjacent to one of r_1, r_2. From the hole $f_1\text{-}\cdots\text{-}f_k\text{-}q_2\text{-}Q\text{-}q_1\text{-}f_1$ it follows that Q has parity $k-1$, and similarly so does R. Suppose first that q_1 is adjacent to r_1. Since $q_1\text{-}Q\text{-}q_2\text{-}f_k\text{-}r_2\text{-}R\text{-}r_1\text{-}q_1$ is not an odd hole, it follows that q_2 is adjacent to r_2, and hence G contains a long prism, since C has length ≥ 6, a contradiction. So q_1 is adjacent to r_2. Since q_1 is a neighbour of f_1 and r_2 of f_k, it follows that q_1, r_2 have opposite wheel-parity, and since they are adjacent, they are both Y-complete. Let q' be the neighbour of q_1 in Q, let $Q' = Q \setminus q_1$, let r' be the neighbour of r_2 in R, and let $R' = R \setminus r_2$. Since in the hole $f_1\text{-}\cdots\text{-}f_k\text{-}q_2\text{-}Q\text{-}q_1\text{-}f_1$ there are only two Y-complete vertices and they are adjacent, it follows that the second is q', and similarly r' is Y-complete. If q_2 is adjacent to r_1 then not both q_2, r_1 are Y-complete since C has length ≥ 6; and so there are exactly three Y-complete edges in C, contrary to 2.3. It follows that q_2 is not adjacent to r_1. From the hole $q_1\text{-}Q\text{-}q_2\text{-}f_k\text{-}r_2\text{-}q_1$ it follows that Q has odd length, and therefore so does R and k is even. But then the path $q'\text{-}Q'\text{-}q_2\text{-}f_k\text{-}\cdots\text{-}f_1\text{-}r_1\text{-}R'\text{-}r'$ has odd length, its ends are Y-complete and its internal vertices are not, and so by 13.6 it has length 3; that is, Q,R have length 1 and $k = 2$. Hence the path $r_1\text{-}f_1\text{-}f_2\text{-}q_2$ is odd, its ends are Y-complete, and its internal vertices are not, so every Y-complete vertex is adjacent to one of f_1, f_2. Let $ab, a'b'$ be two Y-complete edges of C, disjoint and such that there are no edges from $\{a, b\}$ to $\{a', b'\}$. Then each of a, b, a', b' is adjacent to one of f_1, f_2, and since all neighbours of f_1 in C have opposite wheel-parity from all neighbours of f_2 in C, we may assume that a, a' are adjacent to f_1 and b, b' to f_2. But this contradicts 15.3, applied to the cycle $a\text{-}f_1\text{-}a'\text{-}b'\text{-}f_2\text{-}b\text{-}a$. This proves (4).

From (4) we may assume that X_1 has only one member, say p_1. Choose i, j with $2 \leq i, j \leq n$, such that p_i, p_j are adjacent to f_k, with i minimum and j maximum. From the hole $p_1\text{-}f_1\text{-}\cdots\text{-}f_k\text{-}p_i\text{-}p_{i-1}\text{-}\cdots\text{-}p_1$ ($= H_1$ say) we deduce that i, k have the same parity, and from the hole $p_1\text{-}f_1\text{-}\cdots\text{-}f_k\text{-}p_j\text{-}p_{j+1}\text{-}\cdots\text{-}p_n\text{-}p_1$ ($= H_2$ say) that j, k have the same parity. (So either $p_i = p_j$ or p_i, p_j are nonadjacent.) Since p_1, p_i have different wheel-parity, and so do p_1, p_j, there is an odd number of Y-complete edges in each of H_1, H_2; and therefore there is exactly one Y-complete edge and exactly two Y-complete vertices in each of the holes, by 2.3. Suppose that $i = j$. Then there are only two Y-complete edges in C, and therefore they are disjoint, and p_1, p_i are not Y-complete (since H_1, H_2 both have only two Y-complete vertices), contrary to 15.3 applied to C. So $j > i$, and hence $j \geq i + 2$. If p_1 is not Y-complete, then the Y-complete edge in H_1 is disjoint from the path $p_1\text{-}f_1\text{-}\cdots\text{-}f_k$, and so is the one in H_2; but this contradicts 15.3 applied to the hole $p_1\text{-}\cdots\text{-}p_i\text{-}f_k\text{-}p_j\text{-}\cdots\text{-}p_n\text{-}p_1$. So p_1 is Y-complete. Since H_1 contains only two Y-complete vertices and they are adjacent, the other is p_2, and similarly p_n is Y-complete.

(5) *f_k has no neighbour in $\{p_3, \ldots, p_{j-2}\}$.*

For assume it does. We claim there is also a Y-complete vertex in this set; for otherwise the only Y-complete vertices in C are p_n, p_1, p_2 and possibly p_{j-1}, which is impossible since there are two disjoint Y-complete edges and an even number of Y-complete edges in C. Hence there is a path P say from f_k to some x such that x is the unique Y-complete vertex in P and $V(P \setminus f_k) \subseteq \{p_3, \ldots, p_{j-2}\}$. The path p_n-p_{n-1}-$\cdots$-p_j-f_k-P-x is even, since its ends are Y-complete, no internal vertex is Y-complete, and the Y-complete vertex p_1 has no neighbour in its interior. The path p_1-f_1-$\cdots$-f_k-P-x is therefore odd (since k, j have opposite parity), and also its ends are Y-complete and no internal vertex is Y-complete; so it has length 3 by 13.6, and hence $k = 2$ and every Y-complete vertex is adjacent to one of f_1, f_2, by 2.2. So there is no Y-complete vertex in $C \setminus p_1$ with the same wheel-parity as p_1, a contradiction. This proves (5).

Since f_k is adjacent to p_i, and $i < j$ and $j - i$ is even, it follows from (5) that $i = 2$, and similarly f_k has no neighbours in $\{p_{i+2}, \ldots, p_{n-1}\}$ and $j = n$. So f_k has no neighbours in

$$\{p_3, \ldots, p_{j-2}\} \cup \{p_{i+2}, \ldots, p_{n-1}\} = \{p_3, \ldots, p_{n-1}\},$$

and therefore p_2, p_n are its only neighbours, contradicting that there are nonadjacent vertices in X of opposite wheel-parity. This proves 16.2. ∎

The main result of this section is 1.8.7, which we restate, the following.

16.3 *Let $G \in \mathcal{F}_6$. If there is an odd wheel in G then G admits a balanced skew partition. In particular, every recalcitrant graph belongs to $\mathcal{F}_7$.*

Proof. Suppose (C, Y) is an odd wheel with Y maximal, and subject to that, such that the number of Y-complete edges in C is minimum. (We refer to these conditions as the "optimality" of (C, Y).)

(1) *There is no vertex $v \in V(G) \setminus (V(C) \cup Y)$ such that v is not Y-complete and has nonadjacent neighbours in C of opposite wheel-parity.*

Suppose there is such a vertex v. Suppose first that there is an odd $Y \cup \{v\}$-segment in C. From the maximality of Y, $(C, Y \cup \{v\})$ is therefore not a wheel, and so there is a unique $Y \cup \{v\}$-complete edge in C. By 2.10, either v has only two neighbours in C, or some vertex of Y has only three, in either case a contradiction. So there is no odd $Y \cup \{v\}$-segment in C. Define a "line" to be a maximal subpath of C with no internal vertex adjacent to v. It follows that every edge of C is in a unique line. Let C have vertices $p_1, \ldots, p_n$ in order, and let S be an odd Y-segment.

Since there are no odd $Y \cup \{v\}$-segments, it follows that an even number of edges of S are $Y \cup \{v\}$-complete. Hence an odd number are not, and therefore there is a line L containing an odd number of edges of S that are not $Y \cup \{v\}$-complete. In particular it contains at least one edge that is Y-complete and not $Y \cup \{v\}$-complete, so L has length > 1. Let the ends of L be p, q. By 16.1, p and q have the same wheel-parity with respect to (C, Y), and so L contains an odd number of edges of some other Y-segment $S' \neq S$. In particular, there are two disjoint Y-complete edges in the hole v-p-L-q-v ($= H$ say); so H has length ≥ 6 (because v is not Y-complete) and so (H, Y) is a wheel. Moreover it is an odd wheel, for it contains an odd number of edges of S, and those edges form either one or two Y-segments in H, and one of these segments is odd. Since there is a $Y \cup \{v\}$-complete edge in C (by 16.1, since v has neighbours in C of opposite wheel-parity) which therefore does not belong to L, this contradicts the optimality of (C, Y). This proves (1).

Since (C, Y) is an odd wheel, C has at least two segments, and therefore there are vertices u, v in C with different wheel-parity and neither of them Y-complete. Let X be the set of all Y-complete vertices in $V(G)$. Then $|X| > 1$, since $|X \cap V(C)| \geq 4$; so by 15.2, we may assume that $V(G) \setminus (X \cup Y)$ is nonempty and connected ($= Z$ say), and every vertex in X has a neighbour in it, for otherwise G admits a balanced skew partition and the theorem holds. In particular $u, v \in Z$, so there is a minimal connected subset F of Z such that there are two vertices of $C \setminus X$ (say p, q) of opposite wheel-parity, both with neighbours in F. Since p, q have opposite wheel-parity and are not Y-complete, they are not adjacent. From the minimality of F, F is a path, and no vertex of F is in C. By 16.2 and (1), there is a 3-vertex path p_1-p_2-p_3 in C, all Y-complete, and a path p_1-f_1-$\cdots$-f_k-p_3 with interior in F, such that there no edges between $\{f_1, \ldots, f_k\}$ and $\{p_4, \ldots, p_n\}$. But then $C \setminus p_2$ can be completed to a hole C' say, via p_1-f_1-$\cdots$-f_k-p_3; and C' has length ≥ 6. For every odd segment S of (C, Y), either it contained both or neither of the edges p_1p_2, p_2p_3; and so in either case an odd number of edges of S belong to C'. Since (C, Y) has an odd segment and there are an even number of Y-complete edges in C, it has at least two odd segments. It follows that there are two disjoint Y-complete edges in C', and so (C', Y) is a wheel. Since an odd number of edges of the odd segment S belong to C', it follows that (C', Y) is an odd wheel, contrary to the optimality of (C, Y). This proves 16.3. ∎

17 Another extension of the Roussel-Rubio lemma

A "pseudowheel" is a variant of an odd wheel, defined in the next section, and we want to show that Berge graphs containing pseudowheels and nothing better admits balanced skew partitions. The main result of this section is a lemma about graphs in $\mathcal{F}_7$, that will be used in when we handle pseudowheels in section 18.

Let $\{a_1, a_2, a_3\}$ be a triangle in G. A *reflection* of this triangle is another triangle $\{b_1, b_2, b_3\}$ of G, disjoint from the first, such that a_1b_1, a_2b_2, a_3b_3 are edges, and these are the only edges between the two triangles. Hence these six vertices induce a prism. A subset F of $V(G)$ is said to *catch* the triangle $\{a_1, a_2, a_3\}$ if it is connected and disjoint from that triangle and a_1, a_2, a_3 all have neighbours in F. We begin with the following extremely useful little fact.

17.1 *Let A be a triangle in a graph $G \in \mathcal{F}_7$, and let $F \subseteq V(G) \setminus A$ catch A. Then either F contains a reflection of A, or some vertex of F has ≥ 2 neighbours in A.*

Proof. Suppose not, and choose F minimal such that it catches A. Let $A = \{a_1, a_2, a_3\}$ say, and for $i = 1, 2, 3$, let B_i be the set of neighbours of a_i in F. Thus the three sets B_1, B_2, B_3 are pairwise disjoint and nonempty.

(1) *There is no path in F meeting all of B_1, B_2, B_3.*

For assume there is, and choose it minimal. So then we may assume there is a path P from $b_1 \in B_1$ to $b_2 \in B_2$, such that some vertex of P is in B_3, and for $i = 1, 2$, b_i is the only vertex of P in B_i. Since B_3 is disjoint from $B_1 \cup B_2$, every vertex of B_3 in P is an internal vertex of P; and so P has length ≥ 2. But then $(C, \{a_3\})$ is an odd wheel, where C is the hole a_1-b_1-P-b_2-a_2-a_1, contrary to $G \in \mathcal{F}_7$. This proves (1).

Choose $b_1 \in F$ such that $F \setminus \{b_1\}$ is connected; then from the minimality of F, $F \setminus \{b_1\}$ does not catch A, and so we may assume that $B_1 = \{b_1\}$. Since F is connected and $|F| \geq 2$, there is a second vertex $b_2 \neq b_1$ in F such that $F \setminus \{b_2\}$ is connected, and so similarly we may assume $B_2 = \{b_2\}$. Let P be a path in F between b_1, b_2. By (1) no vertex of P is in B_3, so F contains a connected subset F' including $V(P)$ which contains exactly one vertex of B_3. From the minimality of F, $|B_3| = 1$; let $B_3 = \{b_3\}$ say. Let Q be a minimal path in F such that $b_3 \in V(Q)$ and some vertex of P has a neighbour in Q. From the minimality of Q it follows that Q is vertex-disjoint from P, and Q has ends b_3, x say, where x is the unique vertex of Q with neighbours in P. From the minimality of F, x either has one neighbour in P, or just two neighbours and they are adjacent; for if it has two nonadjacent neighbours, any vertex of P between them could be deleted from F, contrary to the minimality of F. If x has just one neighbour (y say) in P, then y can be linked onto the triangle A, contrary to 2.4; so it has two adjacent neighbours. Since G does not contain a long prism it follows that Q has length 0 and P has length 1, and so F contains a reflection of A, as required. This proves 17.1. ∎

We did not use the full strength of $G \in \mathcal{F}_7$ in proving 17.1; we just used that there were no odd wheels with hubs of cardinality 1. This suggests that there should be some generalization of 17.1 whose proof does use the full strength of the hypothesis that there are no odd wheels, and that is true, but not easy — it will be a consequence of the main result of this section.

Before we start on that, let us give a strengthening of 2.10 for graphs in $\mathcal{F}_7$.

17.2 *Let $G \in \mathcal{F}_7$, and let $F, Y \subseteq V(G)$ be disjoint, such that F is connected and Y is anticonnected. Let $a_0, b_0 \in V(G) \setminus (F \cup Y)$ and $a, b \in F$ such that a-a_0-b_0-b is a 3-edge path in G. Suppose that:*

- *a_0, b_0 are both Y-complete, and a, b are not Y-complete,*
- *the only neighbours of a_0, b_0 in F are a and b respectively,*
- *$F \setminus \{a\}$ and $F \setminus \{b\}$ are both connected.*

Then either:

1. *there is a vertex in Y with no neighbour in F, or*
2. *there are two nonadjacent vertices $y_1, y_2 \in Y$, such that a is the only neighbour of y_1 in F, and b is the only neighbour of y_2 in F.*

Proof. We may assume that every vertex in Y has a neighbour in F, for otherwise statement 1 of the theorem holds.

(1) *There exist nonadjacent y_1, y_2 in Y, such that y_1 is adjacent to a and not b, and y_2 is adjacent to b and not a.*

For choose a path P between a and b with $V(P) \subseteq F$. Then the hole a_0-a-P-b-b_0-a_0 ($= C$, say) has length ≥ 6. If there are any Y-complete vertices in P, then they belong to the interior of P since a, b

are not Y-complete, and there is an odd number of Y-complete edges in P, by 2.3; but then (C, Y) is an odd wheel (the path a_0-b_0 is an odd Y-segment), a contradiction. So there are no Y-complete vertices in P. By 2.10 applied to C, Y contains either a hat or a leap. Suppose first it contains a hat, that is, there is a vertex $y \in Y$ with no neighbour in P. By the assumption above, y has a neighbour in F. Consequently F catches the triangle $\{a_0, b_0, y\}$. But y is not adjacent to a or b since it has no neighbour in P, and a is the only vertex in F adjacent to a_0, and the same for b, b_0; and a, b are nonadjacent, so F contains no reflection of the triangle. This contradicts 17.1. Hence there is no such y, and so there is a leap. This proves (1).

(2) *There is no path in F between a and b such that y_1 or y_2 has a neighbour in its interior.*

For suppose there is such a path, P' say. Then the set $\{y_1, y_2\}$ contains neither a leap not a hat for the hole a_0-a-P'-b-b_0-a_0 ($= C$ say), and so by 2.10 there is a vertex in P adjacent to both y_1, y_2. By 2.3 there is an even number of $\{y_1, y_2\}$-complete edges in this hole, and since a, b are not $\{y_1, y_2\}$-complete, $(C, \{y_1, y_2\})$ is an odd wheel, a contradiction. This proves (2).

Now if neither of y_1, y_2 has any more neighbours in F then statement 2 of the theorem holds; so we assume at least one of them has another neighbour in F. Since $F \setminus \{a\}$, $F \setminus \{b\}$ are both connected, there is a connected subset F' of $F \setminus \{a, b\}$, such that both a and b have neighbours in F', and at least one of y_1, y_2 has a neighbour in F'. Choose F' minimal with these properties. At least one of y_1, y_2 has a neighbour (say x) in F'. We claim that $F' \setminus \{x\}$ is connected. For if not, let L be a component of it, and M the union of the other components. From the minimality of F, not both a, b have neighbours in $L \cup \{x\}$, and not both have neighbours in $M \cup \{x\}$; so we may assume all neighbours of a in F' are in L, and all neighbours of b are in M. But then there is a path from a to b with interior in F and with x in its interior, contrary to (2). This proves that $F' \setminus \{x\}$ is connected. There is a path from a to b with interior in F', and x is not in it, by (2), and it has length > 1 since a, b are nonadjacent. So a,b both have neighbours in $F' \setminus \{x\}$. From the minimality of F', y_1 and y_2 both have no neighbours in $F' \setminus \{x\}$. We claim that x is adjacent to both y_1 and y_2. For it is adjacent to at least one, say y_1; let Q be a path from x to b with interior in F'. Then y_1-x-Q-b is a path, since y_1 has no more neighbours in F'. Since b_0-y_1-x-Q-b-b_0 is a hole it follows that Q is odd. Therefore a_0-y_1-x-Q-b-y_2-a_0 is not a hole, and so y_2 has neighbours in Q. Since it has no neighbours in $F' \setminus \{x\}$, this proves our claim that x is adjacent to both y_1, y_2.

With Q as before, and therefore odd, it follows that y_2-x-Q-b-y_2 is not a hole, and therefore Q has length 1, that is, x is adjacent to b. Similarly x is adjacent to a; but then x-a-a_0-b_0-b-x is an odd hole, a contradiction. This proves 17.2. ∎

The following is a variant of 17.2, not so symmetrical, but more useful.

17.3 *Let $G \in \mathcal{F}_7$, and let $F, Y \subseteq V(G)$ be disjoint, such that F is connected and Y is anticonnected. Let $a_0, b_0 \in V(G) \setminus (F \cup Y)$ and $a, b \in F$ such that a-a_0-b_0-b is a 3-edge path in G. Suppose that:*

- *a_0, b_0 are both Y-complete, and a, b are not Y-complete,*
- *the only neighbours of a_0, b_0 in F are a and b respectively,*
- *$F \setminus \{a\}$ is connected.*

Then there is a vertex $y \in Y$ with no neighbour in $F \setminus \{a\}$.

Proof. If $F \setminus \{b\}$ is connected, the result follows from 17.2. So assume it is not, and let F_1' be the component of $F \setminus \{b\}$ that contains a, and F_2' the union of all the other components. For $i = 1, 2$ let $F_i = F_i' \cup \{b\}$. Then $F_1 \setminus \{a\}$, $F_1 \setminus \{b\}$ are both connected, so by 17.2 either there exists $y \in Y$ with no neighbour in F_1, or there exist nonadjacent $y_1, y_2 \in Y$ with no neighbours in F_1 except a, b respectively. Suppose the first. If y has a neighbour in F_2 then b can be linked onto the triangle $\{y, a_0, b_0\}$, a contradiction to 2.4; and if not then y satisfies the theorem. Now suppose the second. If y_1 has neighbours in F_2 then $(F \setminus \{a\}) \cup \{y_2\}$ catches the triangle $\{a, a_0, y_1\}$; the only neighbours of a, a_0, y_1 belong to the disjoint sets $F_1', \{y_2\}, F_2'$; and there is no reflection since there are no edges between y_2 and F_1', contrary to 17.1. So y_1 has no neighbours in F_2. This proves 17.3. ∎

The next result is just a technical lemma for use in proving the main result of this section, which is 17.5.

17.4 *Let $G \in \mathcal{F}_7$ and let P be a path in G with length > 1, with vertices $p_1, \dots, p_n$ in order. Let $X, Y \subseteq V(G) \setminus V(P)$ be anticonnected sets, such that $X \cup Y$ is anticonnected, p_1 is X-complete, and p_n is the unique Y-complete vertex in P. (Note that X, Y need not be disjoint.) Let $z \in V(G) \setminus (X \cup Y \cup V(P))$, complete to $X \cup Y$ and with no neighbours in P. Assume that p_n is not X-complete. Let p_n-x_1-$\cdots$-x_k-y be an antipath with interior in X from p_n to some $y \in Y$. Then p_{n-1} is nonadjacent to x_1.*

Proof. Let $F = \{p_{n-1}, x_1, \dots, x_k\} \cup Y$. Since p_{n-1} is not Y-complete it follows that F is anticonnected, and both $F \setminus \{p_{n-1}\}$, $F \setminus \{x_1\}$ are anticonnected. The only nonneighbour of z in F is p_{n-1}, and the only nonneighbour of p_n in F is x_1; and we may assume that p_{n-1} is adjacent to x_1. Now p_{n-1}-z-p_n-x_1 is a path in $\overline{G}$, and F is connected in $\overline{G}$, and $\{p_1, \dots, p_{n-2}\}$ is anticonnected in $\overline{G}$. Also, z and p_n are $\{p_1, \dots, p_{n-2}\}$-complete in $\overline{G}$, and p_{n-1}, x_1 are not. We may therefore apply 17.2 in $\overline{G}$, and deduce that there is a vertex in $\{p_1, \dots, p_{n-2}\}$ which is complete (in G) to $F \setminus \{p_{n-1}\}$. But then this vertex is Y-complete, a contradiction. This proves 17.4. ∎

We gave in 2.9 an extension of the Roussel-Rubio lemma to two anticonnected sets instead of one (we haven't had much use of that theorem yet, but its time is coming.) In that extension the two sets had to be complete to each other. Now we prove a similar result where the two sets are not complete to each other. Incidentally, unlike 2.9, what we are going to prove here is not true for general Berge graphs — we need the hypothesis that $G \in \mathcal{F}_7$.

17.5 *Let $G \in \mathcal{F}_7$ and let P be an odd path in G with length > 1, with vertices $p_1, \dots, p_n$ in order. Let $X, Y \subseteq V(G) \setminus V(P)$ be anticonnected sets, such that $X \cup Y$ is anticonnected, p_1 is X-complete, and p_n is the unique Y-complete vertex in P. (Note that X, Y need not be disjoint.) Let $z \in V(G) \setminus (X \cup Y \cup V(P))$, complete to $X \cup Y$ and with no neighbours in P. Then an odd number of edges of P are X-complete.*

Proof. If possible choose a counterexample P, X, Y such that

1. P is minimal
2. subject to condition 1, $X \cup Y$ is minimal, and
3. subject to conditions 1 and 2, $|X| + |Y|$ is minimum.

We refer to this property as the "optimality" of P, X, Y.

(1) *No vertex of $P \setminus p_1$ is X-complete.*

If p_n is X-complete, then since P has odd length > 1, and the X-complete vertex z has no neighbour in P, it follows from 2.2 and 2.3 that there are an odd number of X-complete edges in P, and the theorem holds, a contradiction. So p_n is not X-complete. By 17.4, p_{n-1} is not X-complete. Since p_1 is X-complete, we can choose i with $1 \leq i \leq n$ maximum such that p_i is X-complete. So $i \leq n-2$. Since z has no neighbour in the path $p_1\text{-}\cdots\text{-}p_i$, if i is even then there is an odd number of X-complete edges in this path and hence in P, by 2.2 and 2.3. So we may assume that i is odd. Hence the theorem is also false for X, Y and the path $p_i\text{-}\cdots\text{-}p_n$. From the optimality of P, X, Y it follows that $i = 1$. This proves (1).

In view of (1), there is symmetry between X and Y.

(2) *Suppose that $x_1, x_2 \in X$ are distinct and such that $X \setminus \{x_i\}$ is anticonnected for $i = 1, 2$. Then $X \cap Y = \emptyset$, and one of x_1, x_2 is the unique vertex of X with a nonneighbour in Y.*

For if $(X \setminus \{x_i\}) \cup Y$ is not anticonnected for some i, then Y is disjoint from $X \setminus \{x_i\}$ (since both these sets are anticonnected), and Y is complete to $X \setminus \{x_i\}$; and therefore $x_i \notin Y$ (since x_i has a nonneighbour in $X \setminus \{x_i\}$), so $X \cap Y = \emptyset$. But then the statement of (2) holds. So we may assume that $(X \setminus \{x_i\}) \cup Y$ is anticonnected for $i = 1, 2$. From the optimality of P, X, Y it follows that the theorem holds for $X \setminus \{x_i\}, Y, P$; and so, since p_n is the unique Y-complete vertex in P, it follows that there are an odd number of $X \setminus \{x_i\}$-complete edges in P, for $i = 1, 2$. For $i = 1, 2$ let W_i be the set of $X \setminus \{x_i\}$-complete vertices in P. So $W_1 \cap W_2 = \{p_1\}$. Let Q be an antipath in X between x_1 and x_2. We claim that Q is odd. For since $W_1 \cap W_2 = \{p_1\}$, there are nonadjacent vertices p_i, p_j of P, such that $p_i \in W_1 \setminus W_2$ and $p_j \in W_2 \setminus W_1$; and since $p_i\text{-}x_1\text{-}Q\text{-}x_2\text{-}p_j\text{-}p_i$ is an antihole it follows that Q is odd. Let us say a *line* is a minimal subpath of $P \setminus p_1$ meeting both W_1 and W_2. So every line has length ≥ 1, and has one end in W_1 and the other in W_2, and has no more vertices in either W_1 or W_2. If some line L has odd length > 1, then the triple $L, X \setminus \{x_1\}, X \setminus \{x_2\}$ is another counterexample to the theorem, contrary to the optimality of P, X, Y; and if some line has length 1, say $p_i\text{-}p_{i+1}$ where $p_i \in W_1$, then $z\text{-}p_i\text{-}x_1\text{-}Q\text{-}x_2\text{-}p_{i+1}\text{-}z$ is an odd antihole, a contradiction. Hence every line is even. Choose i minimum with $2 \leq i \leq n$ such that $\{p_2, \ldots, p_i\}$ includes a line. (This is possible since both W_1, W_2 meet $P \setminus p_1$.) Since all lines have length ≥ 2 it follows that $i \geq 4$. From the minimality of i, $\{p_2, \ldots, p_{i-1}\}$ does not include a line, and so for some $k \in \{1, 2\}$, the path $p_1\text{-}\cdots\text{-}p_i$ has both ends $X \setminus \{x_k\}$-complete and no internal vertex $X \setminus \{x_k\}$-complete. But this path has length ≥ 2, and z has no neighbour in it, so by 2.2 it is even, that is, i is odd. Choose j with $j \geq 2$ maximum such that $\{p_j, \ldots, p_n\}$ includes a line. Since every line has length ≥ 2 it follows that $2 \leq j \leq n-2$. From the maximality of j it follows that for some $k \in \{1, 2\}$, $W_k \cap \{p_j, \ldots, p_n\} = \{p_j\}$. If the path $p_j\text{-}\cdots\text{-}p_n$ has odd length, then $p_j, \ldots, p_n, X \setminus \{x_k\}, Y$ is a counterexample to the theorem, contrary to the optimality of P, X, Y. So $n - j$ is even, and hence j is even. Now i is odd, so if $i \geq j$ then $p_j\text{-}\cdots\text{-}p_i$ is an odd line, a contradiction. Hence $i < j$, and $j - i$ is odd. Now the edges $p_{i-1}p_i, p_jp_{j+1}$ are in lines. Consequently we may choose r, s with $i \leq r < s \leq j$ such that $p_r, p_s \in W_1 \cup W_2$, and the edges $p_{r-1}p_r, p_sp_{s+1}$ are in lines, and $s - r$ is odd; and therefore we may choose such r, s with $s - r$ minimum. If there is a line contained in the path $p_r\text{-}\cdots\text{-}p_s$, say $p_h\text{-}\cdots\text{-}p_k$, then since $k - h$

is even, one of the paths p_r-$\cdots$-p_h and p_k-$\cdots$-p_s is odd, contrary to the minimality of $s-r$. So we may assume that none of $p_r, \ldots, p_s$ belong to W_2, and in particular $p_r, p_s \in W_1$. Since $p_{r-1}p_r, p_sp_{s+1}$ are in lines, and $p_r, p_s \in W_1$, there exist q, t with $2 \leq q < r < s < t \leq n$ such that p_q-$\cdots$-p_r and p_s-$\cdots$-p_t are lines. All lines are even, so $r-q$ and $t-s$ are even, and therefore $t-q$ is odd. Moreover $p_q, p_t \in W_2$, and none of $p_{q+1}, \ldots, p_{t-1}$ belongs to W_2, and the path p_q-$\cdots$-p_t is odd, and z has no neighbour in it, contrary to 2.2. This proves (2).

(3) *There is an antipath x_1-$\cdots$-x_s-y_1-$\cdots$-y_t such that $s, t > 1$ and $X = \{x_1, \ldots, x_s\}$, and $Y = \{y_1, \ldots, y_t\}$.*

For if $|X| = 1$, $X = \{x\}$ say, then z-x-p_1-$\cdots$-p_n is an odd path of length ≥ 5 between Y-complete vertices, and none of its internal vertices are Y-complete, contrary to 13.6. So $|X| \geq 2$, and similarly $|Y| \geq 2$. Hence there are at least two vertices $x \in X$ such that $X \setminus \{x\}$ is anticonnected, and from (2), $X \cap Y = \emptyset$, and there is a unique vertex $x \in X$ with nonneighbours in Y. By (2), there do not exist two vertices $x' \in X \setminus \{x\}$ such that $X \setminus \{x'\}$ is anticonnected; and therefore X is an antipath with one end x'. Because of the symmetry between X, Y, the same applies for Y, and this proves (3).

Choose t' with $1 \leq t' \leq t$, minimum such that p_1 is nonadjacent to $y_{t'}$. (This is possible since p_1 is not Y-complete.) So x_1-$\cdots$-x_s-y_1-$\cdots$-$y_{t'}$-p_1 is an antipath. Define $W = (X \setminus \{x_1\}) \cup \{y_1, \ldots, y_{t'-1}\}$.

(4) *For every subpath P' of P, if the ends of P' are adjacent to x_1, then there are an even number of W-complete edges in P'.*

For suppose not; then we may choose P' such that no internal vertex of P' is adjacent to x_1. Let P' be p_h-$\cdots$-p_k say, where $1 \leq h < k \leq n$. Choose i, j with $h \leq i \leq j \leq k$ such that p_i, p_j are W-complete, with i minimum and j maximum. Since p_k is not X-complete it follows that p_k is not W-complete (because it is adjacent to x_1), and so $j < k$. Since there are an odd number of W-complete edges in p_h-$\cdots$-p_k, it follows that $k \geq h+2$, and x_1-p_h-$\cdots$-p_k-x_1 is a hole (so $k-h$ is even), containing an odd number of W-complete edges. By 2.3 it contains exactly one, and only two W-complete vertices; so $j = i+1$. The path z-x_1-p_h-$\cdots$-p_i has both ends W-complete, and no internal vertex W-complete, and the W-complete vertex p_j has no neighbour in its interior (since $j < k$); so it is even, by 2.2, and hence $i-h$ is even. Since $k-h$ is even, it follows that p_j-$\cdots$-p_k-x_1-z is an odd path; and again its ends are W-complete and its internal vertices are not. By 13.6 it has length 3, so $k = j+1$; and by 2.2, every W-complete vertex is adjacent to one of p_k, x_1. But no W-complete vertex in P is adjacent to x_1 except p_1, since no other vertex of P is X-complete. So every W-complete vertex in $P \setminus p_1$ is adjacent to p_k, and so must be one of p_{k-1}, p_{k+1}. In particular, since $i < k-1$ it follows that $i = 1$, and so $j = 2, k = 3$, and the W-complete vertices in P are p_1, p_2 and possibly p_4.

By 17.4 (with X and Y exchanged), p_2 is nonadjacent to $y_{t'}$. Choose d with $1 \leq d \leq n$ minimum such that $y_{t'}$ is adjacent to p_d; then $d \geq 3$. Then the path p_1-$\cdots$-p_d-$y_{t'}$-z has length ≥ 4, and its ends are $W \cup \{x_1\}$-complete, and its internal vertices are not, so it is even by 13.6. Hence d is odd, and the path p_1-$\cdots$-p_d-$y_{t'}$ is odd. None of its internal vertices are X-complete, and the X-complete vertex z has no neighbour in its interior, and one end p_1 is X-complete, so the other end $y_{t'}$ is not; and hence $t' = 1$, since all other vertices of Y are X-complete. So $W = X \setminus \{x_1\}$. Let $V = X \setminus \{x_s\}$.

Now the path p_1-$\cdots$-p_d-y_1 is between V-complete vertices, and is odd and has length > 1, and the V-complete vertex z has no neighbour in its interior; so by 2.2, there is a V-complete edge in its interior. Choose c with $2 \le c \le d$ minimum such that p_c is V-complete. Since p_2 is nonadjacent to x_1 it follows that $c \ge 3$. Since p_1-$\cdots$-p_c is between V-complete vertices and its internal vertices are not V-complete and z has no neighbour in it, it is even by 2.2, and so c is odd. We already saw that p_1, p_2 and possibly p_4 are W-complete, and $c \ge 3$, so we may choose b with $2 \le b \le c$ maximum such that p_b is W-complete. Hence $b = 2$ or 4. The path p_b-$\cdots$-p_c is odd, and p_b is W-complete, and p_c is V-complete, and no other vertices of the path are either W- or V-complete. If $c - b > 1$ then p_b-$\cdots$-p_c, W, V is a counterexample to the theorem, contradicting the optimality of X, Y, P. So $c = b + 1$. Then z-p_b-x_1-$\cdots$-x_s-p_c-z is an antihole, so s is odd. But then p_2-x_1-$\cdots$-x_s-y_1-p_2 is an odd antihole, a contradiction. This proves (4).

Choose h with $1 \le h \le n$ maximum such that x_1 is adjacent to p_h. Since x_1-p_h-$\cdots$-p_n is between Y-complete vertices (since $s \ge 2$) and none of its internal vertices are Y-complete, and the Y-complete vertex z has no neighbour in its interior, this path either has length 1 or even length by 2.2. So either $h = n$ or h is odd. From the optimality of P, X, Y, it follows that P, W, Y is not a counterexample to the theorem, and so there are an odd number of W-complete edges in P. Since x_1 is adjacent to p_1, from (4) there are an even number of W-complete edges between p_1 and p_h, so there are an odd number in the path p_h-$\cdots$-p_n, and in particular $h < n$, so h is odd. Choose i, j with $h \le i \le j \le n$ such that p_i, p_j are W-complete, with i minimum and j maximum. Hence $j > i$. Since z-x_1-p_h-$\cdots$-p_i is a path of length ≥ 2 between W-complete vertices, and its internal vertices are not W-complete, and the W-complete vertex p_j has no neighbour in its interior, it follows from 2.2 that $i - h$ is even.

(5) $h > 1$.

For assume $h = 1$; so p_1 is the only neighbour of x_1 in P. Let S be the antipath

$$x_1\text{-}\cdots\text{-}x_s\text{-}y_1\text{-}\cdots\text{-}y'_t\text{-}p_1.$$

Now x_1-S-p_1-z is an antipath, of length ≥ 4; all its internal vertices have neighbours in $P \setminus p_1$, and its ends do not. By 13.6 applied in $\overline{G}$, it follows that this antipath has even length and so S has odd length. Its ends have no neighbours in $P \setminus \{p_1, p_2\}$, and z is complete to its interior and also has no neighbours in $P \setminus \{p_1, p_2\}$; so by 2.2 applied in $\overline{G}$, some internal vertex of S has no neighbour in $P \setminus \{p_1, p_2\}$. But they are all adjacent to p_j or to p_n, so $j = 2$. By 17.4, p_2 is nonadjacent to $y_{t'}$, and also to x_1 since it is not X-complete. Therefore p_2-x_1-$\cdots$-x_s-y_1-$\cdots$-$y_{t'}$-p_2 is an antihole D say. Choose d with $1 \le d \le n$ minimum such that $y_{t'}$ is adjacent to p_d; then $d \ge 3$, and so x_1-p_1-$\cdots$-p_d-$y_{t'}$-x_1 is a hole of length ≥ 6, with three vertices in common with D, namely $p_2, x_1, y_{t'}$. From 15.7, D has length 4, and so $t' = 1$ and $s = 2$. Since $W = \{x_2\}$ and $j = 2$, it follows that the only edges between x_1, x_2 and P are x_1p_1, x_2p_1, x_2p_2. But then the three paths p_1-x_1, x_2-z, p_2-$\cdots$-p_d-y_1 form a long prism, a contradiction. This proves (5).

From (5), since p_h is adjacent to x_1, it follows that p_h is not complete to $X \setminus \{x_1\}$, and therefore $h < i < j$. Choose s' with $1 \le s' \le s$ minimum such that p_h is nonadjacent to $x_{s'}$. So p_j-x_1-$\cdots$-$x_{s'}$-p_h-p_j is an antihole, and so s' is even. Hence x_1-$\cdots$-$x_{s'}$-p_h-z is an odd antipath; all its internal vertices have neighbours in $\{p_{h+1}, \ldots, p_n\}$, and its ends do not, so by 13.6 it has length 3, that is, $s' = 2$. The set $F = \{x_2, p_h, \ldots, p_n\}$ is connected; the only neighbour of x_1 in F is p_h; the only neighbour of z in F is x_2. Since x_1, z are $(X \setminus \{x_1, x_2\}) \cup Y$-complete, and p_h, x_2 are not (for p_h is not Y-complete), it follows from 17.2 that there is a vertex in $(X \setminus \{x_1, x_2\}) \cup Y$ with no neighbour in F except possibly x_2. But every vertex in $(X \setminus \{x_1, x_2\}) \cup Y$ is adjacent to either p_j or to p_n, a contradiction. This proves 17.5. ∎

18 Pseudowheels

Let us say a *pseudowheel* in a graph G is a triple (X, Y, P), satisfying:

- X, Y are disjoint nonempty anticonnected subsets of $V(G)$, complete to each other
- P is a path p_1-$\cdots$-p_n of $G \setminus (X \cup Y)$, where $n \geq 5$
- p_1, p_n are the only X-complete vertices of P
- p_1 is Y-complete, and so is at least one other vertex of P; and p_2, p_n are not Y-complete.

A wheel (C, Y) with a Y-segment S of length one can be viewed as a pseudowheel, taking X to consist of one of the vertices of S. We recommend that the reader think of a general pseudowheel as such an odd wheel, where a vertex of S has "blown up" to become the anticonnected set X.

Our current goal is to prove an analogue of 16.3 for pseudowheels. Fortunately we don't need to generalize 16.3 completely, just the case when there is a segment of the wheel of length 1, and one of its vertices has blown up. We did in fact try to generalize 16.3 completely, but were unable to do it and it gave us a lot of trouble; so eventually we found a way to make do with this special case.

We begin with an even more special case, a form of 15.3 when one vertex is replaced by an anticonnected set.

18.1 *Let $G \in \mathcal{F}_7$, and let X, Y be disjoint nonempty anticonnected subsets of $V(G)$, complete to each other. Let p_1-p_2-p_3-p_4-p_5 be a track in $G \setminus (X \cup Y)$, induced except possibly for the edge p_2p_5. Let X be complete to p_1, p_5 and not to p_2, p_3, p_4. If p_1, p_3, p_4 are Y-complete then so is one of p_2, p_5.*

Proof. Assume not. Then in $\overline{G}$, $\{p_1, p_3, p_5\}$ is a triangle, and the connected set $F = X \cup Y \cup \{p_2, p_4\}$ catches it. In $\overline{G}$, the only neighbours of p_5 in F are in $Y \cup \{p_2\}$, the only neighbours of p_3 in F are in X, and the only neighbour of p_1 in F is p_4. Hence no vertex of F has two neighbours in the triangle, so by 17.1, F contains a reflection of the triangle. So (back in G) there are vertices $b_1 \in X$ and $b_2 \in Y \cup \{p_2\}$ such that b_1, b_2, p_4 are pairwise nonadjacent, and b_1 is adjacent to p_1, p_5 and not p_3, and b_2 is adjacent to p_1, p_3 and not p_5. Since p_4 is Y-complete and b_2, p_4 are nonadjacent it follows that $b_2 \notin Y$, and so $b_2 = p_2$, and p_2 is not adjacent to p_5. Then Y and the six vertices $p_1, \ldots, p_5, b_1$ form an odd wheel, a contradiction. This proves 18.1. ∎

There is a reformulation of 13.7 that we sometimes need:

18.2 *Let $G \in \mathcal{F}_7$, and let X, Y be disjoint nonempty anticonnected subsets of $V(G)$, complete to each other. Let P be a path in G with even length > 0, with vertices $p_1, \ldots, p_n$ in order, such that p_1 is X-complete, p_n is not X-complete and p_n is the unique Y-complete vertex of P. Suppose that there is a Y-complete vertex in G nonadjacent to both p_{n-1}, p_{n-2}. Then either:*

- *there is an odd number of X-complete edges in P, or*
- *$n = 3$ and there is an odd antipath joining p_{n-1} and p_n with interior in X.*

Proof. Choose an X-complete vertex p_i in P with i maximum. Suppose first that i is even. Then the path p_1-$\cdots$-p_i is odd, and we may assume that an even number of its edges are X-complete. So it has length > 1; by 2.3, none of its internal vertices are X-complete; and by 13.6 it has length 3 (that is, $i = 4$), and there is an odd antipath Q joining p_2, p_3 with interior in X. Let R be an antipath joining p_2, p_3 with interior in Y. Since $n \geq i = 4$ and n is odd, it follows that $n \geq 5$, and so one of p_2-R-p_3-Q-p_2, p_n-p_2-R-p_3-p_n is an odd antihole, a contradiction.

Thus i is odd. Hence the path p_i-$\cdots$-p_n is even, and by 13.7 it has length 2, that is, $i = n-2$. Let Q be the antipath between p_{n-2}, p_{n-1} with interior in Y, and R the antipath between p_{n-1}, p_n with interior in X. By hypothesis there is a Y-complete vertex nonadjacent to p_{n-1}, p_{n-2}, and therefore Q is even, so R is odd by 13.7. Hence R cannot be completed to an antihole via p_n-p_1-p_{n-1}; and so $n = 3$ and the theorem holds. This proves 18.2. ∎

We need the following extension of 2.3.

18.3 *Let $G \in \mathcal{F}_7$, and let X, Y be disjoint nonempty anticonnected subsets of $V(G)$, complete to each other. Let P be a path p_1-$\cdots$-p_n of $G \setminus (X \cup Y)$, where $n \geq 5$, such that p_1, p_n are the only X-complete vertices of P. Then P has even length. Assume that at least two vertices of P are Y-complete, and let P' be a maximal subpath of P such that none of its internal vertices are Y-complete. Then the length of P' has the same parity as the number of ends of P' that belong to $\{p_1, p_n\}$ and are not Y-complete. Moreover, the number of Y-complete edges of P has the same parity as the number of elements of $\{p_1, p_n\}$ that are Y-complete.*

Proof. Since P is a path of length ≥ 4, and its ends are X-complete and its internal vertices are not, it follows that P has even length, by 13.6. Let us say a *line* is a maximal subpath P' of P such that no internal vertex of P' is Y-complete. Let P' be a line of length ≥ 2, and assume first that both ends of P' are Y-complete. Suppose P' has odd length, and let its ends be p_i, p_j where $i < j$. Then 13.6 implies that $j - i = 3$, and there is an odd antipath Q joining p_{i+1}, p_{i+2} with interior in Y. Since $n \geq 5$, either $n > j$ or $1 < i$, and from the symmetry between p_1 and p_n we may assume the latter. Since p_{i+1}, p_{i+2} are not X-complete, they are joined by an antipath Q' with interior in X. Since $Q \cup Q'$ is an antihole it follows that Q' is odd. But then p_1-p_{i+1}-Q'-p_{i+2}-p_1 is an odd antihole, a contradiction. So in this case P' has even length. We may therefore assume that an end of P' is not Y-complete, and from the maximality of P', any such end is either p_1 or p_n, and we may assume it is p_n from the symmetry. The other end of P' is therefore not p_1 since at least two vertices of P are Y-complete, and so it is p_i, where i is maximum with $2 \leq i \leq n$ such that p_i is Y-complete. Since $i > 1$, no vertex of P' is X-complete except p_n. Suppose that P' is even; then we may apply 13.7. We deduce that P' has length 2, and so $i = n - 2$. Now the antipath joining p_{n-2}, p_{n-1} with interior in X is even since it can be completed to an antihole via p_{n-1}-p_1-p_{n-2}; and the antipath joining p_{n-1}, p_n with interior in Y is even since it can be completed to an antihole via p_n-p_h-p_{n-1}, where p_h is some Y-complete vertex with $1 \leq h < i$. But this contradicts 13.7. Consequently P' is odd, as required.

We have shown therefore that a line has odd length if and only if either it has length 1, or one of its ends is one of p_1, p_n and is not Y-complete. It follows that the number of odd lines equals $y + z$, where y is the number of Y-complete edges in P, and z is the number of ends of P that are not Y-complete. But since every edge of P belongs to a unique line and P has even length, it follows that the number of odd lines is even, and so y, z have the same parity. This proves the final claim of the theorem, and so proves 18.3. ∎

18.4 *Let (X, Y, P) be a pseudowheel in a graph $G \in \mathcal{F}_7$, where P is $p_1\text{-}\cdots\text{-}p_n$. Then P contains an odd number, at least 3, of Y-complete edges, and P has length ≥ 6.*

Proof. By 18.3, P contains an odd number of Y-complete edges, since an odd number of ends of P are Y-complete. Suppose it only contains one, say $p_i p_{i+1}$. Since p_2, p_n are not Y-complete it follows that $3 \leq i \leq n-2$. So there is an antipath joining p_i, p_{i+1} with interior in X, and by 15.4 applied to the path P, this antipath has length 2, that is, there exists $x \in X$ nonadjacent to both p_i, p_{i+1}. Let C be a hole containing x, p_i, p_{i+1} and with $C \setminus x \subseteq P$. Then (C, Y) is an odd wheel, since C contains the Y-complete vertices x, p_i, p_{i+1} and it also contains p_{i-1}, p_{i+2} which are not Y-complete, contrary to $G \in \mathcal{F}_7$. So at least three edges of P are Y-complete, and therefore P has length ≥ 6. This proves 18.4. ∎

A pseudowheel (X, Y, P) in G is *optimal* if

- there is no pseudowheel (X', Y', P') in G such that the number of Y'-complete vertices in P' is less than the number of Y-complete vertices in P, and
- there is no pseudowheel (X, Y', P) in G such that $Y \subset Y'$.

18.5 *Let $G \in \mathcal{F}_7$, and let (X, Y, P) be an optimal pseudowheel in G, where P is $p_1\text{-}\cdots\text{-}p_n$. Let $v \in V(G) \setminus (X \cup Y \cup V(P))$, not Y-complete. Then there is a subpath P' of P such that*

- *$V(P')$ contains all the neighbours of v in P,*
- *there is no Y-complete vertex in the interior of P', and*
- *if v is X-complete, then either $V(P') = \{p_1\}$, or $p_n \in V(P')$.*

Proof. Choose h, k with $1 \leq h \leq k \leq n$ such that v is adjacent to p_h, p_k, with h minimum and k maximum. (If this is impossible then the theorem holds.) Choose i, j with $2 \leq i \leq j \leq n$ such that p_i, p_j are Y-complete, with i minimum and j maximum. By 18.3 it follows that i is odd and j is even, and $j - i \geq 3$ by 18.4, since all Y-complete edges in P lie in the path $p_i\text{-}\cdots\text{-}p_j$.

(1) *If v is both adjacent to p_1 and X-complete then the theorem holds.*

For from the optimality of (X, Y, P) it follows that $(X, Y \cup \{v\}, P)$ is not a pseudowheel, and so p_1 is the only $Y \cup \{v\}$-complete vertex in P. By 2.11 (with X, Y replaced by $Y \cup \{v\}, X$) we deduce that either there exists $y \in Y \cup \{v\}$ nonadjacent to all $p_2, \ldots, p_n$, or there exist nonadjacent $y_1, y_2 \in Y \cup \{v\}$ such that $y_1\text{-}p_2\text{-}\cdots\text{-}p_n\text{-}y_2$ is a path. But p_i is Y-complete and $3 \leq i \leq n-1$, so the second statement does not hold; and the first holds only if $y = v$. This proves (1).

(2) *We may assume that there is a path Q from v to some vertex q, such that q is the only Y-complete vertex in Q, and $V(Q \setminus v) \subseteq \{p_{i+1}, \ldots, p_{j-1}\}$.*

For by 18.3 and the fact that there is a Y-complete edge in P, it follows that there is a Y-complete vertex in $\{p_{i+1}, \dots, p_{j-1}\}$. If v has a neighbour in this set then the claim holds, so suppose it does not. We may assume v has a neighbour in $\{p_1, \dots, p_i\}$, for otherwise the theorem holds. Suppose it also has a neighbour in $\{p_j, \dots, p_n\}$. Then there is a hole C containing v, with $C \setminus v \subseteq P$, such that p_i-$\cdots$-p_j is a path of C. Since all Y-complete edges in P belong to this path, and there are an odd number of them, it follows that there is an odd number (≥ 3) of Y-complete edges in C, contrary to 2.3. So v has no neighbours in $\{p_j, \dots, p_n\}$, and hence $k \leq i$. We may therefore assume that v is X-complete, so $k > 1$ by (1). The path v-p_k-$\cdots$-p_n has length ≥ 4, and its ends are X-complete, and its internal vertices are not, so by 13.6 it has even length, and therefore the path v-p_k-$\cdots$-p_i is even. But v is the only X-complete vertex in v-p_k-$\cdots$-p_i, and p_i is its only Y-complete vertex (since $k > 1$), so by 13.7, this path has length 2, and so $k = i - 1$. There is no odd antipath joining v, p_k with interior in Y, since the Y-complete vertex p_j is nonadjacent to v, p_k; and there is no odd antipath joining p_k, p_i with interior in X, since the X-complete vertex p_n is nonadjacent to p_k, p_i, contrary to 13.7. This proves (2).

(3) *If v is X-complete then the theorem holds.*

For then we may assume that v is nonadjacent to p_1 by (1). If h is odd then p_1-$\cdots$-p_h-v is an odd path with ends X-complete and its internal vertices not, so it has length 3 by 13.6; but the X-complete vertex p_n has no neighbour in its interior (since $n \geq 5$), contrary to 2.2. So h is even. Suppose that one of $p_2, \dots, p_h$ is Y-complete. Then $h \neq 2$ since p_2 is not Y-complete, so $h \geq 4$, and $h < j$ by (2). Hence $(X, Y, p_1$-$\cdots$-p_h-$v)$ is a pseudowheel, not containing p_j, contrary to the optimality of (P, X, Y). So there are no Y-complete vertices in $\{p_2, \dots, p_h\}$, and so $i > h$. Let Q, q be as in (2). Since the $X \cup Y$-complete vertex p_1 has no neighbours in Q, the pairs $(V(Q), X), (V(Q), Y)$ are balanced by 2.6; so by 2.9, Q has odd length. Hence the path p_1-$\cdots$-p_h-v-Q-q has odd length, and its ends are Y-complete, and its internal vertices are not. By 13.6 it has length 3; so $h = 2$ and v is adjacent to q. Also every Y-complete vertex in P is adjacent to one of v, p_2, by 2.2, so they are all adjacent to v except p_1 and possibly p_3. Suppose p_i is adjacent to v, and is therefore $Y \cup \{v\}$-complete. The path p_1-$\cdots$-p_i has even length; the only X-complete vertex in it is p_1; and the only $Y \cup \{v\}$-complete vertex in it is p_i. By 13.7 it has length 2. But p_n is an X-complete vertex nonadjacent to both p_2, p_3, and p_j is a $Y \cup \{v\}$-complete vertex nonadjacent to both p_1, p_2, since $j - i \geq 3$ and therefore p_j is is necessarily adjacent to v as we already saw. Hence both pairs $(\{p_1, p_2\}, Y \cup \{v\})$ and $(\{p_2, p_3\}, X)$ are balanced by 2.6, contrary to 13.7. This proves that p_i is not adjacent to v, and therefore $i = 3$. Choose $h' > i$ minimum such that v is adjacent to $p_{h'}$. From the hole v-p_2-$\cdots$-$p_{h'}$-v it follows that h' is even. From 18.2 applied to the even path p_3-$\cdots$-$p_{h'}$-v, and using the fact that the $X \cup Y$-complete vertex p_1 has no neighbour in this path, we deduce that there is a Y-complete edge in p_3-$\cdots$-$p_{h'}$-v. Since v is adjacent to every Y-complete vertex in P except p_1, p_3, it follows that the only such edge is p_3p_4, and therefore $h' = 4$. But then the track p_1-$\cdots$-p_4-v violates 18.1. This proves (3).

Henceforth we may therefore assume that v is not X-complete. If $k \leq h + 1$ then the theorem holds, so we assume $k \geq h + 2$.

(4) *If v is not adjacent to p_1 then the theorem holds.*

For let P' be the path p_1-$\cdots$-p_h-v-p_k-$\cdots$-p_n. Suppose that any of $p_2, \ldots, p_h, p_k, \ldots, p_n$ is Y-complete. Then P' has length ≥ 4, since $h > 1$ and p_2, p_n are not Y-complete, and so (X, Y, P') is a pseudowheel. By the optimality of (X, Y, P) it follows that there are no Y-complete vertices among $\{p_{h+1}, \ldots, p_{k-1}\}$; but then the claim holds. So we may assume that none of $p_2, \ldots, p_h, p_k, \ldots, p_n$ is Y-complete, and therefore $h < i \leq j < k$, and since $j - i \geq 3$ it follows that $k - h \geq 5$. Let Q, q be as in (2). Then q-Q-v-p_k-$\cdots$-p_n is a path, R say; the only Y-complete vertex in R is q; the only X-complete vertex in R is p_n; and the $X \cup Y$-complete vertex p_1 has no neighbour in its interior. By 2.9, R is odd. Therefore the paths p_1-$\cdots$-p_h-v-Q-q and p_1-$\cdots$-p_h-v-p_k-$\cdots$-p_n have lengths of opposite parity. For the first path, its ends are Y-complete and its internal vertices are not. For the second, its ends are X-complete and its internal vertices are not. So by 13.6, one of them has length 3, and so $h = 2$, and there is an odd antipath joining v, p_2 with interior in one of X, Y. Since v, p_2 are joined by an antipath with interior in X and by another with interior in Y, and all such pairs of antipaths have the same parity (since their union is an antihole), it follows that v, p_2 are joined by an odd antipath with interior in each of X,Y. Hence every X-complete vertex is adjacent to one of v, p_2, and so is every Y-complete vertex. In particular $k = n$, and v is adjacent to every Y-complete vertex in P except p_1 and possibly p_3. But then R has length 2, contradicting that it has odd length. This proves (4).

Henceforth then we assume that v is adjacent to p_1 and not X-complete.

(5) *p_{n-1} is not $Y \cup \{v\}$-complete.*

For suppose it is. Since $n \geq 7$, it follows from 13.6 applied to $P \setminus p_n$ and $Y \cup \{v\}$ that there is a $Y \cup \{v\}$-complete vertex in $\{p_2, \ldots, p_{n-2}\}$; choose such a vertex, $p_{j'}$ say, with j' maximum. Now $j = n - 1$. If $j' < j - 1$ then $j - j'$ is even from 2.2 applied to $p_{j'}$-$\cdots$-p_j, since p_1 is $Y \cup \{v\}$-complete and has no neighbours in the interior of $p_{j'}$-$\cdots$-p_j; but then the odd path $p_{j'}$-$\cdots$-p_n contains no $Y \cup \{v\}$-complete edges, and p_1 is X-complete, $Y \cup \{v\}$-complete and has no neighbours in the path $p_{j'}$-$\cdots$-p_n, contrary to 17.5. So $j' = j - 1$. Let $F = X \cup Y \cup \{v, p_{n-1}\}$. Then F is anticonnected, and each of p_1, p_{n-2}, p_n has a nonneighbour in F; the only nonneighbour of p_1 in F is p_{n-1}; all nonneighbours of p_{n-2} in F belong to X; and all nonneighbours of p_n in F belong to $Y \cup \{v\}$. So in $\overline{G}$, the connected set F catches the triangle $\{p_1, p_{n-2}, p_n\}$, and by 17.1 it contains a reflection of the triangle, which is impossible since p_{n-1} is complete (in G) to $Y \cup \{v\}$. This proves (5).

(6) *v is not adjacent to p_n.*

For suppose it is. By 18.4 there are at least three Y-complete edges in P, and so there is a Y-complete vertex p_a in P with $a \geq 3$, even and different from p_{n-1}. Thus $j - a$ is even, and so by 2.3 there is an even number of Y-complete edges in the even path p_a-$\cdots$-p_j, and hence in the odd path p_a-$\cdots$-p_n. But p_a is Y-complete, and p_n is the unique $X \cup \{v\}$-complete vertex in this path, contrary to 17.5. This proves (6).

(7) *There is no neighbour p_m of v in P with $1 \leq m \leq n$ such that v, p_m are joined by an odd antipath with interior in Y.*

For suppose such a neighbour exists. So $1 < m < n$ by (6), and there is an antipath joining v, p_m with interior in X, which therefore is also odd, since its union with the antipath through Y is an antihole. Since it cannot be completed to an odd antihole via p_m-p_n-v, it follows that $m = n-1$, and in particular m is even. Since j is even, either $p_j = p_m$ or p_j is nonadjacent to p_m; and in either case it follows that p_j is adjacent to v, since every Y-complete vertex is adjacent to one of v, p_m. By (5) $n - j \geq 3$ and odd, and the path p_j-$\cdots$-p_n (with anticonnected sets X and $Y \cup \{v\}$) violates 17.5. This proves (7).

Suppose that $j \geq k$, and let P' be the path p_1-v-p_k-$\cdots$-p_n. Then P' has length ≥ 4, since p_{n-1} is not $Y \cup \{v\}$-complete, and so (X, Y, P') is a pseudowheel; and by the optimality of (X, Y, P) it follows that there are no Y-complete vertices in p_2-$\cdots$-p_{k-1}, contrary to (2). So $j < k$. Let Q, q be as in (2), and assume first that Q is even. Then the path p_1-v-Q-q has odd length; its ends are Y-complete, and its internal vertices are not, so by 13.6 it has length 3, and its internal vertices are joined by an odd antipath with interior in Y, contrary to (7). So Q is odd.

Next assume that k is even. Then the path p_1-v-p_k-$\cdots$-p_n is odd, and its ends are X-complete, and its internal vertices are not, so by 13.6 it has length 3, and $k = n-1$, and its internal vertices v, p_{n-1} are joined by an odd antipath with interior in X. Since p_{n-1} is not $Y \cup \{v\}$-complete, they are also joined by an odd antipath with interior in Y, contrary to (7). This proves that k is odd. Hence the path q-Q-v-p_k-$\cdots$-p_n is even, and by (6) it has length > 2 contrary to 13.7. This proves 18.5. ∎

18.6 *Let $G \in \mathcal{F}_7$, and let (X, Y, P) be an optimal pseudowheel in G, where P is p_1-$\cdots$-p_n. Let $F \subseteq V(G) \setminus (X \cup Y \cup V(P))$ be connected, such that no vertex in F is Y-complete. Then there is a subpath P' of P such that*

- *$V(P')$ contains all the attachments of F in P,*
- *there is no Y-complete vertex in the interior of P', and*
- *if some vertex of F is X-complete then either $V(P') = \{p_1\}$ or $p_n \in V(P')$.*

Proof. Suppose the theorem is false, and choose a minimal counterexample F. From 18.5 $|F| \geq 2$.

(1) *Some vertex in F is X-complete.*

For suppose not. Since F is a counterexample, it has attachments p_a, p_c such that there is a Y-complete vertex p_b with $a < b < c$. From the minimality of F, F is the interior of a path p_a-f_1-$\cdots$-f_k-p_c. Let W_1 be the set of attachments of $F \setminus \{f_k\}$ in P, and W_2 the set of attachments of $F \setminus \{f_1\}$ in P. From the minimality of F, for $i = 1, 2$ there is a subpath p_{a_i}-$\cdots$-p_{b_i} of P ($= P_i$ say), such that no internal vertex of P_i is Y-complete, and $W_i \subseteq V(P_i)$. Choose P_1, P_2 minimal; then p_{a_1} is a neighbour of some member of $F \setminus \{f_k\}$, and therefore of f_1 from the minimality of F, and similarly p_{b_2} is a neighbour of f_k, and p_1-$\cdots$-p_{a_1}-f_1-$\cdots$-f_k-p_{b_2}-$\cdots$-p_n is a path P' say. Suppose that there is a Y-complete vertex in P' different from p_1. Then P' has length ≥ 4, and (X, Y, P') is a pseudowheel, contrary to the optimality of (X, Y, P). So there are no Y-complete vertices in P' different from p_1. But also there are none in $\{p_{a_1+1}, \ldots, p_{b_1-1}\}$ and none in $\{p_{a_2+1}, \ldots, p_{b_2-1}\}$, so all the Y-complete vertices of P belong to $\{p_{b_1}, \dots, p_{a_2}\}$, except for p_1. By 18.4 there are an odd number, at least 3, of Y-complete edges in this path. From the minimality of F, f_1-$\cdots$-f_k-p_{a_2}-p_{a_2-1}-$\cdots$-p_{b_1}-f_1 is a hole, which therefore also contains an odd number ≥ 3 of Y-complete edges. But this contradicts 2.3. This proves (1).

(2) *There do not exist a, b with $1 < a < b \leq n$ such that p_a is an attachment of F and p_b is Y-complete.*

For suppose that such a, b exist. From (1), there is an X-complete vertex in F; and from the minimality of F, there is a path p_a-f_1-$\cdots$-f_k such that $F = \{f_1, \dots, f_k\}$ and f_k is the unique X-complete vertex in F. Let W_1 be the set of attachments of $F \setminus \{f_k\}$ in P, and W_2 the set of attachments of $F \setminus \{f_1\}$ in P. From the minimality of F, for $i = 1, 2$ there is a subpath p_{a_i}-$\cdots$-p_{b_i} of P ($= P_i$ say), such that no internal vertex of P_i is Y-complete, and $W_i \subseteq V(P_i)$, and either $b_2 = n$ or $a_2 = b_2 = 1$.

First assume that $b_2 = n$. Choose P_1, P_2 minimal; then p_{a_1} is a neighbour of f_1, and p_1-$\cdots$-p_{a_1}-f_1-$\cdots$-f_k is a path P' say. Suppose that there is a Y-complete vertex in P' different from p_1. Then P' has length ≥ 4, and (X, Y, P') is a pseudowheel, contrary to the optimality of (X, Y, P). So there are no Y-complete vertices in P'. But also there are none in $\{p_{a_1+1}, \dots, p_{b_1-1}\}$ and none in $\{p_{a_2+1}, \dots, p_{b_2-1}\}$, so all the Y-complete vertices of P belong to $\{p_{b_1}, \dots, p_{a_2}\}$, except for p_1. By 18.4 there are an odd number, at least 3, of Y-complete edges in this path. From the minimality of F, f_1-$\cdots$-f_k-p_{a_2}-p_{a_2-1}-$\cdots$-p_{b_1}-f_1 is a hole, which therefore also contains an odd number ≥ 3 of Y-complete edges. But this contradicts 2.3.

So we may assume that $a_2 = b_2 = 1$, and that $p_1 \in W_2$, and therefore $b_1 > 1$. From the minimality of F there are no edges between $F \setminus \{f_1\}$ and $V(P \setminus p_1)$. Choose P_1 minimal. So p_{b_1} is adjacent to f_1, and either $a_1 = 1$ or p_{a_1} is adjacent to f_1. Suppose first that an odd number of edges of the path p_1-$\cdots$-p_{a_1} are Y-complete. Hence p_1 has no neighbours in $F \setminus \{f_k\}$, and so f_1-$\cdots$-f_k-p_1-$\cdots$-p_{a_1}-f_1 is a hole. It contains an odd number of Y-complete edges, and at least three Y-complete vertices, because p_1 is Y-complete and p_2 is not, a contradiction to 2.3. So there are an even number of Y-complete edges in the path p_1-$\cdots$-p_{a_1}, and therefore an odd number in p_{b_1}-$\cdots$-p_n, since there are an odd number in P, and none in P_1. Therefore there are an odd number in the path f_k-$\cdots$-f_1-p_{b_1}-$\cdots$-p_n ($= R$ say). But an edge of p_{b_1}-$\cdots$-p_n is Y-complete and p_n is not, so $b_2 \leq n-2$; and since $k \geq 2$, it follows that R has length ≥ 4. Also, at least two vertices of R are Y-complete, and its ends are not Y-complete, and its ends are its only X-complete vertices. This contradicts 18.3, So there is no such F. This proves (2).

Choose b with $1 \leq b \leq n$ maximum such that p_b is Y-complete. By (2), none of $p_2, \dots, p_{b-1}$ are attachments of F, and since F is a counterexample, it follows that p_1 is an attachment of F and also there exists c with $b \leq c \leq n$ such that p_c is an attachment of F. Choose c with c minimum, and let Q be a path between p_1, p_c with interior in F. Then p_1-$\cdots$-p_c-Q-p_1 is a hole, and the Y-complete edges in it are precisely the Y-complete edges in P. But there are an odd number of such edges and at least 3, by 18.4, contrary to 2.3. Thus there is no such F. This proves 18.6. ∎

Now we come to the main result of this section, 1.8.8, which we restate, the following.

18.7 *Let $G \in \mathcal{F}_7$. If it contains a pseudowheel then it admits a balanced skew partition. In particular, every recalcitrant graph belongs to $\mathcal{F}_8$.*

Proof. Suppose G contains a pseudowheel; then it contains an optimal pseudowheel, say (X, Y, P), where P is p_1-$\cdots$-p_n. Let Z be the set of all Y-complete vertices in G. So Y, Z are disjoint, nonempty, and complete to each other, and $|Z| \geq 2$. Let $F_0 = V(G) \setminus (Y \cup Z)$. By 15.2, we may assume that F_0 is connected and every vertex in Z has a neighbour in F_0, for otherwise the theorem holds. Choose $i > 1$ such that $p_i p_{i+1}$ is Y-complete, and let A,B be the two components of $V(P \setminus p_i)$. Since p_1, p_{i+1} both have neighbours in F_0, it follows that F_0 contains a minimal connected set F such that there are vertices in A and in B with neighbours in F. From the minimality of F it is disjoint from $V(P)$; and disjoint from $X \cup Y$ since $X \subseteq Z$, contrary to 18.6. This proves 18.7. ∎

19 Wheel systems

Henceforth, therefore, we can exclude pseudowheels, and so our graph G belongs to $\mathcal{F}_8$. Please note that G might still contain wheels; not every wheel can be converted to a pseudowheel. Our next goal is to show that if there is a wheel in a member of $\mathcal{F}_8$ then the graph admits a balanced skew partition, and in particular that there is no wheel in a recalcitrant graph. Assuming there is no balanced skew partition, the strategy is to show that there is no anticonnected set which is maximal such that there is a wheel of which it is a hub. In other words, we want to show that given any wheel, there is a second wheel whose hub is a proper superset of the hub of the first wheel. The proof of this is quite complex, and we begin with an overview before we launch into the details. But before the overview we need some definitions.

Let G be a graph. A *frame* in G is a pair (z, A_0), where $z \in V(G)$, and A_0 is a nonnull connected subset of $V(G) \setminus \{z\}$, containing no neighbours of z. For the moment, fix a frame (z, A_0). With respect to the given frame, a *wheel system* in G of *height* $t \geq 1$ is a sequence $x_0, \ldots, x_t$ of distinct vertices of $G \setminus (A_0 \cup \{z\})$, satisfying the following conditions:

1. A_0 contains neighbours of x_0 and of x_1, and no vertex in A_0 is $\{x_0, x_1\}$-complete.
2. For $2 \leq i \leq t$, there is a connected subset of $V(G)$ including A_0, containing a neighbour of x_i, containing no neighbour of z, and containing no $\{x_0, \ldots, x_{i-1}\}$-complete vertex.
3. For $1 \leq i \leq t$, x_i is not $\{x_0, \ldots, x_{i-1}\}$-complete.
4. z is adjacent to all of $x_0, \ldots, x_t$.

Note that this definition is symmetric between x_0, x_1, so $x_1, x_0, x_2, \ldots, x_t$ is another wheel system.

A wheel system is defined with respect to a given frame, but it is convenient usually to leave the dependence on the frame implicit. Until 23.3 we shall always be working with a fixed frame, and all wheel systems are with respect to that frame.

Let $x_0, \ldots, x_t$ be a wheel system of height t. For $1 \leq i \leq t$ we define $X_i = \{x_0, \ldots, x_i\}$, and we define A_i to be the maximal connected subset of $V(G)$ that includes A_0, contains no neighbour of z, and contains no X_i-complete vertex. So for each i, $A_{i-1} \subseteq A_i$. Note that condition 2 above just says that x_i has a neighbour in A_{i-1}.

Let $x_0, \ldots, x_t$ be a wheel system, and let Y be a nonempty anticonnected subset of $V(G) \setminus (A_0 \cup \{z\})$. We say Y is a *hub* for the wheel system if $z, x_0, \ldots, x_{t-1}$ are all Y-complete and x_t is not.

Now we can begin the overview. Suppose there is a wheel system with hub Y. We would like to infer that there is a wheel with hub Y. This is not in general true, but our main theorem about wheel systems, the following, asserts that this is true under some mild extra hypotheses:

19.1 *Let $G \in \mathcal{F}_8$, let (z, A_0) be a frame, and let $x_0, \ldots, x_{t+1}$ be a wheel system with hub Y, and with $t \geq 1$. Define A_i, X_i as usual, and assume that at most one member of Y has no neighbour in A_1. Suppose that for all r with $1 \leq r \leq t$, if $x_0, x_1, \ldots, x_r, x_{t+1}$ is a wheel system, then every member of Y has a neighbour in $A_r \cup \{x_{t+1}\}$. Then there is a wheel with hub Y.*

The proof of this is lengthy, but here is the idea. Choose r with $1 \leq r \leq t$, minimum such that x_{t+1} has a neighbour in A_r and a nonneighbour in X_r. By hypothesis, every member of Y has a neighbour in $A_r \cup \{x_{t+1}\}$. From the minimality of r, either

- $r = 1$, or
- $r > 1$ and x_{t+1} has a neighbour in A_{r-1}, and x_{t+1} is X_{r-1}-complete, or
- $r > 1$ and x_{t+1} has no neighbour in A_{r-1}.

We handle these three cases separately; they are the results 19.2, 20.1, and 21.2 respectively. In the second case, we call the wheel system $x_0, x_1, \ldots, x_r, x_{t+1}$ a "Y-diamond", and prove the claim by induction on its height; and in this case, it turns out that the hypothesis that every member of Y has a neighbour in $A_r \cup \{x_{t+1}\}$ is redundant (and indeed, so is the hypothesis that z is Y-complete), and there is an advantage to relaxing these hypotheses, to strengthen the inductive hypothesis. The proof of 19.1 is completed in section 21.

Now let us sketch how 19.1 will be applied. The first application is to prove that no recalcitrant graph contains a wheel. For suppose that (C, Y) is a wheel, with Y maximal. Since it is not an odd wheel, there are three consecutive Y-complete vertices x_0, z, x_1 of C. Let $A_0 = V(C) \setminus \{x_0, z, x_1\}$; then (z, A_0) is a frame, and x_0, x_1 is a wheel system with respect to it. Since G admits no balanced skew partition, there is a path T from z to A_0 so that no internal vertex of T belongs to Y or is Y-complete. Let y be the neighbour of z in T. If we choose the rim C carefully, then because G contains no pseudowheels, it can be shown (in the proof of 23.2) that y is adjacent to x_0, x_1. Enlarge x_0, x_1 to a wheel system $x_0, \ldots, x_t$ such that $x_0, \ldots, x_t$ are all $Y \cup \{y\}$-complete, with t maximum. Since we may assume that G admits no balanced skew partition, there is a path P from z to A_0 so that no internal vertex of T is in X_t or is X_t-complete. Let x_{t+1} be the neighbour of z in P; then $x_0, \ldots, x_{t+1}$ is a wheel system, so from the maximality of t, x_{t+1} has a nonneighbour in $Y \cup \{y\}$. Hence $Y \cup \{y\}$ is a hub for the wheel system $x_0, \ldots, x_{t+1}$. From the maximality of Y, there is no wheel with hub $Y \cup \{y\}$, and since every member of Y has a neighbour in A_0, we deduce from 19.1 that there exists r with $1 \leq r \leq t$, such that $x_0, x_1, \ldots, x_r, x_{t+1}$ is a wheel system, and y has no neighbour in $A_r \cup \{x_{t+1}\}$. In particular, y has only three neighbours in C. On the other hand, recall that y was the second vertex of the path T between z and A_0. We deduce that the other neighbour of y in T does not belong to $A_r \cup \{x_{t+1}\}$, and therefore there are vertices of $T \setminus \{y, z\}$ that are X_r-complete. Since G contains no pseudowheels, this turns out to be impossible, as we show in the proof of 22.4.

There is another application of 19.1, to prove that in a recalcitrant graph, if C is a hole of length at least 6 then no vertex has three consecutive neighbours in C. But this application (in the proof of 23.3) is much less convoluted, since at that stage we know there are no wheels, and we do not sketch it here.

The result of this section is the following. (Incidentally, we will not need the hypothesis that there is no pseudowheel in G for several more sections. What we are proving here is true also for graphs in $\mathcal{F}_7$, and we formulate it that way, although we only need it for graphs in $\mathcal{F}_8$.)

19.2 *Let $G \in \mathcal{F}_7$, and let (z, A_0) be a frame. Let x_0, x_1, x_2 be a wheel system with respect to this frame, and define A_1 as usual. Let $Y \subseteq V(G) \setminus \{z, x_0, x_1, x_2\}$ be anticonnected, such that*

- *x_0, x_1 are Y-complete and x_2 is not, and*
- *every vertex in Y that is nonadjacent to x_2 has a neighbour in A_1 and is adjacent to z.*

Then z is Y-complete and there is a wheel (C, Y) in G with $x_0, x_1, z \in V(C) \subseteq \{x_0, x_1, z\} \cup A_1$.

Proof. If possible, choose Y not satisfying the theorem, with $|Y|$ minimum. For fixed Y choose $A \subseteq A_1$ minimal with the properties that

- A is connected
- x_0, x_1, x_2 all have neighbours in A, and
- every vertex in Y that is nonadjacent to x_2 has a neighbour in A.

It follows from the hypotheses that A, Y are both nonempty.

(1) *There exists $y \in Y$ adjacent to z and with a neighbour in A, such that $Y \setminus \{y\}$ is empty or anticonnected.*

For if $|Y| = 1$, let $Y = \{y\}$; then since x_2 is not Y-complete it follows that y is nonadjacent to x_2, and therefore is adjacent to z and has a neighbour in A and the claim holds. So assume $|Y| > 1$, and choose distinct $y_1, y_2 \in Y$ such that $Y \setminus \{y_i\}$ is anticonnected $(i = 1, 2)$. Not both y_1, y_2 is the unique nonneighbour of x_2 in Y; so we may assume that x_2 is not $Y \setminus \{y_2\}$-complete. By the minimality of $|Y|$, z is $Y \setminus \{y_2\}$-complete and there is a $Y \setminus \{y_2\}$-complete vertex in A; and in particular, y_1 is adjacent to z and has a neighbour in A, so we may set $y = y_1$. This proves (1).

Let y be as in (1), and let $Y' = Y \setminus \{y\}$.

(2) *Either x_2 is Y'-complete and nonadjacent to y, or z is Y-complete and there is a path x_0-p_1-$\cdots$-p_n-x_1 from x_0 to x_1 with interior in A, containing at least two Y'-complete edges.*

For if x_2 is Y'-complete the first assertion holds, so we assume not; and in particular Y' is nonempty. From the minimality of $|Y|$, z is Y'-complete and therefore Y-complete, and there is a path as in the claim. This proves (2).

(3) *There is no connected $F \subseteq A$ containing neighbours of all of x_0, x_1, x_2, y except A itself.*

For suppose there is. From the minimality of A, some member of Y has no neighbour in F and is nonadjacent to x_2. In particular, x_2 is not Y'-complete, so Y' is nonempty and by (2), at least two vertices of A are Y'-complete. Since $F \neq A$, there exists $f \in A \setminus F$ such that $A \setminus \{f\}$ is connected. But every vertex in $Y \cup \{x_0, x_1, x_2\}$ has a neighbour in $A \setminus \{f\}$; for all members of Y' have at least two neighbours in A (since A contains two Y'-complete vertices), and x_0, x_1, x_2, y have neighbours in F. This contradicts the minimality of A, and therefore proves (3).

Let x_0-p_1-$\cdots$-p_n-x_1 be a path from x_0 to x_1 with interior in A, and let C be the hole z-x_0-p_1-$\cdots$-p_n-x_1-z.

(4) *If any vertex of $p_1, \ldots, p_n$ is $Y \cup \{x_2\}$-complete then z is Y-complete; and if z is Y-complete then no edge of x_0-p_1-$\cdots$-p_n-x_1 is Y-complete. In particular, neither of p_1, p_n is $Y \cup \{x_2\}$-complete.*

For let p_i be $Y \cup \{x_2\}$-complete, say, and suppose z is not Y-complete. By (2), x_2 is Y'-complete and nonadjacent to y. Let Q be an antipath between z, y with interior in Y', and let R be an antipath between x_2, p_i with interior in $\{x_0, x_1\}$. Then z-Q-y-x_2-R-p_i-z is an antihole, meeting the hole C in at least three vertices, contrary to 15.7. This proves the first assertion. The second is immediate, for otherwise (C, Y) satisfies the theorem. For the third, note that if say p_n is $Y \cup \{x_2\}$-complete, then $p_n x_1$ is a Y-complete edge, a contradiction. This proves (4).

(5) *With $p_1, \ldots, p_n$ and C as in (4), if x_0 is adjacent to x_2, then x_2 is nonadjacent to all of $p_2, \ldots, p_n$.*

For suppose x_2 is adjacent to one of $p_2, \ldots, p_n$, and choose i with $2 \le i \le n$ maximum such that x_2 is adjacent to p_i. Suppose first that $i = n$. Since x_0, x_1, p_n belong to C, there is no antihole of length ≥ 5 containing them by 15.7. By (4), p_n is not Y-complete, and hence there is an antipath between p_n, x_2 with interior in this set, and it can be completed via x_2-x_1-x_0-p_n to an antihole of length ≥ 5 containing x_0, x_1, p_n, a contradiction. So $i < n$.

Since the hole C is even, it follows that n is odd. From the hole z-x_2-p_i-$\cdots$-p_n-x_1-z it follows that i is odd. Since $i > 1$, x_0-x_2-p_i-$\cdots$-p_n-x_1 is an odd path of length ≥ 5. Its ends are $Y \cup \{z\}$-complete, and its internal vertices are not, so by 13.6, $Y \cup \{z\}$ is not anticonnected. Hence z is Y-complete. The ends of the same path are both Y-complete, so by 13.6, some edge of the path is Y-complete. Since x_2 is not Y-complete, this edge belongs to C, contrary to (4). This proves (5).

Let us choose $p_1, \ldots, p_n$ and C such that either x_2 is Y'-complete or (C, Y') is a wheel (this is possible by (2)).

(6) *If x_0 is adjacent to x_2, then not both x_2, y have neighbours in $\{p_1, \ldots, p_n\}$.*

For if they do, then by (5) p_1 is the only neighbour of x_2 in $\{p_1, \ldots, p_n\}$. Suppose first that x_2 is adjacent to y. By (2), z is Y-complete, and (C, Y') is a wheel, and so every vertex in Y' has a neighbour in $\{p_2, \ldots, p_n\}$. By (4) p_1 is not Y-complete. Therefore z, x_0 are the only $Y \cup \{x_2\}$-complete vertices in C, and by 2.10 there is a hat or a leap. Since all vertices in Y' have a neighbour in $\{p_2, \ldots, p_n\}$, and y is adjacent to x_1, it follows that there is no hat, and so y, x_2 form a leap, a contradiction since they are adjacent. So x_2 is nonadjacent to y. Choose j with $1 \le j \le n$ minimum such that y is adjacent to p_j. From the hole z-x_2-p_1-$\cdots$-p_j-y-z we deduce that j is odd, and therefore x_0-p_1-$\cdots$-p_j-y-x_0 is not a hole, that is, $j = 1$, and hence p_1 is adjacent to y. By (4) p_1 is not Y'-complete. If x_2 is Y'-complete, then an antipath between p_1 and y with interior in Y' can be extended to an antihole via y-x_2-x_1-p_1, and this antihole shares the vertices p_1, x_1, x_2 with the hole z-x_2-p_1-$\cdots$-p_n-x_1-z, contrary to 15.7. So x_2 is not Y'-complete. By (2), z is Y-complete, and (C, Y') is a wheel. By 16.1 applied to the wheel (C, Y') and vertex x_2, it follows that p_1 is Y'-complete and therefore $Y \cup \{x_2\}$-complete, contrary to (4). This proves (6).

(7) *Not both x_2, y have neighbours in $\{p_1, \ldots, p_n\}$.*

For by (6) we may assume that x_2 is nonadjacent to x_0, and similarly nonadjacent to x_1. Choose i with $1 \le i \le n$ maximum such that x_2 is adjacent to p_i. From the hole z-x_2-p_i-$\cdots$-p_n-x_1-z it follows that i is odd. Suppose first that x_2 is not Y'-complete. By (2), z is Y-complete and (C, Y') is a wheel. By 16.1, p_i, z have the same wheel-parity, and so there are an odd number of Y'-complete edges in p_i-$\cdots$-p_n-x_1. By (4) no edge of the path x_0-p_1-$\cdots$-p_n-x_1 is Y-complete. Consequently zx_1 is the unique Y-complete edge of the hole z-x_2-p_i-$\cdots$-p_n-x_1-z ($= C_1$ say). Suppose that y is nonadjacent to all $x_2, p_i, \ldots, p_n$. Now y has a neighbour in $\{p_1, \ldots, p_n\}$ by hypothesis, so $\{p_1, \ldots, p_n, x_2\}$ ($= F$ say) catches the triangle $\{z, x_1, y\}$. The only neighbour of z in F is x_2; the only neighbour of x_1 in F is p_n; and y is nonadjacent to both x_2, p_n by assumption. By 17.1, F includes a reflection of the triangle; but then $i = n$ and there is an antihole of length 6 using z, x_1, p_n, contrary to 15.7. This proves that y is adjacent to one of $x_2, p_i, \ldots, p_n$. Since there is an odd number of Y'-complete edges in the path p_i-$\cdots$-p_n-x_1, it follows that every member of Y is adjacent to one of $x_2, p_i, \ldots, p_n$. Consequently Y contains no hat for C_1. Assume that C_1 has length ≥ 6. By 2.10, Y contains a leap, so there are nonadjacent $y_1, y_2 \in Y$ such that y_1-x_2-p_i-$\cdots$-p_n-y_2 is a path, of odd length ≥ 5. But the ends of this path are $\{x_0, x_1\}$-complete and its internal vertices are not, contrary to 13.6. So C_1 has length 4, that is, $i = n$, and p_n is Y'-complete. By (4) it follows that p_n is nonadjacent to y, and therefore y is adjacent to x_2 (since we already showed that y is adjacent to one of $x_2, p_i, \ldots, p_n$). From the symmetry between x_0, x_1 we deduce that the same holds for p_1, that is, p_1 is $Y' \cup \{x_2\}$-complete and nonadjacent to y. Let Q be an antipath between x_2, y with interior in Y'; then the three antipaths p_1-x_1, p_n-x_0 and y-Q-x_2 form a long prism in $\overline{G}$ with triangles $\{p_1, p_n, y\}$ and $\{x_1, x_0, x_2\}$, a contradiction. This proves (7) assuming that x_2 is not Y'-complete.

We therefore assume that x_2 is Y'-complete, and consequently nonadjacent to y. Now $\{x_2, p_1, \ldots, p_n\}$ is connected and catches the triangle $\{z, x_1, y\}$. By 15.7, it contains no reflection of the triangle, since as before that would give an antihole of length 6 with three vertices in C. So by 17.1, there is a vertex in $\{x_2, p_1, \ldots, p_n\}$ with two neighbours in the triangle. The only neighbour of z in it is x_2, which is nonadjacent to both x_1, y. The only neighbour of x_1 in it is p_n, and therefore y is adjacent to p_n. We recall that i is maximum such that x_2 is adjacent to p_i. Since y is adjacent to p_n, we may choose j with $i \le j \le n$ minimum such that y is adjacent to p_j. From the hole z-x_2-p_i-$\cdots$-p_j-y-z we see that j is odd. Suppose $j \ne i$. Then the path x_2-p_i-$\cdots$-p_j-y is even and has length ≥ 4. By 13.7 with anticonnected sets $\{x_0, x_1\}, Y' \cup \{z\}$ we deduce that $Y' \cup \{z\}$ is not anticonnected, and hence z is Y-complete. Consequently, by (4), no edge of x_0-p_1-$\cdots$-p_n-x_1 is Y-complete, and in particular p_n is not Y-complete, and therefore not Y'-complete (since p_n is adjacent to y). Since there is no Y-complete edge in the odd path p_j-$\cdots$-p_n-x_1, and the Y-complete vertex z has no neighbour in its interior, it follows from 2.2 that p_j is not Y-complete and hence not Y'-complete. By 18.2 with sets $\{x_0, x_1\}, Y'$, since the $\{x_0, x_1\} \cup Y'$-complete vertex z has no neighbours in A, it follows that there are an odd number of Y'-complete edges in the path x_2-p_i-$\cdots$-p_j-y. Since y is not Y'-complete, they all belong to the path x_2-p_i-$\cdots$-p_j. Since x_2z, zx_1 are both Y'-complete edges and x_1p_n is not, it follows that p_j, p_n have opposite wheel-parity with respect to the wheel (C_1, Y'), where C_1 is z-x_2-p_i-$\cdots$-p_n-x_1-z. But p_j, p_n are both not Y'-complete, and so (C_1, Y') is an odd wheel, contrary to $G \in \mathcal{F}_7$. This proves that $j = i$, that is, y is adjacent to p_i.

Suppose that $i < n$. If p_i is not Y-complete then an antipath between p_i and y with interior in Y' can be extended via y-x_2-x_1-p_i to an antihole sharing the vertices p_i, x_1, x_2 with the hole z-x_2-p_i-$\cdots$-p_n-x_1-z ($= C_1$ say), contrary to 15.7. So p_i is Y-complete, and therefore so is z, by (4).

But then (C_1, Y) is an odd wheel, since z, x_1, p_i are Y-complete and x_2, p_n are not (by (4)), contrary to $G \in \mathcal{F}_7$. So $i = n$, and hence p_n is adjacent to both x_2, y. From the symmetry between x_0, x_1 it follows that p_1 is adjacent to both x_2, y. By (4), p_1, p_n are not Y-complete. So in $\overline{G}$, the connected set $Y \cup \{p_1, p_n\}$ catches the triangle $\{x_0, x_1, x_2\}$; x_0, x_1, x_2 all have unique neighbours in it, namely p_n, p_1, y respectively; and these three vertices do not form a triangle since yp_1 is not an edge (of $\overline{G}$), contrary to 17.1. This proves (7).

(8) *If x_2 is nonadjacent to y then it is nonadjacent to both x_0, x_1.*

For assume x_2 is nonadjacent to y and adjacent to x_0 say. Now $A \cup \{x_1\}$ catches the triangle $\{z, x_0, x_2\}$; it contains no reflection of this triangle, since x_0, x_1 have no common neighbour in A; and the unique neighbour of z in this set is nonadjacent to both x_0, x_2. So by 17.1 it follows that there is a vertex in A adjacent to both x_0, x_2. Also, $A \cup x_2$ catches the triangle $\{z, x_1, y\}$. Suppose that $A \cup \{x_2\}$ contains a reflection of this triangle; then there exists $f \in A$ adjacent to x_1, x_2 and not to y. Since $f \in A$ it follows that f is nonadjacent to x_0; but then f-x_2-x_0-y-x_1-f is an odd hole, a contradiction. Hence by 17.1 there is a vertex in A adjacent to both x_1, y. Consequently from (3), A is the vertex set of a path f_1-$\cdots$-f_k, where f_1 is adjacent to x_0, x_2, and f_k to x_1, y. Since $f_1 \in A$ it follows that f_1 is not adjacent to x_1.

Now assume that f_1 is not the unique neighbour of x_2 in A. From (3), f_1 is the unique neighbour of x_0 in A. By (7), f_k is not the unique neighbour of x_1 in A, and so from (3) it is the unique neighbour of y in A. In particular y is not adjacent to f_1. Both x_0, z have unique neighbours in $A \cup \{x_1\} = F$ say, namely f_1, x_1 respectively. Now x_0, z are both $\{x_2, y\}$-complete, and f_1, x_1 are not. Since $F \setminus \{x_1\}$ is connected, this contradicts 17.3. So f_1 is the unique neighbour of x_2 in A. Suppose that f_k is the unique neighbour of y in A. Then both z, y have unique neighbours in $A \cup \{x_2\}$, namely x_2, f_k respectively; and z, y are $\{x_0, x_1\}$-complete, and x_2, f_k are not. Once again this contradicts 17.3. So f_k is not the unique neighbour of y in A, and therefore it is the unique neighbour of x_1 in F.

Suppose that f_k is Y-complete. Since $f_k = p_n$, it follows from (4) that z is not Y-complete; and so x_2 is Y'-complete by (2), and an antipath between z, y with interior in Y' can be extended to an antihole via y-x_2-f_k-z, which shares the vertices z, x_2, f_k with the hole z-x_2-f_1-$\cdots$-f_k-x_1-z ($= C_1$ say), contrary to 15.7. So f_k is not Y-complete and therefore not Y'-complete (and in particular, Y' is nonempty).

Suppose that z is not Y-complete; and therefore $Y' \cup \{z\}$ is anticonnected, and x_2 is Y'-complete by (2). Choose h with $1 \leq h < k$ minimum such that f_h is adjacent to y (this exists since f_k is not the unique neighbour of y in A). The path x_2-f_1-$\cdots$-f_h-y is even, since it can be completed to a hole via y-z-x_2, and therefore the path x_2-f_1-$\cdots$-f_h-y-x_1 is odd (this is a path since f_k is the unique neighbour of x_1 in A); and the ends of this path are $Y' \cup \{z\}$-complete, and its internal vertices are not. By 13.6 it has length 3. So f_1 is adjacent to y and x_2. If f_1 is not Y'-complete, then an antipath between f_1, y with interior in Y' can be completed to an antihole via y-x_2-x_1-f_1, which shares the vertices x_1, x_2, f_1 with the hole C_1, contrary to 15.7; while if f_1 is Y-complete, then an antipath between z, y with interior in Y' can be completed to an antihole via y-x_2-x_1-f_1-z, again contrary to 15.7. This proves that z is Y-complete.

In the hole C_1, z, x_1 are Y-complete and x_2, f_k are not; so since $G \in \mathcal{F}_7$, no other vertex of C_1 is Y-complete. By 2.10, Y contains a leap or hat for C_1. From a hypothesis of the theorem, every vertex in Y has a neighbour in $A \cup \{x_2\}$, so there is no hat, and hence there exist nonadjacent y_1, y_2 in Y such that y_1-x_2-f_1-$\cdots$-f_k-y_2 is a path. Since both ends of this path are $\{x_0, x_1\}$-complete, and no internal vertex is $\{x_0, x_1\}$-complete, this contradicts 13.6. This proves (8).

(9) *There is no connected $F \subseteq A$ containing neighbours of all of x_0, x_1, x_2 except A itself.*

For suppose that such a set F exists with $F \neq A$, and choose $f \in A \setminus F$ such that $A \setminus \{f\}$ is connected. From the minimality of A, there exists $y' \in Y$ nonadjacent to x_2 with no neighbour in $A \setminus \{f\}$, and therefore f is the unique neighbour of y' in A. If $y' \in Y'$, then x_2 is not Y'-complete, and therefore by (2) there are two Y'-complete vertices in A, a contradiction. So $y' = y$, and therefore y is not adjacent to x_2. Suppose that x_2 is not adjacent to f. Then both z, y have unique neighbours in $A \cup \{x_2\}$, namely x_2, f; z, y are $\{x_0, x_1\}$-complete, and x_2, f are not; f-y-z-x_2 is a path; and x_0, x_1 both have neighbours in A, contrary to 17.3. So x_2 is adjacent to f. By (8) x_2 is nonadjacent to both x_0, x_1. Since f is not $\{x_0, x_1\}$-complete, we may assume from the symmetry that f is nonadjacent to x_1. Now $A \cup \{x_2\}$ catches the triangle $\{z, y, x_1\}$; the only neighbour of z in $A \cup \{x_2\}$ is x_2; the only neighbour of y in $A \cup \{x_2\}$ is f; x_2, f are both nonadjacent to x_1; and so by 17.1, $A \cup \{x_2\}$ contains a reflection of the triangle. Hence there exists $f_1 \in A \setminus \{f\}$, adjacent to x_1, x_2, f and not to y (and therefore not to x_0). Since every path between x_0, x_1 with interior in A has length ≥ 4 it follows that x_0 is nonadjacent to f, f_1, and this restores the symmetry between x_0, x_1; and consequently by the same argument there exists $f_0 \in A \setminus \{f\}$ adjacent to x_2, f, x_0 and not to y, x_1. Since z-x_0-f_0-f_1-x_1-z is not an odd hole, f_0 is nonadjacent to f_1; but then x_0-f_0-f-f_1-x_1 violates (7). This proves (9).

From (7) and (9), it follows that there exists $f \in A$ such that $A \setminus \{f\}$ is connected, f does not belong to C, and f is the unique neighbour of x_2 in A.

(10) *x_2 is nonadjacent to both of x_0, x_1.*

For suppose that x_2 is adjacent to x_0 say. Suppose first that x_0 is not adjacent to f. Then $A \cup \{x_1\}$ catches the triangle $\{z, x_2, x_0\}$; the only neighbour of z in $A \cup \{x_1\}$ is x_1; the only neighbour of x_2 in $A \cup \{x_1\}$ is f; x_1, f are both nonadjacent to x_0; and $A \cup \{x_1\}$ contains no reflection of the triangle since that would give a 6-antihole with 3 vertices in common with C, contradicting 17.1. So x_0 is adjacent to f, and therefore x_1 is nonadjacent to both x_2, f. By (8) x_2 is adjacent to y, and therefore not Y'-complete. By (2) z is Y'-complete and (C, Y') is a wheel. Let x_2-q_1-$\cdots$-q_k-x_1 be a path between x_1, x_2 with interior in A (so $f = q_1$) and let C_1 be the hole z-x_2-q_1-$\cdots$-q_k-x_1-z. From (9), $A = \{q_1, \ldots, q_k\}$. Since $q_k = p_n$ and z is Y-complete, it follows from (4) that q_k is not Y-complete. Since (C_1, Y) is not an odd wheel, it follows that (C_1, Y) is not a wheel, and so z, x_1 are the only Y-complete vertices in C_1, by 2.3. By 2.10, Y contains a leap or hat for C_1. But y is adjacent to x_2, and all other vertices of Y have at least two neighbours in $\{p_1, \ldots, p_n\}$, which is a subset of $\{q_1, \ldots, q_k\}$, a contradiction. This proves (10).

From (9) one of x_0, x_1 has a unique neighbour in A, and from the symmetry we may assume it is x_1. Let its neighbour be f_1. By (7) and (9), x_2 has no neighbour in $\{p_1, \ldots, p_n\}$, and in particular $f \neq f_1$. Let Q be a path in A between f, f_1, say $f = q_1$-$\cdots$-$q_k = f_1$, so z-x_2-q_1-$\cdots$-q_k-x_1-z is a hole (C_1 say).

(11) *z is not Y'-complete, and x_2 is Y'-complete and nonadjacent to y.*

For assume z is Y'-complete. So z, x_1 both have unique neighbours in $A \cup \{x_2\}$, namely x_2, f_1. By (4), f_1 is not Y-complete. So z, x_1 are Y-complete, and x_2, f_1 are not. By 17.3, it follows that some vertex in Y has no neighbour in A. But y has a neighbour in A by (1), and so some vertex in Y' has no neighbour in A. In particular, there is no Y'-complete vertex in A, and so by (2), x_2 is Y'-complete and nonadjacent to y. From 17.3 applied to the path x_2-z-x_1-f_1 and the anticonnected set $\{y\}$, it follows that y is adjacent to f_1. Since (C_1, Y) is not an odd wheel, it follows from 2.10 that Y contains a leap or a hat for C_1. Since all members of Y' are adjacent to x_2 and y is adjacent to f_1, there is no hat, and the leap must use y; so we may assume y, y' is a leap for some $y' \in Y'$. Hence y-f_1-Q-f-x_2-y' is a path. Since this path has odd length ≥ 5, and its ends are $\{x_0, x_1\}$-complete and its internal vertices are not, this contradicts 13.6. So z is not Y'-complete. The claim follows from (2). This proves (11).

(12) *y is nonadjacent to all of $q_1, \ldots, q_{k-1}$.*

For suppose not, and choose i with $1 \leq i < k$ minimum such that y is adjacent to q_i. From the hole z-x_2-q_1-$\cdots$-q_i-y-z it follows that i is odd. So by (10) x_2-q_1-$\cdots$-q_i-y-x_1 is an odd path. Its ends are $Y' \cup \{z\}$-complete, its internal vertices are not, and $Y' \cup \{z\}$ is anticonnected by (11); so it has length 3 by 13.6, that is, $i = 1$ and y is adjacent to f. If f is not Y'-complete, an antipath between f, y with interior in Y' can be completed to an antihole via y-x_2-x_1-f, sharing the vertices x_1, x_2, f with C_1, contrary to 15.7. So f is Y'-complete. Since z is not, an antipath between z, y with interior in Y' can be completed to an antihole via y-x_2-x_1-f-z, again contrary to 15.7. This proves (12).

To conclude, $A \cup \{x_2\}$ catches $\{y, z, x_1\}$, and so by 17.1, y is adjacent to $f_1 = q_k$. Suppose that x_0 is adjacent to one of $q_1, \ldots, q_k$. Then $\{p_1, \ldots, p_n\} \subseteq \{q_1, \ldots, q_k\}$ from the minimality of A, and so the neighbours of y in C are precisely $x_0, z, x_1, q_k = p_n$, contrary to 2.3 applied to C and y. So x_0 is nonadjacent to all of $q_1, \ldots, q_k$; but then x_2-q_1-$\cdots$-q_k-y-x_0 is an odd path of length ≥ 5, its ends are $Y' \cup \{z\}$-complete, and its internal vertices are not, contrary to 13.6. Thus there is no such choice of Y. This proves 19.2. ∎

20 Diamond and square wheel systems

Now we turn to the second of the three steps of the proof of 19.1. We need two special kinds of wheel systems. Let $x_0, \ldots, x_t$ be a wheel system, and define X_i, A_i as usual. Let $Y \subseteq V(G)$ be nonempty and anticonnected, such that Y is disjoint from $\{z, x_0, \ldots, x_t\}$, and $x_0, \ldots, x_{t-1}$ are all Y-complete and x_t is not. We say $x_0, \ldots, x_t$ is a

- *Y-diamond* if $t \geq 3$, x_t is X_{t-2}-complete, and x_t has a neighbour in A_{t-2}
- *Y-square* if $t \geq 3$, x_t is adjacent to x_{t-1}, x_t has no neighbour in A_{t-2}, and there is a vertex in A_{t-1} adjacent to x_t with a neighbour in A_{t-2}

A Y-diamond $x_0, \dots, x_t$ is said to be *polished* if $t \ge 4$, x_{t-1} is not X_{t-3}-complete, x_t has no neighbour in A_{t-3}, x_{t-1} has a neighbour in A_{t-3}, and there is a vertex in A_{t-2} adjacent to both x_t, x_{t-1} with a neighbour in A_{t-3}.

We need four lemmas to prove the main result of this section, which is the following.

20.1 *Let $G \in \mathcal{F}_7$ and let (z, A_0) be a frame. For all $Y \subseteq V(G) \setminus (A_0 \cup \{z\})$, if Y is nonempty and anticonnected, and there is either a Y-diamond or a Y-square in G, then z is Y-complete and G contains a wheel with hub Y.*

Proof of 20.1, assuming 20.2, 20.3, 20.4, and 20.5.

We shall prove by induction on t that for any nonempty anticonnected $Y \subseteq V(G) \setminus (A_0 \cup \{z\})$, if there is a Y-diamond or Y-square in G of height t, then z is Y-complete and G contains a wheel with hub Y. Certainly $t \ge 3$, and if $t = 3$ then the result holds by 20.2, so we may assume that $t \ge 4$. By 20.3 and 20.4, we may assume that there is an anticonnected set Y' with $Y' \subseteq V(G) \setminus (A_0 \cup \{z\})$ such that either $Y \subseteq Y'$ or z is not Y'-complete, and such that either:

- there is a Y'-diamond in G of height $t-1$, or
- there is a Y'-square in G of height $t-1$, or
- there is a polished Y'-diamond in G of height t.

In the first two cases, it follows from the inductive hypothesis that z is Y'-complete, and there is a wheel with hub Y'. Since z is Y'-complete, it follows that $Y \subseteq Y'$, and so z is Y-complete and there is a wheel with hub Y, as required. Thus we may assume that the third case holds. By 20.2 it follows that $t \ge 5$; and by 20.5, there is an anticonnected set Y'' with $Y'' \subseteq V(G) \setminus (A_0 \cup \{z\})$ such that either $Y' \subseteq Y''$ or z is not Y''-complete, and either

- there is a Y''-diamond in G of height $t-2$, or
- there is a Y''-square in G of height $t-2$, or
- there is a polished Y''-diamond in G of height $t-1$.

In each case it follows from the inductive hypothesis that z is Y''-complete and there is a wheel with hub Y''. Consequently $Y' \subseteq Y''$, and so z is Y'-complete; and therefore $Y \subseteq Y'$, and so z is Y-complete, and there is a wheel with hub Y. This proves 20.1. ∎

Now we turn to the proofs of the lemmas. First we show:

20.2 *Let $G \in \mathcal{F}_7$, let (z, A_0) be a frame, and let $Y \subseteq V(G) \setminus (A_0 \cup \{z\})$ be nonempty and anticonnected. There is no Y-square of height 3 or polished Y-diamond of height 4 in G; and if $x_0, \dots, x_3$ is a Y-diamond of height 3, then z is Y-complete and G contains a wheel $(C, Y \cup \{x_3\})$.*

Proof. Let $x_0, \dots, x_t$ be a wheel system in G, and let X_i, A_i be defined as before. Suppose first that $x_0, \dots, x_t$ is a Y-square of height 3. So $t = 3$, x_3 is adjacent to x_2, x_3 has no neighbour in A_1, and there is a vertex q in A_2 adjacent to x_3 with a neighbour in A_1. From the maximality of A_1 it follows that q is X_1-complete, and therefore nonadjacent to x_2 (since it belongs to A_2 and so is not X_2-complete). Let Q be a path from q to x_2 with interior in A_1; so Q has length ≥ 2. But

Q is even since it can be completed to a hole via x_2-x_3-q, and so q-Q-x_2-z is an odd path; its ends are X_1-complete, and its internal vertices are not. By 13.6 it has length 3, and there is an antipath with interior in X_1, joining its middle vertices (x_2 and r say). This antipath can be completed via r-z-q-x_2 to an antihole of length ≥ 6, containing x_0, x_1 and z. But let P be a path from x_0 to x_1 with interior in A_0; then it has length ≥ 3 since A_0 contains no vertex adjacent to both x_0, x_1, and hence z-x_0-P-x_1-z is a hole of length ≥ 6 containing x_0, x_1 and z. But this contradicts 15.7, as required.

Now suppose $x_0, \dots, x_t$ is a polished Y-diamond of height 4. So $t = 4$, x_4 is X_2-complete, x_3 is not X_1-complete, x_4 has no neighbour in A_1, x_3 has a neighbour in A_1, and there is a vertex q in A_2 adjacent to both x_4, x_3 with a neighbour in A_1. As before q is X_1-complete, and therefore not adjacent to x_2; let Q be a path from q to x_2 with interior in A_1. The proof is completed exactly as in the previous paragraph.

So now we may assume that $x_0, \dots, x_t$ is a Y-diamond of height 3. So $t = 3$, x_3 is X_1-complete (and therefore nonadjacent to x_2), and x_3 has a neighbour in A_1. But then from 19.2 with $A = A_1$, $v = x_2$ and anticonnected set $Y \cup \{x_3\}$, the result follows. This proves 20.2. ∎

We remark that the pieces of this jigsaw do not seem to fit well together. There is some annoying wastage in 20.2; we produce a wheel with hub $Y \cup \{x_3\}$, and all we use in proving 20.1 is that there is a wheel with hub Y. Perhaps there is a better way to organize it, but so far it eludes us.

20.3 *Let $G \in \mathcal{F}_7$, let (z, A_0) be a frame, and let $Y \subseteq V(G) \setminus (A_0 \cup \{z\})$ be nonempty and anticonnected. Let $x_0, \dots, x_t$ be a Y-diamond in G of height $t \geq 4$. Suppose that there is no anticonnected set Y' with $Y \subseteq Y' \subseteq V(G)$ such that either:*

- *there is a Y'-diamond in G of height $t-1$, or*
- *there is a Y'-square in G of height $t-1$, or*
- *there is a polished Y'-diamond in G of height t.*

Then z is Y-complete and G contains a wheel (C, Y).

Proof. Assume that either z is not Y-complete or G contains no wheel (C, Y). Define X_i, A_i as usual. So x_t is X_{t-2}-complete, and x_t has a neighbour in A_{t-2}, and Y is complete to X_{t-1} and not to x_t.

(1) *Not both x_t and x_{t-1} have neighbours in A_{t-3}.*

For suppose they do. If x_{t-1} is X_{t-3}-complete, then

$$x_0, \dots, x_{t-1}$$

is a $Y \cup \{x_t\}$-diamond of height $t-1$, while if x_{t-1} is not X_{t-3}-complete, then

$$x_0, \dots, x_{t-3}, x_{t-1}, x_t$$

is a Y-diamond of height $t-1$, in both cases a contradiction. This proves (1).

(2) *There is a vertex q in A_{t-2} adjacent to both x_t and x_{t-1}, and a path R in A_{t-2} from q to*

A_{t-3} *such that not both* x_t *and* x_{t-1} *have neighbours in* $A_{t-3} \cup V(R \setminus q)$.

For let F be a minimal connected subgraph of A_{t-2} including A_{t-3} and containing neighbours of both x_t and x_{t-1}. If x_t, x_{t-1} have a common neighbour in F, then the claim is satisfied (from the minimality of F), so we assume not. Let P be a path between x_t and x_{t-1} with interior in F, say x_t-p_1-$\cdots$-p_n-x_{t-1}. Hence P has length > 2, and the hole z-x_1-P-x_2-z $(= C$ say) it follows that P is even. The only X_{t-2}-complete vertices in C are z and x_t, so by 2.10, X_{t-2} contains a leap or a hat for C. Suppose it contains a leap; then there are nonadjacent $x_i, x_j \in X_{t-2}$ such that x_i-p_1-$\cdots$-p_n-x_{t-1}-x_j is an odd path. Since x_i, x_j are $Y \cup \{x_t\}$-complete, it follows from 13.6 that this path contains another $Y \cup \{x_t\}$-complete vertex, which must be p_1 since no others are adjacent to x_t. Its ends are also $Y \cup \{x_t, z\}$-complete, and no internal vertex is $Y \cup \{x_t, z\}$-complete, so by 13.6, $Y \cup \{x_t, z\}$ is not anticonnected, that is, z is Y-complete. But then let C_1 be the hole z-x_i-p_1-$\cdots$-p_n-x_{t-1}-z; then (C_1, Y) is a wheel, a contradiction.

So X_{t-2} contains a hat for C; that is, there exists $x_i \in X_{t-2}$ with no neighbours in C except x_t, z. Hence the path x_i-x_t-p_1-$\cdots$-p_n-x_{t-1} is odd and has length ≥ 5, and its ends are $Y \cup \{z\}$-complete, and no internal vertex is $Y \cup \{z\}$-complete, so by 13.6, z is Y-complete. Let S be a path between x_i and x_{t-1} with interior in F. Then $V(S \cup P) \setminus \{x_i, x_t\}$ ($= F'$ say) is connected and catches the triangle $\{z, x_i, x_t\}$. The only neighbour of z in F' is x_{t-1}, which is nonadjacent to both x_i, x_t. If F' contains a reflection of the triangle, there is an antihole of length 6 containing z, x_{t-1}, x_t, which is impossible by 15.7 since these three vertices belong to C. So by 17.1, there is a vertex in F' adjacent to both x_i, x_t. Since x_i has no neighbour in $P \setminus x_t$, it follows that both x_t, x_{t-1} have neighbours in the interior of S, and so there is a path P' between x_t, x_{t-1} with $P' \setminus x_t$ a subpath of $S \setminus x_i$. As before P' has length ≥ 4, and so S has length ≥ 4, and P', S both have even length since they can be completed to holes through z. Since the X_{t-2}-complete vertex z has no neighbours in the interior of P', from 18.2 (applied to P' with anticonnected sets Y and X_{t-2}) it follows that there is a Y-complete edge in P', and since x_t is not Y-complete, there is therefore one in S. But since the edges zx_{t-1}, zx_i are also Y-complete, we deduce that there are at least three Y-complete edges in the hole z-x_i-S-x_{t-1}-z, and such that hole is the rim of a wheel with hub Y, a contradiction. This proves (2).

Choose q, R as in (2) with R minimal, and let R be r_1-$\cdots$-r_n, where $r_1 = q$ and r_n is the only vertex of R in A_{t-3} .

(3) x_{t-1} *has neighbours in* A_{t-3}.

For assume not. Since x_{t-1} has no neighbours in A_{t-3} it follows that $q \notin A_{t-3}$, and so R has length > 0. Suppose first that every antipath between x_{t-1} and q with interior in X_{t-2} is odd, and let Q be such an antipath. Since all internal vertices of Q have neighbours in A_{t-3}, and z is complete to its interior and anticomplete to A_{t-3}, it follows from 2.2 applied in $\overline{G}$ that one end of Q has a neighbour in A_{t-3}. By hypothesis, x_{t-1} does not, so q does. From the maximality of A_{t-3} it follows that q is X_{t-3}-complete; and since $q \in A_{t-2}$ and is therefore not X_{t-2}-complete, q is nonadjacent to x_{t-2}. Now by assumption, every every antipath between x_{t-1} and q with interior in X_{t-2} is odd, and so x_{t-2} is adjacent to x_{t-1}. But then

$$x_0, \ldots, x_{t-1}$$

is a $Y \cup \{x_t\}$-square of height $t-1$, a contradiction. So we may assume some antipath Q between between x_{t-1} and q with interior in X_{t-2} is even.

From (2), not both x_t, x_{t-1} have neighbours in $A_{t-3} \cup V(R \setminus q)$. Suppose that x_{t-1} has such a neighbour, and so x_t does not. Since by assumption x_{t-1} has no neighbours in A_{t-3}, it follows that all neighbours of x_{t-1} in $A_{t-3} \cup V(R \setminus q)$ lie in the interior of R, and in particular R has length ≥ 2. The antipath x_t-x_{t-1}-Q-q is odd, and its ends have no neighbours in the connected set $A_{t-3} \cup \{r_3, \ldots, r_n\}$. Since z is complete to its interior and anticomplete to $A_{t-3} \cup \{r_3, \ldots, r_n\}$, it follows from 2.2 applied in $\overline{G}$ that some internal vertex of this antipath has no neighbours in $A_{t-3} \cup \{r_3, \ldots, r_n\}$. But all internal vertices of Q lie in X_{t-2} and therefore have neighbours in A_{t-3}; so x_{t-1} has no neighbour in $A_{t-3} \cup \{r_3, \ldots, r_n\}$. Hence r_2 is its only neighbour in $A_{t-3} \cup V(R \setminus q)$. Suppose that every antipath between x_{t-1} and r_2 with interior in X_{t-2} is odd, and let Q' be such an antipath. All internal vertices of Q' have neighbours in the connected set A_{t-3}, and z is complete to the interior of Q' and anticomplete to A_{t-3}; so by 2.2 applied in $\overline{G}$, it follows that r_2 has neighbours in A_{t-3}. From the maximality of A_{t-3}, r_2 is X_{t-3}-complete, and therefore not adjacent to x_{t-2}. Since by assumption every antipath between x_{t-1} and r_2 with interior in X_{t-2} is odd, it follows that x_{t-1} is adjacent to x_{t-2}. But then

$$x_0, \ldots, x_{t-1}$$

is a $Y \cup \{x_t\}$-square of height $t-1$, a contradiction. So some antipath Q' between x_{t-1} and r_2 with interior in X_{t-2} is even. Hence the antipath x_{t-1}-Q'-r_2-z is odd. All its internal vertices have neighbours in the connected set $A_{t-3} \cup \{r_3, \ldots, r_n\}$ and its ends do not, so by 13.6 this antipath has length 3, that is, Q' has length 2. Let x_i be its middle vertex. Then the connected set $A_{t-3} \cup V(R \setminus \{r_1, r_2\}) \cup \{x_i, x_t, z\}$ ($= F$ say) catches the triangle $\{r_1, r_2, x_{t-1}\}$; the only neighbours of r_1 in F are x_t and possibly x_i; the neighbours of r_2 in F lie in $A_{t-3} \cup \{r_3\}$; and the only neighbour of x_{t-1} in F is z. This contradicts 17.1, since z has no neighbour in $A_{t-3} \cup \{r_3\}$.

So x_{t-1} has no neighbours in $A_{t-3} \cup V(R \setminus q)$. Now the antipath z-q-Q-x_{t-1} is odd, and all its internal vertices have neighbours in $A_{t-3} \cup V(R \setminus q)$, and its ends do not, so by 13.6 it has length 3, that is, Q has length 2 (let its middle vertex be x_i); and there is an odd path P between q, x_i with interior in $A_{t-3} \cup V(R \setminus q)$. Let C be the hole z-x_{t-1}-q-P-x_i-z; then C has length ≥ 6. By 15.7 there is no antihole of length ≥ 6 containing q, x_i, x_{t-1}. If q is not Y-complete then an antipath between q, x_t with interior in Y can be completed to such an antihole via x_t-x_{t-1}-x_i-q, so q is Y-complete; and if z is not Y-complete, an antipath between z and x_t with interior in Y can be extended to such an antihole, via x_t-x_{t-1}-x_i-q-z. So z is also Y-complete. Hence the hole C contains at least three Y-complete edges, namely $x_i z, z x_{t-1}$ and $x_{t-1} q$, a contradiction. This proves (3).

From (3) and the choice of R it follows that x_t has no neighbours in $A_{t-3} \cup V(R \setminus q)$. Let Q be an antipath between q and x_{t-1} with interior in X_{t-2}. Then z-q-Q-x_{t-1}-x_t is an antipath of length ≥ 4, and its ends have no neighbours in the connected set $A_{t-3} \cup V(R \setminus q)$, and its internal vertices do, so by 13.6 it has even length, that is, Q is even. The antipath x_t-x_{t-1}-Q-q is therefore odd, and its internal vertices have neighbours in A_{t-3}, and z is complete to its interior and anticomplete to A_{t-3}, so by 2.2 applied in $\overline{G}$, it follows that one of its ends, and hence q has a neighbour in A_{t-3}. From the maximality of A_{t-3} it follows that q is X_{t-3}-complete and therefore nonadjacent to x_{t-2}. If x_{t-1} is not X_{t-3}-complete, then

$$x_0, \ldots, x_t$$

is a polished Y-diamond of height t; while if x_{t-1} is X_{t-3}-complete, then

$$x_0, \dots, x_{t-1}$$

is a $Y \cup \{x_t\}$-diamond of height $t-1$, in both cases a contradiction. This proves 20.3. ∎

20.4 *Let $G \in \mathcal{F}_7$, let (z, A_0) be a frame, and let $Y \subseteq V(G) \setminus (A_0 \cup \{z\})$ be nonempty and anticonnected. Let $x_0, \dots, x_t$ be a Y-square in G of height $t \geq 4$. Then there is a nonempty anticonnected set Y' with $Y' \subseteq V(G) \setminus (A_0 \cup \{z\})$ such that either $Y = Y'$ or z is not Y'-complete, and such that either:*

- *there is a Y'-diamond in G of height $t-1$, or*
- *there is a Y'-square in G of height $t-1$, or*
- *there is a polished Y'-diamond in G of height t.*

Proof. Assume that no such Y' exists. Define X_i, A_i as usual. So x_t is adjacent to x_{t-1}, x_t has no neighbour in A_{t-2}, there is a vertex q in A_{t-1} adjacent to x_t with a neighbour in A_{t-2}, and Y is complete to X_{t-1} and not to x_t. From the maximality of A_{t-2} it follows that q is X_{t-2}-complete. Since $q \in A_{t-1}$, it is not X_{t-1}-complete, and so q is nonadjacent to x_{t-1}.

(1) *x_{t-1} has neighbours in A_{t-3}.*

For suppose not. Let R be a path between q and x_{t-1} with interior in A_{t-2}. Then R has length ≥ 2, and from the hole q-R-x_{t-1}-x_t-q it follows that R has even length. So the path q-R-x_{t-1}-z is odd, and its ends are X_{t-2}-complete, and its interior vertices are not, so by 13.6 it has length 3, that is, R has length 2. Let its middle vertex be r. Since x_{t-1} has no neighbour in A_{t-3}, it follows that $r \in A_{t-2} \setminus A_{t-3}$. Let Q be an antipath between r and x_{t-1} with interior in X_{t-2}. Since r-Q-x_{t-1}-q-z-r is an antihole, it follows that Q is odd. All its internal vertices have neighbours in A_{t-3}, and one end x_{t-1} does not, and z is complete to its interior and anticomplete to A_{t-3}. By 2.2 applied in $\overline{G}$, it follows that r has neighbours in A_{t-3}. Hence r is X_{t-3}-complete, and nonadjacent to x_{t-2}. Since z-x_{t-1}-r-q-x_{t-2}-z is not an odd hole it follows that x_{t-2} is adjacent to x_{t-1}. But then

$$x_0, \dots, x_{t-1}$$

is a $\{q\}$-square of height $t-1$, and yet z is not $\{q\}$-complete, a contradiction. This proves (1).

(2) *q has neighbours in A_{t-3}.*

For suppose not. Let S be an antipath between x_t and x_{t-1} with $V(S) \subseteq X_t$, that is, with interior in X_{t-2}. Then x_t-S-x_{t-1}-q is an antipath with length ≥ 3; by (1), all its internal vertices have neighbours in A_{t-3}, and its ends do not, and z is complete to its interior and anticomplete to A_{t-3}; so by 2.2 applied in $\overline{G}$ it follows that S has odd length. But then x_t-S-x_{t-1}-q-z has odd length ≥ 5, and its internal vertices have neighbours in A_{t-2} and its ends do not, contrary to 13.6 applied in $\overline{G}$. This proves (2).

If x_{t-1} is X_{t-3}-complete, then

$$x_0, \dots, x_{t-1}$$

is a $\{q\}$-diamond of height $t-1$, and yet z is not $\{q\}$-complete, a contradiction. So x_{t-1} is not X_{t-3}-complete. It follows from (2) that if x_t is X_{t-3}-complete then

$$x_0, \dots, x_{t-3}, x_{t-1}, x_{t-2}, x_t$$

is a polished Y-diamond of height t, while if x_t is not X_{t-3}-complete then

$$x_0, \dots, x_{t-3}, x_{t-1}, x_t$$

is a Y-square of height $t-1$, in either case a contradiction. This proves 20.4. ∎

20.5 *Let $G \in \mathcal{F}_7$, let (z, A_0) be a frame, and let $Y \subseteq V(G) \setminus (A_0 \cup \{z\})$ be nonempty and anticonnected. Let $x_0, \dots, x_{t+1}$ be a polished Y-diamond in G of height $t+1 \geq 5$. Then there is a nonempty anticonnected set Y' with $Y' \subseteq V(G) \setminus (A_0 \cup \{z\})$ such that either $Y \subseteq Y'$ or z is not Y'-complete, and such that either:*

- *there is a Y'-diamond in G of height $t-1$, or*
- *there is a Y'-square in G of height $t-1$, or*
- *there is a polished Y'-diamond in G of height t.*

Proof. Suppose that no such Y' exists. Let $x_0, \dots, x_{t+1}$ be a polished Y-diamond in G, and define X_i, A_i as usual. So x_{t+1} is X_{t-1}-complete, x_t is not X_{t-2}-complete, x_{t+1} has no neighbour in A_{t-2}, x_t has a neighbour in A_{t-2}, there is a vertex q in A_{t-1} adjacent to both x_{t+1}, x_t with a neighbour in A_{t-2}, and Y is complete to X_t and not to x_{t+1}. From the maximality of A_{t-2} it follows that q is X_{t-2}-complete, and therefore nonadjacent to x_{t-1}.

Choose a path v_1-$\cdots$-v_s with s minimum such that $v_1, \dots, v_s \in A_{t-2}$, and v_1 is adjacent to q, and $v_s \in A_{t-3}$. (If q has a neighbour in A_{t-3} then $s = 1$.) Let R be a path between q and x_{t-1} with interior in A_{t-2}, and if possible with interior in $A_{t-3} \cup \{v_1, \dots, v_s\}$. Then R has length ≥ 2, and from the hole q-R-x_{t-1}-x_{t+1}-q it follows that R has even length. So the path q-R-x_{t-1}-z is odd, and its ends are X_{t-2}-complete, and its internal vertices are not, so by 13.6 it has length 3, that is, R has length 2. Let its middle vertex be r.

(1) *x_{t-1} has neighbours in A_{t-3}.*

For suppose not. It follows that $r \in A_{t-2} \setminus A_{t-3}$. Let Q be an antipath between r and x_{t-1} with interior in X_{t-2}. Since r-Q-x_{t-1}-q-z-r is an antihole, it follows that Q is odd. All its internal vertices have neighbours in A_{t-3}, and one end x_{t-1} does not, and z is complete to its interior and anticomplete to A_{t-3}. By 2.2 applied in $\overline{G}$, it follows that r has neighbours in A_{t-3}. Hence r is X_{t-3}-complete, and nonadjacent to x_{t-2}. Since z-x_{t-1}-r-q-x_{t-2}-z is not an odd hole it follows that x_{t-2} is adjacent to x_{t-1}. But then

$$x_0, \dots, x_{t-1}$$

is a $\{q\}$-square of height $t-1$, and yet z is not $\{q\}$-complete, a contradiction. This proves (1).

From (1) it follows that it is possible to choose R with interior in $A_{t-3} \cup \{v_1, \dots, v_s\}$, and therefore we have done so.

(2) *q has neighbours in A_{t-3}, and therefore $r \in A_{t-3}$.*

For suppose it does not. Then $s \geq 2$ and $r = v_1$. Let Q be an antipath between x_{t-1} and r with interior in X_{t-2}. From the antihole x_{t-1}-Q-r-z-q-x_{t-1} it follows that Q is odd. Hence the antipath q-x_{t-1}-Q-r-x_{t+1} is odd with length ≥ 5; and its internal vertices have neighbours in $A_{t-3} \cup \{v_2, \dots, v_s\}$, and its ends do not, contrary to 13.6 applied in $\overline{G}$. This proves (2).

(3) *x_{t-1} is not X_{t-3}-complete.*

For if it is, then

$$x_0, \dots, x_{t-1}$$

is a $\{q\}$-diamond of height $t-1$, and yet z is not $\{q\}$-complete, a contradiction. This proves (3).

(4) *x_t has no neighbour in A_{t-3}.*

For suppose x_t has a neighbour in A_{t-3}. If x_t is X_{t-3}-complete then since it is not X_{t-2}-complete, it is nonadjacent to x_{t-2}, and therefore

$$x_0, \dots, x_{t-2}, x_t$$

is a $Y \cup \{x_{t+1}\}$-diamond of height $t-1$; while if x_t is not X_{t-3}-complete then

$$x_0, \dots, x_{t-3}, x_{t-1}, x_t, x_{t+1}$$

is a polished Y-diamond of height t, in either case a contradiction. This proves (4).

In particular, x_t is not adjacent to r. Since z-x_t-q-r-x_{t-1}-z is not an odd hole it follows that x_t is adjacent to x_{t-1}. If x_t is X_{t-3}-complete, then

$$x_0, \dots, x_{t-3}, x_{t-1}, x_{t-2}, x_t$$

is a polished $Y \cup \{x_{t+1}\}$-diamond of height t; while if x_t is not X_{t-3}-complete, then

$$x_0, \dots, x_{t-3}, x_{t-1}, x_t$$

is a $Y \cup \{x_{t+1}\}$-square of height $t-1$, in either case a contradiction. This proves 20.5. ∎

21 From wheel systems to wheels

Now we complete the proof of 19.1. First we need a lemma.

21.1 *Let $G \in \mathcal{F}_7$, and let X, Y be disjoint nonempty anticonnected subsets of $V(G)$, complete to each other. Let $p_1, \dots, p_n$ be a path in $G \setminus (X \cup Y)$ of length ≥ 4, such that p_1, p_n are X-complete and $p_2, \dots, p_{n-1}$ are not. Suppose that either:*

1. *p_1, p_2, p_3 are Y-complete, or*

2. *there exists i with $1 \leq i \leq n-3$ such that $p_i, p_{i+1}, p_{i+2}, p_{i+3}$ are all Y-complete, or*

3. *there exists i with $1 \leq i \leq n-3$ such that p_{i+1}, p_{i+2} are Y-complete and p_i, p_{i+3} are not.*

Then there is a wheel in G with hub Y.

Proof. In the second and third case let i be as given, and in the first case let $i = 1$. Let Q be an antipath joining p_{i+1}, p_{i+2} with interior in X. Since $1 < i+1, i+2 < n$, and $n \geq 5$, and p_1, p_n are both complete to the interior of Q, it follows from 15.4 that Q has length 2, that is, there exists $x \in X$ nonadjacent to both p_{i+1}, p_{i+2}. Choose h with $1 \leq h \leq i$ maximum such that x is adjacent to p_h, and choose j with $i+3 \leq j \leq n$ minimum such that x is adjacent to p_j. Then x-p_h-$\cdots$-p_j-x is a hole of length ≥ 6, say C, and $x, p_i, p_{i+1}, p_{i+2}, p_{i+3}$ are all vertices of it, and x, p_{i+1}, p_{i+2} are Y-complete. In the first case xp_1, p_1p_2, p_2p_3 are all Y-complete edges of the hole, so (C, Y) is a wheel. In the second case, the three edges of p_i-p_{i+1}-p_{i+2}-p_{i+3} are all Y-complete edges of C, so again (C, Y) is a wheel. In the third case, 2.3 implies that (C, Y) is a wheel (and in this case it is in fact an odd wheel, a contradiction). This proves 21.1. ▮

The final step of the proof of 19.1 is given by the following. (In this paper we only apply it to graphs in containing no pseudowheels, that is, graphs in $\mathcal{F}_8$, so the first hypothesis could be simplified; but it is convenient to present it this way for a future application.)

21.2 *Let $G \in \mathcal{F}_7$, and let $Y \subseteq V(G)$, such that there do not exist X, P so that (X, Y, P) is a pseudowheel. Let (z, A_0) be a frame, and let $x_0, \ldots, x_{t+1}$ be a wheel system with hub Y, and with $t \geq 2$. Define X_i, A_i as usual. Suppose that x_{t+1} has no neighbour in A_{t-1}; and moreover that at most one member of Y has no neighbour in $A_{t-1} \cup \{x_{t+1}\}$, and any such vertex has a neighbour in A_t. Then there is a wheel in G with hub Y.*

Proof.

(1) *There do not exist $x_i, x_j \in X_t$ joined by an odd path x_i-x_{t+1}-P-x_j of length ≥ 5 such that $x_i, x_j \in X_t$ and P has interior in A_t.*

For assume such a path exists, and let P have vertices x_{t+1}-p_1-$\cdots$-p_n-x_j. Thus $n \geq 4$. There is an even path S between x_i and x_j with interior in A_{t-1}. Since x_i-x_{t+1}-P-x_j-S-x_i is not an odd hole, and x_{t+1} has no neighbours in A_{t-1}, it follows that $\{p_1, \ldots, p_n\} \cup A_{t-1}$ is connected. Since $p_1 \notin A_{t-1}$, there exists k such that $p_k \notin A_{t-1}$ and p_k has a neighbour in A_{t-1}; and since p_k is not adjacent to z, it follows from the maximality of A_{t-1} that p_k is X_{t-1}-complete. Since at least one of x_i, x_j is in X_{t-1}, it follows that $k = n$ and $i = t$. But $\{p_1, \ldots, p_n, x_j\} \cup A_{t-1}$ ($= F$ say) catches the triangle $\{z, x_{t+1}, x_t\}$; the only neighbour of z in F is x_j; the only neighbour of x_{t+1} in F is p_1; and x_j, p_1 are nonadjacent (since $n \geq 4$), and are both nonadjacent to x_t, contrary to 17.1. This proves (1).

Since x_{t+1} has a neighbour in A_t and none in A_{t-1}, there is a path from x_{t+1} to A_{t-1} with interior in $A_t \setminus A_{t-1}$. Hence there is a path x_{t+1}-p_1-$\cdots$-p_m such that $p_1, \ldots, p_m \in A_t \setminus A_{t-1}$ and p_m is the unique vertex of this path with a neighbour in A_{t-1}. (Hence $m \geq 1$, and p_m is

X_{t-1}-complete.) Choose such a path such that if possible, every member of Y has a neighbour in $A_{t-1} \cup \{x_{t+1}, p_1, \ldots, p_m\}$.

(2) *We may assume that one of $x_0, \ldots, x_t$ is nonadjacent to both x_{t+1}, p_1.*

For certainly there is an antipath Q joining x_{t+1}, p_1 with interior in X_t, since x_{t+1}, p_1 are not X_t-complete. Suppose that Q is odd. Every vertex of the interior of Q has neighbours in the connected set A_{t-1}, and x_{t+1} does not, and z is complete to the interior of Q and anticomplete to A_{t-1}; so by 2.2 applied in $\overline{G}$ it follows that p_1 has a neighbour in A_{t-1}. Hence $m = 1$, and p_1 is X_{t-1}-complete, and therefore not adjacent to x_t. If x_t is also nonadjacent to x_{t+1} then the claim holds, and if x_t is adjacent to x_{t+1}, then

$$x_0, \ldots, x_{t+1}$$

is a Y-square, and the theorem holds by 20.1. Now assume that Q is even. The antipath z-p_1-Q-x_{t+1} is therefore odd and has length ≥ 3; all its internal vertices have neighbours in the connected set $A_{t-1} \cup \{p_2, \ldots, p_m\}$, and its ends do not. So it has length 3, by 13.6 applied in $\overline{G}$, and hence Q has length 2. This proves (2).

(3) *Every vertex in Y has a neighbour in $A_{t-1} \cup \{x_{t+1}, p_1, \ldots, p_m\}$.*

For suppose some $y \in Y$ has no such neighbour. By hypothesis y has a neighbour in A_t. Consequently there is a connected subset F of A_t including $A_{t-1} \cup \{p_1, \ldots, p_m\}$ which contains a neighbour of y, and we may choose F minimal with this property. Since y has no neighbour in $A_{t-1} \cup \{p_1, \ldots, p_m\}$, it follows from the minimality of F that y has a unique neighbour in F, say f, and therefore $f \in A_t \setminus A_{t-1}$. There is a path R between y and x_{t+1} with interior in F, and therefore z-x_{t+1}-R-y-z is a hole (C say), and so R has even length. Suppose it has length ≥ 4. The only X_t-complete vertices in C are z, y, so by 2.10, X_t contains a hat or leap. By (1) there is no leap, so there exists $x \in X_t$ with no neighbours in C except y, z. But $F \cup \{x_{t+1}\}$ catches the triangle $\{x, y, z\}$; the only neighbour of z in $F \cup \{x_{t+1}\}$ is x_{t+1}; the only neighbour of y in $F \cup \{x_{t+1}\}$ is f; and x_{t+1}, f are nonadjacent, and both nonadjacent to x, contrary to 17.1. So R has length 2, and therefore x_{t+1} is adjacent to f.

Since y has no neighbour in $\{x_{t+1}\} \cup A_{t-1}$, it follows from the hypothesis that all other members of Y have neighbours in $\{x_{t+1}\} \cup A_{t-1}$. We recall that initially we chose the path x_{t+1}-p_1-$\cdots$-p_m such that p_m is the unique vertex of it with a neighbour in A_{t-1}, and if possible every member of Y has a neighbour in $A_{t-1} \cup \{x_{t+1}, p_1, \ldots, p_m\}$. Since f is adjacent to both of y, x_{t+1}, it follows that f has no neighbours in A_{t-1}, and f is nonadjacent to $p_2, \ldots, p_m$, since otherwise there would be a better choice of path using f. Let Q be an antipath between f, x_{t+1} with interior in X_t. Every internal vertex of Q has a neighbour in A_{t-1}, and its ends do not, and z is complete to the interior of Q and anticomplete to A_{t-1}; so by 2.2 applied in $\overline{G}$, it follows that Q is even. So the antipath y-x_{t+1}-Q-f is odd, and all its internal vertices have neighbours in $A_{t-1} \cup \{p_1, \ldots, p_m\}$, and y does not; and z is complete to the interior of the antipath and anticomplete to $A_{t-1} \cup \{p_1, \ldots, p_m\}$. By 2.2 applied in $\overline{G}$, it follows that f has a neighbour in $A_{t-1} \cup \{p_1, \ldots, p_m\}$, and therefore f is adjacent to p_1. By (2) there exists $x \in X_t$ nonadjacent to x_{t+1}, p_1. Consequently, $\{z, y, x, p_2, \ldots, p_m\} \cup A_{t-1}$ ($= F'$ say) catches the triangle $\{x_{t+1}, f, p_1\}$. The only neighbour of x_{t+1} in F' is z; the only neighbours of f in F' are y and possibly x; and x, y, z are all nonadjacent to p_1. By 17.1, F' contains a reflection of the triangle, and hence there is a vertex in F' adjacent to both of z, p_1. But the only neighbours of z in

F' are x, y, and they are both nonadjacent to p_1, a contradiction. This proves (3).

Since p_m is X_{t-1}-complete it follows that $x_0, \dots, x_{t-1}$ all have neighbours in $p_1, \dots, p_m$. Since x_t, p_m have neighbours in A_{t-1} and none of $x_{t+1}, p_1, \dots, p_{m-1}$ have neighbours in A_{t-1}, we can extend the path x_{t+1}-p_1-$\cdots$-p_m to a path x_{t+1}-p_1-$\cdots$-p_m-p_{m+1}-$\cdots$-p_n containing neighbours of all members of X_t. By (2), we can choose i with $2 \leq i \leq n$ maximum such that some vertex of X_t is nonadjacent to all of $x_{t+1}, p_1, \dots, p_{i-1}$; and choose s with $0 \leq s \leq t$ such that x_s is nonadjacent to all of $x_{t+1}, p_1, \dots, p_{i-1}$. Since every vertex in X_t has a neighbour in $\{x_{t+1}, p_1, \dots, p_n\}$, it follows from the maximality of i that every vertex in X_t is adjacent to one of $x_{t+1}, p_1, \dots, p_i$, and in particular, x_s is adjacent to p_i. Note that if $i > m$ then $s = t$, since p_m is X_{t-1}-complete.

(4) *i is odd, and p_i is Y-complete.*

For z-x_{t+1}-p_1-$\cdots$-p_i-x_s-z is a hole C say, and so i is odd. Suppose p_i is not Y-complete. Now C has length ≥ 6, and z, x_s are Y-complete (since Y is a hub), and x_{t+1}, p_i are not. Since (C, Y) is not an odd wheel, 2.10 implies that Y contains a leap or hat for C. Suppose it contains a leap; then there are nonadjacent $y_1, y_2 \in Y$ such that y_1-x_{t+1}-p_1-$\cdots$-p_i-y_2 is a path. This path is odd and has length ≥ 5, and its ends are X_t-complete and its internal vertices are not, contrary to 13.6. So Y contains a hat, that is, there exists $y \in Y$ nonadjacent to $x_{t+1}, p_1, \dots, p_i$. By (3), y has a neighbour in $A_{t-1} \cup \{p_j : i+1 \leq j \leq m\}$.

Suppose first that $i \leq m$, and let p_i-r_1-$\cdots$-r_k-y be a path from p_i to y with interior in $A_{t-1} \cup \{p_{i+1}, \dots, p_m\}$. Then z-x_{t+1}-p_1-$\cdots$-p_i-r_1-$\cdots$-r_k-y-z is a hole of length ≥ 6, and the only X_t-complete vertices in this hole are z, y. Since this hole is not the rim of an odd wheel, 2.10 implies that X contains a hat or leap, and so some $x \in X_t$ has no neighbour in $\{x_{t+1}, p_1, \dots, p_i\}$, contrary to the choice of i.

Now suppose that $i > m$, and so $s = t$. Let p_m-r_1-$\cdots$-r_k-y be a path from p_m to y with interior in A_{t-1}. Again, z-x_{t+1}-p_1-$\cdots$-p_m-r_1-$\cdots$-r_k-y-z is a hole of length ≥ 6, and its only X_t-complete vertices are z, y. By 2.10 X_t contains a hat or leap. By (1) it contains no leap, so there exists $x \in X_t$ nonadjacent to all $x_{t+1}, p_1, \dots, p_m, r_1, \dots, r_k$. Since p_m is X_{t-1}-complete, it follows that $x = x_t$. Now $\{x_{t+1}, p_1, \dots, p_i, r_1, \dots, r_k\}$ ($= F$ say) is connected, and catches the triangle $\{y, z, x_t\}$; the only neighbour of z in F is x_{t+1}; the only neighbour of y in F is r_k (because y is nonadjacent to $x_{t+1}, p_1, \dots, p_i$); and the only neighbour of x_t in F is p_i (because x_t is a hat). Since x_{t+1} is not adjacent to p_i, this contradicts 17.1. This proves (4).

(5) *Let R be a path from x_t to some vertex r, such that r is the unique X_{t-1}-complete vertex in R, and $V(R \setminus x_t) \subseteq A_{t-1} \cup \{p_1, \dots, p_m\}$. Then R is odd, and has length ≥ 3. In particular, x_t is nonadjacent to p_m, p_{m-1}.*

For assume that R is even. Then the path z-x_t-R-r is odd, and its ends are X_{t-1}-complete, and its internal vertices are not, so by 13.6, it has length 3, that is, R has length 2. Let q be the middle vertex of R. By 13.6 there is an odd antipath Q joining q, x_t with interior in X_{t-1}. Now p_m is X_{t-1}-complete and nonadjacent to x_t, and since Q cannot be completed to an antihole via x_t-p_m-q, it follows that p_m is adjacent to q. Suppose first that $q \in \{p_1, \dots, p_m\}$; then it follows that $q = p_{m-1}$. Hence q-Q-x_t-p_m is an even antipath of length ≥ 4; q is its only vertex that is anticomplete to A_{t-1}, and p_m is its only vertex that is anticomplete to $\{z, x_{t+1}, p_1, \dots, p_{m-2}\}$. Since the sets A_{t-1}, $\{z, x_{t+1}, p_1, \dots, p_{m-2}\}$ are each connected and anticomplete to each other, this contradicts 13.7 applied in $\overline{G}$. So $q \in A_{t-1}$, and in particular x_t is nonadjacent to p_m, p_{m-1}. Let R' be a path between x_t, p_m with interior in $\{z, x_{t+1}, p_1, \ldots, p_m\}$; then x_t-R-p_m-R'-x_t is a hole of length ≥ 6 sharing the vertices x_t, q, p_m with the antihole q-Q-x_t-p_m-z-q, contrary to 15.7. So R is odd. Since r is not X_t-complete, it follows that R has length ≥ 3. The last assertion of the claim is immediate. This proves (5).

(6) *We may assume that none of $x_{t+1}, p_1, \ldots, p_{i-1}$ is X_{t-1}-complete, and in particular $i \leq m$.*

For suppose first that one of $p_1, \ldots, p_{i-1}$ is X_{t-1}-complete, and choose h with $1 \leq h < i$ maximum such that p_h is X_{t-1}-complete. Since p_h is not adjacent to x_s it follows that $s = t$, and therefore p_i is not X_{t-1}-complete (because p_i is not X_t-complete and is adjacent to x_s). By (5), $i - h$ is even, and so the path p_h-$\cdots$-p_i-x_t-z is even and has length ≥ 4. Since its only X_{t-1}-complete vertices are its ends, and since z, x_t, p_i are Y-complete by (4), it follows from 21.1 that there is a wheel with hub Y, and the theorem holds. So we may assume that none of $p_1, \ldots, p_{i-1}$ is X_{t-1}-complete, and in particular $i \leq m$, since p_m is X_{t-1}-complete. Now assume that x_{t+1} is X_{t-1}-complete. Since x_{t+1} is nonadjacent to x_s it follows that $s = t$. Let R be a path between x_t, p_m with interior in A_{t-1}. By (5), R is odd, and so the path x_{t+1}-p_1-$\cdots$-p_i-x_t-R-p_m is odd, of length ≥ 5, its ends are X_{t-1}-complete, and its internal vertices are not, contrary to 13.6. This proves (6).

Choose k with $i \leq k \leq m$ minimum such that p_k is X_{t-1}-complete.

(7) *None of $x_{t+1}, p_1, \ldots, p_{k-1}$ is X_{t-1}-complete, and k is odd.*

The first assertion follows from (6) and the choice of k. Hence the path z-x_{t+1}-p_1-$\cdots$-p_k has length ≥ 4, and its ends are X_{t-1}-complete, and its internal vertices are not; so by 13.6, it has even length. This proves (7).

(8) *x_t is adjacent to one of $p_1, \ldots, p_k$.*

For suppose x_t is nonadjacent to all of $p_1, \ldots, p_k$. From the definition of i it follows that x_t is adjacent to x_{t+1}. Let S be a path between x_t, p_k with interior in $A_{t-1} \cup \{p_{k+1}, \ldots, p_m\}$, and let C be the hole x_t-x_{t+1}-p_1-$\cdots$-p_k-S-x_t. Since C is even and k is odd, it follows that S is even, and so by (5), some internal vertex of S is X_{t-1}-complete. The path z-x_t-S-p_k is odd, and its ends are X_{t-1}-complete, so by 2.3 it contains an odd number of X_{t-1}-complete edges. Since x_t is not X_{t-1}-complete, all these X_{t-1}-complete edges belong to S and hence to C, and there are no further X_{t-1}-complete edges in C. Thus an odd number of edges of C are X_{t-1}-complete, and so by 2.3 there is exactly one, and exactly two X_{t-1}-complete vertices. Since p_k is X_{t-1}-complete, the second such vertex is the neighbour of p_k in S. This therefore does not belong to A_{t-1}, and so $k < m$, and p_{k+1} is the second X_{t-1}-complete vertex of C. By 2.10 applied to C, X_{t-1} contains a leap or hat, and in either case some $x \in X_{t-1}$ is nonadjacent to all of x_t, x_{t+1}, p_1, and adjacent to p_k. Hence $(V(C) \setminus \{x_t, x_{t+1}\}) \cup \{x\}$ ($= F$ say) catches the triangle $\{z, x_t, x_{t+1}\}$; the only neighbour of z in F is x; the only neighbour of x_{t+1} in F is p_1; and x, p_1 are nonadjacent, and are both nonadjacent to x_t, contrary to 17.1. This proves (8).

(9) *p_k is Y-complete.*

For suppose not. Then $i < k$, by (4). But then z, X_{t-1} are Y-complete and x_{t+1}, p_k are not, and some vertex of the path x_{t+1}-p_1-$\cdots$-p_k is Y-complete (namely p_i); and so $(X_{t-1}, Y, z\text{-}x_{t+1}\text{-}p_1\text{-}\cdots\text{-}p_k)$ is a pseudowheel, contrary to $G \in \mathcal{F}_8$. This proves (9).

By (8), we may choose j with $1 \le j \le k$ maximum such that x_t is adjacent to p_j. By (5), $k - j$ is even and ≥ 2. Suppose that p_j is Y-complete. The path z-x_t-p_j-$\cdots$-p_k has even length ≥ 4, and its only X_{t-1}-complete vertices are its ends, and z, x_t, p_j, X_{t-1} are all Y-complete, so by 21.1, there is a wheel with hub Y and the theorem holds. So we may assume that p_j is not Y-complete. Now the path x_t-p_j-$\cdots$-p_k has odd length ≥ 3, and both its ends are Y-complete, and the Y-complete vertex z has no neighbour in its interior, so by 2.2 and 2.3, an odd number of its edges are Y-complete. Since p_j is not Y-complete, an odd number of edges of p_j-$\cdots$-p_k are Y-complete. The path z-x_{t+1}-p_1-$\cdots$-p_k ($= P$ say) is even, by (7), and since its ends are Y-complete, it follows that an even number of its edges are Y-complete, by 2.3. We deduce that an odd number of edges of z-x_{t+1}-p_1-$\cdots$-p_j are Y-complete. There is therefore a Y-segment P' of this path that has odd length. Since p_j is not Y-complete, it follows that P' is also a Y-segment of P. If P' has length > 1 then 21.1 applied to P implies that there is a wheel with hub Y, and the theorem holds. So we may assume that P' has length 1. But both vertices of P' are internal vertices of P, since x_{t+1}, p_j are not Y-complete, and again 21.1 applied to P implies there is a wheel with hub Y. This proves 21.2. ∎

Now we can deduce our main theorem about wheel systems, 19.1, which we restate:

21.3 *Let $G \in \mathcal{F}_8$, let (z, A_0) be a frame, and let $x_0, \dots, x_{t+1}$ be a wheel system with hub Y, and with $t \ge 1$. Define A_i, X_i as usual, and assume that at most one member of Y has no neighbour in A_1. Suppose that for all r with $1 \le r \le t$, if $x_0, x_1, \dots, x_r, x_{t+1}$ is a wheel system, then every member of Y has a neighbour in $A_r \cup \{x_{t+1}\}$. Then there is a wheel with hub Y.*

Proof. Suppose there is no such wheel. Choose r with $1 \le r \le t$, minimum such that x_{t+1} has a neighbour in A_r and a nonneighbour in X_r. By hypothesis, every member of Y has a neighbour in $A_r \cup \{x_{t+1}\}$. By 19.2, $r > 1$. Since at most one member of Y has no neighbour in A_{r-1} (because at most one has no neighbour in A_1), it follows from 21.2 that x_{t+1} has a neighbour in A_{r-1}. Since no wheel has hub Y, 20.1 implies that

$$x_0, \dots, x_r, x_{t+1}$$

is not a Y-diamond, and so x_{t+1} is not X_{r-1}-complete. But that contradicts the minimality of r. Thus there is a wheel with hub Y. This proves 19.1. ∎

22 Wheels and tails

We continue with the proof that recalcitrant graphs do not contain wheels. Now we come to apply 19.1, as explained at the start of section 19. We use the following lemma.

22.1 *Let $G \in \mathcal{F}_8$, not admitting a balanced skew partition, let (z, A_0) be a frame, and let $x_0, \dots, x_s$ be a wheel system. Let $Y \subseteq V(G) \setminus (A_0 \cup \{z, x_0, \dots, x_s\})$ be nonempty and anticonnected, such that $z, x_0, \dots, x_s$ are Y-complete. Then there is a sequence $x_{s+1}, \dots, x_{t+1}$ with $t \ge s$ such that $x_0, \dots, x_{t+1}$ is a wheel system with respect to the frame (z, A_0), with hub Y.*

Proof. Choose a sequence $x_{s+1}, \ldots, x_t$, all Y-complete and such that $x_0, \ldots, x_t$ is a wheel system with respect to (z, A_0), with t maximum. So $t \geq s \geq 1$. Define X_i and A_i as usual. From 15.2, there is a path P from z to A_t, disjoint from X_t and containing no X_t-complete vertex except z. Let v be the neighbour of z in this path. From the maximality of A_t, it follows that P has length 2. So v has a neighbour in A_t, and therefore $x_0, \ldots, x_t, v$ is a wheel system. From the maximality of t it follows that v is not Y-complete, and therefore Y is a hub for this wheel system. This proves 22.1. ∎

We combine 19.1 and 22.1 to prove the following.

22.2 *Let $G \in \mathcal{F}_8$, not admitting a balanced skew partition, let (z, A_0) be a frame, and let $x_0, \ldots, x_s$ be a wheel system. Let $Y \subseteq V(G) \setminus (A_0 \cup \{z, x_0, \ldots, x_s\})$ be nonempty and anticonnected, such that $z, x_0, \ldots, x_s$ are Y-complete. Define A_i, X_i as usual, and assume that every member of Y has a neighbour in A_s, and at most one member of Y has no neighbour in A_1. Suppose there is no wheel with hub Y. Then there exists r with $1 \leq r < s$, and a member $y \in Y$, and a vertex $v \notin Y \cup \{z, x_0, \ldots, x_s\}$ with the following properties:*

- *y has no neighbour in $A_r \cup \{v\}$*
- *v is adjacent to z, and has a neighbour in A_r, and a non-neighbour in X_r.*

Proof. By 22.1, there is a sequence $x_{s+1}, \ldots, x_{t+1}$ with $t \geq s$ such that $x_0, \ldots, x_{t+1}$ is a wheel system with respect to the frame (z, A_0), with hub Y. By 19.1, there exists r with $1 \leq r \leq t$, and a member $y \in Y$, such that y has no neighbour in $A_r \cup \{x_{t+1}\}$, and x_{t+1} has a neighbour in A_r, and a non-neighbour in X_r. Since every member of Y has a neighbour in A_s, it follows that $r < s$, and the result holds (taking $v = x_{t+1}$). This proves 22.2. ∎

If (C, Y) is a wheel in G, and there is no wheel (C', Y') with $Y \subset Y'$, we say (C, Y) is an *optimal* wheel. Let (C, Y) be a wheel in G. A *kite* for (C, Y) is a vertex $y \in V(G) \setminus (Y \cup V(C))$, not Y-complete, that has at least four neighbours in C, three of which are consecutive and Y-complete.

22.3 *Let $G \in \mathcal{F}_8$, not admitting a balanced skew partition, and let (C, Y) be an optimal wheel in G. Then there is no kite for (C, Y).*

Proof. Assume y is a kite for (C, Y). Let x_0-z-x_1 be a subpath of C, all Y-complete and adjacent to y. Let $A_0 = V(C) \setminus \{z, x_0, x_1\}$, so x_0, x_1 is a wheel system with respect to (z, A_0), and x_0, x_1 are $Y \cup \{y\}$-complete. Thus every member of $Y \cup \{y\}$ has a neighbour in A_0, and yet there is no wheel with hub $Y \cup \{y\}$, contrary to 22.2 with $s = 1$. This proves 22.3. ∎

Let (C, Y) be a wheel in G, let $z \in V(C)$, and let x_0, x_1 be the neighbours of z in C. A path T of $G \setminus \{x_0, x_1\}$ from z to $V(C) \setminus \{z, x_0, x_1\}$ is called a *tail* for z (with respect to the wheel (C, Y)) if

- x_0, z, x_1 are all Y-complete, and there is a Y-complete edge in $C \setminus \{x_0, z, x_1\}$
- the neighbour of z in T is adjacent to x_0, x_1, and
- no internal vertex of T is in Y or is Y-complete.

22.4 *Let $G \in \mathcal{F}_8$, and let (C, Y) be an optimal wheel, such that no vertex is a kite for (C, Y). Let $z \in V(C)$, and let x_0, x_1 be the neighbours of z in C. Let T be a tail for z, and let y be the neighbour of z in T. Let $A_0 = V(C) \setminus \{z, x_0, x_1\}$, and let $x_0, \ldots, x_{t+1}$ be a wheel system with respect to the frame (z, A_0), with hub $Y \cup \{y\}$. Define $A_1, \ldots, A_{t+1}$ as usual. Then y has a neighbour in $A_t \cup \{x_{t+1}\}$.*

Proof. We assume for a contradiction that y has no neighbour in $A_t \cup \{x_{t+1}\}$. Let y-u_1-$\cdots$-u_n be a minimal subpath of $T \setminus z$ such that u_n has a neighbour in A_t; so $n > 0$. From the maximality of A_t it follows that u_n is X_t-complete and therefore X_1-complete since $t \geq 1$; and since T is a tail it follows that none of $u_1, \ldots, u_n$ are Y-complete. Let P be a path with vertex set in $A_t \cup \{u_n\}$, from u_n to some Y-complete vertex p say, such that no vertex of $P \setminus p$ is Y-complete.

(1) *P is odd.*

For P has length ≥ 1 since no vertex of $T \setminus z$ is Y-complete; and the only X_t-complete vertex of P is u_n, and the only Y-complete vertex of P is p. Since z is complete to X_t and to Y, and anticomplete to $V(P)$, it follows from 2.9 that P has odd length. This proves (1).

Since $y, u_1, \ldots, u_{n-1}$ have no neighbours in A_t it follows that z-y-u_1-$\cdots$-u_n-P-p is a path, Q say.

(2) *We may assume that Q has even length ≥ 4, and so n is even.*

For the ends of Q are Y-complete, and since none of $y, u_1, \ldots, u_n$ are Y-complete, it follows that no internal vertex of Q is Y-complete. Suppose that Q has length 3. So $n = 1$, and there is an odd antipath joining y, u_1 with interior in Y. Hence every Y-complete vertex in G is adjacent to one of y, u_1. In particular, since y has no neighbour in A_t, it follows that u_1 is adjacent to all the Y-complete vertices in C except z (for we already showed that it is X_t-complete and therefore adjacent to x_0, x_1). Since (C, Y) is not an odd wheel, it follows that u_1 is a kite for (C, Y), a contradiction. So we may assume that Q does not have length 3. Hence by 13.6, Q has even length. From (1), it follows that n is even. This proves (2).

(3) *x_{t+1} is adjacent to one of $u_1, \ldots, u_{n-1}$.*

For suppose not. Choose a path N from x_{t+1} to u_n with interior in A_t (possibly of length 1). Then z-y-u_1-$\cdots$-u_n-N-x_{t+1}-z is a hole, and since n is even it follows that N is even. Hence z-x_{t+1}-N-u_n is an odd path; its ends are X_t-complete, its internal vertices are not, and the X_t-complete vertex y has no neighbour in its interior, contrary to 2.2. This proves (3).

(4) *x_{t+1} is not Y-complete.*

For suppose it is. Since $G \in \mathcal{F}_8$, the triple (Y, X_{t+1}, Q) is not a pseudowheel. Since y, p are not X_{t+1}-complete, it follows that no internal vertex of Q is X_{t+1}-complete. By 2.11 applied to Q, Y and X_{t+1}, it follows that there exists $x \in X_{t+1}$ with no neighbour in $Q \setminus z$ except possibly p. But x_{t+1} is adjacent to one of $u_1, \ldots, u_{n-1}$ by (3), and all other members of X_{t+1} are adjacent to y, a contradiction. This proves (4).

Since x_{t+1} has a neighbour in A_t, there is a path R from x_{t+1} to some Y-complete vertex r in A_t with $V(R \setminus x_{t+1}) \subseteq A_t$ such that no vertex of $R \setminus r$ is Y-complete.

(5) *R has odd length.*

For certainly R has length ≥ 1; suppose it has length 2, and let its middle vertex be a say. There is an antipath joining x_{t+1}, a with interior in Y, and it is odd since it can be completed to an antihole via a-z-r-x_{t+1}. Now x_{t+1}, a are not X_t-complete (since $a \in A_t$) and so there is an antipath joining x_{t+1}, a with interior in X_t, which is therefore also odd, since its union with the antipath with interior in Y is an antihole. But y is X_t-complete and nonadjacent to both x_{t+1} and a (since it has no neighbour in A_t), and so this antipath can be completed to an odd antihole via a-y-x_{t+1}, a contradiction. This proves that R does not have length 2. Hence the path z-x_{t+1}-R-r does not have length 3; its ends are Y-complete and its internal vertices are not, and it has length > 1, so by 13.6 it has even length, that is, R has odd length. This proves (5).

(6) *If x_{t+1} is adjacent to u_1 then u_1 is X_t-complete.*

For suppose not; then there is an antipath L say joining x_{t+1}, u_1 with interior in X_t. So z-u_1-L-x_{t+1}-y is an antipath of length ≥ 4; all its internal vertices have neighbours in $A_t \cup \{u_2, \ldots, u_n\}$, and its ends do not. By 13.6 applied in $\overline{G}$, it has even length, and so u_1-L-x_{t+1}-y is an odd antipath. But all its internal vertices have neighbours in A_t, and its ends do not (for $n \geq 2$ since n is even), and z is complete to its interior and has no neighbours in A_t, contrary to 2.2 applied in $\overline{G}$. This proves (6).

(7) *None of $u_1, \ldots, u_{n-1}$ is X_t-complete.*

For suppose that one of $u_1, \ldots, u_{n-1}$ is X_t-complete, and let S be a path from x_{t+1} to some X_t-complete vertex s say, with $V(S \setminus x_{t+1}) \subseteq \{u_1, \ldots, u_{n-1}\}$, such that s is the only X_t-complete vertex in S. Certainly S has length ≥ 1. Suppose it has even length. Then the path z-x_{t+1}-S-s is odd, and its ends are X_t-complete, and its internal vertices are not; so by 2.2, the X_t-complete vertex y has a neighbour in its interior, contrary to (6). So S has odd length. The path s-S-x_{t+1}-R-r therefore has even length; its only X_t-complete vertex is s, and its only Y-complete vertex is r, so by 13.7, the path has length 2, that is, both R, S have length 1. Moreover, either x_{t+1}, r are joined by an odd antipath with interior in X_t, or x_{t+1}, s are joined by an odd antipath with interior in Y. The first is impossible since the antipath could be completed to an odd antihole via r-y-x_{t+1}, so the second holds. In particular, every Y-complete vertex is adjacent to one of x_{t+1}, s, and therefore all such vertices in A_t are adjacent to x_{t+1}. In particular, x_{t+1} is adjacent to all the Y-complete vertices in C except possible x_0, x_1. Since there is a Y-complete edge in $C \setminus \{x_0, z, x_1\}$ from the definition of a tail, it follows that x_{t+1} has two adjacent neighbours in C of opposite wheel-parity, and at least one other neighbour in C; but it is not a kite, and the wheel is optimal, contrary to 16.1. This proves (7).

By (3) we may choose i with $1 \leq i \leq n-1$ minimum such that x_{t+1} is adjacent to u_i. By (7), the only X_t-complete vertices in the hole z-y-u_1-$\cdots$-u_i-x_{t+1}-z are z,y, and therefore by (6) this hole has length ≥ 6. By 2.10 X_t contains a leap or a hat. If it contains a leap, there are nonadjacent vertices in X_t, joined by an odd path of length ≥ 5 with interior in $\{u_1, \ldots, u_i, x_{t+1}\}$, and consequently with no internal vertex Y-complete. Since both its ends are Y-complete, this contradicts 13.6. So there is a hat, that is, there exists $x \in X_t$ with no neighbours in $\{u_1, \ldots, u_i, x_{t+1}\}$. Then $A_t \cup \{u_1, \ldots, u_n, x_{t+1}\}$ ($= F$ say) catches the triangle $\{z, y, x\}$; the only neighbour of z in F is x_{t+1}; the only neighbour of y in F is u_1; and both x_{t+1}, u_1 are nonadjacent to x. Moreover x_{t+1} is nonadjacent to u_1, and so F contains no reflection of the triangle. This contradicts 17.1, and therefore proves 22.4. ∎

We combine the previous result with 19.1 to prove the following.

22.5 *Let $G \in \mathcal{F}_8$, not admitting a balanced skew partition, and let (C, Y) be an optimal wheel in G. Then no vertex of C has a tail.*

Proof. Suppose $z \in V(C)$ has a tail T; let y be the neighbour of z in T, and let x_0, x_1 be the neighbours of z in C. Let $A_0 = V(C) \setminus \{z, x_0, x_1\}$, so x_0, x_1 is a wheel system with respect to (z, A_0), and x_0, x_1 are $Y \cup \{y\}$-complete. By 22.1 there exist $x_2, \ldots, x_{t+1}$ with $t \geq 1$ such that $x_0, \ldots, x_{t+1}$ is a wheel system with respect to (z, A_0), with hub $Y \cup \{y\}$. Define A_i, X_i as usual. From the construction, all members of Y have a neighbour in A_0. By 19.1, there exists r with $1 \leq r \leq t$, such that $x_0, \ldots, x_r, x_{t+1}$ is a wheel system and y has no neighbour in $A_r \cup \{x_{t+1}\}$. But $Y \cup \{y\}$ is a hub for this wheel system, and T is a tail for z. By 22.3, there is no kite for (C, Y); and so by 22.4 applied to this wheel system, y has a neighbour in $A_r \cup \{x_{t+1}\}$, a contradiction. This proves 22.5. ∎

23 The end of wheels

In this section we complete the proof that there is no wheel in a recalcitrant graph. We need the following:

23.1 *Let $G \in \mathcal{F}_8$, not admitting a balanced skew partition, and let (C, Y) be an optimal wheel in G. Then there is a subpath c_1-c_2-c_3 of C such that c_1, c_2, c_3 are all Y-complete, and a path c_1-p_1-$\cdots$-p_k-c_3 such that none of $p_1, \ldots, p_k$ are in $V(C) \cup Y$, none of them is Y-complete, and none of them has a neighbour in $V(C) \setminus \{c_1, c_2, c_3\}$.*

Proof. There are two nonadjacent Y-complete vertices in C with opposite wheel-parity, say a, b, and by 15.2, there is a path P in G joining them such that none of its interior vertices is in Y or is Y-complete. There may be internal vertices of P that belong to C, but we may choose a subpath P' of P, with ends a', b' say, such that $a', b' \in V(C)$ have opposite wheel-parity and P' has minimum length. It follows that no vertex of the interior of P' is in C. Suppose a', b' are adjacent; then since they are in C and have opposite wheel-parity, they are both Y-complete, and therefore neither is in the interior of P, and so a, b are adjacent, a contradiction. So a', b' are nonadjacent. Let F be the interior of P'; then no vertex of F is in $Y \cup V(C)$, no vertex of F is Y-complete, and there are attachments of F in C which are nonadjacent and have opposite wheel-parity. The result follows from 22.3 and 16.2 applied to F. This proves 23.1. ∎

Now we can prove 1.8.9, which we restate.

23.2 *Let $G \in \mathcal{F}_8$, not admitting a balanced skew partition; then there is no wheel in G. In particular, every recalcitrant graph belongs to $\mathcal{F}_9$.*

Proof. Suppose there is a wheel in G, and choose an optimal wheel (C, Y) such that C contains as few Y-complete edges as possible.

(1) *Exactly 4 edges of C are Y-complete.*

For by 23.1 there is a subpath c_1-c_2-c_3 of C such that c_1, c_2, c_3 are all Y-complete, and a path c_1-p_1-$\cdots$-p_k-c_3 such that none of $p_1, \ldots, p_k$ are in $V(C) \cup Y$, none of them is Y-complete, and none of them has a neighbour in $V(C) \setminus \{c_1, c_2, c_3\}$. Let C' be the hole formed by the union of the paths $C \setminus c_2$, c_1-p_1-$\cdots$-p_k-c_3. Then it has length ≥ 6, and it contains fewer Y-complete edges than C. From the choice of (C, Y) it follows that (C', Y) is not a wheel, and since C has at least 4 Y-complete edges, and C' has only two fewer, it follows that exactly 4 edges of C are Y-complete. This proves (1).

Since (C, Y) is not an odd wheel, there are vertices $x_0, z, x_1, c_1, c_2, c_3$ of C, in order, and all distinct except possibly $x_1 = c_1$ or $c_3 = x_0$, such that the Y-complete edges in C are $x_0z, zx_1, c_1c_2, c_2c_3$. Let $A_0 = V(C) \setminus \{z, x_0, x_1\}$. Since G does not admit a skew partition, there is a path T of $G \setminus \{x_0, x_1\}$ from z to A_0, such that no vertex in its interior is in Y or Y-complete. Let y be the neighbour of z in T.

(2) *y is not adjacent to both x_0, x_1.*

For assume it is. By 22.3 there is no kite for (C, Y), and with respect to the wheel (C, Y), T is a tail for z (because at least one of the Y-complete edges c_1c_2, c_2c_3 belongs to $C \setminus \{x_0, z, x_1\}$). This contradicts 22.5, and therefore proves (2).

(3) *y has no neighbour in A_0.*

For suppose first that it has a neighbour in $A_0 \setminus \{c_2\}$, say c. Then c, z are nonadjacent and have opposite wheel-parity in the wheel (C, Y); it is not the case that c and both its neighbours in C are Y-complete, by (1) and the fact that $c \in A_0$; not both neighbours of z in C are adjacent to y, by (2); so 16.1 implies that $(C, Y \cup \{y\})$ is a wheel, a contradiction. So y has no neighbour in $A_0 \setminus \{c_2\}$. Next suppose that y is adjacent to c_2. From the symmetry we may assume that $x_0 \neq c_3$. Let Q be the path of $C \setminus z$ between x_0, c_3; so Q has length > 0, and even length by 2.3. Since x_0-Q-c_3-c_2-y-x_0 is not an odd hole, it follows that y is not adjacent to x_0. But then the hole x_0-Q-c_3-c_2-y-z-x_0 is the rim of an odd wheel with hub Y, contrary to $G \in \mathcal{F}_8$. So y is not adjacent to c_2. This proves (3).

Let T have vertices z-y-v_1-$\cdots$-v_{n+1}, where $v_{n+1} \in A_0$. From (3), $n \geq 1$. By choosing T of minimum length we may assume that none of $y, v_1, \ldots, v_{n-1}$ have neighbours in A_0.

(4) *If $n = 1$ then no neighbour of v_1 in A_0 is Y-complete.*

For otherwise we may assume v_2 is Y-complete. From the symmetry we may assume that $x_0 \neq c_3$. Let Q be the path of $C \setminus z$ between x_0, c_3; so Q has length > 0, and even length by 2.3. Since y, v_1 are not Y-complete, there is an antipath joining them with interior in Y, and it is odd since it can be completed to an antihole via v_1-z-v_2-y. Hence every Y-complete vertex is adjacent to one of y, v_1, and since c_2, c_3 are Y-complete and not adjacent to y by (3), it follows that v_1 is adjacent to c_2, c_3. By (2), v_1 is adjacent to one of x_0, x_1, and so it has two nonadjacent neighbours in C, and two neighbours in C of opposite wheel-parity. By 16.1, there are three consecutive vertices in C, all Y-complete and adjacent to v_1. By 22.3, v_1 has no other neighbours in C. Hence $x_1 = c_1$ and the neighbours of v_1 in C are c_1, c_2, c_3. Consequently x_0 is adjacent to y; but then x_0-Q-c_3-v_1-y-x_0 is an odd hole, a contradiction. This proves (4).

(5) *One of* x_0, x_1 *has no neighbours in* $\{y, v_1, \ldots, v_n\}$.

For let P be a path y-p_1-$\cdots$-p_k from y to some Y-complete vertex $p_k \in A_0$, with interior in $A_0 \cup \{v_1, \ldots, v_n\}$, such that p_k is the only Y-complete vertex in P. Since none of $y, v_1, \ldots, v_{n-1}$ have neighbours in A_0 it follows that $\{y, v_1, \ldots, v_n\} \subseteq \{y, p_1, \ldots, p_{k-1}\}$. From (4), $k \geq 3$. Since $G \in \mathcal{F}_8$, $(Y, \{x_0, x_1\}, z$-y-p_1-$\cdots$-$p_k)$ is not a pseudowheel. But the ends of the path z-y-p_1-$\cdots$-p_k are Y-complete and its internal vertices are not; the path has length ≥ 4 (and therefore has even length by 13.6); Y, z are $\{x_0, x_1\}$-complete, and y, p_k are not. So no other vertices of the path are $\{x_0, x_1\}$-complete. By 2.11, applied to the same path and the same anticonnected sets, it follows that one of x_0, x_1 is nonadjacent to all of $y, p_1, \ldots, p_{k-1}$. Since $\{y, v_1, \ldots, v_n\} \subseteq \{y, p_1, \ldots, p_{k-1}\}$, this proves (5).

Let $F = \{y, v_1, \ldots, v_n\}$. From the symmetry we may assume that x_0 has no neighbours in F. Let S be a path from y to x_0 with interior in $F \cup A_0$. It follows that S has length ≥ 3. Let C' be the hole z-y-S-x_0-z; so C' has length ≥ 6. Suppose that x_0 is different from c_3. Since (C', Y) is not an odd wheel, it follows that (C', Y) is not a wheel, and so x_0, z are the only Y-complete vertices in C'. By 2.10, Y contains a leap or a hat. A leap would imply there are two vertices in Y, joined by an odd path of length ≥ 5 with interior in $F \cup A_0$. Hence its ends are $\{x_0, x_1\}$-complete, and its internal vertices are not, contrary to 13.6. So Y contains a hat, that is, there exists $y' \in Y$ with no neighbour in C' except z, x_0. But $F \cup A_0$ catches the triangle $\{x_0, y', z\}$; the only neighbour of x_0 in $F \cup A_0$ is its neighbour in S, say s; the only neighbour of z in $F \cup A_0$ is y; and s, y are nonadjacent, and both nonadjacent to y', contrary to 17.1. This proves that $x_0 = c_3$, and therefore $x_1 \neq c_1$. By exchanging x_0, x_1, we deduce that x_1 has a neighbour in F. There are therefore two attachments of F in C with opposite wheel-parity, and two that are nonadjacent. By (1), 16.2, 22.3 and the optimality of the wheel, and since $x_0 = c_3$ has no neighbour in F, it follows that there is a path R between z, c_2 with interior in F, and no vertex of C has neighbours in the interior of R except z, c_2. But then the hole formed by the union of R and the path $C \setminus x_0$ is the rim of an odd wheel with hub Y, a contradiction. This proves 23.2. ∎

23.3 *Let* $G \in \mathcal{F}_9$*, admitting no balanced skew partition, let* (z, A_0) *be a frame and* $x_0, \ldots, x_s$ *a wheel system with respect to it, and define* X_i, A_i *as usual. Then there is no vertex* $y \in V(G) \setminus \{z, x_0, \ldots, x_s\}$ *that is* $\{z, x_0, \ldots, x_s\}$*-complete and has a neighbour in* A_s.

Proof. Suppose there is such a frame, wheel system, and y, and choose them with s minimum (it is important here that we minimize over all choices of the frame, not just of the wheel system); say (z, A_0), $x_0, \ldots, x_s$ and y respectively. By 22.2, there exists r with $1 \leq r < s$, and a vertex v such that y has no neighbour in $A_r \cup \{v\}$, and v is adjacent to z, and has a neighbour in A_r, and a non-neighbour in X_r. Then (y, A_0) is a frame, and $x_0, \ldots, x_r$ is a wheel system with respect to it, and z is $\{y, x_0, \ldots, x_r\}$-complete, and has a neighbour in A'_r (namely v), where A'_r is the maximal connected subset of $V(G)$ including A_0 and containing no neighbour of y and no X_r-complete vertex. But this contradicts the minimality of s. This proves 23.3. ∎

Now we can prove 1.8.10, the following.

23.4 *Let $G \in \mathcal{F}_9$, admitting no balanced skew partition, and let C be a hole in G of length ≥ 6. Then there is no vertex of $G \setminus V(C)$ with three consecutive neighbours in C. In particular, every recalcitrant graph belongs to $\mathcal{F}_{10}$.*

Proof. Suppose that there is such a vertex, say y, and let it be adjacent to $x_0, z, x_1 \in V(C)$, where x_0-z-x_1 is a path. Let $A_0 = V(C) \setminus \{z, x_0, x_1\}$. By 23.3 applied to (z, A_0) and x_0, x_1, it follows that y has no other neighbour in C. Choose t maximum such that there is a sequence $x_2, \ldots, x_t$ with the following properties:

- for $2 \leq i \leq t$, there is a connected subset A_{i-1} of $V(G)$ including A_{i-2}, containing a neighbour of x_i, containing no neighbour of z or y, and containing no $\{x_0, \ldots, x_{i-1}\}$-complete vertex,
- for $1 \leq i \leq t$, x_i is not $\{x_0, \ldots, x_{i-1}\}$-complete, and
- $x_0, \ldots, x_t$ are $\{y, z\}$-complete.

Since G admits no skew partition by 15.1, there is a path P from $\{z, y\}$ to A_0, disjoint from $\{x_0, \ldots, x_t\}$ and containing no $\{x_0, \ldots, x_t\}$-complete vertex in its interior. Choose such a path of minimum length. From the symmetry between z, y we may assume its first vertex is y; say the path is y-p_1-$\cdots$-p_{k+1}, where $p_{k+1} \in A_0$. From the minimality of the length of P it follows that z is not adjacent to any of $p_2, \ldots, p_k$. If z is adjacent to p_1 then we may set $x_{t+1} = p_1$, contrary to the maximality of t. So $p_1, \ldots, p_{k+1}$ are all nonadjacent to z. Hence (z, A_0) is a frame, and $x_0, \ldots, x_t$ is a wheel system with respect to it, and y is adjacent to all of $z, x_0, \ldots, x_t$, and there is a connected subset of $V(G)$ including A_0, containing a neighbour of y, containing no neighbour of z, and containing no $\{x_0, \ldots, x_t\}$-complete vertex. But this contradicts 23.3. This proves 23.4. ∎

This has the following useful corollary, which is 1.8.11.

23.5 *Let $G \in \mathcal{F}_{10}$; then G does not contain both a hole of length ≥ 6 and an antihole of length ≥ 6. In particular, for every recalcitrant graph G, one of $G, \overline{G}$ belongs to $\mathcal{F}_{11}$.*

Proof. Let C be a hole and D an antihole, both of length ≥ 6. Let $W = V(C) \cap V(D)$, $A = V(C) \setminus W$, and $B = V(D) \setminus W$. Let W, A, B have cardinality w, a, b respectively. Let there be p edges between A and W, q edges between B and W, r edges between A and B, and s edges with both ends in W. Let there be p' nonedges between A and W, q' nonedges between B and W, r' nonedges between A and B, and s' nonedges with both ends in W. By 2.3, and since $G \in \mathcal{F}_{10}$, every vertex in B has at most $\frac{1}{2}(a + w)$ neighbours in C, so $q + r \leq \frac{1}{2}(a + w)b$. Also, every vertex in W has at most two neighbours in $A \cup W$, so $p + 2s \leq 2w$. Summing, we obtain

$$p + q + r + 2s \leq \frac{1}{2}ab + \frac{1}{2}bw + 2w.$$

By the same argument in the complement we deduce that

$$p' + q' + r' + 2s' \leq \frac{1}{2}ab + \frac{1}{2}aw + 2w.$$

But

$$p + p' + q + q' + r + r' + 2s + 2s' = ab + aw + bw + w(w - 1),$$

so

$$4w \geq \frac{1}{2}aw + \frac{1}{2}bw + w(w-1),$$

that is,

$$w(a+b+2w-10) \leq 0.$$

Since $a+w, b+w \geq 6$, it follows that $w = 0$, and so C, D are disjoint. Moreover, equality holds throughout this calculation, so every vertex in D is adjacent to exactly half the vertices of C and vice versa. By 2.3, and since $G \in \mathcal{F}_{10}$, it follows that for each $v \in D$, its neighbours in C are pairwise nonadjacent. Let C have vertices $c_1, \dots, c_m$ in order, and let D have vertices $d_1, \dots, d_n$. So for every vertex of D, its set of neighbours in $V(C)$ is either the set of all c_i with i even, or the set with i odd, and the same with C, D exchanged. We may assume that c_1 is adjacent to d_1. Hence the edges between $\{c_1, c_2, c_4, c_5\}$ and $\{d_1, d_2, d_4, d_5\}$ are $c_1d_1, c_1d_5, c_2d_2, c_2d_4, c_4d_2, c_4d_4, c_5d_1, c_5d_5$; and so the subgraph induced on these eight vertices is a double diamond, contrary to $G \in \mathcal{F}_{10}$. This proves 23.5. ∎

Let us mention a theorem of [12], which could be applied at this stage as an alternative to the next section, the following (and see also [8] for some related material):

23.6 *Let $G \in \mathcal{F}_5$. Suppose that for every hole C in G of length ≥ 6, and every vertex $v \in V(G) \setminus V(C)$, either:*

- *v has ≤ 3 neighbours in C, or*
- *v has exactly 4 neighbours in C, say a, b, c, d, where ab and cd are edges, or*
- *v is $V(C)$-complete, or*
- *no two neighbours of v in C are adjacent.*

Suppose also that the same holds in $\overline{G}$. Then either one of $G, \overline{G}$ is bipartite or a line graph of a bipartite graph, or G admits a loose skew partition.

The method we give below is somewhat shorter than the proof of 23.6 in [12], however.

24 The end

We recall that we are trying to prove 13.5. In view of 23.5, it suffices to show the following, which is 1.8.12, and the objective of the remainder of the paper:

24.1 *Let $G \in \mathcal{F}_{11}$; then either G is complete, or G is bipartite, or G admits a balanced skew partition.*

We begin with a further strengthening of 13.6, as follows.

24.2 *Let $G \in \mathcal{F}_{11}$, and let P be a path in G with odd length. Let $X \subseteq V(G)$ be anticonnected, such that both ends of P are X-complete. Then some edge of P is X-complete.*

Proof. Suppose not; then from 13.6, P has length 3 (let its vertices be p_1, p_2, p_3, p_4 in order) and p_2, p_3 are joined by an antipath Q with interior in X. But then p_2-Q-p_3-p_1-p_4-p_2 is an antihole of length > 4, a contradiction. This proves 24.2. ∎

24.3 *Let $G \in \mathcal{F}_{11}$. Let $X \subseteq V(G)$ be nonempty and anticonnected, and let p_1-$\cdots$-p_n be a path of $G \setminus X$ with $n \geq 4$, such that p_1, p_n are X-complete and $p_2, \ldots, p_{n-1}$ are not. There is no vertex $y \in V(G) \setminus (X \cup \{p_1, \ldots, p_n\})$ such that y is X-complete and adjacent to p_1, p_2.*

Proof. Suppose such a vertex y exists. By 24.2, n is odd, and therefore $n \geq 5$. Let Q be an antipath joining p_2, p_3 with interior in X. Since Q can be completed to an antihole via p_3-p_n-p_2, it follows that Q has length 2, and so there exists $x \in X$ nonadjacent to p_2, p_3. Since x is adjacent to p_n, we may choose i with $2 \leq i \leq n$ minimum such that x is adjacent to p_i. Hence x-p_1-$\cdots$-p_i-x is a hole of length ≥ 6, and y has three consecutive neighbours in it, contrary to $G \in \mathcal{F}_{11}$. This proves 24.3. ∎

The next is a strengthening of 17.1.

24.4 *Let $G \in \mathcal{F}_{11}$. Let X_1, X_2, X_3 be disjoint nonempty anticonnected sets, complete to each other. Let $F \subseteq V(G) \setminus (X_1 \cup X_2 \cup X_3)$ be connected, such that for $i = 1, 2, 3$ there is an X_i-complete vertex in F. Then there is a vertex in F complete to two of X_1, X_2, X_3.*

Proof. Suppose not; then we may assume F is minimal with this property.

(1) *If $p_1, \ldots, p_n$ is a path in F, and p_1 is its unique X_1-complete vertex and p_n is its unique X_2-complete vertex then n is even.*

For $n > 1$, since no vertex is both X_1-complete and X_2-complete. Assume n is odd; then by 13.7, $n = 3$. But there is an antipath Q_1 between p_2, p_3 with interior in X_1, and an antipath Q_2 between p_1, p_2 with interior in X_2; and then p_2-Q_1-p_3-p_1-Q_2-p_2 is an antihole of length > 4, a contradiction. This proves (1).

From the minimality of F, there are (up to symmetry) three cases:

1. For $i = 1, 2, 3$ there is a unique X_i-complete vertex $v_i \in F$; there is a vertex $u \in F$ different from v_1, v_2, v_3, and three paths P_1, P_2, P_3 in F, all of length ≥ 1, such that each P_i is from v_i to u, and for $1 \leq i < j \leq 3$, $V(P_i \setminus u)$ is disjoint from $V(P_j \setminus u)$ and there is no edge between them.

2. For $i = 1, 2, 3$ there is a unique X_i-complete vertex $v_i \in F$; there are three paths P_1, P_2, P_3 in F, where each P_i is from v_i to some u_i say, possibly of length 0; and for $1 \leq i < j \leq 3$, $V(P_i)$ is disjoint from $V(P_j)$ and the only edge between $V(P_i), V(P_j)$ is $u_i u_j$.

3. For $i = 1, 2$ there is a unique X_i-complete vertex $v_i \in F$, and there is a path P in F between v_1, v_2 containing at least one X_3-complete vertex.

Suppose that the first holds, and let P_1, P_2, P_3 be as in the first case. Then some two of P_1, P_2, P_3 have lengths of the same parity, and their union violates (1).

Now suppose the second holds, and for $i = 1, 2, 3$ let u_i, v_i, P_i be as in the second case. Let Q_1 be an antipath joining u_2, u_3 with interior in X_1, and define Q_2, Q_3 similarly. If P_1, P_2, P_3 all have length 0, then the union of Q_1, Q_2, Q_3 is an antihole of length > 4, a contradiction. So we may assume that P_1 has length > 0, and hence $u_1 \neq v_1$. Since v_1-u_2-Q_1-u_3-v_1 is an antihole, Q_1 has length 1. Since u_1, u_3 are not X_1-complete, they are joined by an antipath with interior in X_1, and its union with Q_2 is an antihole; so Q_2 has length 2, and similarly so does Q_3. For $i = 1, 2, 3$ let x_i be the middle vertex of Q_i. So

$$V(P_1 \setminus u_1) \cup V(P_2 \setminus u_2) \cup V(P_3 \setminus u_3) \cup \{x_1, x_2, x_3\}$$

is connected, and catches the triangle $\{u_1, u_2, u_3\}$; and none of its vertices have two neighbours in the triangle, and it contains no reflection of the triangle since there is no antihole of length 6. This is contrary to 17.1.

Now suppose the third holds, and let v_1, v_2, P be as in the third case. Let P have vertices $p_1, \ldots, p_n$ where $v_1 = p_1$ and $v_2 = p_n$. Since one of its vertices is X_3-complete and p_1, p_n are not, it follows that $n \geq 3$; and by (1), n is odd, so $n \geq 4$. Choose i minimum and j maximum with $1 \leq i, j \leq n$ such that p_i, p_j are X_3-complete. So $i > 1$, and i is even by (1), and similarly $j < n$ and j is odd. So the path p_i-$\cdots$-p_j has odd length, and so by 24.2 one of its edges is X_3-complete, say $p_k p_{k+1}$ where $2 \leq k \leq n-2$. Now p_k, p_{k+1} are joined by an antipath with interior in X_1, and by another with interior in X_2, and the union of these is an antihole; so they both have length 2. Hence for $i = 1, 2$ there exist $x_i \in X_i$ nonadjacent to both p_k, p_{k+1}. Let R be a path between p_{k+2}, p_{k-1} with interior in $(V(P) \setminus \{p_k, p_{k+1}\}) \cup \{x_1, x_2\}$. Then R can be completed to a hole C via p_{k-1}-p_k-p_{k+1}-p_{k+2}, and C has length ≥ 6, and at least one edge of C is X_3-complete, namely $p_k p_{k+1}$, and at least one more vertex of it is X_3-complete, since R uses at least one of x_1, x_2. But this contradicts 2.3, and the hypothesis that $G \in \mathcal{F}_{11}$.

This proves 24.4. ∎

24.5 *Let $G \in \mathcal{F}_{11}$, admitting no balanced skew partition. Let X, Y be disjoint anticonnected subsets of $V(G)$, complete to each other, and let p_1-$\cdots$-p_n be a path of $G \setminus (X \cup Y)$, with $n \geq 2$, such that p_1 is the unique X-complete vertex in the path, and p_n is the unique Y-complete vertex. Then there is no $z \in V(G) \setminus (X \cup Y \cup \{p_1, \ldots, p_n\})$, complete to $X \cup Y$ and nonadjacent to p_1, p_n.*

Proof. Suppose that z exists, and choose X maximal. By 15.2, there is a path Q in G from z to p_1, such that none of its internal vertices is in X or is X-complete. Since no vertex of $\{p_2, \ldots, p_n\}$ is X-complete, we may choose Q such that if z has a neighbour in $\{p_2, \ldots, p_n\}$ then $V(Q) \subseteq \{z, p_1, \ldots, p_n\}$. The connected subset $V(Q \setminus z) \cup \{p_1, \ldots, p_n\}$ ($= F$ say) contains an X-complete vertex, a Y-complete vertex, and a $\{z\}$-complete vertex. The only X-complete vertex in F is p_1, and that is not Y-complete or $\{z\}$-complete; so by 24.4 some vertex in F is Y-complete and adjacent to z. If z has a neighbour in $\{p_1, \ldots, p_n\}$, then $V(Q) \subseteq \{z, p_1, \ldots, p_n\}$, and so p_n is the only vertex of F that is Y-complete; and it is not adjacent to z, a contradiction. So z has no neighbour in $\{p_1, \ldots, p_n\}$, and therefore only one vertex in F is adjacent to z, the neighbour of z in Q, say q. Hence q is nonadjacent to p_1, for otherwise we could add q to X, contrary to the maximality of X. Consequently Q has length > 2. This contradicts 24.3 applied to Q,X and any vertex $y \in Y$. This proves 24.5. ∎

We deduce

24.6 *Let $G \in \mathcal{F}_{11}$, admitting no balanced skew partition, and let C be a hole. If $z \in V(G) \setminus V(C)$ has two neighbours in C that are adjacent, then C has length 4 and z has a third neighbour in C. In particular, G has no antipath of length 4.*

Proof. Let C be the hole with vertices $p_1, \ldots, p_{n+2}$ in order, and assume some $z \in V(G) \setminus V(C)$ is adjacent to p_{n+1}, p_{n+2}. By 24.5, taking $X = \{p_{n+1}\}$ and $Y = \{p_{n+2}\}$ we deduce that z is adjacent to at least one of p_1, p_n. Since $G \in \mathcal{F}_{11}$ it follows that C has length 4. This proves 24.6. ∎

24.7 *Let $G \in \mathcal{F}_{11}$, admitting no balanced skew partition. Let X_1, X_2, X_3 be pairwise disjoint, nonempty, anticonnected subsets of $V(G)$, complete to each other. Let $F \subseteq V(G) \setminus (X_1 \cup X_2 \cup X_3)$ be connected, such that for at least two values of $i \in \{1,2,3\}$, every member of X_i has a neighbour in F. Then some vertex of F is complete to two of X_1, X_2, X_3.*

Proof. Assume not, and choose a counterexample with $X_1 \cup X_2 \cup X_3 \cup F$ minimal. Suppose F contains an X_i-complete vertex for two values of $i \in \{1,2,3\}$, say $i = 1,2$; and choose a path p_1-$\cdots$-p_n of F such that p_1 is X_1-complete and p_n is X_2-complete, with n minimum. So $n \geq 2$. From the minimality of F, $F = V(P)$, and there is a vertex $x_1 \in X_1$ such that p_1 is its only neighbour in F, and there exists $x_2 \in X_2$ such that p_n is its only neighbour in F. By 24.6 applied to the hole x_2-x_1-p_1-$\cdots$-p_n-x_2 and any $x_3 \in X_3$, it follows that $n = 2$. Let Q be an antipath between p_1, p_2 with interior in X_3; since p_1 has a nonneighbour $x_2 \in X_2$, and p_2 has a nonneighbour $x_1 \in X_1$, it follows that x_1-p_2-Q-p_1-x_2 is an antipath of length ≥ 5, contrary to 24.5.

So there is at most one i such that F contains X_i-complete vertices, and from the symmetry we may assume that F contains no X_1- or X_2-complete vertices. We may also assume that all members of X_1 have neighbours in F, and therefore $|X_1| \geq 2$; choose distinct $x_1, x_1' \in X_1$ such that $X_1 \setminus \{x_1\}, X_1 \setminus \{x_1'\}$ are both anticonnected. From the minimality of X_1, there is a vertex f of F complete to two of $X_1 \setminus \{x_1\}, X_2, X_3$, and therefore complete to $X_1 \setminus \{x_1\}$ and X_3, and similarly a vertex f' of F complete to $X_1 \setminus \{x_1'\}$ and X_3. Let P be a path in F between f, f'. Since all vertices of $X_1 \cup X_3$ have neighbours in $V(P)$, the minimality of F implies that $F = V(P)$; and moreover, since all vertices of $(X_1 \setminus \{x_1\}) \cup X_3$ are adjacent to f, the minimality of F implies that f' is the unique neighbour of x_1 in F. Similarly f is the unique neighbour of x_1' in F. Let Q be an antipath in X_1 joining x_1, x_1'. Since f has a nonneighbour $x \in X_2$, x-f-x_1-Q-x_1' is an antipath, and so Q has length 1, and hence x_1, x_1' are nonadjacent. From the minimality of F, there exists $x_2 \in X_2$ with no neighbour in $F \setminus \{f\}$. If x_2 is also nonadjacent to f, then x_2-x_1-f'-P-f-x_1'-x_2 is a hole of length ≥ 6, and any member of X_3 has three consecutive neighbours on it, contrary to $G \in \mathcal{F}_{11}$. But then x_1 has two consecutive neighbours on the hole x_1'-f-P-f'-x_1-x_2-x_1', and this hole has length > 4, contrary to 24.6. This proves 24.7. ∎

Now we can complete the proof of 24.1, and hence of 13.5 and therefore of 1.3 and 1.2, as follows.
Proof of 24.1. Let $G \in \mathcal{F}_{11}$, admitting no balanced skew partition. We may assume that G is not bipartite, and therefore has a triangle. Consequently we may choose disjoint nonempty anticonnected sets $X_1, \ldots, X_k$, complete to each other, with $k \geq 3$, with maximal union. Suppose first that $X_1 \cup \cdots \cup X_k \neq V(G)$, and let $F = V(G) \setminus (X_1 \cup \cdots \cup X_k)$. By 15.2 (applied to X_k and $X_1 \cup \cdots \cup X_{k-1}$), F is connected and every vertex of $X_1 \cup X_2$ has a neighbour in it. By 24.7, some vertex $v \in F$ is complete to two of X_1, X_2, X_3. We may assume that for some i with $2 \leq i \leq k$, v is X_j-complete for $1 \leq j \leq i$ and not X_j-complete for $i < j \leq k$. Define

$$X_{i+1}' = X_{i+1} \cup \cdots \cup X_k \cup \{v\};$$

then the sets $X_1, \ldots, X_i, X_{i+1}'$ violate the optimality of the choice of $X_1, \ldots, X_k$.

Hence $X_1 \cup \cdots \cup X_k = V(G)$, and therefore $\overline{G}$ has at least three components. From 15.2 it follows that G is complete. This proves 24.1. ∎

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