Five-Connected Toroidal Graphs Are Hamiltonian

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Abstract

We prove that every edge in a 5-connected graph embedded in the torus is contained in a Hamilton cycle. Our proof is constructive and implies a polynomial time algorithm for finding a Hamilton cycle.

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1 Introduction

It is well known that not all 3-connected planar graphs are hamiltonian. Whitney [10] proved that every triangulation of the sphere with no separating triangles is hamiltonian. Tutte [9] proved that every 4-connected planar graph has a Hamilton cycle. Extending Tutte's technique, Thomassen [8] proved that every 4-connected planar graph is in fact Hamilton connected. (A small omission in [8] was corrected by Chiba and Nishizeki [3].) With some additional techniques, Thomas and Yu [7] proved that every edge in a 4-connected projective-planar graph is contained in a Hamilton cycle, which establishes a conjecture of Grünbaum [4].

Grünbaum [4] and Nash-Williams [5] also conjectured that every 4-connected toroidal graph has a Hamilton cycle. While this remains open, Brunet and Richter [2] proved that every 5-connected triangulation of the torus is hamiltonian. Also Altshuler [1] showed that all 6-connected toroidal graphs are Hamiltonian. In this paper we prove the following result.

(1.1) *Let G be a 5-connected toroidal graph. Then every edge of G is contained in a Hamilton cycle.*

On the other hand, for 4-connected graphs embedded in the torus, certain edges may not be contained in any Hamilton cycle. The following example was provided by Thomassen [8]. Embed the product of two even cycles (of length at least 4) in the torus so that every face is bounded by a cycle of length 4, and add an edge joining two non-adjacent vertices in some facial cycle. Then this new edge is not contained in any Hamilton cycle of the new graph.

Only simple graphs will be considered. Let G be a graph, and X a subset of vertices or edges. Then $G - X$ denotes the graph obtained from G by deleting X and all edges incident with X. When $X = \{x\}$, we write $G - x$ instead of $G - \{x\}$. Let x, y be two vertices in a graph G. Then $G + xy = G$ if xy is an edge in G, otherwise $G + xy$ denotes the graph obtained from G by adding the edge xy. Let G and H be two graphs. Then $G \cap H$ (respectively, $G \cup H$) denotes the *intersection* (respectively, *union*) of G and H. A *block* in a graph is a maximal 2-connected subgraph. (We view K_2 as 2-connected.) For a path P and two vertices $x, y \in V(P)$, xPy denotes the subpath of P between x and y. Let S be a subset of $V(G)$, and A and B two subgraphs of G. Then we say that S *separates* A *from* B if $A - S \neq \emptyset$, $B - S \neq \emptyset$, and no component of $G - S$ contains vertices from both A and B.

Let P be a subgraph of a graph G. A *P-bridge* of G is either an edge of $G - E(P)$ with both ends on P or a subgraph of G induced by the edges in a component of $G - V(P)$ and all edges from that component to P. For a P-bridge B of G, the vertices in $B \cap P$ are the *attachments* of B (on P). A path P in a graph G is a *Tutte path* if every P-bridge of G contains at most three attachments.

A *plane graph* is a graph embedded in the plane (with no edge-crossings). Given a plane graph G, a *plane subgraph* H of G is a subgraph of G inheriting the embedding of G. Given a connected plane graph, the boundary of the infinite face is called the *outerwalk* of the graph, or *outercycle* if the graph is 2-connected. (Note that faces are open subsets of the plane.) A walk is a *facial walk* in a plane graph if it bounds a face of the graph. Given two vertices x and y on a facial walk C in a plane graph, we use xCy to denote the minimal subwalk of C from x to y in the clockwise order. Note that xCy is unique because of planarity. Two vertices are said to be *cofacial* if they are contained in a common facial walk.

Our proof technique relies on generalizations of the concept of a Tutte path. We first prove several results (in Section 2) about disjoint paths in plane graphs. Then in Section 3 we prove our main result by an inductive argument and reducing the problem to one about disjoint paths in plane graphs.

2 Lemmas

In this section we prove a few results about Tutte paths in plane graphs. The first lemma (which will be most frequently used) was proved independently by Sanders [6] and Yu (unpublished).

(2.1) *Let G be a connected plane graph with outerwalk C, and let $e \in E(C)$ and $x, y \in V(G)$ with $x \neq y$ such that G contains a path from x to y through e. Then G has a Tutte path P from x to y through e such that every P-bridge of G containing an edge of C has at most* *two attachments.* ∎

One can prove the following result using (2.1): in a 4-connected planar graph, there is a Hamilton path between any two vertices through any given edge; moreover, the deletion of any vertex results in a Hamilton connected graph.

In order to state the next lemma, we need the following definition. Let H be a plane graph with outerwalk W and $x, y \in V(W)$, such that xWy is a simple path. We use J to denote the union of blocks of H containing an edge of xWy and write $J = v_0B_1v_1B_2 \cdots v_{m-1}B_mv_m$, where each B_i is a block of H containing an edge of xWy, $v_i \in V(B_i \cap B_{i+1})$ for $i \in \{1, \cdots, m-1\}$, and $v_0 = x \in V(B_1 - v_1)$ and $v_m = y \in V(B_m - v_{m-1})$. This J is said to be the *plane chain of blocks in H along xWy*. If $H = J$, then we just say H is a plane chain of blocks (along xWy).

(2.2) *Let U be a graph and $A \subset V(U)$, such that $U - A$ is a plane graph with outerwalk W. Let $x, y \in V(W)$ such that xWy is a simple path, and let $J = v_0B_1v_1B_2 \cdots v_{m-1}B_mv_m$ be the plane chain of blocks in $U - A$ along xWy. Let D_i be the outerwalk of B_i, let H_i be the union of B_i and $(J \cup A)$-bridges of U containing a vertex of $V(B_i) - \{v_{i-1}, v_i\}$, and let $a(B_i) = V(H_i) \cap A$. Suppose all attachments of $(B_i \cup a(B_i))$-bridges of H_i are contained in $V(v_iD_iv_{i-1}) \cup a(B_i)$ and that H_i is a plane graph with $v_{i-1}D_iv_i \cup a(B_i)$ contained in its outerwalk. Let k be an integer with $0 \leq k \leq n$, and $e \in E(D_k)$ if $k \geq 1$. For $i = 1, \cdots, m$, let $a_i = \max(3, |a(B_i)| + 1)$ if $i \neq k$, and $a_i = \max(3, |a(B_i)| + 2)$ otherwise.*

Then J contains a path P from x to y with the following properties.
(i) Every $(P \cup A)$-bridge of U containing an edge of xWy has exactly two attachments,
(ii) every $(P \cup A)$-bridge of U containing a vertex of $V(B_i) - \{v_{i-1}, v_i\}$ has at most a_i attachments, and
(iii) if $k \geq 1$ then $e \in E(P)$.

Note that the $(P \cup A)$-bridges of U not mentioned in (i) or (ii) are the $(J \cup A)$-bridges of U with no attachments in $J - \{v_0, \cdots, v_m\}$, and such $(P \cup A)$-bridges have the same number of attachments on $P \cup A$ as on $J \cup A$.

Proof. We note that each $(J \cup A)$-bridge of U has at most one attachment in J.

If $|a(B_i)| \leq 1$, then let w_i be any vertex on $v_iD_iv_{i-1}$; otherwise by planarity, let $w_i \in V(v_iD_iv_{i-1})$ such that all $(B_i \cup a(B_i))$-bridges of H_i containing a vertex of $v_iD_iw_i - w_i$ contains a common vertex $u_i \in a(B_i)$ and any $(B_i \cup a(B_i))$-bridges of H_i containing a vertex of $w_iD_iv_{i-1} - w_i$ does not contain u_i.

For $i \neq k$, we use (2.1) in B_i to find a Tutte path P_i from v_{i-1} to v_i through w_i, such that every P_i-bridge of B_i containing an edge of D_i has just two attachments. Hence every $(P_i \cup a(B_i))$-bridge of H_i has at most $a_i = \max(3, |a(B_i)| + 1)$ attachments.

If $k \geq 1$, then we use (2.1) in B_k to find a Tutte path P_k from v_{k-1} to v_k through e, such that every P_k-bridge of B_k containing an edge of D_k has just two attachments. Hence every $(P_k \cup a(B_k))$-bridge of H_k has at most $a_k = \max(3, |a(B_k)| + 2)$ attachments.

Now let $P = \bigcup_{i=1}^{m} P_i$, and it is easy to see that P is the desired path. ∎

We point out here that (2.2) will be used frequently as a technical lemma. It is helpful to be familiar with H_i and $a(B_i)$. In later proofs we sometimes need Tutte paths through two prescribed edges in a specified order. Hence we need the following result.

(2.3) *Let G be a connected plane graph with outerwalk C, and let $e, f \in E(C)$, $y \in V(C)$ and $x \in V(G - y)$, such that G contains a path from x to y through e first and then f.*

Then G contains a Tutte path P from x to y through e and then f; moreover, if $x \in V(C)$ and $e, f \in E(xCy)$, then every P-bridge of G containing an edge of xCy has at most two attachments.

Proof. We only prove (2.3) for the case that G is 2-connected. The general case may be proved by using induction and (2.1).

We may assume that $e = st$ and $f = uv$ such that s, t, u, v, y are on C in this clockwise order. By planarity and since G has a path from x to y through e and then f, $x \notin vCy$ and $G - vCy$ contains a component H which contains a path from x to u through e. We use (2.1) in H to find a Tutte path Q from x to u through e such that every Q-bridge of H containing an edge of the outerwalk of H has at most two attachments.

Let U be the union of vCy, all $(H \cup vCy)$-bridges of G, and all Q-bridges of H containing a vertex which is not on Q but is in some $(H \cup vCy)$-bridge of G. Let $A = V(U \cap Q)$ and let $J = vB_1v_1B_2 \cdots v_{m-1}B_my$ be the plane chain of blocks in $U - A$ along vCy. Define H_i

and $a(B_i)$ as in (2.2). Hence $|a(B_i)| \leq 2$ for each i. By (2.2), we can find a path R in J from v to y such that every $(R \cup A)$-bridge of U has at most three attachments, and every $(R \cup A)$-bridge of U containing an edge of vCy has just two attachments. Now $Q \cup R \cup \{uv\}$ gives the desired Tutte path. ∎

We also need the following lemma to deal with several cases in the proofs of (2.6) and (3.1).

(2.4) *Let H be a 2-connected plane graph with outercycle D and another facial cycle C. Let $w, z \in V(D)$ and $a, b, c \in V(C)$ (in this clockwise order on C) such that $c \neq a, b, z$ and $a \neq b$. Then H contains a subgraph L which either is a path in $H - c$ from a to b through z or consists of two disjoint paths in H from $\{a, b\}$ to $\{c, z\}$, such that*
(i) every $(L \cup \{c\})$-bridge of H has at most four attachments,
(ii) every $(L \cup \{c\})$-bridge of H containing an edge of $aCb \cup D$ has at most three attachments, and
(iii) if $w \notin V(L) \cup \{c\}$ then the $(L \cup \{c\})$-bridge of H containing w has just two attachments.

Proof. Suppose first there is a path in $H - c$ from a to b through z. Then we use (2.1) in $H - c$ to find a Tutte path L from a to b through z such that every L-bridge of $H - c$ containing an edge of D has at most two attachments. Note that if $w \in V(L)$, or if $w \notin V(L)$ and the L-bridge of $H - c$ containing w has just one attachment, or if $w \notin V(L)$ and the L-bridge of $H - c$ containing w contains no neighbor of c (except as an attachment), then L is the desired subgraph. So assume $w \notin V(L)$, and assume the L-bridge of $H - c$ containing w has two attachments and contains some neighbor of c (not as an attachment). Clearly in this case $H - c$ has a 2-cut $\{p, q\} \subset D$ separating w from $aCb \cup \{z\}$. Let p, w, q be on D in this clockwise order. By planarity, L is from b to a through p, z, q in this order. Let S be the $\{p, q, c\}$-bridge of H containing w, and let T be the $\{p, q\}$-bridge of $H - c$ containing L. Because of the existence of L, $T + pa$ contains a path from b to z through pa and then q (unless $p = z$) or $T + qb$ contains a path from a to z through qb and then p (unless $q = z$). By symmetry we may assume that $p \neq z$ and $T + pa$ contains a path from b to z through pa and then q. In $T + pa$ we use (2.3) to find a Tutte path P from b to z through pa and then q. In $S + qp$ we use (2.1) to find a Tutte path Q from c to q through qp such that every Q-bridge of S containing an edge of D has just two attachments. Clearly $(P - pa) \cup (Q - qp)$ gives the desired subgraph L.

Hence we may assume that $H - c$ contains no path from a to b through z. Then H contains a 2-cut $\{c, t\}$ separating $\{a, b\}$ from z. Let S and T be the $\{c, t\}$-bridges of H containing C and D, respectively. In $S + ab$, we use (2.1) to find a Tutte path P from c to t through ab. In $T - c$ we use (2.1) to find a Tutte path Q from z to t such that every Q-bridge of $T - c$ containing an edge of D has at most two attachments, and if $w \notin V(Q)$ then the Q-bridge of $T - c$ containing w has just one attachment. Clearly $L = (P - ab) \cup Q$ is as desired. ∎

The next two results, (2.5) and (2.6), are special cases in the proof of (3.1). In particular, (2.5) deals with a case resulting from cutting a graph in the torus with representativity 2 (Case 1 in proof of (3.1)), and (2.5) and (2.6) will be used to deal with two cases resulting from cutting a graph in the torus with representativity 3 (Case 2 in proof of (3.1)).

(2.5) *Let G be a 2-connected plane graph with outercycle C' and another facial cycle C. Let $a', b' \in V(C')$ and $a, b \in V(C)$ with $a' \neq b'$ and $a \neq b$. Suppose $e \in E(a'C'b')$ and $a'C'b' \cap C = \emptyset$. Then G contains two disjoint paths, P from a' to b' through e and Q from a to b, such that*

(i) every $(P \cup Q)$-bridge of G has at most four attachments,

(ii) every $(P \cup Q)$-bridge of G containing an edge of aCb has at most three attachments, and

(iii) every $(P \cup Q)$-bridge of G containing an edge of $a'C'b'$ has just two attachments.

Proof. Let H be the block of $G - a'C'b'$ containing C, and let D be the outercycle of H. We proceed by finding a path Q in H from a to b, and then finding a path P from a' to b' disjoint from Q.

If $|V(H) \cap V(b'C'a')| \leq 1$, let L be the union of all $(H \cup a'C'b')$-bridges of G containing an edge of $b'C'a'$, and let $Z = V(L \cap H)$; otherwise let $Z = \emptyset$. Note that $|Z| \leq 1$. Let $w_1, \cdots, w_n$ be all the attachments (in $V(D) - Z$) of $(H \cup a'C'b')$-bridges of G in this clockwise order on D.

For $i = 1, \cdots, n$, we define $s_i, t_i \in V(a'C'b')$ by saying that $s_iC't_i$ is a maximal subpath of $a'C'b'$ such that s_i and t_i are contained in some $(H \cup a'C'b')$-bridges of G with w_i as an attachment. We may assume that the notation is chosen so that $a', s_1, t_1, s_2, t_2, \cdots, s_n, t_n, b'$ occur on C' in this clockwise order. Let $k \in \{0, 1, \cdots, n\}$ be defined as follows. If $e \in E(a'Cs_1 \cup t_nC'b')$ then let $k = 0$ (in this case $|V(H) \cap V(b'C'a')| \leq 1$), and if $e \in E(s_1C't_1)$ then let $k = 1$; otherwise let k be such that $e \in E(t_{k-1}C't_k)$. Let $Z = \{w_0\}$ if $Z \neq \emptyset$; otherwise let w_0 be an arbitrary vertex on w_nDw_1.

We claim that there is a Tutte path Q in H from a to b with the following properties:
(a) every Q-bridge of H containing an edge of D has just two attachments,
(b) if $w_k \notin V(Q)$, then $w_0 \in V(Q)$, and the Q-bridge of H containing w_k contains no edge of aCb, and
(c) if $w_0 \notin V(Q)$, then $w_k \in V(Q)$, and the Q-bridge of H containing w_0 contains no edge of aCb.
Indeed, by (2.1), H contains a Tutte path Q from a to b through w_k such that every Q-bridge of H containing an edge of D has just two attachments, and H contains a Tutte path Q' from a to b through w_0 such that every Q'-bridge of H containing an edge of D has just two attachments. Since Q satisfies (a) and (b), we may assume that Q does not satisfy (c). Then $w_0 \notin Q$ and the Q-bridge of H containing w_0 contains edges of both D and aCb (and thus has two attachments). Then it follows that Q' satisfies (a), (b) and (c). This proves the claim.

Now let Q be a path as in the above claim. We define G_1 and G_2 as follows. If $|V(H) \cap V(b'C'a')| \leq 1$ and $w_0 \in V(Q)$, then there exist $a'', b'' \in V(a'C'b')$ and graphs G_1, G_2, such that $G_1 \cup G_2 = G$, $E(G_1) \cap E(G_2) = \emptyset$, $V(G_1) \cap V(G_2) = \{a'', b''\} \cup Z$, $V(H) \subset V(G_1)$, and $V(b'C'a') \subset V(G_2)$. If $|V(H) \cap V(b'C'a')| \geq 2$ and $w_nDw_1 \cap Q \neq \emptyset$, then let $a'' = a'$, $b'' = b'$, $G_1 = G$, $E(G_2) = \emptyset$, and $V(G_2) = \{a'', b''\}$. If $|V(H) \cap V(b'C'a')| \leq 1$ and $w_0 \notin V(Q)$ or if $|V(H) \cap V(b'C'a')| \geq 2$ and $w_nDw_1 \cap Q = \emptyset$, then let B be the Q-bridge of H containing w_0 or w_nDw_1. There exist $a'', b'' \in V(a'C'b')$ and graphs G_1, G_2, such that $G_1 \cup G_2 = G$, $E(G_1) \cap E(G_2) = \emptyset$, $V(G_1) \cap V(G_2) = \{a'', b''\} \cup V(Q \cap B)$, $V(H) - V(B) \subset V(G_1)$, and $V(b'C'a') \subset V(G_2)$. Moreover, if $w_nDw_1 \cap Q = \emptyset$, then by (c), $w_k \in V(Q)$, and hence we may select a'', b'', G_1, G_2 so that $e \notin E(G_2)$. This completes the descriptions of G_1 and G_2.

Let G_2' be obtained from $G_2 - V(Q)$ by adding an edge $b'a'$ such that $b'C'a'$ does not belong to the outerwalk of G_2'. If $e \notin G_2$, then we use (2.1) in G_2' to find a Tutte path P_1 from a'' to b'' through $b'a'$ such that every P_1-bridge of G_2' containing an edge of its outerwalk has at most two attachments. If $e \in E(G_2)$, then $w_n D w_1 \cap Q \neq \emptyset$, and hence $w_0 \in V(Q)$ (and thus $|V(G_1 \cap G_2)| \leq 3$). In this case, we use (2.3) to find a Tutte path P_1 in G_2' from a'' to b'' through e and $b'a'$ such that every P_1-bridge of G_2' containing an edge of $a'C'a'' \cup b''C'b'$ has at most two attachments.

Let U be the union of $a''C'b''$, all $(H \cup a''C'b'')$-bridges of G_1, and all Q-bridges of H (other than B in the case when B exists) containing a vertex of $\{w_1, \cdots, w_n\} - V(Q)$. Let $A = V(Q \cap U)$, and let $J = a''B_1v_1B_2 \cdots v_{m-1}B_mb''$ be the plane chain of blocks in $U - A$ along $a''C'b''$. Define H_i and $a(B_i)$ as in (2.2). It follows from the choice of Q and the planarity of G that $|a(B_i)| \leq 2$. By (2.2) there is a path P_2 in J from a'' to b'' (through e if $e \in E(a''C'b'')$), such that every $(P_2 \cup A)$-bridge of U containing an edge of $a''C'b''$ has just two attachments and every $(P_2 \cup A)$-bridge of U has at most three attachments, unless it contains an edge of B_i and $e \in E(B_i)$ (in which case it has four attachments). Let B' be a $(P_2 \cup A)$-bridge of U with four attachments; then $B' \cap H$ is a subgraph of the Q-bridge B'' of H containing w_k. By (b), B'' contains no edge of aCb, and hence neither does B'. Since B' has four attachments, B' contains no edge of $a''C'b''$. Now let $P = (P_1 - b'a') \cup P_2$, and it follows that P and Q are the desired paths. ∎

(2.6) *Let $G = a'G_1u_1G_2 \cdots u_{s-1}G_sc'$ be a plane chain of blocks (if $s = 1$, let $u_1 = c' \in V(G_1)$) with outerwalk C' and another facial cycle C in G_1. Let $b' \in V(G_1) - \{u_1, a'\}$ such that $u_1 \in b'C'a'$, and let a, b, c be three distinct vertices on C in this clockwise order. Let G^* be obtained from G by identifying a with a' as a^*, b with b' as b^*, c with c' as c^*, respectively. Suppose $e \in E(a'C'b')$ and suppose the following are satisfied.*

(a) In G, c and c' are not cofacial and no vertex of $a'C'b'$ is cofacial with a vertex of aCb,

(b) G contains no cutset $\{p, q\} \subset V(aCb)$ or $\{p, q\} \subset V(a'C'b')$ separating c from c',

(c) $a'C'b' \cap C = \emptyset$,

(d) G contains no cutset $\{p, q, z\}$ with $p, q \in V(a'C'b')$ separating $C \cup \{e\}$ from $b'C'a'$, and

(e) G contains no cutset $\{p, q\}$ with $p \in V(a'C'b')$ and $q \in V(b'C'c' - c'C'a')$ separating $c'C'a'$ from $C \cup \{e, b'\}$, and G contains no cutset $\{p, q\}$ with $p \in V(a'C'b')$ and $q \in V(c'C'a' - b'C'c')$ separating $b'C'c'$ from $C \cup \{e, a'\}$.

Then G^* contains a cycle C^* through e and exactly one edge at each of a, a', b, b' in G, such that
(i) every C^*-bridge of G^* has at most four attachments,
(ii) every C^*-bridge of G^* containing an edge of aCb has at most three attachments, and
(iii) every C^*-bridge of G^* containing an edge of $a'C'b'$ has just two attachments.

Proof. Let H be the block of $G - a'C'b'$ containing C, and let D be the outercycle of H. By (b), C and c' belong to a common component of $G - a'C'b'$. Let $z = u_1$ if $u_1 \in V(H)$; otherwise let $z \in V(H)$ be the cutvertex of $G - a'C'b'$ separating C from u_1.

(2.6.1) We may assume $z \neq c$.

Otherwise, by (a) and (c), $z \notin V(C')$ and G has a cutset $\{a'', b'', c\}$ separating C from u_1 (such that a', a'', b'', b' are on C' in this clockwise order). In H we use (2.1) to find a Tutte path P from a to b through c such that every P-bridge of H containing an edge of D has just two attachments. Let T be the $\{a'', b'', c\}$-bridge of G containing c'. Then by (d), $e \in E(T)$. Let $T' = T - \{c, c'\} + a''b''$ (so that $b'C'a' - c'$ is on the outerwalk of T'). In T' we use (2.3) to find a Tutte path Q from a' to b' through $a''b''$ and e such that every Q-bridge of T' containing an edge of $a'C'a'' \cup b''C'b'$ has just two attachments. Note that there are possibly $(Q \cup \{c, c'\})$-bridges of T with five attachments, but two of these attachments are c and c' and such bridges contain no edge of $a'C'b' \cup aCb$. Also note that any $(Q \cup \{c, c'\})$-bridge of T containing an edge of $a'C'b'$ contains neither c nor c'. Now let U be the union of $a''C'b''$, all $(H \cup a'C'b')$-bridges of G outside T, and all P-bridges of H containing a vertex which is not on P but is in some $(H \cup a'C'b')$-bridge of G. Let $A = V(U \cap P)$ and let $J = a''B_1v_1B_2 \cdots v_{m-1}B_mb''$ be the plane chain of blocks in $U - A$ along $a''C'b''$. Define H_i and $a(B_i)$ as in (2.2). Then $|a(B_i)| \leq 2$ for each i. We use (2.2) to find a path R in J from a'' to b'' such that every $(R \cup A)$-bridge of U has at most three attachments and every $(R \cup A)$-bridge of U containing an edge of $a''C'b''$ has just two attachments. By (a), $D \cap aCb = \emptyset$. Hence every $(P \cup (Q - a''b'') \cup R)$-bridge of G containing an edge of aCb is also a P-bridge of H. Thus $E(P \cup (Q - a''b'') \cup R)$ induces the desired C^*. Hence we may assume $z \neq c$.

Let $w_1, \cdots, w_n \in V(H - z)$ be the attachments of $(H \cup a'C'b')$-bridges of G in this clockwise order on D such that $z \in V(w_nDw_1)$. Let $x_iC'y_i$ be the maximal subpath of

$a'C'b'$ such that x_i and y_i are in some $(H \cup a'C'b')$-bridges of G with w_i as an attachment.

(2.6.2) We may assume $e \in E(x_1C'y_n)$.

Suppose $e \notin E(x_1C'y_n)$. By (2.6.1) we use (2.1) in $H + ab$ to find a Tutte path L from z to c through ab. Let T be the z-bridge of $G - a'C'b'$ containing c'. In T we use (2.1) to find a Tutte path R from z to c' such that every R-bridge of T containing an edge of its outerwalk has at most two attachments. Now let U be the union of $a'C'b'$, all $(H \cup a'C'b' \cup T)$-bridges of G, and all $((L - ab) \cup R)$-bridges of $H \cup T$ containing a vertex which is not on $(L - ab) \cup R$ but is in some $(H \cup a'C'b' \cup T)$-bridge of G. Let $A = V((R \cup (L - ab)) \cap U)$ and let $J = a'B_1v_1B_2 \cdots v_{m-1}B_mb'$ be the plane chain of blocks in $U - A$ along $a'C'b'$. Define H_i and $a(B_i)$ as in (2.2) (here $(L - ab) \cup R$ is the Q in (2.2)). Clearly $|a(B_i)| \leq 3$, and if $e \in E(B_i)$ then $|a(B_i)| \leq 2$ (since $e \notin E(x_1C'y_n)$). Thus by (2.2) we can find a path P in J from a' to b' through e such that every $(P \cup A)$-bridge of U has at most four attachments and every $(P \cup A)$-bridge of U containing an edge of $a'C'b'$ has just two attachments. By (a), $E(P \cup (L - ab) \cup R)$ induces the desired C^* in G^*.

Now by (2.6.2), (d) and (e), we have $z = u_1$. Let $e \in E(y_{k-1}C'y_k)$, where $1 \leq k < n$ and $y_0 = x_1$. If $z \neq c'$, then we use (2.1) in H to find a Tutte path L from a to b through z such that every L-bridge of H containing an edge of D has just two attachments. If $z = c'$, then in H we use (2.4) to find a subgraph L which either is a path in $H - c$ from a to b through z or consists of two disjoint paths in H from $\{c, z\}$ to $\{a, b\}$, such that every $(L \cup \{c\})$-bridge of H has at most four attachments, every $(L \cup \{c\})$-bridge of H containing an edge of $D \cup aCb$ has at most three attachments, and if $w_k \notin V(L) \cup \{c\}$ then the $(L \cup \{c\})$-bridge of H containing w_k has just two attachments. Now let U be the union of $a'C'b'$, all $(H \cup a'C'b')$-bridges of G, and all $(L \cup \{c\})$-bridges of H containing a vertex of $\{w_1, \cdots, w_n\} - (L \cup \{c\})$. Let $A = V(U \cap (L \cup \{c\}))$ and let $J = a'B_1v_1B_2 \cdots v_{m-1}B_mb'$ be the plane chain of blocks in $U - A$ along $a'C'b'$. Define H_i and $a(B_i)$ as in (2.2). Clearly $|a(B_i)| \leq 3$, and if $e \in E(B_i)$ then $|a(B_i)| \leq 2$. Thus by (2.2) we find a path P in J from a' to b' through e such that every $(P \cup A)$-bridge of U has at most four attachments and every $(P \cup A)$-bridge of U containing an edge of $a'C'b'$ has just two attachments. Clearly $E(P \cup L \cup R)$ induces the desired C^*. ∎

3 The main result

Now we consider graphs which can be embedded in the torus, that is, toroidal graphs. For a graph G embedded in a surface, the *representativity* of G is the minimum number k such that every non-null homotopic simple closed curve intersects G at least k times.

Suppose that G is a graph embedded in the torus. A *plane subgraph* of G is a subgraph of G which is contained in a closed disc. A *facial cycle* of G is a cycle which bounds a (open) face. Let C be a facial cycle of G. We define $\rho_G(C)$ to be $\min |N \cap G|$, where N is a non-null homotopic simple closed curve in the torus intersecting the face bounded by C and $N \cap G \subset V(G)$.

(3.1) *Let G^* be a 2-connected graph embedded in the torus, and C_1 a facial cycle of G^* containing an edge e. Then G^* contains a cycle C^* through e such that*
(i) every C^*-bridge of G^* has at most four attachments, and
(ii) every C^*-bridge of G^* containing an edge of C_1 is a plane subgraph of G^* and has at most three attachments.

Proof. Since C_1 is a facial cycle, $\rho_{G^*}(C_1) \geq 2$. We first treat the case when $\rho(G^*) \leq 1$. If $\rho(G^*) = 0$, then G^* is a contained in a closed disc in the torus, and so G^* is a plane graph. In this case the existence of C^* follows from (2.1). Now suppose $\rho(G^*) = 1$. Then there is a non-null homotopic simple closed curve in the torus intersecting G^* only at a vertex t^*. We cut along this curve and obtain a plane graph G from G^* with t^* being split as t and t'. Note that C_1 is a facial cycle of G. Since G^* is 2-connected, G contains a path from t to t' through e. Now applying (2.1) to G, we find a Tutte path in G from t to t' through e. Clearly this Tutte path gives the desired C^*.

Thus, we may assume that $\rho(G^*) \geq 2$. We use induction on $|V(G^*)|$ and consider three cases: $\rho_{G^*}(C_1) = 2$, $\rho_{G^*}(C_1) = 3$, and $\rho_{G^*}(C_1) \geq 4$.

Case 1. $\rho_{G^*}(C_1) = 2$.

In this case, there is a non-null homotopic simple closed curve N in the torus through the face bounded by C_1 only intersecting G^* at two vertices a^* and b^*. Cut the torus along the curve N. We obtain a plane graph G with outerwalk C' and another facial walk C such that $a', b' \in V(C')$ and $a, b \in V(C)$, and we get G^* from G by identifying a and

b with a' and b' as a^* and b^*, respectively. Clearly we may select the notation so that $E(a'C'b') \cup E(aCb) = E(C_1)$ and a', e, b' are on C' in this clockwise order. Since $\rho(G^*) \geq 2$, a and a' are not cofacial in G, and b and b' are not cofacial in G. In particular, $C \neq C'$.

Next we show that we may choose N so that after cutting the torus along N we have a situation in which (2.5) may be applied.

(1a) We may choose N and the notation so that C is a cycle.

We choose N and a^* and b^* so that $|E(C)|$ is minimum. Suppose that C is not a cycle. Then G has a cutvertex $z \in V(C)$. Since $C \neq C'$, let Z be a z-bridge of G not containing C'. Since G^* is 2-connected, every z-bridge of G must contain at least one of a, a', b, b'. Hence Z contains one of a or b but not both (otherwise $\rho(G^*) = 1$).

By symmetry assume that $a \in V(Z)$ and $b \notin V(Z)$. There is a non-null homotopic simple closed curve in the torus through the face bounded by C_1 intersecting G^* only at z and b^*. If we cut the torus along this curve (and view z as new a^*), then we have a new C such that the new $|E(C)|$ is smaller, a contradiction. Hence C is a cycle.

(1b) We may choose N so that subject to (1a), $a'C'b' \cap C = \emptyset$.

We choose N, a^* and b^* (subject to (1a)) so that $|V(a'C'b' \cap C)|$ is minimum. Let $t \in V(a'C'b' \cap C)$. Since C_1 is a cycle in G^*, $t \in V(bCa)$. By symmetry let $e \in E(tC'b')$. Choose t so that $|V(a'C't)$ is minimum. Then there is a non-null homotopic simple closed curve N' in the torus through the face bounded by C_1 intersecting G^* only at t and b^*. We view t as new a^*, and cut the torus along N'. It is easy to see that $tC'b'$ becomes new $a'C'b'$, and that $aCt \cup a'C't$ becomes new C (which is a cycle by the choice of t). Clearly the new $|V(a'C'b' \cap C)|$ is smaller, a contradiction. Hence $a'C'b' \cap C = \emptyset$.

If C' is also a cycle, then (2.5) gives two paths P and Q which induces the desired C^*. Note that all $(P \cup Q)$-bridges of G containing an edge of C_1 are plane subgraphs of G^* except possibly a $(P \cup Q)$-bridge of G containing an edge of $a'C'b' \cup aCb$ and both a and a' (or both b and b'). But it is easy to see that a and a' or b and b' are cofacial in G, contradicting $\rho(G^*) \geq 2$.

So we may assume that C' is not a cycle. Let H be the block of G containing C. Since G^* is 2-connected, G has at most two cutvertices contained in H, each separates exactly one of a', b' from C. Let $a'' = a'$ if $a' \in V(H)$; otherwise let $a'' \in V(H)$ be the cutvertex of G separating a' from C. Define b'' similarly. Let $A = \{a'\}$ if $a'' = a'$; otherwise let A be the a''-bridge of G containing a'. Let $B = \{b'\}$ if $b'' = b'$; otherwise let B be the b''-bridge of G containing b'. Now apply (2.5) to H we get two disjoint paths P and Q, with P from a'' to b'' (through e if $e \notin E(A \cup B)$) and Q from a to b, such that every $(P \cup Q)$-bridge of H has at most four attachments and every $(P \cup Q)$-bridge of H containing an edge of $aCb \cup a'C'b'$ has at most three attachments. In A we use (2.1) to find a Tutte path R from a' to a'' (through e if $e \in E(A)$). Similarly we find a Tutte path S in B from b' to b''. Then $E(P \cup Q \cup R \cup S)$ induces the desired cycle C^*. Again, since $\rho(G^*) \geq 2$, all C^*-bridges of G^* containing an edge of C_1 are plane subgraphs of G^*.

Case 2. $\rho_{G^*}(C_1) = 3$.

Let N be a non-null homotopic simple closed curve in the torus through the face bounded by C_1 intersecting G^* at a^*, b^*, c^* with $a^*, b^* \in V(C_1)$. If we cut the torus along N, we get a plane graph G with a', b', c' on the outerwalk C' and a, b, c on another facial walk C such that $E(a'C'b' \cup aCb) = E(C_1)$, $c' \in V(b'C'a')$ and $c \in V(bCa)$, and G^* is obtained from G by identifying a with a' as a^*, b with b' as b^*, and c with c' as c^*, respectively. By symmetry we may assume that $e \in E(a'C'b')$.

Next we show that we may select N so that after cutting the torus along N, G has properties (2a)-(2g) below (and hence we will have situations as in or similar to (2.6)). Note that (2a)-(2d) are true for any curve N as above.

(2a) In G, c and c' are not cofacial, and no vertex of $a'C'b'$ is cofacial with a vertex of aCb.

If c and c' are are cofacial, then $\rho(G^*) = 1$, a contradiction. Now suppose that G has a face F containing a vertex $s \in a'C'b'$ and a vertex $t \in aCb$. Let N' be a simple closed curve in the torus through the face bounded by C_1 intersecting G^* only at s and t and dividing F into two regions. This N' is non-null homotopic. Hence we have $\rho_{G^*}(C_1) = 2$, a contradiction.

(2b) G does not have a cutvertex on C' separating $\{a', c'\}$ from $C \cup \{b'\}$, or $\{b', c'\}$ from $C \cup \{a'\}$), and G does not have a cutvertex on C separating $\{a, c\}$ from $C' \cup \{b\}$ (or $\{b, c\}$ from $C' \cup \{a\}$).

Otherwise, we may assume by symmetry that t is a cutvertex of G separates $\{a', c'\}$ from $C \cup \{b'\}$. If $t \neq b'$ and c and b' belong to a common component of $G - t$, then there is a non-null homotopic simple closed curve in the torus through the face bounded by C_1 intersecting G^* only at t and b^*, and so $\rho_{G^*}(C_1) = 2$, a contradiction. If $t = b'$ or t and b' belong to different components of $G - t$, then there is a non-null homotopic simple closed curve in the torus through the face bounded by C_1 intersecting G^* only at t, and so $\rho(G^*) = 1$, a contradiction.

(2c) G contains no cutset $\{p, q\} \subset V(a'C'b')$ or $\{p, q\} \subset V(aCb)$ separating c from c'.

Otherwise, let $\{p, q\} \subset V(a'C'b')$ separate c from c'. Since $\rho(G^*) \geq 2$ and by (2b), $p \neq q$. Thus there is a non-null homotopic simple closed curve in the torus through the face bounded by C_1 only intersecting G^* at p and q, and so $\rho_{G^*}(C_1) = 2$, a contradiction.

(2d) $a'C'b' \cap C = \emptyset$.

Since C_1 is a cycle, $a'C'b' \cap aCb = \emptyset$. Suppose $z \in V(a'C'b' \cap C)$. If $z \in V(bCc)$, then there is a non-null homotopic simple closed curve in the torus through the face bounded by C_1 only intersecting G^* at z and b^*, and so $\rho_{G^*}(C_1) = 2$, a contradiction. If $z \in V(cCa)$, then there is a non-null homotopic simple closed curve in the torus through the face bounded by C_1 intersecting G^* only at z and a^*, and so $\rho_{G^*}(C_1) = 2$, a contradiction. Hence $a'C'b' \cap C = \emptyset$.

(2e) We may select N so that G contains no cutset $\{p, q, z\}$ with $p, q \in V(a'C'b')$ separating $C \cup \{e\}$ from $b'C'a'$.

Suppose that $K = \{r^*, s^*, t^*\}$ is a cutset as in the statement of (2e) with r^*, s^* on $a'C'b'$ and $r^* \in a'C's^*$, such that no other 3-cut $\{p, q, z\}$ separates $K \cup b'C'a'$ from $C \cup \{e\}$. There is a non-null homotopic simple closed curve N' in the torus through the face bounded by C_1 and intersecting G^* only at r^*, s^*, and t^*. Now cut the torus along N' (as we cut along N), we obtain a new plane graph G' and two facial walks D and D' with $r', s', t' \in V(D')$ and $r, s, t \in V(D)$, where $e \in E(r'D's')$, $t' \in V(s'D'r')$ and r, s, t are on D in this clockwise order. Then G' has no cutset as in the statement of (2e) separating $s'D'r'$ from $D \cup \{e\}$, for such a cutset is also a cutset in G separating $K \cup b'C'a'$ from $C \cup \{e\}$, contradicting the choice of K.

(2f) We may select N so that subject to (2e), G contains no cutset $\{p, q\}$ with $p \in V(a'C'b')$ and $q \in V(b'C'c' - c'C'a')$ separating $c'C'a'$ from $C \cup \{e, b'\}$ and G contains no cutset $\{p, q\}$ with $p \in V(a'C'b')$ and $q \in V(c'C'a' - b'C'c')$ separating $b'C'c'$ from $C \cup \{e, a'\}$.

Otherwise by symmetry let $K = \{r^*, s^*\}$ be a cutset in G with $r^* \in V(a'C'b')$ and $s^* \in V(b'C'c' - c'C'a')$ separating $c'C'a'$ from $C \cup \{e, b'\}$. We select K so that no other cutset as in the statement of (2f) separates $K \cup c'C'a'$ from $C \cup \{e, b'\}$. Now there is a non-null homotopic simple closed curve N' in the torus through the face bounded by C_1 and intersecting G^* only at r^*, b^*, s^*. Cutting the torus along N' we obtain a plane graph G' and two facial walks D and D' with $r', b', s' \in V(D')$ and $r, b, s \in V(D)$, where $e \in E(r'D'b')$, $s' \in V(b'D'r')$, and r, b, s are on D in this clockwise order. Note that no cutset in G' as in the statement of (2e) separates $b'D'r'$ from $D \cup \{e\}$; otherwise such a cutset would also be a cutset in G separating $b'C'a'$ from $C \cup \{e\}$, contradicting (2e). No cutset $\{p, q\}$ with $p \in V(r'D'b')$ and $q \in V(b'D's' - s'D'r')$ separating $s'D'r'$ from $D \cup \{e, b'\}$, for such a cutset would also be a cutset in G separating $K \cup c'C'a'$ from $C \cup \{e, b'\}$, contradicting the choice of K. Finally, no cutset $\{p, q\}$ with $p \in V(r'D'b')$ and $q \in V(s'D'r' - b'D's')$ separating $b'D's'$ from $D \cup \{e, r'\}$; otherwise $\{p, q, r^*\}$ is a cutset in G contradicting (2e).

(2g) We may select N so that subject (2e) and (2f), c is not a cutvertex, and every cutvertex of G on C separates aCb from $C' \cup \{c\}$

We choose N so that subject to (2e) and (2f), the number of cutvertices on C is minimum. Suppose G has a cutvertex $z \in V(C)$. Since $\rho(G^*) \geq 2$, z cannot separate $\{a, b, c\}$ from C'. Now c cannot be a cutvertex of G; otherwise by (2b) and since G^* is 2-connected, c separates $\{a, b, c\}$ from C', a contradiction. Hence $z \neq c$.

If $z \neq a$ separates b from $C' \cup \{a, c\}$, then we may select N to pass through a^*, c^*, z such that the new G has fewer cutvertices on the new C. Note that (2e) and (2f) hold for the new G (with z replacing b^*). Hence no cutvertex $z \neq a$ of G separates b from $C' \cup \{a, c\}$. Similarly, G has no cutvertex $z \neq b$ separating a from $C' \cup \{b, c\}$. Also G has no cutvertex on C separating c from $C' \cup \{a, b\}$. This proves (2g).

By (2g), c is not a cutvertex of G. Hence let H be the unique block of G containing c. Now H is a plane subgraph of G with outercycle a subgraph of C' and another facial cycle a subgraph of C. Let $a'' = a'$ if $a' \in V(H)$, otherwise let $a'' \in V(H)$ be the cutvertex of G separating a' from c. Similarly we define b'' and c'' with respect to b' and c', respectively.

The rest of Case 2 is divided into four subcases.

Subcase 1. $a'' \neq b''$ and G has no cutvertex on C.

Let $H' = H$ if $c' = c''$; otherwise let H' be the the union of H and the c''-bridge of G containing c'. Let H^* be obtained from H' by identifying a'', b'', c' with a, b, c, respectively. We apply (2.6) to H^* (with a'', b'' replacing a', b', respectively), to get a cycle F in H^* through e, such that F uses exactly one edge at each of a, a'', b, b'' in H', every F-bridge of H' has at most four attachments, and every F-bridge of H' containing an edge of $a''C'b'' \cup aCb$ has at most three attachments. In the a''-bridge of G containing a' we use (2.1) to find a Tutte path P from a'' to a' (through e if it contains e), and in the b''-bridge of G containing b' we use (2.1) to find a Tutte path Q from b' to b'' (through e if it contains e). Clearly $E(F \cup P \cup Q)$ induces the desired C^* in G^*.

Subcase 2. $a'' \neq b''$ and G contains a cutvertex t on C.

In this case, $tCc \cap b'C'c' = \emptyset$; otherwise $\rho(G^*) = 1$. We may assume by symmetry that in the t-bridge of G containing aCb, b does not separate a from t. We now distinguish two cases: $e \in E(a'C'b'')$ and $e \in E(b''C'b')$.

Suppose first $e \in E(a'C'b'')$. If $a'C'c' \cap C = \emptyset$, then we use (2.5) in H (with a'', c'', t, c as a', b', a, b, respectively) to find two disjoint paths, P from c'' to a'' (through e if $e \in E(a''C'b'')$) and Q from t to c, such that every $(P \cup Q)$-bridge of H has at most four attachments, every $(P \cup Q)$-bridge of H containing an edge of $a''C'c''$ has just two attachments. In the a''-bridge of G containing a' we use (2.1) to find a Tutte path R_1 from a'' to a' (through e if $e \in E(a'C'a'')$). In the c''-bridge of G containing c' we use (2.1) to find a Tutte path R_2 from c'' to c'. Let T be the t-bridge of G containing aCb. If $b' \in V(P \cup Q)$, then we find a Tutte path R_3' in $T + ab$ from t to b through ab, and let $R_3 = R_3' - ab$; otherwise, by (2.1) and (2g) we can find a Tutte path R_3 in T from a to t through b (or if a is a cutvertex of T separating b from t, then we find a Tutte path R_3 in T from a to t). Then $E(P \cup Q \cup R_1 \cup R_2 \cup R_3)$ induces the desired C^*.

Therefore we may assume that $a'C'c' \cap C \neq \emptyset$. Hence by (2d) and since $tCc \cap b'C'c' = \emptyset$, there is a vertex $w \in V(cCt - \{c, t\}) \cap V(b'C'c')$. Let K be the component of $G - tCc$ containing $a''C'c''$. In K we use (2.3) to find a Tutte path P from c'' to a'' through w (and e if $e \in E(a''C'b'')$) such that every P-bridge of K containing an edge of $a''C'c''$ has just two attachments. Now let U be the union of tCc, all $(K \cup tCc)$-bridges of G, and all P-bridges of K containing a vertex which is not on P but is in some $(K \cup tCc)$-bridge of G. Let $A = V(U \cap P)$ and let $J = tB_1v_1B_2 \cdots v_{m-1}B_mc$ be the plane chain of blocks in $U - A$ along

tCc. Define H_i and $a(B_i)$ as in (2.2). Clearly $|a(B_i)| \leq 3$. Hence by (2.2) we find a path Q in J from t to c such that every $(Q \cup A)$-bridge of U has at most four attachments. We now find R_1, R_2 and R_3 as in the previous paragraphs, and then $E(P \cup Q \cup R_1 \cup R_2 \cup R_3)$ induces the desired C^*.

Hence we may assume that $e \in E(b''C'b')$. We may also assume that a separates b from c, that is, a is a cutvertex of G; otherwise the above argument can be applied with the roles of a' and b', as well as those of a and b, interchanged.

Let P_1 be a Tutte path in the a''-bridge of G containing a' from a'' to a', and P_2 a Tutte path in the b''-bridge of G containing b' from b'' to b' through e. Also let Q_1 be a Tutte path in the t-bridge of G containing aCb from a to b.

If $H - \{a, c'\}$ contains a path from a'' to b'' through c, then we use (2.1) to find a Tutte path Q_2 from a'' to b'' through c such that every Q_2-bridge of $H - \{a, c'\}$ containing an edge of the facial walk of $H - \{a, c'\}$ which bounds the face containing C has at most two attachments. Note that every Q_2-bridge of $H - \{a, c'\}$ containing an edge of $a''C'b''$ does not contain any neighbor of c' (except as its attachments). Hence $E(P_1 \cup P_2 \cup Q_1 \cup Q_2)$ induces the desired C^*. So we may assume that such a path does not exist. Then H contains a cutset S separating $a''C'b''$ from c such that $|S| \leq 3$, $S \cap \{a, c'\} \neq \emptyset$, and if S does not contain $\{a, c'\}$ then $|S| = 2$. Let T be the union of the S-bridges of H containing an edge of the outercycle of H. We select S so that T is minimal.

If $c' \in S$, then in $T - a$ we use (2.1) to find a Tutte path Q_2 from a'' to b'' through c' such that every Q_2-bridge of $T - a$ containing an edge of the facial walk of $T - a$ which bounds the face containing C has at most two attachments. By (2a), for every Q_2-bridge B of $T - a$ containing an edge of $a''C'b''$, $V(B - Q_2)$ does not contain a neighbor of a. Thus $E(P_1 \cup P_2 \cup Q_1 \cup Q_2)$ induces the desired C^*.

Hence we may assume $c' \notin S$. Then $a = t$. Let $S = \{a, t'\}$, and let t'' be a neighbor of t' on the outerwalk of T. By minimality of T, in $T - a + a''b''$ (so that $b''C'c' \cup c'C'a''$ is in its outerwalk), we use (2.1) to find a Tutte path Q_2 from t' to t'' through $a''b''$ such that every Q_2-bridge of $T - a + a''b''$ containing an edge of $b''C'a''$ has just two attachments. By (2a), no Q_2-bridge of $T - a + a''b''$ containing an edge of $a''C'b''$ contains a neighbor of a (except as an attachment). Hence $E(P_1 \cup P_2 \cup Q_1 \cup (Q_2 - a''b'' + t't''))$ induces the desired C^*.

Subcase 3. $a'' = b''$ and G contains a cutvertex z on C.

Then $z \notin V(C')$ (otherwise $\rho(G^*) = 1$). Let S be the component of $G - \{z, c, c'\}$ containing $a'C'b'$, and let T be the z-bridge of G containing aCb. In S we use (2.1) to find a Tutte path P from a' to b' through e such that every P-bridge of S containing an edge of the outerwalk of S has at most two attachments. Now in T we use (2.1) to find a Tutte path Q from a to b such that if $z \notin V(Q)$ then the Q-bridge of T containing z has just one attachment. Clearly $E(P \cup Q)$ induces the desired C^* in G^*.

Subcase 4. $a'' = b''$ and there is no cutvertex on C.

Note that $a'' \notin V(C)$; otherwise $\rho(G^*) = 1$. Also $c'' \notin V(C)$; otherwise $\rho_{G^*}(C_1) = 1$, a contradiction.

Suppose first that $a'' \notin V(a'C'b')$. Let D be the outercycle of H. By (2.4), we find a subgraph L of H which either consists of two disjoint paths in H from $\{a, b\}$ to $\{c, c''\}$ or is a path in $H - c$ from a to b through c'', such that every $(L \cup \{c\})$-bridge of H has at most four attachments, every $(L \cup \{c\})$-bridge of H containing an edge of $D \cup aCb$ has at most three attachments, and either $a'' \in V(L)$ or the $(L \cup \{c\})$-bridge of H containing a'' has just two attachments. Let S be the a''-bridge of G containing $a'C'b'$. Then in $S - a''$ we find a Tutte path R from a' to b' through e such that every R-bridge of $S - a''$ containing an edge of the outerwalk of $S - a''$ has at most two attachments. If L is a single path in H, then let $R' = \emptyset$; otherwise, in the c''-bridge of G containing c', we use (2.1) to find a Tutte path R' from c'' to c'. Clearly $E(L \cup R \cup R')$ induces the desired C^*.

Hence we may assume $a'' \in V(a'C'b')$. In the union of the a''-bridges of G containing a' or b', we use (2.1) to find a Tutte path P from a' to b' through e. We also use (2.1) in the c''-bridge of G containing c' to find a Tutte path R' from c'' to c'. If in $H' = H - a'' + ab$ there is a path from c'' to c through ab, then we use (2.1) to find a Tutte path Q in H' from c'' to c through ab. Then $E(P \cup (Q - ab) \cup R')$ induces the desired C^*. Hence H' does not contain a path from c to c'' through ab. In this case $\{a'', c\}$ is a 2-cut of H separating aCb from c''. Therefore we use (2.1) in $H - a''$ to find a Tutte path Q from a to b through c such that every Q-bridge of $H - a''$ containing an edge of C has just two attachments. Clearly $E(P \cup Q)$ induces the desired C^*.

Case 3. $\rho_{G^*}(C_1) \geq 4$.

In this case $G^* - C_1$ has a cycle which bounds an open disc containing C_1. Let H^* be the block of $G^* - C_1$ containing such a cycle, and let C_2 be the new facial cycle of H^*.

We may assume G^* contains no 2-cut separating C_1 from H^*. Otherwise, let $\{s,t\}$ be a 2-cut of G^* separating C_1 from H^*, and let G_1, G_2 be the $\{s,t\}$-bridges of G^* containing C_1 and C_2, respectively. By (2.1), G_1 contains a Tutte path P from s to t through e. By induction, $G_2 + st$ contains a cycle C through st such that every C-bridge of G_2 contains at most four attachments. Clearly $E(P \cup (C - st))$ induces the desired C^*.

Now let $u_1, u_2, \cdots, u_n$ be the attachments of $(C_1 \cup H^*)$-bridges of G^* on C_2 in this clockwise order. For two vertices x and y on C_i $(i = 1, 2)$, we use xC_iy to denote the subpath of C_i from x to y in the clockwise order. For each u_i, let $s_i, t_i \in V(C_1)$ with $s_iC_1t_i$ maximal, such that s_i, t_i are contained in $(C_1 \cup H^*)$-bridges of G^* with u_i as an attachment, and no $(C_1 \cup H^*)$-bridges of G^* containing u_j $(j \neq i)$ contains a vertex of $s_iC_1t_i - \{s_i, t_i\}$. Define J_i to be the union of $s_iC_1t_i$ and all $(C_1 \cup H^*)$-bridges of G^* whose attachments are contained in $s_iC_1t_i \cup \{u_i\}$.

Without loss of generality we may assume that $e \in E(t_nC_1t_1)$ and let $t_k \neq t_1$ but $t_1 = \cdots = t_{k-1}$. (We may assume such a k exists, possibly by reflecting the graph so that counterclockwise becomes clockwise.) Let K be obtained from H^* by adding a vertex t_1 and joining t_1 to $u_1, \cdots, u_k$ such that $u_kC_2u_1 \cup \{u_1t_1, t_1u_k\}$ is a facial cycle C_2' of K. Applying induction to K we find a cycle C in K through t_1u_k such that every C-bridge of K has at most four attachments, every C-bridge of K containing an edge of C_2' is a plane subgraph of K and has at most three attachments. We now extend $C - t_1$ to the desired cycle C^*.

Let t_1u_l be the edge in C other than t_1u_k. Note that if $l \neq 1$ and $u_1 \notin V(C)$, then u_1 is in a C-bridge of K with at most three attachments (one of which is t_1).

In J_k we use (2.1) to find a Tutte path P from u_k to t_k through s_k if $s_k \neq t_1$; otherwise in $J_k + u_ks_k$ we use (2.1) to find a Tutte path P' from t_k to s_k through u_ks_k, and let $P = P' - u_ks_k$.

Suppose first $l = 1$. In the union of $t_nC_1t_1$ and all $(H^* \cup C_1)$-bridges of G^* with attachments contained in $t_nC_1t_1 \cup \{u_1\}$, we use (2.3) to find a Tutte path Q from u_1 to t_n through e and t_1. Let U be the union of $t_kC_1t_n$, all $(H^* \cup C_1)$-bridges of G^* with attachments in $V(t_kC_1t_n) \cup \{u_{k+1}, \cdots, u_n\}$, and all C-bridges of H^* containing a vertex of $\{u_{k+1}, \cdots, u_n\} - C$. Let $A = V((C - t_1) \cap U)$. Note that $U - A$ is a plane subgraph of G^*

and $t_kC_1t_n$ is a simple path in its outerwalk. Let $J = t_kB_1v_1B_2\cdots v_{m-1}B_mt_n$ be the plane chain of blocks in $U - A$ along $t_kC_1t_n$. Define H_i and $a(B_i)$ as in (2.2). Clearly $|a(B_i)| \le 3$. We use (2.2) to find a path R in J from t_k to t_n such that every $(R \cup A)$-bridge of U has at most four attachments and every $(R \cup A)$-bridge of U containing an edge of $t_kC_1t_n$ has just two attachments. Now $E((C - t_1) \cup P \cup Q \cup R)$ induces the desired C^*.

So $l \ne 1$. In J_l we find a Tutte path Q by (2.1) from u_l to t_1. Let U be the union of $t_kC_1t_1$, all $(H^* \cup C_1)$-bridges of G^* with attachments contained in $t_kC_1t_1 \cup H^*$ (except those contained in J_i for $l \le i \le k$), and all C-bridges of H^* containing a vertex of $\{u_1, \cdots, u_{l-1}, u_{k+1}, \cdots, u_n\} - C$. Let $A = V((C - t_1) \cap U)$. Note that $U - A$ is a plane graph and $t_kC_1t_1$ is a simple path in its outerwalk. Let $J = t_kB_1v_1B_2\cdots v_{m-1}B_mt_1$ be the plane chain of blocks in $U - A$ along $t_kC_1t_1$. Define H_i and $a(B_i)$ as in (2.2). Clearly $|a(B_i)| \le 3$. Note that if $e \in E(B_i)$ then $|a(B_i)| \le 2$, since as noted above the C-bridge of K containing u_1 has at most three attachments (including t_1). Hence we use (2.2) to find a path R in J from t_k to t_1 through e such that every $(R \cup A)$-bridge of U has at most four attachments and every $(R \cup A)$-bridge of U containing an edge of $t_kC_1t_1$ has just two attachments. Clearly $E((C - t_1) \cup P \cup Q \cup R)$ induces the desired C^*.

∎

We now use (3.1) to prove our main result (1.1).

Proof of (1.1). Let G be a 5-connected graph embedded in the torus, and let e be an edge of G. We first claim that $\rho(G) \ge 2$. Otherwise there is a non-null homotopic simple closed curve in the torus intersecting G only at one vertex, say a. We cut the torus along this curve, and obtain a plane graph G' with a being split to two vertices a' and a''. In G' we use (2.1) to find a Tutte path P from a' to a'' through e. Clearly $E(P)$ induces a Hamilton cycle in G.

Hence we may assume that e is contained in a facial cycle C_1 of G. Then by (3.1) we find a cycle C^* in G through e satisfying the conclusion of (3.1). Clearly C^* is a Hamilton cycle in G. ∎

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