SUB-EXPONENTIALLY MANY 3-COLORINGS OF TRIANGLE-FREE PLANAR GRAPHS

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ABSTRACT

Thomassen conjectured that every triangle-free planar graph on n vertices has exponentially many 3-colorings, and proved that it has at least $2^{n^{1/12}/20000}$ distinct 3-colorings. We show that it has at least $2^{\sqrt{n/362}}$ distinct 3-colorings.

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1 Introduction

All graphs in this paper are finite, and have no loops or multiple edges. Our terminology is standard, and may be found in [2] or [3]. In particular, cycles and paths have no repeated vertices. The following is a well-known theorem of Grötzsch [7].

Theorem 1.1 *Every triangle-free planar graph is* 3*-colorable.*

Theorem 1.1 has been the subject of extensive research. Thomassen [11] gave several short proofs [11, 12, 13] of Grötzsch's theorem and extended it to projective planar and toroidal graphs. The theorem does not extend verbatim to any non-planar surface, but Thomassen proved that every graph of girth at least five embedded in the projective plane or the torus is 3-colorable. Gimbel and Thomassen [6] found an elegant characterization of 3-colorability for triangle-free projective planar graphs. There does not seem to be a corresponding counterpart for other surfaces, but Král' and Thomas [9] found a characterization of 3-colorability for toroidal and Klein bottle graphs that are embedded with all faces even. It was an open question for a while whether a 3-coloring of a triangle-free planar graph can be found in linear time. First Kowalik [8] designed an almost linear time algorithm, and then a linear-time algorithm was found by Dvořák, Kawarabayashi and Thomas in [4]. For a general surface Σ, Dvořák, Král' and Thomas [5] found a linear-time algorithm to decide whether a triangle-free graph in Σ is 3-colorable.

In this paper we study how many 3-colorings a triangle-free planar graph must have. Thomassen conjectured in [15] that exponentially many:

Conjecture 1.2 *There exists an absolute constant $c > 0$, such that if G is a triangle-free planar graph on n vertices, then G has at least 2^{cn} distinct* 3*-colorings.*

Thomassen gave a short proof of this conjecture under the additional hypothesis that G has girth at least five. We use that argument in Lemma 2.3 below; Thomassen's original proof may be recovered by taking $\mathcal{F}$ to be the set of all facial cycles. Thomassen [15] then extended this result by showing that every planar graph of girth at least five has exponentially many list-colorings for every list assignment that gives each vertex a list of size at least three. For triangle-free graphs Thomassen [15] proved a weaker version of Conjecture 1.2, namely that every triangle-free planar graph on n vertices has at least $2^{n^{1/12}/20000}$ distinct 3-colorings. Our main result is the following improvement.

Theorem 1.3 *Every triangle-free planar graph on n vertices has at least $2^{\sqrt{n/362}}$ distinct* 3*-colorings.*

In closely related work Thomassen [14] proved that every (not necessarily triangle-free) planar graph has exponentially many list colorings provided every vertex has at least five available colors.

Our paper is organized as follows. In the next section we investigate non-crossing families of 5-cycles, and reduce Theorem 1.3 to Lemma 2.4, which states that if a triangle-free planar graph has k nested 5-cycles, then it has at least $2^{k/12}$ 3-colorings. The rest of the paper is devoted to a proof of Lemma 2.4, which we complete in Section 4. In Section 3 we prove an auxiliary result stating that some entries in the product of certain matrices grow exponentially in the number of matrices.

We end this section by stating two useful theorems of Thomassen [11].

Theorem 1.4 *Let G be a plane graph of girth at least five and $C = v_1v_2 \dots v_k$ be an induced facial cycle of G of length $k \leq 9$. Then a 3-coloring Φ of C extends to a 3-coloring of G, unless $k = 9$ and there exists a vertex $v \in V(G) - V(C)$ such that v is adjacent to three vertices of C that received three different colors under Φ.*

Theorem 1.5 *Let G be a triangle-free plane graph with facial cycle C of length at most five. Then every 3-coloring of C extends to a 3-coloring of G.*

We would like to acknowledge that an extended abstract of this paper appeared in [1].

2 Laminar Families of 5-Cycles

First we define some terminology. Let A and B be two subsets of $\mathbb{R}^2$. We say that A and B *cross* if $A \cap B$, $A \cap B^c$, $A^c \cap B$, $A^c \cap B^c$ are all non-null. Then we say that a family $\mathcal{F}$ of subsets of $\mathbb{R}^2$ is *laminar* if for every two sets $A, B \in \mathcal{F}$, A and B do not cross. Now let G be a plane graph and C be a cycle in G. Then we let $Int(C)$ denote the bounded component of $\mathbb{R}^2 - C$ and $Ext(C)$ denote the unbounded component of $\mathbb{R}^2 - C$. Now we say that a family $\mathcal{F}$ of cycles of G is *laminar* if the corresponding family of sets $\bigcup_{C \in \mathcal{F}} Int(C)$ is laminar. We call a family $\mathcal{F}$ of cycles an *antichain* if $Int(C_1) \cap Int(C_2) = \emptyset$ for every distinct $C_1, C_2 \in \mathcal{F}$, and we call it a *chain* if for every two cycles $C_1, C_2 \in \mathcal{F}$, either $Int(C_1) \subseteq Int(C_2)$ or $Int(C_2) \subseteq Int(C_1)$.

Let G be a triangle-free plane graph, and let $v \in V(G)$. We define G_v to be the graph obtained from G by deleting v, identifying all the neighbors of v to one vertex, and deleting resulting parallel edges. We also let $D_k(G)$ denote the set of vertices of G with degree at most k.

Lemma 2.1 *If G is a triangle-free plane graph and $k \geq 0$ is an integer, then either*

(i) there exists $v \in D_k(G)$ such that G_v is triangle-free or,

(ii) there exists a laminar family $\mathcal{F}$ of 5-cycles such that every $v \in D_k(G)$ belongs to some member of $\mathcal{F}$.

Proof. We proceed by induction on the number of vertices of G. Suppose condition (i) does not hold. Notice that if $v \in V(G)$ and G_v is not triangle-free, this implies, since G is triangle-free, that v is in a 5-cycle in G. Hence if condition (i) does not hold, every $v \in D_k(G)$ must be in a 5-cycle in G.

Now suppose there does not exist a separating 5-cycle in G. Then we let $\mathcal{F}$ be the set of all 5-cycles in G. The second condition then holds since the absence of separating cycles implies that $\mathcal{F}$ is laminar.

Thus we may assume that there exists a 5-cycle C that separates G into two triangle-free plane graphs G_1 and G_2, where both G_1 and G_2 include C. By induction, the lemma holds for G_1 and G_2. Suppose that both G_1 and G_2 satisfy condition (ii) with laminar families $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively. Then let $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$. Note that $\mathcal{F}$ is laminar. Now G satisfies condition (ii) since every $v \in D_k(G)$ is contained in either $D_k(G_1)$ or $D_k(G_2)$. Thus we may assume without loss of generality that G_1 satisfies condition (i). That is, there exists $v \in D_k(G_1)$ such that $(G_1)_v$ is triangle-free. This implies that v is not in a 5-cycle in G_1. In particular, $v \notin V(C)$, and hence $v \in D_k(G)$. Yet since G_v is not triangle-free by assumption, v must be in a 5-cycle in G, say C'. It follows that C' intersects C. Since G is triangle-free, C and C' intersect in exactly two vertices u_1 and u_2. Now the path from u_1 to u_2 along C' that includes v must have t edges, where $t \in \{2, 3\}$. But then there is another path from u_1 to u_2 along C with $5 - t$ edges. Hence v is in a 5-cycle in G_1, a contradiction. □

Lemma 2.2 *If G is a triangle-free plane graph on n vertices, then either*

(i) there exists $v \in D_k(G)$ such that G_v is triangle-free, or

(ii) G has an antichain $\mathcal{F}$ of 5-cycles such that $|\mathcal{F}| \geq \sqrt{\frac{(k-3)n}{10(k-1)}}$, or

(iii) G has a chain $\mathcal{F}$ of 5-cycles such that $|\mathcal{F}| \geq \sqrt{\frac{2(k-3)n}{5(k-1)}}$.

Proof. Since G is triangle-free and planar, it satisfies $2|V(G)| \geq |E(G)|$. We may assume that (i) does not hold and hence every vertex of G has degree at least two. It follows that

$$4|V(G)| \geq 2|E(G)| = \sum_{v \in V(G)} deg(v) \geq (k+1)\left(|V(G)| - |D_k(G)|\right) + 2|D_k(G)|,$$

and hence $|D_k(G)| \geq \frac{k-3}{k-1}|V(G)|$. Since (i) does not hold, we deduce from Lemma 2.1 that there exists a laminar family of 5-cycles $\mathcal{G}$ of size at least $|D_k(G)|/5 \geq \frac{k-3}{5(k-1)}n$. By Dilworth's theorem applied to the partial order on $\mathcal{G}$ defined by $Int(C_1) \subseteq Int(C_2)$ we deduce that $\mathcal{G}$ has either an antichain of size at least $\sqrt{|\mathcal{G}|/2}$, in which case condition (ii) holds, or a chain of size at least $\sqrt{2|\mathcal{G}|}$, in which case condition (iii) holds. □

Lemma 2.3 *Let G be a triangle-free plane graph. If G has an antichain $\mathcal{F}$ of 5-cycles, then G has at least $2^{|\mathcal{F}|/6}$ distinct 3-colorings.*

Proof. Let G' be obtained from G by deleting the vertices in $\bigcup_{C\in\mathcal{F}} Int(C)$. Now G' has at least $|\mathcal{F}|$ facial 5-cycles. By Euler's formula $|E(G')| \leq 2|V(G')| - |\mathcal{F}|/2$. By Theorem 1.1 the graph G' has a 3-coloring Φ. For $i, j \in \{1, 2, 3\}$ with $i < j$ let G_{ij} denote the subgraph of G induced by the vertices colored i or j. Since $\sum_{i<j}(|V(G_{ij}| - |E(G_{ij}|) = 2|V(G')| - |E(G')| \geq |\mathcal{F}|/2$, there exist $i, j \in \{1, 2, 3\}$ such that $i < j$ and G_{ij} has at least $|\mathcal{F}|/6$ components. But then there are at least $2^{|\mathcal{F}|/6}$ distinct 3-colorings of G' since switching the colors on any subset of the components of G_{ij} gives rise to a distinct coloring of G'. Furthermore, every 3-coloring of G' extends to a 3-coloring of G by Theorem 1.5. □

Lemma 2.4 *Let G be a triangle-free plane graph. If G has a chain $\mathcal{F}$ of 5-cycles, then G has at least $24 \cdot 2^{|\mathcal{F}|/12}$ distinct 3-colorings.*

We will prove Lemma 2.4 in Section 4, but now we deduce the main theorem from it.

Proof of Theorem 1.3, assuming Lemma 2.4. We proceed by induction on the number of vertices. If $n \leq 362$, then the conclusion clearly holds. We may therefore assume that $n \geq 363$ and that the theorem holds for all graphs on fewer than n vertices. If there exists $v \in D_{363}(G)$ such that the graph G_v (defined prior to Lemma 2.1) is triangle-free, then by induction G_v has at least $2^{\sqrt{(n-\deg(v))/362}}$ distinct 3-colorings. Hence G has at least $2 \times 2^{\sqrt{(n-\deg(v))/362}}$ distinct 3-colorings, which is greater than $2^{\sqrt{n/362}}$ since $\deg(v) \leq 363$. So we may assume by Lemma 2.2 applied to $k = 363$ that G has either an antichain of 5-cycles of size at least $\sqrt{36n/362}$, in which case the theorem holds by Lemma 2.3; or a chain of 5-cycles of size at least $\sqrt{720n/1810}$, in which case the theorem holds by Lemma 2.4. □

3 Two matrix lemmas

Let the matrices A_1, A_2 be defined by

$$A_1 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

By a *cyclic permutation matrix* we mean a permutation matrix such that the corresponding permutation is cyclic. Let A and B be two 5×5 matrices. We say that A *majorizes* B if every entry in A is greater than or equal to the corresponding entry of B. We say that A *dominates* B if there exist cyclic permutation matrices P, Q such that A majorizes PBQ.

Lemma 3.1 *Let $n \geq 0$ be an integer, and let $M_1, \ldots, M_n$ be 5×5 matrices such that each dominates the matrix A_1. Then $M_1 M_2 \ldots M_n$ dominates the matrix $A_1^{\lceil \frac{n}{5} \rceil}$.*

Proof. We may assume that $M_1 M_2 \ldots M_n = A_1 P_1 A_1 P_2 \ldots A_1 P_n$, where $P_1, \ldots, P_n$ are cyclic permutation matrices. Let $m = \lceil \frac{n}{5} \rceil$. Since there are only five 5×5 cyclic permutation matrices and the product of two cyclic permutation matrices is a cyclic permutation matrix, we deduce that there exist integers $1 \leq i_1 < i_2 \cdots < i_m \leq n$ such that the matrices $P_{i_j} P_{i_j+1} \ldots P_n$ are equal for all $j = 1, \ldots, m$. It follows that for all $j = 1, \ldots, m-1$ the matrix $P_{i_j} P_{i_j+1} \ldots P_{i_{j+1}-1}$ is the identity matrix, and hence the matrix $B_j = P_{i_j} A_1 P_{i_j+1} A_1 \ldots A_1 P_{i_{j+1}-1}$ majorizes the identity matrix. Let $B_0 = A_1 P_1 A_1 P_2 \ldots A_1 P_{i_1 - 1}$ and $B_m = P_{i_m} A_1 P_{i_m+1} \ldots A_1 P_n$. Then $M_1 M_2 \ldots M_n = B_0 A_1 B_1 A_1 B_2 \ldots A_1 B_{m-1} A_1 B_m$. Now, B_0 majorizes some cyclic permutation, and so does B_m, and $A_1 B_1 A_1 B_2 \ldots A_1 B_{m-1} A_1$ majorizes A_1^m, as desired. □

We denote the vector of all ones by $\mathbf{1}$.

Lemma 3.2 *Let $n \geq 2$ be an integer, and let $M_1, M_2, \ldots, M_{n-1}$ be 5×5 matrices with non-negative entries such that each of them dominates A_1 or A_2. Let $M = M_1 M_2 \cdots M_{n-1}$, and let $\mathbf{1}^T M = (x_0, x_1, \ldots, x_4)$. Then there exist four distinct indices $0 \leq i, j, k, l \leq 4$ such that* $\min\{x_i, x_j\} \cdot \min\{x_k, x_l\} \geq 2^{n/6}$.

Proof. We prove the lemma by induction on n. If $n = 2$ then we may assume that $M_1 = A_1$ or $M_1 = A_2$, and hence $x_0, x_1 \geq 2$, $x_2, x_3, x_4 \geq 1$, and hence the lemma holds. We may therefore assume that $n \geq 3$, and that the lemma holds for all smaller values of n. If M_i dominates A_1 for all $i = 1, 2, \ldots, n-1$, then by Lemma 3.1 the product $M_1 \ldots M_{n-1}$ dominates $A_1^{\lceil n/5 \rceil}$. Thus there exist distinct indices $i, j = 0, 1, \ldots, 4$ such that $\min\{x_i, x_j\} \geq 2^{\lceil n/5 \rceil}$. Let $k, l \in \{0, 1, \ldots, 4\} - \{i, j\}$ be distinct. Then $x_k, x_l \geq 1$, and hence the indices i, j, k, l satisfy the conclusion of the lemma. This completes the case when each M_i dominates A_1.

So we may select the largest integer $p \in \{1, 2, \ldots, n-1\}$ such that M_p dominates A_2. Without loss of generality we may assume that $M_p = A_2$. Let $\mathbf{1}^T (M_1 \ldots M_{p-1}) = (y_0, \ldots, y_4)$. Since $p < n$, the induction hypothesis implies that there exist four distinct indices $0 \leq i, j, k, l \leq 4$ such that $\min\{y_i, y_j\} \cdot \min\{y_k, y_l\} \geq 2^{p/6}$. Without loss of generality we may assume that $0 \leq i \leq 1$.

We first dispose of the case $p = n - 1$. Then $x = (x_0, \ldots, x_4) = (y_0, \ldots, y_4) M_p$. Thus $x_0 = x_1 \geq y_k + y_l \geq 2 \min\{y_k, y_l\}$ and $x_2 = x_3 = x_4 \geq y_i$. Therefore,

$$\min\{x_0, x_1\} \cdot \min\{x_2, x_3\} \geq 2 \min\{y_i, y_j\} \cdot \min\{y_k, y_l\} \geq 2 \cdot 2^{(n-1)/6} \geq 2^{n/6},$$

as desired. This completes the case $p = n - 1$.

We may therefore assume that $p < n-1$. The choice of p implies that $M_{p+1}, M_{p+2}, \dots, M_{n-1}$ all dominate A_1. Let $B = M_{p+1}M_{p+2}\dots M_{n-1}$ and let $u = \sum_{i=0}^{4} y_i$ and let $v = y_0 + y_1$. Thus $(x_0, \dots, x_4) = (y_0, \dots, y_4)M_pB = (u, u, v, v, v)B$.

By Lemma 3.1, B dominates $A_1^{\lceil (n-p-1)/5 \rceil}$. Thus there exist distinct indices $s, t \in \{0, 1, \dots, 4\}$ and distinct indices $s', t' \in \{0, 1, \dots, 4\}$ such that $b := \min\{B_{ss'}, B_{st'}, B_{ts'}, B_{tt'}\} \geq 2^{\lceil (n-p-1)/5 \rceil - 1}$. It follows that $x_{s'}, x_{t'} \geq 2bv$, and if $\{s, t\} \cap \{0, 1\} \neq \emptyset$, then $x_{s'}, x_{t'} \geq bu$.

Recall that the matrix B dominates $A_1^{\lceil (n-p-1)/5 \rceil}$. If $\{s, t\} \cap \{0, 1\} = \emptyset$, then there exist distinct indices $r, r' \in \{0, 1, \dots, 4\} - \{s', t'\}$ such that $B_{1r}, B_{2r'} \geq 1$, and hence $x_r, x_{r'} \geq u$. If $\{s, t\} \cap \{0, 1\} \neq \emptyset$, then we select $r, r' \in \{0, 1, \dots, 4\} - \{s', t'\}$ arbitrarily; in that case $x_r, x_{r'} \geq v$.

The results of the previous two paragraphs imply that $\min\{x_{s'}, x_{t'}\} \cdot \min\{x_r, x_{r'}\} \geq buv$. But $v \geq y_i \geq \min\{y_i, y_j\}$ and $u \geq y_k + y_l \geq 2\min\{y_k, y_l\}$. We conclude that

$$\min\{x_{s'}, x_{t'}\} \cdot \min\{x_r, x_{r'}\} \geq buv \geq 2b\min\{y_i, y_j\} \cdot \min\{y_k, y_l\} \geq 2^{p/6 + \lceil (n-p-1)/5 \rceil} \geq 2^{n/6},$$

as desired. □

4 Chains of 5-Cycles

In order to prove Lemma 2.4, we will first characterize how the 3-colorings of an outer 5-cycle of a plane graph G extend to the 3-colorings of another 5-cycle. If C is a 5-cycle in a graph G and Φ a 3-coloring of C, then there exists a unique vertex $v \in V(C)$ such that v is the only vertex of C colored $\Phi(v)$. We call such a vertex the *special vertex of C for Φ*. Let G be a triangle-free plane graph and C_1, C_2 be 5-cycles in G such that $C_1 \neq C_2$ and $Int(C_2) \subseteq Int(C_1)$. Let us choose a fixed orientation of the plane, and let $C_1 := u_1 \dots u_5, C_2 := v_1 \dots v_5$ be both numbered in clockwise order. Then we define a *color transition matrix M* of G with respect to C_1 and C_2 as follows. Let G' be the subgraph of G consisting of all the vertices and edges of G drawn in the closed annulus bounded by $C_1 \cup C_2$. We let M_{ij} equal one sixth the number of 3-colorings Φ of G' such that u_i is the special vertex of C_1 for Φ and v_j is the special vertex of C_2 for Φ. The following lemma is straightforward.

Lemma 4.1 *Let G be a triangle-free graph and $\mathcal{F} = \{C_1, \dots, C_n\}$ be a family of 5-cycles such that $Int(C_i) \supseteq Int(C_j)$ if $1 \leq i < j \leq n$. Let M_i be a color transition matrix of G with respect to C_i and C_{i+1}. Then $M_1M_2\dots M_{n-1}$ is a color transition matrix of G with respect to C_1 and C_n.*

Let us recall that the matrices A_1, A_2 were defined at the beginning of Section 3.

Lemma 4.2 *Let G be a graph isomorphic to one of the graphs shown in Fig. 1. If C_1, C_2 are the two cycles of G shown in Fig. 1, and M is a color transition matrix of G with respect to C_1 and C_2, then M dominates either A_1 or A_2.*

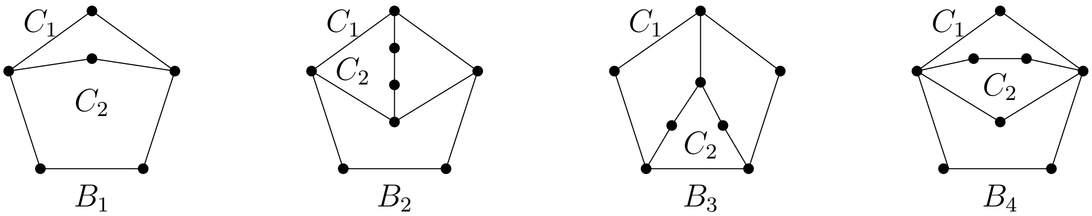

Figure 1: Basic Graphs

Proof. Let M_i be a color transition matrix of B_i with respect to C_1, C_2, where $1 \leq i \leq 4$. Determining the various valid colorings of the B_i's gives the following matrices up to cyclic permutations of rows and columns:

$$M_1 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, M_2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}, M_3 = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix},$$

$$M_4 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

Therefore, M_1, M_3 and M_4 dominate A_1 and M_2 dominates A_2. □

In the rest of the paper, we call the graphs shown in Figure 1 *basic graphs*.

Definition 4.3 Let G be a triangle-free plane graph and C_1, C_2 be two distinct 5-cycles in G. Let $i \in \{1, 2\}$, and let $C_i = v_1w_1w_2w_3w_4$. Further suppose that there exist two vertices of degree three $w_5, w_6 \in V(G) - V(C_i)$ and three facial 5-cycles distinct from C_1 and C_2: $v_1v_4w_6w_3w_4$, $v_1v_3w_5w_2w_1$, and $v_2w_5w_2w_3w_6$. Finally suppose that either v_1 has degree four and does not belong to C_{3-i}, or that v_2 has degree three and does not belong to C_{3-i}. Then we say that G has an *H-structure* around C_i, or simply an *H-structure*. An illustration is shown in Fig. 2.

We denote the 5×5 matrix of all ones by J. If G is a graph and X is a vertex or a set of vertices of G, then we denote by $G \backslash X$ the graph obtained from G by deleting X.

Lemma 4.4 *Let G be a triangle-free plane graph and let C_1, C_2 be two distinct 5-cycles in G. If G has an H-structure and every 4-cycle and every 6-cycle in G separates C_1 from C_2, then every color transition matrix of G with respect to C_1, C_2 dominates the matrix J.*

Proof. Let G have an H-structure around C_2 with its vertices labeled as in Definition 4.3. Let $W := \{w_1, w_2, \ldots, w_6\}$. We will need the following claim.

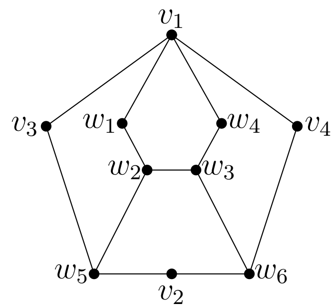

Figure 2: An H-structure

(1) *Every 3-coloring of C_1 can be extended to a 3-coloring of $G \backslash W$ such that v_1 and v_2 are colored the same.*

Let us first deduce the lemma from the claim. It is straightforward to verify that for every $v \in V(C_2)$, every 3-coloring of $G \backslash W$ in which v_1 and v_2 are colored the same color can be extended to a 3-coloring Φ of G such that v is the special vertex of C for Φ. This and (1) imply the conclusion of the lemma.

Thus it remains to prove (1). To that end assume first that v_1 has degree four and does not belong to C_1. Let G' be the graph obtained from $G \backslash W \backslash v_1$ by adding the edges v_2v_3 and v_2v_4. We claim that G' is triangle-free. Indeed, otherwise either v_2 and v_3 have a common neighbor, or v_2 and v_4 have a common neighbor in $G \backslash W \backslash v_1$. From the symmetry we may assume that v_2 and v_3 have a common neighbor $z \notin W \cup \{v_1\}$. But then either $zv_3w_5v_2$ is a 4-cycle in G which does not separate C_1 from C_2 or $zv_2w_6v_4v_1v_3$ is a 6-cycle in G which does not separate C_1 from C_2, a contradiction in either case. Thus G' is triangle-free, and hence every 3-coloring of C_1 extends to a 3-coloring Φ of G' by Theorem 1.5. By letting $\Phi(v_1) := \Phi(v_2)$ we obtain a coloring desired for (1).

We may therefore assume that v_2 has degree three and does not belong to C_1. Let $v_5 \neq w_5, w_6$ be the third neighbor of v_2. Notice that $v_5 \neq v_1$ as then there would be a 4-cycle which does not separate C_1 from C_2. Moreover, $v_5 \neq v_3, v_4$ as G is triangle-free. Now let G_1 be the graph obtained from $G \backslash W \backslash v_2$ by identifying v_3 and v_5. Similarly, let G_2 be the graph obtained from $G \backslash W \backslash v_2$ by identifying v_4 and v_5.

Now we claim that at least one of the graphs G_1, G_2 is triangle-free. To prove this claim, suppose that G_1 and G_2 are not triangle-free. Since G_1 is not triangle-free, there exists a path $v_5z_1z_2v_3$. Moreover, the 6-cycle $v_5z_1z_2v_3w_5v_2$ must separate C_1 and C_2. Furthermore, $z_1 \neq v_4$ as otherwise $v_5v_2w_6v_4$ is a 4-cycle not separating C_1 from C_2. Also $z_1 \neq v_1$ and $z_2 \neq v_4$ since G is triangle-free. Similarly $z_2 \neq v_1$ as otherwise $v_1v_4w_6v_2v_5z_1$ is a 6-cycle not separating C_1 from C_2.

Since G_2 is not triangle-free, there exists a path $v_5z_1'z_2'v_4$. By a similar reasoning as for G_1, we find that z_1', z_2' are distinct from v_4 and v_1 and that the 6-cycle $v_5z_1'z_2'v_4w_6v_2$ must separate C_1 and C_2. Since G is a plane graph, this implies that $\{z_1, z_2\} \cap \{z_1', z_2'\} \neq \emptyset$. Notice that $z_1 \neq z_1'$ as G is simple and both 6-cycles described above separate C_1 from C_2. If $z_1' = z_2$ or if $z_1 = z_2'$, then G has a triangle, a contradiction. Thus $z_2 = z_2'$. But then $v_4 z_2 v_3 v_1$ is a 4-cycle which does not separate C_1 from C_2, a contradiction. This proves our claim that one of G_1, G_2, say G', is triangle-free.

Since G' is triangle-free, every 3-coloring of C_1 extends to a 3-coloring Φ of G' by Theorem 1.5. Since one neighbor of v_1 has the same color as v_5, we may set $\Phi(v_2) := \Phi(v_1)$ and thus obtain a coloring desired for (1). This completes the proof of (1), and hence the proof of the lemma. □

Definition 4.5 Let G be a triangle-free plane graph, let C_1, C_2 be two 5-cycles in G, and let f be a face of G bounded by a 5-cycle C_3, where C_1, C_2, C_3 are pairwise distinct. We say that f is a *good face* if one of the following conditions hold:

1. At least four vertices of C_3 have degree three and $E(C_3) \cap (E(C_1) \cup E(C_2)) = \emptyset$, or
2. all five vertices of C_3 have degree three and either $E(C_3) \cap E(C_1) = \emptyset$ or $E(C_3) \cap E(C_2) = \emptyset$.

If the first condition holds, then we say that f is a *good face of the first kind*, and if the second condition holds, then we say that f is a *good face of the second kind.*

Lemma 4.6 *Let G be a triangle-free plane graph and C_1, C_2 be two distinct 5-cycles in G. Assume that all vertices of G of degree two are on C_1 and C_2; for every integer $k \in \{4, 6, 7\}$ every k-cycle in G separates C_1 from C_2; every 5-cycle of G bounds a face; and every face in G is bounded by a cycle of length five. If G has a good face, then there exists a triangle-free graph G' and two 5-cycles $C_1' \neq C_2'$ in G' such that $|V(G')| < |V(G)|$ and every color transition matrix of G with respect to C_1 and C_2 dominates some color transition matrix of G' with respect to C_1' and C_2'.*

Proof. We say that a cycle C in G is an *important cycle* if $C = C_1$; or $C = C_2$; or C has length four, six or seven; or C has length nine and no vertex of G has three or more neighbors on C. Let us assume for a moment that some important cycle C does not separate C_1 from C_2. Then C has length nine by hypothesis, and hence no vertex of G has three or more neighbors on C. Let G' be the subgraph of G obtained by deleting all vertices and edges drawn in the face of C that is disjoint from C_1 and C_2. Then $|V(G')| < |V(G)|$, because every face of G is bounded by a 5-cycle. By Theorem 1.4 every 3-coloring of G' extends to a 3-coloring of G, and hence G' satisfies the conclusion of the lemma. Thus we may assume that every important cycle in G separates C_1 from C_2. It follows that

(∗) *for every subgraph H of G, at most two facial cycles of H are important.*

Good face of the first kind. Let f be a good face of the first kind bounded by the cycle $C_3 := v_1, \ldots, v_5$ where $v_1, \ldots, v_4$ are vertices of degree three on C_3. For $i = 1, 2, 3, 4$ let w_i be the neighbor of v_i which is not on C_3. Let $S = \{v_1, v_2, \ldots, v_5, w_1, w_2, w_3, w_4\}$. These vertices are pairwise distinct, because G is triangle-free and has no separating 5-cycles. Similarly, no v_i is adjacent to w_j for $i \neq j$. Finally, we claim that w_i is not adjacent to w_j for $i, j \in \{1, 2, 3, 4\}$ with $i \neq j$. Indeed, this follows similarly if $j = i + 2$, or $i = 1$ and $j = 4$, and so we may assume that $i \in \{1, 2, 3\}$ and $j = i + 1$. But w_i and w_{i+1} have a common neighbor, because the path $w_i v_i v_{i+1} w_{i+1}$ is a subpath of a facial cycle (of length five), and hence w_i and w_j are not adjacent, because G is triangle-free. Thus we have shown that

(0) *the vertices of S are pairwise distinct, and the only edges of G with both ends in S are the edges of C_3 and the edges $v_i w_i$ for $i = 1, 2, 3, 4$.*

Let G_1 be the graph obtained from G by deleting the vertices $v_1, \ldots, v_4$, identifying w_2 with w_3, and adding the edge $w_1 w_4$. Let G_2 be the graph obtained from G by deleting the vertices $v_1, \ldots, v_4$, identifying w_3 with v_5, and identifying w_1 with w_2. Let G_3 be the graph obtained from G by deleting the vertices $v_1, \ldots, v_4$, identifying w_2 with v_5, and identifying w_3 with w_4.

We will prove that at least one of the graphs G_1, G_2 or G_3 is triangle-free. The lemma then follows, for if G' is one of the above three graphs that is triangle-free, then we may assume that C_1 and C_2 are 5-cycles in G'. It is well-known [7] that every 3-coloring of G' can be converted to a 3-coloring of G, and hence every color transition matrix of G' with respect to C_1 and C_2 dominates some color transition matrix of G with respect to C_1 and C_2, as desired.

Thus it remains to prove the claim that one of G_1, G_2, G_3 is triangle-free. To that end we may assume the contrary. From the fact that G_1 is not triangle-free, we deduce that there exist vertices $z_1, z_2 \in V(G) - \{v_1, v_2, v_3, v_4\}$ such that either

(1a) $w_1 z_1 w_4 v_4 v_5 v_1$ *is a cycle in* G

or

(1b) $w_2 z_1 z_2 w_3 v_3 v_2$ *is a cycle in* G.

In either case, let C_4 denote the corresponding cycle. Similarly, the fact that G_2 is not triangle-free implies that there exist vertices $x_1, x_2 \in V(G) - \{v_1, v_2, v_3, v_4\}$ such that either

(2a) $w_1 x_1 x_2 w_2 v_2 v_1$ *is a cycle in* G

or

(2b) $v_5 x_1 x_2 w_3 v_3 v_4$ *is a cycle in* G.

In either case, let C_5 denote the corresponding cycle. Finally, the fact that G_3 is not triangle-free implies that there exist vertices $x_3, x_4 \in V(G) - \{v_1, v_2, v_3, v_4\}$ such that either

(3a) $w_3x_3x_4w_4v_4v_3$ *is a cycle in* G

or

(3b) $v_5x_3x_4w_2v_2v_1$ *is a cycle in* G.

In either case, let C_6 denote the corresponding cycle. From (0) we deduce that

(4) $x_1, x_2, x_3, x_4, z, z_1, z_2 \notin S$.

If (2a) and (3a) hold, then in both cases (1a) and (1b) the subgraph $C_3 \cup C_4 \cup C_5$ has at least three important faces, contrary to $(*)$. (This requires some checking. For instance, in case (1b) it is possible that $z_1 = x_4$ or $z_2 = x_1$.) Thus from the symmetry we may assume that (2b) holds.

Assume now that (1a) holds. Then by planarity and (4) either $z_1 = x_1$ or $z_1 = x_2$. In the former case the subgraph of G induced by $\{v_1, v_4, v_5, w_1, w_4, x_1\}$ has three important faces, contrary to $(*)$, and in the latter case the 5-cycles $v_5x_1x_2w_4v_4$ and $v_4w_4x_2w_3v_3$ are facial and not equal to C_1 or C_2 (because each shares an edge with C_3), and hence w_4 has degree two, contrary to hypothesis. We conclude that (1a) does not hold.

Thus (1b) and (2b) hold. By considering the subgraph $C_3 \cup C_4 \cup C_5$ of G we deduce from $(*)$ that $v_5v_1v_2w_2z_1z_2w_3x_2x_1$ is a cycle of length nine, and some vertex of G has at least three neighbors on it. That vertex is w_1, and hence w_1 has degree three and is adjacent to v_1, z_1, x_2. Since every face of G is bounded by a 5-cycle, there is a vertex u such that $w_2v_2v_3w_3u$ is a facial 5-cycle. Then $w_2z_1z_2w_3u$ is also a 5-cycle, and hence also bounds a face. It follows that u has degree two, and so one of the incident faces is C_1 or C_2; but C_3 shares no edges with C_1 or C_2, and hence we may assume that $C_2 = w_2z_1z_2w_3u$. It follows that G has an H-structure around C_2, where the degree three vertex v_1 plays the role of the vertex v_2 from the definition of H-structure. Thus the subgraph $C_1 \cup C_2$ of G satisfies the conclusion of the lemma by Lemma 4.4.

Good face of the second kind. Let f now be a good face of the second kind bounded by the cycle $C_3 := v_1v_2 \dots v_5$, where each v_i has degree three. For $i = 1, 2, \dots, 5$ let w_i be the neighbor of v_i which is not on C_3, and let $W := \{v_1, v_2, \dots, v_5, w_1, w_2, \dots, w_5\}$. We have the following analogue of (0):

(5) *the vertices of* W *are pairwise distinct, and the only edges of* G *with both ends in* W *are the edges of* C_3 *and the edges* v_iw_i *for* $i = 1, 2, 3, 4, 5$.

We may assume that G does not have a good face of the first kind, and hence we may assume from the symmetry that C_2 shares an edge with C_3. Thus we may assume without loss of generality that $C_2 := v_4v_5w_5zw_4$ for some vertex $z \in V(G)$. Let G_1 be the graph obtained from $G \backslash \{v_1, v_2, v_3, v_4\}$ by identifying w_3 and v_5, and identifying w_1 with w_2. Let G_2 be the graph obtained from $G \backslash \{v_5, v_1, v_2, v_3\}$ by identifying w_1 with v_4, and identifying w_2 with w_3.

We will prove below that one of G_1, G_2 is triangle-free, but let us first deduce the lemma from this assertion. From the symmetry we may assume that G_1 is triangle-free. Let x be the fifth vertex in the facial cycle $w_4v_4v_3w_3x$ of G, and let y be the vertex of G_1 obtained by identifying w_3 and v_5. Then C_1 is a cycle of G_1, and let $C_2' = w_5zw_4yx$.

Now every 3-coloring of G_1 extends to a 3-coloring of G by coloring v_4 the same as x and then coloring v_3. The vertices w_1 and w_2 are colored the same, and v_5 and v_3 are colored differently. It follows that this coloring can be extended to v_1 and v_2, as desired. Thus every color transition matrix of G with respect to C_1 and C_2 dominates some color transition matrix of G_1 respect to C_1 and C_2' and Lemma 4.6 holds.

It remains to prove the claim that at least one of G_1 or G_2 is triangle-free. To that end we may assume the contrary. Since G_1 is not triangle free, there exist vertices $x_1, x_2 \in V(G) - \{v_1, v_2, v_3, v_4\}$ such that either

(6a) $w_1x_1x_2w_2v_2v_1$ *is a cycle in* G

or

(6b) $w_3x_1x_2v_5v_4v_3$ *is a cycle in* G.

In either case, let D_1 denote the corresponding cycle. Since G_2 is not triangle free, there exist vertices $y_1, y_2 \in V(G) - \{v_1, v_2, v_3, v_5\}$ such that either

(7a) $w_3y_1y_2w_2v_2v_3$ *is a cycle in* G

or

(7b) $w_1y_1y_2v_4v_5v_1$ *is a cycle in* G.

In either case, let D_2 denote the corresponding cycle. It follows from (5) that

(8) $z, x_1, x_2, y_1, y_2 \notin W$.

If (6a) and (7a) hold, then the graph $H := C_2 \cup C_3 \cup D_1 \cup D_2$ has at least three important faces, contrary to $(*)$. Next we show that if (7b) holds, then w_1 is adjacent to z. To that end assume that (7b) holds. Since v_4 has degree three it follows that $y_2 = w_4$. If $y_1 \neq z$, then H has at least three important faces, contrary to $(*)$. Thus $y_1 = z$ and hence w_1 is adjacent to z if (7b) holds. Similarly, if (6b) holds, then w_3 is adjacent to z. We conclude that if (6b) and (7b) hold, then z is adjacent to w_1 and w_3, and it follows that G has an H-structure around C_2, where the degree three vertex v_2 plays the role of the vertex v_2 from the definition of H-structure. Therefore, the subgraph $C_1 \cup C_2$ of G satisfies the conclusion of the lemma by Lemma 4.4.

Finally, by symmetry we may assume that (6a) and (7b) hold, and that (6b) does not. In particular, w_3 is not adjacent to z. Then, as we have shown above, w_1 is adjacent to z. It follows that $z \notin \{x_1, x_2\}$, and hence $D := w_1zw_4v_4v_3v_2w_2x_2x_1$ is a cycle of length nine. We deduce from $(*)$ that some vertex of G has three neighbors on D. This vertex must be w_3, and its three neighbors are v_3, z, x_2, contrary to the fact that w_3 is not adjacent to z.

This completes the proof of the fact that one of the graphs G_1, G_2 is triangle-free, and hence completes the proof of the lemma. $\square$

Let us recall that the matrices A_1, A_2 were defined at the beginning of Section 3.

Lemma 4.7 *Let G be a triangle-free plane graph and C_1, C_2 be two distinct 5-cycles in G. Then every color transition matrix of G with respect to C_1 and C_2 dominates either A_1 or A_2.*

Proof. We use an argument similar to the proof of Grötzsch's Theorem given in [13]. Let us assume for a contradiction that the lemma is false, and choose a counterexample G with cycles C_1 and C_2 with $|V(G)|$ minimum. Let M be a color transition matrix of G with respect to C_1 and C_2.

(1) *If a cycle C in G of length at most seven does not bound a face, then it separates C_1 and C_2.*

To prove (1) suppose for a contradiction that a cycle C of length at most seven is not facial and does not separate C_1 from C_2. Then some component J of $G \backslash V(C)$ is disjoint from $C_1 \cup C_2$, and hence every 3-coloring of $G \backslash V(J)$ extends to G by Theorem 1.5. Thus M is a color transition matrix of $G \backslash V(J)$ with respect to C_1 and C_2, and hence M dominates A_1 or A_2 by the minimality of G, a contradiction. This proves (1).

(2) *G is 2-connected.*

To prove (2) we may assume that G is not 2-connected. If C_1 and C_2 belong to the same block B of G, then M is a color transition matrix of B and we obtain contradiction as above. If C_1 and C_2 are in different blocks, then M dominates the matrix of all ones, as is easily seen, a contradiction. This proves (2).

(3) *Every vertex of G of degree two belongs to $C_1 \cup C_2$.*

Claim (3) follows similarly by deleting a vertex of degree two not in $C_1 \cup C_2$.

(4) *Every 5-cycle in G bounds a face.*

To prove (4) let C be a 5-cycle in G that does not bound a face. By (1) it separates C_1 from C_2. Let M_1 be a color transition matrix of G with respect to C_1 and C, and let M_2 be a color transition matrix of G with respect to C and C_2. By Lemma 4.1 the matrix $M_1 M_2$ is a color transition matrix of G with respect to C_1 and C_2. By the minimality of G the matrices M_1 and M_2 dominate A_i and A_j, respectively, where $i, j \in \{1, 2\}$. It follows that M dominates $A_i A_j$. Notice that $A_1^2, A_1 A_2$ and $A_2 A_1$ dominate A_1 and A_2^2 dominates A_2, and so M dominates A_1 or A_2, a contradiction. This proves (4).

(5) *G has no facial 4-cycle.*

To prove (5) suppose for a contradiction that $C := v_1v_2v_3v_4$ is a facial 4-cycle in G. Let G_1 be the graph obtained from G identifying v_1 and v_3 and let G_2 be the graph obtained from G by identifying v_2 and v_4. At least one of the graphs G_1, G_2 is a triangle-free plane graph. From the symmetry we may assume that G_1 is triangle-free. Let C_1', C_2' be the cycles in G_1 that correspond to C_1 and C_2, respectively. As every 3-coloring of G_1 extends to a 3-coloring of G, a color transition matrix of G with respect to C_1, C_2 dominates a color transition matrix of G_1 with respect to C_1', C_2'. If $C_1' \neq C_2'$, then G_1 satisfies the hypotheses of lemma 4.7, and so we obtain contradiction to the minimality of G. Thus $C_1' = C_2'$. Now G must be isomorphic to the basic graph B_1. Then by Lemma 4.2 a color transition matrix of G with respect to C_1, C_2 dominates A_1, a contradiction. This proves (5).

(6) *G has no facial cycle of length six or more.*

To prove (6) suppose for a contradiction that $C := v_1v_2 \dots v_k$ is a facial cycle in G of length $k \geq 6$. Let G_1 be the graph obtained from G identifying v_1 and v_3 and let G_2 be the graph obtained from G by identifying v_2 and v_4. If G_1 is triangle-free, let $G' = G_1$. If G_1 is not triangle-free, then there exists a path $v_1u_1u_2v_3$ in G. Since $v_1v_2v_3u_2u_1$ is not a separating 5-cycle, it must be facial. Hence v_2 is degree two in G. This implies that G_2 is a triangle-free plane graph, for otherwise there exists a path $v_2v_1w_1v_4$ in G, in which case $v_1v_2v_3v_4w_1$ is a separating 5-cycle, a contradiction. In this case let $G' = G_2$.

Let C_1', C_2' be the cycles in G' that correspond to C_1 and C_2, respectively. Moreover, the cycles cannot be equal as there are at least three faces in G'. As every 3-coloring of G' extends to a 3-coloring of G, a color transition matrix of G with respect to C_1, C_2 dominates a color transition matrix of G' with respect to C_1', C_2', contrary to the minimality of G. This proves (6).

(7) *Every cycle in G of length four, six or seven separates C_1 from C_2.*

Claim (7) follows immediately from (1), (5) and (6).

It follows from (3), (4) and (7) that G satisfies the hypotheses of Lemma 4.6. In particular, every facial cycle in G has length exactly five. Let us recall that good faces were defined in Definition 4.5. Let f_1 and f_2 be the faces bounded by C_1 and C_2, respectively. Thus f_1, f_2 are never good. We may assume that

(8) *G has no good face,*

because otherwise the lemma follows from Lemma 4.6.

Now we use a standard discharging argument. Let the charge of a vertex v be $ch(v) = 4 - deg(v)$ and the charge of a face f be $ch(f) = 4 - |f|$. Then by Euler's formula the sum of the charges of all vertices and faces is 8. Now we discharge the vertices as follows. Suppose v is a vertex of G. If the degree of v is at least three, distribute the charge of it uniformly over the faces incident with it. Thus if v has degree $d \geq 5$, it will receive $1/d$ from each adjacent face. If the degree of v is two, v must be on C_1 or C_2. If v is incident with both f_1 and f_2 then distribute the charge of v uniformly over f_1 and f_2. Otherwise, let $f_3 \notin \{f_1, f_2\}$ be the other face incident with v. In this case let v send $+5/3$ to f_i and $+1/3$ to f_3. We denote the new charge of a face f by $ch'(f)$. The new charge of every vertex is zero. Let us recall that every face of G is bounded by a 5-cycle. The discharging rules imply that for every face $f \notin \{f_1, f_2\}$ of G:

(9) *if f is incident with five vertices of degree at most three, then $ch'(f) = 2/3$; otherwise $ch'(f) \leq 1/3$,*

(10) *$ch'(f) > 0$ if and only if f is incident with at least four vertices of degree at most three.*

Let $\mathcal{F}_1, \mathcal{F}_2$ be the set of faces other than f_1 and f_2 which are adjacent to f_1 and f_2, respectively. Since the sum of the new charges of all faces is 8, we have either

- $N_1 = ch'(f_1) + \sum_{f\in\mathcal{F}_1-\mathcal{F}_2} ch'(f) + \frac{1}{2}\sum_{f\in\mathcal{F}_1\cap\mathcal{F}_2} ch'(f) \geq +4$, or
- $N_2 = ch'(f_2) + \sum_{f\in\mathcal{F}_2-\mathcal{F}_1} ch'(f) + \frac{1}{2}\sum_{f\in\mathcal{F}_1\cap\mathcal{F}_2} ch'(f) \geq +4$, or
- there exists a face $f_3 \neq f_1, f_2$ which is not adjacent to f_1 or f_2, such that $ch'(f_3) > 0$.

The last case does not happen, because the face f_3 would be good by (9), contrary to (8). By the symmetry between f_1 and f_2 we may therefore assume that $N_2 \geq 4$. Let $C_2 := v_1v_2\dots v_5$.

(11) *At least two vertices of C_2 have degree two.*

To prove (11) we may assume for a contradiction that C_2 has at most one vertex of degree two. Thus $ch'(f_2) \leq 2$. If for every face $f \in \mathcal{F}_2$ either $f \in \mathcal{F}_1$ or $ch'(f) \leq +1/3$, then $N_2 \geq 4$ implies $|\mathcal{F}_2| = 5$. But then C_2 has no vertex of degree two, implying $ch'(f_2) \leq 2/3$, and hence $N_2 \leq 8/3$, a contradiction. Thus there exists a face $f \in \mathcal{F}_2 - \mathcal{F}_1$ such that $ch'(f_3) > 1/3$. But then f is good by (9), contrary to (8). This proves (11).

(12) *If C_2 has exactly two vertices of degree two, then they are not consecutive on C_2.*

To prove (12) we may assume for a contradiction that v_1 and v_2 are the only vertices of degree two on C_2. Since all of the faces of G are bounded by 5-cycles, there exists a vertex w such that w is adjacent to v_3 and v_5. Since the 4-cycle $wv_3v_4v_5$ separates C_1 from C_2, we deduce that $C_3 := v_1v_5wv_3v_2$ is facial 5-cycle.

Suppose that the degree of v_3 or of v_5 is three. Since there must be a facial 5-cycle incident with this vertex, v_4 and w, there must exist a path v_4xyw. However, the 5-cycles v_4xywv_3 and v_4xywv_5 must be facial. Hence G is isomorphic to B_2 and Lemma 4.7 follows from Lemma 4.2.

Now we may assume that v_3 and v_5 have degree at least four. Thus, $ch'(f_2) \leq +8/3$. Moreover, $|\mathcal{F}_2| = 3$. Notice that every face f in $\mathcal{F}_2$ has a vertex of degree at least four so that $ch'(f) \leq 1/3$. Hence, $N_2 \leq 11/3$, a contradiction. This proves (12).

(13) *C_2 has at least three vertices of degree two.*

To prove (13) we may assume by (11) and (12) that C_2 has exactly two vertices of degree two, and that they are not consecutive. Thus we may assume that v_1 and v_3 are the vertices of degree two on C_2. First suppose that v_2 has degree three and let $z \neq v_1, v_3$ be a neighbor of v_2. Since v_1 and v_3 have degree two, there exist facial 5-cycles $v_5v_1v_2zw_1$ and $v_4v_3v_2zw_2$. Moreover, $w_1 \neq w_2$ since G is triangle-free. But then $zw_1v_5v_4w_2$ is a 5-cycle, and so it is C_1 by (4). Thus G is isomorphic to the basic graph B_3 and Lemma 4.7 follows from Lemma 4.2.

So we may assume that v_2 has degree at least four. Thus, $ch'(f_2) \leq +3$. Let $f_3 \neq f_2$ be the face incident with v_1, $f_4 \neq f_2$ be the face incident with v_2, and let $f_5 \neq f_2$ be the face incident with the edge v_4v_5. Since the degree of v_2 is at least four, $ch'(f_3), ch'(f_4) \leq +1/3$. If $\deg(v_2) \geq 5$, then $ch'(f_2) \leq +3 - 1/5$ and $ch'(f_3), ch'(f_4) \leq +1/3 - 1/5$. Thus $N_2 \leq 13/3 - 3/5 < 4$, a contradiction.

So we may assume that $\deg(v_2) = 4$. Now if v_5 has degree at least four, then $ch'(f_2) \leq 8/3$ and $ch'(f_3) \leq 0$. In that case, $N_2 \leq +11/3$, a contradiction. Thus v_5 has degree three. Similarly we find that v_4 has degree three. Let y_1 be the neighbor of v_5 not on C_2 and let y_2 be the neighbor of v_4 not on C_2. Note that $y_1 \neq y_2$. Now f_5 must be incident with y_1 and y_2. If y_1 has degree at least four, then $ch'(f_3) \leq 0$ and $ch'(f_5) \leq +1/3$. In that case, $N_2 \leq +11/3$, a contradiction. Thus y_1 has degree three. Similarly we find that y_2 also has degree three. Let z_1 be the neighbor of y_1 incident with $f_3 = v_5v_1v_2z_1y_1$ and let z_2 be the neighbor of y_2 incident with $f_4 = v_4v_3v_2z_2y_2$. Finally, let z_3 be the common neighbor of y_1 and y_2 incident with $f_5 = v_4v_5y_1z_3y_2$. Thus G has an H-structure and Lemma 4.7 follows from Lemma 4.4. This proves (13).

(14) *If C_2 has exactly three vertices of degree two, then they are not consecutive on C_2.*

To prove (14) we may assume for a contradiction that v_1, v_2 and v_3 have degree two, and v_4, v_5 have degree at least three. Let $G' = G \setminus \{v_1, v_2, v_3\}$. Notice that v_1, v_2 and v_3 do not belong to C_1 (because $C_1 \neq C_2$), so $V(C_1) \subseteq V(G')$. Obviously for every $1 \leq i \leq 5$ and any 3-coloring Φ of G', we can extend Φ to a 3-coloring of G such that v_i is the special vertex on C_2 for that coloring. Since by Theorem 1.4 any 3-coloring of C_1 can be extended to a 3-coloring of G', a color transition matrix of G with respect to C_1 and C_2 dominates the matrix J. This proves (14).

We are now ready to complete the proof of the lemma. By (2) and (13) there are exactly three vertices of degree two on C_2, and by (14) we may assume that they are v_1, v_2 and v_4. Let $f_3 = v_1v_2v_3z_1v_5$ be the face distinct from f_2 that is incident with v_1 and v_2 and let $f_4 = v_3v_4v_5z_2z_3$ be the face distinct from f_2 incident with v_4. Note that $z_1 \neq z_2, z_3$ as G is triangle-free. Since the 5-cycle $v_3z_1v_5z_2z_3$ does not separate C_1 from C_2, it must be C_1. Hence G is isomorphic to the basic graph B_4 and Lemma 4.7 follows from Lemma 4.2. □

Proof of Lemma 2.4. Suppose $n = |\mathcal{F}| \geq 2$ and let $C_1, C_2, \ldots, C_n$ be the elements of $\mathcal{F}$ such that $Int(C_i) \supseteq Int(C_j)$ if and only if $1 \leq i < j \leq n$. For $i = 1, 2, \ldots, n-1$ let M_i be a color transition matrix of G with respect to C_i, C_{i+1}. Lemma 4.1 implies that $M = M_1 M_2 \ldots M_{n-1}$ is a color transition matrix of G with respect to C_1, C_n. Hence the number of 3-colorings of G is at least six times $\mathbf{1}^T M \mathbf{1}$. For all $1 \leq i \leq n-1$, Lemma 4.2 implies that M_i dominates either A_1 or A_2, the matrices defined in Section 3. It follows from Lemma 3.2 that the number of 3-colorings of G is at least $24 \cdot 2^{n/12}$, as desired. $\square$

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