STRUCTURAL DESCRIPTIONS OF LOWER IDEALS OF TREES

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ABSTRACT. A *lower ideal* of trees is a set $\mathcal{I}$ of finite trees such that if $T \in \mathcal{I}$ and T topologically contains S then $S \in \mathcal{I}$. We prove that every lower ideal of trees $\mathcal{I}$ has a structural description, a finite set of rules which describes how to construct an arbitrary element of $\mathcal{I}$.

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1. INTRODUCTION

Trees in this paper are non-null, finite, rooted, directed away from the root, and for technical reasons the vertices are assumed to be integers. More precisely, a *tree* is a triple $T = (V, E, r)$, where V, the set of *vertices*, is a non-empty finite set of integers, E, the set of *edges*, is a subset of $V \times V$, and r, the *root* of T, is a member of V, such that for every $t \in V$ there is a unique directed walk from r to t. (A sequence $t_0, t_1, \ldots, t_n$ is a *directed walk from* t_0 *to* t_n if $(t_{i-1}, t_i) \in E$ for all $i = 1, 2, \ldots, n$.) We write $V(T) = V$, $E(T) = E$, and $\mathrm{root}(T) = r$. The *height* of T is the maximum number of edges in a directed walk in T. For $s, t \in V(T)$, let $s \wedge t$ denote the last vertex of the directed walk from $\mathrm{root}(T)$ to s which belongs to the directed walk from $\mathrm{root}(T)$ to t. We say that T_2 *topologically contains* T_1, or that T_1 is *topologically contained* in T_2 if there exists a 1-1 mapping $f : V(T_1) \to V(T_2)$, called a *tree–embedding*, with the property that $f(s \wedge t) = f(s) \wedge f(t)$ for every two elements $s, t \in V(T_1)$. Let $\mathcal{I}$ be a set of trees such that if $T \in \mathcal{I}$ and T topologically contains S then $S \in \mathcal{I}$. We say that $\mathcal{I}$ is a *lower ideal of trees*, or a *tree ideal* for short. A tree ideal $\mathcal{I}$ is *proper* if some tree does not belong to $\mathcal{I}$. A tree ideal $\mathcal{I}$ is *coherent* if $\mathcal{I} \neq \emptyset$ and for every $T_1, T_2 \in \mathcal{I}$ there exists $T \in \mathcal{I}$ such that T topologically contains both T_1 and T_2. For notational convenience we introduce a formal symbol Γ, called the *null tree*. We define $V(\Gamma) = E(\Gamma) = \emptyset$, and if T is a tree or Γ we say that Γ is topologically contained in T.

If $T_1, T_2, \ldots, T_n$ $(n \geq 0)$ are vertex-disjoint trees or Γ we define a new tree $\mathrm{Tree}(T_1, T_2, \ldots, T_n)$. Its vertices are $V(T_1) \cup V(T_2) \cup \cdots \cup V(T_n) \cup \{t_0\}$, where $t_0 \notin V(T_1) \cup V(T_2) \cup \cdots \cup V(T_n)$ is a new vertex. $\mathrm{Tree}(T_1, T_2, \ldots, T_n)$ has root t_0 and edges $E(T_1) \cup E(T_2) \cup \cdots \cup E(T_n) \cup \{(t_0, \mathrm{root}(T_i)) : 1 \leq i \leq n, T_i \neq \Gamma\}$.

We say that $B = (\mathcal{I}_1, \ldots, \mathcal{I}_n; k; \mathcal{I}_0)$ is a *bit* if $n, k \geq 0$ are integers, $\mathcal{I}_1, \ldots, \mathcal{I}_n$ are coherent tree ideals and $\mathcal{I}_0$ is a tree ideal. The tree ideals $\mathcal{I}_0, \mathcal{I}_1, \ldots, \mathcal{I}_n$ will be called the *lower ideals of* B, and k will be called the *width* of B. Two bits are considered identical if they differ only by a permutation of $\mathcal{I}_1, \ldots, \mathcal{I}_n$. Let $\mathcal{B}$ be a set of bits. We define $I(\mathcal{B})$ to be the intersection of all tree ideals $\mathcal{I}$ that satisfy the following condition:

(*) If $(\mathcal{I}_1, \ldots, \mathcal{I}_n; k; \mathcal{I}_0) \in \mathcal{B}$, $T_i \in \mathcal{I}_i$ for $i = 1, 2, \ldots, n$, $T_{n+i} \in \mathcal{I} \cup \{\Gamma\}$ for $i = 1, 2, \ldots, k$, $j \geq 0$ and $T_{k+n+1}, \ldots, T_{k+n+j} \in \mathcal{I}_0$, then $\mathrm{Tree}(T_1, \ldots, T_{k+n+j})$ belongs to $\mathcal{I}$.

We remark that $I(\mathcal{B})$ is a tree ideal satisfying (*), and that if $\mathcal{B} \neq \emptyset$ then every one-vertex tree belongs to $I(\mathcal{B})$. We offer the following examples. Let $\mathcal{J}$ be the tree ideal consisting of all one-vertex trees. Then $\mathcal{J} = I(\{(; 0; \emptyset)\})$. Further, $I(\{(; 2; \emptyset)\})$ is the tree ideal of all trees with out-degree at most two, and $I(\{(\mathcal{J}; 1; \emptyset)\})$ is the tree ideal of all trees obtained from a directed walk by gluing a directed edge to some of its vertices. As a last example we consider

$I(\{(;2;\emptyset),(;0;\mathcal{J})\})$. Every element of this tree ideal is obtained from an arbitrary tree with out-degree at most two by gluing stars onto a set of vertices of out-degree zero.

Let $B = (\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n; k; \mathcal{I}_0)$, $B' = (\mathcal{I}'_1, \mathcal{I}'_2, \ldots, \mathcal{I}'_{n'}; k'; \mathcal{I}'_0)$ be bits and let $\mathcal{I}$ be a tree ideal. We say that B is *$\mathcal{I}$-dominated* by B' if there exist a set $S \subseteq \{1,2,\ldots,n\}$ and a mapping $f : \{1,2,\ldots,n\} - S \to \{0,1,\ldots,n'\}$ such that

(D1) $|S| \le k' - k$ (and hence $k \le k'$),

(D2) $\mathcal{I}_i \subseteq \mathcal{I}$ for every $i \in S$,

(D3) for $i, j \in \{1,2,\ldots,n\} - S$, if $f(i) = f(j) > 0$ then $i = j$, and

(D4) $\mathcal{I}_0 \subseteq \mathcal{I}'_0$, and $\mathcal{I}_i \subseteq \mathcal{I}'_{f(i)}$ for every $i \in \{1,2,\ldots,n\} - S$.

Let $\mathcal{I}$ be a tree ideal. A *name* of $\mathcal{I}$ is a finite set $\mathcal{B}$ of bits such that

(N1) if $(\mathcal{I}_1, \ldots, \mathcal{I}_n; k; \mathcal{I}_0) \in \mathcal{B}$ then $\mathcal{I}_i \not\subseteq \mathcal{I}_0$ for all $i = 1,2,\ldots,n$ and $\mathcal{I}_i$ is a proper subset of $\mathcal{I}$ for all $i = 0,1,\ldots,n$,

(N2) $I(\mathcal{B}) = \mathcal{I}$,

(N3) no bit of $\mathcal{B}$ is $\mathcal{I}$-dominated by a bit of $\mathcal{B}$ other than itself, and

(N4) if $B \in \mathcal{B}$ has width zero and $I(\{B\}) \subseteq \mathcal{I}'$ for a lower ideal $\mathcal{I}'$ of a bit $B' \in \mathcal{B}$, then $B = B' = (\mathcal{I}'; 0; \emptyset)$.

The following is the main result of this paper.

(1.1) *Every proper lower ideal of trees has a unique name.*

There is a trivial difficulty that the lower ideal of all trees does not have a name. This could be overcome for instance by allowing k to be ∞. However, we chose not to do so. A related result, in terms of finite automata, for lower ideals in the minor containment relation was obtained by Gupta [1].

We now put this result into the context of well-quasi-ordering. A *quasi-ordering* is a reflexive and transitive relation. Let Q be a set and let $\le$ be a quasi-ordering on Q. We say that Q is *quasi–ordered*. We say that Q is *well-quasi-ordered (wqo)* if for every infinite sequence $q_1, q_2, \ldots$ of elements of Q there are indices i, j such that $i < j$ and $q_i \le q_j$. The following is a theorem of Kruskal [3]; for a simple proof see [4].

(1.2) *The set of all trees quasi-ordered by topological containment is well-quasi-ordered.*

A *lower ideal* in Q is a set $I \subseteq Q$ such that if $q \in I$ and $q' \le q$ then $q' \in I$. We say that a sequence $q_1, q_2, \ldots$ of elements of Q is *nondecreasing* if $q_i \le q_j$ for all $i < j$. The following is easy to see.

(1.3) *Let Q be wqo. Then*

(i) every infinite sequence of elements of Q contains an infinite nondecreasing subsequence, and

(ii) there is no infinite strictly decreasing sequence $I_1 \supset I_2 \supset \cdots$ *of lower ideals in* Q.

It should be noted that if $B = (\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n; k; \mathcal{I}_0)$ belongs to the name of a lower ideal of trees, then each $\mathcal{I}_i$ also has a name, expressed in terms of finitely many smaller lower ideals. Thus the name of a lower ideal of trees can be regarded as a finitely branching structured tree. It follows from (1.2), (1.3ii) and König's lemma that this structured tree is finite. This is the *structural description* we were referring to in the title and in the abstract. Given the structural description of two lower ideals of trees $\mathcal{I}_1, \mathcal{I}_2$ one would like to be able to test if $\mathcal{I}_1 \subseteq \mathcal{I}_2$. It follows from (2.3) and (2.6) that this can be done. We remark that conditions (N3) and (N4) are only needed to guarantee uniqueness.

Another remark that needs to be made is that it follows from (1.2) that for every lower ideal of trees $\mathcal{I}$ there exists a finite set of trees Ω such that $T \notin \mathcal{I}$ if and only if T topologically contains some member of Ω. Thus both Ω and the name are "finite descriptions" of $\mathcal{I}$. However, there is a difference in how Ω and the name describe $\mathcal{I}$; namely, Ω describes the structure of non-members of $\mathcal{I}$, whereas the name describes the structure of members of $\mathcal{I}$.

In the rest of this section we state several lemmas. Let Q be quasi-ordered. A lower ideal I in Q is said to be *coherent* if $I \neq \emptyset$ and for every $q, q' \in I$ there exists $q'' \in I$ such that $q \leq q''$ and $q' \leq q''$. Thus $\emptyset$ is a lower ideal, but not a coherent lower ideal. We need the following lemma.

(1.4) *Let* Q *be wqo and let* I *be a lower ideal in* Q. *Then there exists a unique finite set* $\{I_1, I_2, \ldots, I_n\}$ *of coherent lower ideals such that* $I_1 \cup I_2 \cup \cdots \cup I_n = I$ *and* $I_i \not\subseteq I_j$ *for all* $i, j = 1, 2, \ldots, n$ *with* $i \neq j$.

Proof. We first prove that every lower ideal in Q can be represented as a finite union of coherent lower ideals. Suppose to the contrary that there is a lower ideal I in Q which is not expressible as a finite union of coherent lower ideals. By (1.3ii) we may choose I in such a way that every proper lower subideal of I is expressible as a finite union of coherent lower ideals. Since, in particular, I is not coherent, there are $q_1, q_2 \in I$ such that there is no $q \in I$ with $q_1 \leq q$ and $q_2 \leq q$. For $i = 1, 2$ let I_i be the set of all $q \in I$ such that $q_i \not\leq q$. Then $I_1 \cup I_2 = I$, and both I_1 and I_2 are proper lower subideals of I. Since both I_1, I_2 can be expressed as a finite union of coherent lower ideals, so can I, a contradiction.

To prove uniqueness we assume that there are coherent lower ideals I_1, $I_2, \ldots, I_n, I'_1, I'_2, \ldots, I'_{n'}$ such that $I_1 \cup I_2 \cup \ldots \cup I_n = I'_1 \cup I'_2 \cup \ldots \cup I'_{n'} = I$ and $I_i \not\subseteq I_j$ for all $i, j = 1, 2, \ldots, n$ with $i \neq j$ and $I'_i \not\subseteq I'_j$ for all $i, j = 1, 2, \ldots, n'$ with $i \neq j$. Let $i \in \{1, 2, \ldots, n\}$. Since I_i is coherent there exists an integer $\pi(i) \in \{1, 2, \ldots, n'\}$ such that $I_i \subseteq I'_{\pi(i)}$. Similarly there exists an integer

$j \in \{1, 2, \ldots, n\}$ such that $I'_{\pi(i)} \subseteq I_j$, and so $i = j$ and thus $I_i = I'_{\pi(i)}$. We deduce that $n = n'$, and that π is a permutation of $\{1, 2, \ldots, n\}$, as desired. $\square$

If $I_1, I_2, \ldots, I_n$ are as in the above lemma we say that $\{I_1, I_2, \ldots, I_n\}$ is the *coherent lower ideal decomposition* of I.

Let $Q_1, Q_2, \ldots, Q_n$ be quasi–ordered. The Cartesian product $Q_1 \times Q_2 \times \ldots \times Q_n$ is quasi–ordered by the coordinatewise quasi–ordering, that is, $(q_1, q_2, \ldots, q_n) \le (q'_1, q'_2, \ldots, q'_n)$ if $q_i \le q'_i$ in Q_i for every $i = 1, 2, \ldots, n$. The following lemma follows easily from (1.3i).

(1.5) *If $Q_1, Q_2, \ldots, Q_n$ are wqo, then $Q_1 \times Q_2 \times \ldots \times Q_n$ is wqo.*

Let Q be quasi-ordered. We denote by $Q^{<\omega}$ the set of all finite sequences of elements of Q quasi-ordered by the rule that $(q_1, q_2, \ldots, q_n) \le (q'_1, q'_2, \ldots, q'_{n'})$ if there exists a strictly increasing function $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n'\}$ such that $q_i \le q'_{f(i)}$. The following lemma is due to Higman [2].

(1.6) *If Q is wqo then $Q^{<\omega}$ is wqo.*

We say that a set $I \subseteq Q$ is *generated* by a sequence $q_1, q_2, \ldots$, or that the sequence $q_1, q_2, \ldots$ *generates* I if I is the set of all $q \in Q$ for which there exists an integer $i \ge 1$ with $q \le q_i$. The following is easy to see.

(1.7) *Let Q be a countable quasi-ordered set and let $I \subseteq Q$. Then I is a coherent lower ideal if and only if I is generated by a nondecreasing sequence.*

2. UNIQUENESS

In this section we prove that every proper lower ideal of trees has at most one name. Let T be a tree, and let $\mathcal{B}$ be a set of bits. We say that T *conforms to* $\mathcal{B}$ if there exists a bit $B = (\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n; k; \mathcal{I}_0) \in \mathcal{B}$ such that either $T \in \mathcal{I}_0 \cup \mathcal{I}_1 \cup \ldots \cup \mathcal{I}_n$, or there exist an integer $m \ge 0$ and trees $T_1, \ldots, T_{n+k+m}$ such that $T_i \in \mathcal{I}_i \cup \{\Gamma\}$ for $i = 1, 2, \ldots, n$, $T_{n+i} \in I(\mathcal{B}) \cup \{\Gamma\}$ for $i = 1, 2, \ldots, k$, $T_{n+k+i} \in \mathcal{I}_0$ for $i = 1, 2, \ldots, m$ and such that T is isomorphic to $\mathrm{Tree}(T_1, T_2, \ldots, T_{k+n+m})$. We also say that T *conforms to B in $\mathcal{B}$*. We omit the (easy) proof of the following lemma.

(2.1) *Let $\mathcal{B}$ be a set of bits, and let T be a tree. Then T belongs to $I(\mathcal{B})$ if and only if it conforms to some B in $\mathcal{B}$.*

We say that a set $\mathcal{B}$ of bits is *coherent* if $\mathcal{B}$ is nonempty, finite, and either $\mathcal{B}$ contains at most one bit of width zero or $\mathcal{B}$ contains a bit of width at least two. We need the following lemma.

(2.2) *Let $\mathcal{B}$ be a set of bits. If $\mathcal{B}$ is coherent, then $I(\mathcal{B})$ is a coherent lower ideal of trees.*

Proof. Let $\mathcal{B}$ be coherent. Since $\mathcal{B}$ is non-empty we see that $I(\mathcal{B})$ is non-empty. Suppose for a contradiction that $T, T' \in I(\mathcal{B})$ are such that there is no tree in $I(\mathcal{B})$ that topologically contains both T, T', and, subject to that, the sum of the heights of T and T' is minimum. Since Tree(T, T') topologically contains both T and T', we deduce that

(1) *Tree$(T, T') \notin I(\mathcal{B})$.*

From (1) we deduce that $\mathcal{B}$ contains no bit of width two or more, and therefore contains at most one bit of width zero.

(2) *If $B = (\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_k; k; \mathcal{I}_0) \in \mathcal{B}$ and $T \in \mathcal{I}_0 \cup \mathcal{I}_1 \cup \cdots \cup \mathcal{I}_n$, then $T' \notin \mathcal{I}_0 \cup \mathcal{I}_1 \cup \cdots \cup \mathcal{I}_n$ and $k = 0$.*

This follows from (1) and the fact that $\mathcal{I}_i$ is a coherent tree ideal for all $i = 1, 2, \ldots, n$.

Let $B = (\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n; k; \mathcal{I}_0), B' = (\mathcal{I}'_1, \mathcal{I}'_2, \ldots, \mathcal{I}'_{n'}; k'; \mathcal{I}'_0) \in \mathcal{B}$ be such that T conforms to B in $\mathcal{B}$ and T' conforms to B' in $\mathcal{B}$. Then $k, k' \leq 1$. Let $T =$Tree$(T_1, T_2, \ldots, T_{n+k+m})$, where $m \geq 0$, and for $i = 1, 2, \ldots, n+k+m$, T_i is a tree or Γ in such a way that either $T \in \mathcal{I}_0 \cup \mathcal{I}_1 \cup \cdots \cup \mathcal{I}_n$, or $T_i \in \mathcal{I}_i \cup \{\Gamma\}$ for $i = 1, 2, \ldots, n$, $T_{n+i} \in I(\mathcal{B}) \cup \{\Gamma\}$ for $i = 1, 2, \ldots, k$, and $T_{n+k+i} \in \mathcal{I}_0$ for $i = 1, 2, \ldots, m$. Let $T' =$Tree$(T'_1, T'_2, \ldots, T'_{n'+k'+m'})$ similarly.

If $k = 1$, then $T \notin \mathcal{I}_0 \cup \mathcal{I}_1 \cup \cdots \cup \mathcal{I}_n$ by (2). By the choice of T and T' there exists a tree $T'' \in I(\mathcal{B})$ which topologically contains both T_{n+1} and T'. Then

$$\text{Tree}(T_1, T_2, \ldots, T_n, T'', T_{n+2}, T_{n+3}, \ldots, T_{n+k+m})$$

belongs to $I(\mathcal{B})$ and topologically contains both T and T', a contradiction. Because of the symmetry between k and k' we may therefore assume that $k = k' = 0$. Then $B = B'$, and we deduce from (2) that $\{T, T'\} \not\subseteq \mathcal{I}_0 \cup \mathcal{I}_1 \cup \cdots \cup \mathcal{I}_n$. From the symmetry we may assume that $T \notin \mathcal{I}_0 \cup \mathcal{I}_1 \cup \cdots \cup \mathcal{I}_n$. We wish to define T''_i for every $i = 1, 2, \ldots, n$. We choose $T''_i \in \mathcal{I}_i$ in such a way that if $R \in \{T_i, T'_i, T'\} \cap \mathcal{I}_i$, then T''_i topologically contains R. Such a choice is clearly possible, because $\mathcal{I}_i$ is a coherent tree ideal for every $i = 1, 2, \ldots, n$. Let $T_0 = T'$ if $T' \in \mathcal{I}_0$, and let $T_0 = \Gamma$ otherwise. If $T' \in \mathcal{I}_0 \cup \mathcal{I}_1 \cup \cdots \cup \mathcal{I}_n$ then

$$\text{Tree}(T''_1, T''_2, \ldots, T''_n, T_{n+1}, T_{n+2}, \ldots, T_{n+k+m}, T_0)$$

belongs to $I(\mathcal{B})$ and topologically contains both T and T', and if $T' \notin \mathcal{I}_0 \cup \mathcal{I}_1 \cup \cdots \cup \mathcal{I}_n$ then

$$\text{Tree}(T''_1, T''_2, \ldots, T''_n, T_{n+1}, T_{n+2}, \ldots, T_{n+k+m}, T'_{n'+1}, T'_{n'+2}, \ldots, T'_{n'+k'+m'})$$

belongs to $I(\mathcal{B})$ and topologically contains both T and T', a contradiction in both cases. □

(2.3) *Let $\mathcal{B}, \mathcal{B}'$ be finite sets of bits such that $\mathcal{B}$ is coherent. Then $I(\mathcal{B}) \subseteq I(\mathcal{B}')$ if and only if for every bit $B \in \mathcal{B}$ there exists a bit $B' \in \mathcal{B}'$ such that either*

(i) B is $I(\mathcal{B}')$-dominated by B', or

(ii) $I(\{B\}) \subseteq \mathcal{I}'$ for some lower ideal $\mathcal{I}'$ of B', and if B has positive width then $I(\mathcal{B}) \subseteq \mathcal{I}'$.

Proof. We first prove the "if" part. Let $T \in I(\mathcal{B})$, and assume that no tree of smaller height belongs to $I(\mathcal{B}) - I(\mathcal{B}')$. By (2.1) T conforms to some B in $\mathcal{B}$; let $B = (\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n; k; \mathcal{I}_0)$. Then either $T \in \mathcal{I}_0 \cup \mathcal{I}_1 \cup \ldots \cup \mathcal{I}_n$, or T has the form $\mathrm{Tree}(T_1, T_2, \ldots, T_{n+k+m})$, where $T_i \in \mathcal{I}_i \cup \{\Gamma\}$ for $i = 1, 2, \ldots, n$, $T_{n+i} \in I(\mathcal{B}) \cup \{\Gamma\}$ for $i = 1, 2, \ldots, k$, and $T_{n+k+i} \in \mathcal{I}_0$ for $i = 1, 2, \ldots, m$. By the minimality of the height of T it follows that $T_{n+i} \in I(\mathcal{B}') \cup \{\Gamma\}$ for $i = 1, 2, \ldots, k$. By the hypothesis there exists a bit $B' \in \mathcal{B}'$ such that (i) or (ii) holds.

We assume first that (i) holds. Then there exist a set S and mapping f satisfying (D1)-(D4). It is now routine to verify that $T \in I(\mathcal{B}')$. So we may assume that (ii) holds. Let $\mathcal{I}'$ be as in (ii). If B has positive width, then $T \in I(\mathcal{B}) \subseteq \mathcal{I}' \subseteq I(\mathcal{B}')$, and if B has width zero then $T \in I(\{B\}) \subseteq \mathcal{I}' \subseteq I(\mathcal{B}')$. This completes the proof of the "if" part.

To prove "only if" let $B = (\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n; k; \mathcal{I}_0) \in \mathcal{B}$. For $j = 1, 2, \ldots, n$ let, by (1.7), $\{T_j^i\}_{i \geq 1}$ be a nondecreasing sequence that generates $\mathcal{I}_j$, for $j = n+1, n+2, \ldots, n+k$ let, by (1.7) and (2.2), $\{T_j^i\}_{i \geq 1}$ be a nondecreasing sequence that generates $I(\mathcal{B})$, let $T_{n+k+1}^1, T_{n+k+2}^1, \ldots$ be a sequence that contains infinitely many isomorphic copies of every element of $\mathcal{I}_0$, for $i \geq 2$ and $j \geq n+k+1$ let $T_j^i = T_j^1$, and for $i \geq 1$ let $T^i = \mathrm{Tree}(T_1^i, T_2^i, \ldots, T_{n+k+i}^i)$. Then $T^i \in I(\mathcal{B}) \subseteq I(\mathcal{B}')$, and hence T^i conforms to $\mathcal{B}'$. Since $\mathcal{B}'$ is finite, by (2.1) we may assume (by taking a subsequence) that there exists a bit $B' \in \mathcal{B}'$ such that T^i conforms to B' in $\mathcal{B}'$ for all $i = 1, 2, \ldots$. Let $B' = (\mathcal{I}_1', \mathcal{I}_2', \ldots \mathcal{I}_{n'}'; k'; \mathcal{I}_0')$. There are two cases.

We assume first that $T^i \in \mathcal{I}_0' \cup \mathcal{I}_1' \cup \cdots \cup \mathcal{I}_{n'}'$ for all $i \geq 1$. Since the sequence $T^1, T^2, \ldots$ is nondecreasing, we deduce that there exists $j \in \{0, 1, \ldots, n'\}$ such that $T^i \in \mathcal{I}_j'$ for all $i \geq 1$. Since the sequence $\{T^i\}_{i \geq 1}$ generates an ideal that contains $I(\{B\})$ it follows that $I(\{B\}) \subseteq \mathcal{I}_j'$. Moreover, if $k > 0$ then $\{T^i\}_{i \geq 1}$ generates $I(\mathcal{B})$, and hence $I(\mathcal{B}) \subseteq \mathcal{I}_j'$. Thus (ii) holds.

We now assume that (ii) does not hold. In particular, $T^i \notin \mathcal{I}_0' \cup \mathcal{I}_1' \cup \cdots \cup \mathcal{I}_{n'}'$ for some $i \geq 1$. By taking a subsequence we may assume that $T^i \notin \mathcal{I}_0' \cup \mathcal{I}_1' \cup \cdots \cup \mathcal{I}_{n'}'$ for all $i \geq 1$. We claim that

(1) $\quad \mathcal{I}_0 \subseteq \mathcal{I}_0'$.

Indeed, let $T \in \mathcal{I}_0$ and let i be so big that at least $n+k+1$ of the trees $T^i_{n+k+1}, T^i_{n+k+2}, \ldots, T^i_{n+k+i}$ are isomorphic to T. Since T^i conforms to B' in $\mathcal{B}'$ and $T^i \notin \mathcal{I}'_0 \cup \mathcal{I}'_1 \cup \ldots \cup \mathcal{I}'_{n'}$ we deduce that $T \in \mathcal{I}'_0$. This proves (1).

Since T^i conforms to B' in $\mathcal{B}'$ and $T^i \notin \mathcal{I}'_0 \cup \mathcal{I}'_1 \cup \cdots \cup \mathcal{I}'_{n'}$ there exists a function $f_i : \{1, 2, \ldots, n+k\} \to \{-1, 0, 1, \ldots, n'\}$ such that

(a) $|f_i^{-1}(-1)| \le k'$,

(b) $T^i_j \in I(\mathcal{B}')$ for all $j = 1, 2, \ldots, k+n$ with $f_i(j) = -1$,

(c) for all $j, j' = 1, 2, \ldots, k+n$, if $f_i(j) = f_i(j') > 0$ then $j = j'$, and

(d) $T^i_j \in \mathcal{I}'_{f_i(j)}$ for all $j = 1, 2, \ldots, k+n$ with $f_i(j) \ge 0$.

By taking a subsequence we may assume that $f_i(j)$ is constant for every fixed $j = 1, 2, \ldots, n+k$. Let $f(j)$ denote this common value.

(2) *$f(j) = -1$ for $j = n+1, n+2, \ldots, n+k$.*

Indeed, let $f(j) \ge 0$ for some $j \in \{n+1, n+2, \ldots, n+k\}$. Since $\{T^i_j\}_{i \ge 1}$ generates $I(\mathcal{B})$, it follows from (d) that $I(\mathcal{B}) \subseteq \mathcal{I}'_{f(j)}$, contrary to our assumption that (ii) does not hold. This proves (2).

Let S be the set of all integers j such that $1 \le j \le n$ and $f(j) = -1$. From (1), (2) and (a)-(d) it follows that S and the restriction of f to $\{1, 2, \ldots, n\} - S$ satisfy (D1)-(D4), because $\{T^i_j\}_{i \ge 1}$ generates $\mathcal{I}_j$ for $j = 1, 2, \ldots, n$. Thus (i) holds. □

(2.4) *Let $\mathcal{B}, \mathcal{B}'$ be names of some tree ideals, let $B \in \mathcal{B}$, $B' \in \mathcal{B}'$ and let $\mathcal{I}$ be a tree ideal. If B is $\mathcal{I}$–dominated by B' and B' is $\mathcal{I}$–dominated by B then $B = B'$.*

Proof. Let $B = (\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n; k; \mathcal{I}_0)$ and $B = (\mathcal{I}'_1, \mathcal{I}'_2, \ldots, \mathcal{I}'_{n'}; k; \mathcal{I}'_0)$. From the fact that B is $\mathcal{I}$-dominated by B' there exist a set $S \subseteq \{1, 2, \ldots, n\}$ and a mapping $f : \{1, 2, \ldots, n\} \to \{0, 1, \ldots, n'\}$ such that (D1)–(D4) hold. Similarly, from the fact that B' is $\mathcal{I}$–dominated by B there exist a set $S' \subseteq \{1, 2, \ldots, n'\}$ and a mapping $f' : \{1, 2, \ldots, n'\} \to \{0, 1, \ldots, n\}$ such that (D1)–(D4) hold with B, S, f replaced by B', S', f'. Thus $k = k'$, $\mathcal{I}_0 = \mathcal{I}'_0$, $S = S' = \emptyset$, $f(i) > 0$ for every $i = 1, 2, \ldots, n$ (because otherwise $\mathcal{I}_i \subseteq \mathcal{I}'_0 \subseteq \mathcal{I}_0$, contrary to (N1)), $f'(i) > 0$ for every $i = 1, 2, \ldots, n'$ and $n = n'$. Since bits that differ by a permutation of $\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n$ are considered equal, it follows that $B = B'$, as desired. □

(2.5) *Let $\mathcal{B}, \mathcal{B}'$ be names of a tree ideal $\mathcal{I}$, and assume that both $\mathcal{B}, \mathcal{B}'$ are coherent. Then $\mathcal{B} = \mathcal{B}'$.*

Proof. Let $\mathcal{B}_1$ denote the subset of $\mathcal{B}$ consisting of all bits of $\mathcal{B}$ of positive width, and let $\mathcal{B}'_1$ be defined similarly. We first prove the following.

(1) $\mathcal{B}_1 = \mathcal{B}'_1$.

Indeed, let $B \in \mathcal{B}$ have positive width. By (2.3) there exists $B' \in \mathcal{B}'$ such that either B is $\mathcal{I}$–dominated by B', or $I(\mathcal{B})$ is a subset of one of the lower ideals of B', say $\mathcal{I}'$. The latter case is impossible, because then $\mathcal{I} = I(\mathcal{B}) \subseteq \mathcal{I}' \subseteq I(\mathcal{B}') = \mathcal{I}$, and hence $\mathcal{I}' = \mathcal{I}$, contrary to (N1). Thus B is $\mathcal{I}$–dominated by B', and, in particular, B' has positive width. Similarly there exists $B'' \in \mathcal{B}$ such that B' is $\mathcal{I}$–dominated by B''. Since $\mathcal{I}$–domination is transitive we see that B is $\mathcal{I}$–dominated by B'', and hence $B = B''$ by (N3). Thus B is $\mathcal{I}$–dominated by B' and B' is $\mathcal{I}$–dominated by B, and so $B = B'$ by (2.4). This proves (1).

(2) *For every bit $B \in \mathcal{B}$ of width zero there exists a bit $B' \in \mathcal{B}'$ as in (2.3) of width zero.*

Indeed, let $B \in \mathcal{B}$ have width zero, let $B' \in \mathcal{B}'$ and suppose that B' has positive width. Then $B' \in \mathcal{B}$ by (1). From (N3) we deduce that B is not $\mathcal{I}$–dominated by B', and from (N4) it follows that $I(\{B\})$ is a subset of no lower ideal of B'. Hence B' satisfies neither (i) nor (ii) of (2.3). Thus if $B' \in \mathcal{B}'$ is as in (2.3) we see that B' has width zero. By (2.3) at least one such bit exists, and (2) follows.

From the symmetry we deduce that

(3) *For every bit $B' \in \mathcal{B}'$ of width zero there exists a bit $B'' \in \mathcal{B}$ as in (2.3) of width zero.*

(4) *For every bit $B \in \mathcal{B}$ there exists a bit $B' \in \mathcal{B}'$ such that B is $\mathcal{I}$–dominated by B'.*

Indeed, let $B \in \mathcal{B}$. We may assume that B has width zero, because otherwise the result follows from (1). Let $B' \in \mathcal{B}'$ be as in (2); we may assume that $I(\{B\})$ is a subset of one of the lower ideals of B', because otherwise we are done. Let B'' be as in (3). Since B'' has width zero, it follows that $I(\{B\})$ is a subset of one of the lower ideals of B'', say $\mathcal{I}_1$, and so $I(\{B\}) = \mathcal{I}_1$, and $B'' = B = (\mathcal{I}_1; 0; \emptyset)$ by (N4). Similarly $B' = (\mathcal{I}_1; 0; \emptyset)$, and so B is $\mathcal{I}$–dominated by B', as desired. This proves (4).

It follows similarly that

(5) *for every bit $B' \in \mathcal{B}'$ there exists a bit $B'' \in \mathcal{B}$ such that B' is $\mathcal{I}$–dominated by B''.*

Now we are finally ready to finish the proof of (2.5). Let $B \in \mathcal{B}$, let $B' \in \mathcal{B}'$ be as in (4), and let $B'' \in \mathcal{B}$ be as in (5). Then B is $\mathcal{I}$–dominated by B'', and so $B = B''$ by (N3). Consequently, B is $\mathcal{I}$–dominated by B' and B' is $\mathcal{I}$–dominated by B, and hence $B = B'$ by (2.4). Thus $\mathcal{B} = \mathcal{B}'$, as desired. □

Let $\mathcal{B}$ be a finite set of bits. We define a set $\{\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_m\}$ of finite sets of bits, called the *coherent decomposition* of $\mathcal{B}$, as follows. If $\mathcal{B}$ is coherent then we define the coherent decomposition to be $\{\mathcal{B}\}$. Otherwise we let $B_1, B_2, \ldots, B_m$

be all the elements of $\mathcal{B}$ of width zero, and for $i = 1, 2, \ldots, m$ we put $\mathcal{B}_i = (\mathcal{B} - \{B_1, B_2, \ldots, B_m\}) \cup \{B_i\}$ and define the coherent decomposition of $\mathcal{B}$ to be $\{\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_m\}$.

(2.6) *Let $\mathcal{B}$ be a name of a tree ideal $\mathcal{I}$, and let $\{\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_m\}$ be the coherent decomposition of $\mathcal{B}$. Then $I(\mathcal{B}_1) \cup I(\mathcal{B}_2) \cup \cdots \cup I(\mathcal{B}_m)$ is a coherent lower ideal decomposition of $\mathcal{I}$.*

Proof. It follows from the definition that each $\mathcal{B}_i$ is coherent, and hence each $I(\mathcal{B}_i)$ is a coherent lower ideal by (2.2). To prove that $I(\mathcal{B}_i) \not\subseteq I(\mathcal{B}_j)$ for i, j with $1 \leq i, j \leq m$ and $i \neq j$ we suppose the contrary and note that (since then $m > 1$) $\mathcal{B}_i$ contains exactly one bit of width zero, say B and that $B \notin \mathcal{B}_j$. Then $I(\{B\}) \subseteq I(\mathcal{B}_i) \subseteq I(\mathcal{B}_j) \subseteq \mathcal{I}$. By (2.3) there exists a bit $B' \in \mathcal{B}_j \subseteq \mathcal{B} - \{B\}$ such that either B is $I(\mathcal{B}_j)$-dominated by B', or $I(\{B\})$ is contained in some lower ideal of B'. The latter is impossible by (N4), and so B is $\mathcal{I}$-dominated by B', contrary to (N3). This proves that $I(\mathcal{B}_i) \not\subseteq I(\mathcal{B}_j)$ for $i \neq j$.

Finally, we must show that $\mathcal{I} = \mathcal{I}'$, where $\mathcal{I}' = I(\mathcal{B}_1) \cup I(\mathcal{B}_2) \cup \cdots \cup I(\mathcal{B}_m)$. Since obviously $\mathcal{I}' \subseteq \mathcal{I}$, it remains to prove that $\mathcal{I} \subseteq \mathcal{I}'$. We may assume that $m > 1$, for otherwise the equality $\mathcal{I} = \mathcal{I}'$ is trivial. Then $\mathcal{B}$ contains only bits of width zero and one. Let $T \in \mathcal{I}$ and assume that no tree of smaller height belongs to $\mathcal{I} - \mathcal{I}'$. Since $T \in \mathcal{I} = I(\mathcal{B})$ it conforms to some B in $\mathcal{B}$; let $B = (\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n; k; \mathcal{I}_0)$, where $k = 0$ or 1. Then either $T \in \mathcal{I}_0 \cup \mathcal{I}_1 \cup \cdots \cup \mathcal{I}_n$, or T has the form $\mathrm{Tree}(T_1, T_2, \ldots, T_{n+k+p})$, where $p \geq 0$, $T_i \in \mathcal{I}_i \cup \{\Gamma\}$ for $i = 1, 2, \ldots, n$, $T_{n+k} \in \mathcal{I} \cup \{\Gamma\}$ and $T_{n+k+j} \in \mathcal{I}_0$ for $j = 1, 2, \ldots, p$. We define $l \in \{1, 2, \ldots, m\}$ as follows. If either $k = 0$ or $T \in \mathcal{I}_0 \cup \mathcal{I}_1 \cup \cdots \cup \mathcal{I}_n$ we let l be such that $B \in \mathcal{B}_l$. Otherwise $T_{n+1} \in \mathcal{I}' \cup \{\Gamma\}$ by the induction hypothesis, and we let l be such that $T_{n+1} \in I(\mathcal{B}_l) \cup \{\Gamma\}$. We remark that $B \in \mathcal{B}_l$, because if $k = 1$ then B belongs to every $\mathcal{B}_i$ for $i = 1, 2, \ldots, m$. In either case we see that T conforms to B in $\mathcal{B}_l$, and hence $T \in \mathcal{I}'$, as desired. □

(2.7) *Every lower ideal of trees $\mathcal{I}$ has at most one name.*

Proof. Let $\mathcal{B}, \mathcal{B}'$ be two names of a lower ideal of trees $\mathcal{I}$, and let $\{\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_k\}$, $\{\mathcal{B}'_1, \mathcal{B}'_2, \ldots, \mathcal{B}'_{k'}\}$ be their respective coherent decompositions. From (1.4) and (2.6) we deduce that $k = k'$ and that we may choose our notation so that $I(\mathcal{B}_i) = I(\mathcal{B}'_i)$ for all $i = 1, 2, \ldots, k$. By (2.5) $\mathcal{B}_i = \mathcal{B}'_i$ for all $i = 1, 2, \ldots, k$, and hence $\mathcal{B} = \mathcal{B}'$, as desired. □

3. EXISTENCE

In this section we complete the proof of (1.1) by showing that every proper lower ideal of trees has at least one name. We need the following lemma.

(3.1) *Let $\mathcal{B}$ be a set of bits, let $B \in \mathcal{B}$ have width zero and assume that $I(\{B\}) \subseteq \mathcal{I}'$ for a lower ideal $\mathcal{I}'$ of a bit $B' \in \mathcal{B}$. If $B = B'$ let $\mathcal{B}' = (\mathcal{B} - \{B\}) \cup \{(\mathcal{I}'; 0; \emptyset)\}$, and otherwise let $\mathcal{B}' = \mathcal{B} - \{B\}$. Then $I(\mathcal{B}) = I(\mathcal{B}')$.*

Proof. Assume first that $B \neq B'$. Then clearly $I(\mathcal{B}') \subseteq I(\mathcal{B})$. To prove the converse let $T \in I(\mathcal{B})$ and assume that every tree in $I(\mathcal{B})$ of smaller height belongs to $I(\mathcal{B}')$. By (2.1) T conforms to some B_1 in $\mathcal{B}$. If $B_1 \neq B$ it follows that $T \in I(\mathcal{B}')$; if $B_1 = B$ then (since B has width zero) $T \in I(\{B\}) \subseteq \mathcal{I}' \subseteq I(\{B'\}) \subseteq I(\mathcal{B}')$, because $B \neq B'$. This completes the proof in the case when $B \neq B'$.

We may therefore assume that $B = B'$. We first show that $I(\mathcal{B}') \subseteq I(\mathcal{B})$. Let $T \in I(\mathcal{B}')$ and assume that every tree in $I(\mathcal{B}')$ of smaller height belongs to $I(\mathcal{B})$. By (2.1) T conforms to some B_2 in $\mathcal{B}'$. If $B_2 \neq (\mathcal{I}'; 0; \emptyset)$ then it follows that $T \in I(\mathcal{B})$; otherwise $T \in I(\{(\mathcal{I}'; 0; \emptyset)\}) \subseteq I(\{B'\}) = I(\{B\}) \subseteq I(\mathcal{B})$. This proves that $I(\mathcal{B}') \subseteq I(\mathcal{B})$. To prove the converse inequality let $T \in I(\mathcal{B})$, and assume that every tree in $I(\mathcal{B})$ of smaller height belongs to $I(\mathcal{B}')$. By (2.1) T conforms to some B_3 in $\mathcal{B}'$. If $B_3 \neq B$ then it follows that $T \in I(\mathcal{B}')$; otherwise (since B has width zero) $T \in I(\{B\}) = I(\{B'\}) \subseteq I(\{(\mathcal{I}'; 0; \emptyset)\}) \subseteq I(\mathcal{B}')$, as desired. □

We say that $(T_1, T_2, \ldots, T_n; k; M)$ is a *germ* if $n, k \geq 0$ are integers, T_1, $T_2, \ldots, T_n$ are trees, and M is a finite set of trees. If $g = (T_1, T_2, \ldots, T_n; k; M)$ is a germ and $\mathcal{I}$ is a lower ideal of trees, we denote by $H(g, \mathcal{I})$ the set of all trees isomorphic to $\text{Tree}(T'_1, T'_2, \ldots, T'_{n+k+m})$, where $m \geq 0$, T'_i is Γ or a tree topologically contained in T_i for $i = 1, 2, \ldots, n$, $T'_{n+i} \in \mathcal{I} \cup \{\Gamma\}$ for $i = 1, 2, \ldots, k$, and T'_{n+k+i} is a tree topologically contained in some member of M for $i = 1, 2, \ldots, m$. A germ g is a *germ of a lower ideal* of trees $\mathcal{I}$ if $H(g, \mathcal{I}) \subseteq \mathcal{I}$.

We order germs as follows: $(T_1, T_2, \ldots, T_n; k; M) \leq (T'_1, T'_2, \ldots, T'_{n'}; k'; M')$ if there exists a set $S \subseteq \{1, 2, \ldots, n\}$ and a mapping $f : \{1, 2, \ldots, n\} - S \to \{0, 1, \ldots, n'\}$ such that

(i) $|S| \leq k' - k$,

(ii) every element of M is topologically contained in some element of M',

(iii) for $i, j \in \{1, 2, \ldots, n\} - S$, if $f(i) = f(j) > 0$ then $i = j$,

(iv) for every $i \in \{1, 2, \ldots, n\} - S$, if $f(i) > 0$ then T_i is topologically contained in $T'_{f(i)}$, and if $f(i) = 0$ then T_i is topologically contained in some member of M'.

It follows from (1.5) and (1.6) that germs are well–quasi–ordered. We say that a germ $g = (T_1, T_2, \ldots, T_n; k; M)$ of a lower ideal of trees $\mathcal{I}$ is *reduced* if there is no germ $g' = (T'_1, T'_2, \ldots, T'_{n'}; k'; M')$ of $\mathcal{I}$ with $g \leq g'$ and $n > n'$. Clearly for every germ g of $\mathcal{I}$ there exists a reduced germ g' of $\mathcal{I}$ with $g \leq g'$.

(3.2) *Every proper lower ideal of trees has at least one name.*

Proof. Let $\mathcal{I}$ be a proper lower ideal of trees and let $\mathcal{G}$ be the set of all germs of $\mathcal{I}$. Let $\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2 \cup \cdots \cup \mathcal{G}_p$ be the coherent lower ideal decomposition of $\mathcal{G}$. Let us fix $l \in \{1, 2, \ldots, p\}$. By (1.7) $\mathcal{G}_l$ is generated by a nondecreasing sequence, say $g_1, g_2, \ldots$, where $g_i = (T_1^i, T_2^i, \ldots, T_{n_i}^i; k_i; M_i)$. We may assume without loss of generality that each g_i is reduced. (To see this, we may assume by taking a subsequence that $g_1 \notin \bigcup_{i \neq l} \mathcal{G}_i$ and then consider $\tilde{g}_i = (\tilde{T}_1^i, \tilde{T}_2^i, \ldots, \tilde{T}_{\tilde{n}_i}^i; \tilde{k}_i; \tilde{M}_i) \in \mathcal{G}$ with $g_i \leq \tilde{g}_i$ and subject to that with $\tilde{n}_i$ minimum.) Let S_i, f_i be a set and a mapping witnessing that $g_i \leq g_{i+1}$. The sequence $\{k_i\}$ is bounded (because $\mathcal{I}$ is proper), and so we may assume (by taking a subsequence) that $\{k_i\}$ is constant; let k denote the common value. It follows that $S_i = \emptyset$ for every $i \geq 1$, and since each g_i is reduced we deduce that $f_i(n) > 0$ for all $i = 1, 2, \ldots$ and all $n = 1, 2, \ldots, n_i$. We may therefore assume that each f_i is the identity. We now claim that

(1) *the sequence $\{n_i\}_{i \geq 1}$ is bounded.*

To prove (1) suppose that the sequence $\{n_i\}_{i \geq 1}$ is unbounded. By taking a subsequence we may assume that $n_1 \geq 1$, that $\{n_i\}_{i \geq 1}$ is strictly increasing and that, by (1.2) and (1.3i), $\{T_{n_i}^i\}_{i \geq 1}$ is a nondecreasing sequence of trees. We claim that $g = (T_1^1, T_2^1, \ldots, T_{n_1-1}^1; k; M_1 \cup \{T_{n_1}^1\})$ is a germ of $\mathcal{I}$. Indeed, let $T \in H(g, \mathcal{I})$. Then $T = \text{Tree}(T_1, T_2, \ldots, T_{n_1+k+m+t})$, where $m, t \geq 0$, for $j = 1, 2, \ldots, n_1 - 1$, T_j is Γ or a tree topologically contained in T_j^1, $T_{n_1} = \Gamma$, $T_{n_1+j} \in \mathcal{I} \cup \{\Gamma\}$ for $j = 1, 2, \ldots, k$, T_{n_1+k+j} is topologically contained in $T_{n_1}^1$ for $j = 1, 2, \ldots, m$ and $T_{n_1+k+m+j}$ is topologically contained in some member of M_1 for $j = 1, 2, \ldots, t$. Let $j \in \{1, 2, \ldots, m\}$ and put $n(j) = n_1 + k + j$. In the sequence $T_{n_1+k+j}, T_{n_1}^1, T_{n_{(j)}}^{n(j)}, T_{n_{(j)}}^{n(m)}$ each term is topologically contained in the next (the last containment holds because $g_{n(j)} \leq g_{n(m)}$ and all the f_i's are assumed to be the identity). It follows that $T \in H(g_{n(m)}, \mathcal{I})$, and hence $T \in \mathcal{I}$. Thus g is a germ of $\mathcal{I}$, contrary to the fact that g_1 is reduced, since obviously $g_1 \leq g$. This proves (1).

By (1) we may assume by choosing a subsequence that $\{n_i\}_{i \geq 1}$ is constant; let n denote the common value. We have thus arrived at a sequence $g_1, g_2 \ldots$ of elements of $\mathcal{G}_l$. This sequence will be later referred to as a *fundamental sequence* of $\mathcal{G}_l$, and the notation $g_i = (T_1^i, T_2^i, \ldots, T_n^i; k; M_i)$ will be assumed. Let $\mathcal{I}_0$ be the lower ideal generated by $\bigcup_{i \geq 1} M_i$, and for $j = 1, 2, \ldots, n$ let $\mathcal{I}_j$ be the coherent lower ideal generated by $T_j^1, T_j^2, \ldots$.

(2) *$\mathcal{I}_0, \mathcal{I}_1, \ldots, \mathcal{I}_n$ are proper lower subideals of $\mathcal{I}$, and $\mathcal{I}_i \not\subseteq \mathcal{I}_0$ for all $i = 1, 2, \ldots, n$.*

The proof of (2) is very similar to the proof of (1), and so we just sketch it. If $\mathcal{I}_i = \mathcal{I}$ for some $i = 1, 2, \ldots, n$, say for $i = 1$, then it can be shown that $(T^1_2, T^1_3, \ldots, T^1_{n_1}; k+1; M_1)$ is a germ of $\mathcal{I}$, contrary to the fact that g_1 is reduced. Similarly, if $\mathcal{I}_i \subseteq \mathcal{I}_0$ for some $i = 1, 2, \ldots, n$, say for $i = 1$, then it can be shown that $(T^1_2, T^1_3, \ldots, T^1_{n_1}; k; M_1 \cup \{T^1_1\})$ is a germ of $\mathcal{I}$. Finally, if $\mathcal{I}_0 = \mathcal{I}$, then it can be shown that every tree belongs to $\mathcal{I}$, contrary to our assumption that $\mathcal{I}$ is a proper lower ideal of trees. This completes the sketch of the proof of (2).

We now define $B_l = (\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n; k; \mathcal{I}_0)$ and put $\mathcal{B}_0 = \{B_1, B_2, \ldots, B_p\}$. We will modify $\mathcal{B}_0$ to obtain a name of $\mathcal{I}$. We need the following three claims.

(3) *For $l, l' \in \{1, 2, \ldots, p\}$, if B_l is $\mathcal{I}$-dominated by $B_{l'}$ then $\mathcal{G}_l \subseteq \mathcal{G}_{l'}$.*

To prove (3) let $B_l = (\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n; k; \mathcal{I}_0)$ and let $B_{l'} = (\mathcal{I}'_1, \mathcal{I}'_2, \ldots, \mathcal{I}'_{n'}; k'; \mathcal{I}'_0)$. By the assumption there exist a set $S \subseteq \{1, 2, \ldots n\}$ and a mapping $f : \{1, 2, \ldots, n\} \to \{0, 1, \ldots, n'\}$ such that conditions (D1)-(D4) hold. We may assume that there are integers m, t such that $f(i) = i$ for $i = 1, 2, \ldots, m$, $f(i) = 0$ for $i = m+1, m+2, \ldots, t$, and $\{t+1, t+2, \ldots, n\} = S$. Let $g_1, g_2, \ldots$ be a fundamental sequence of $\mathcal{G}_l$; we must show that $g_i \in \mathcal{G}_{l'}$ for every integer $i \geq 1$. To this end let $i \geq 1$ be an integer. Then $g_i = (T^i_1, T^i_2, \ldots, T^i_n; k; M_i)$, where $T^i_j \in \mathcal{I}_j$ for $j = 1, 2, \ldots, n$ and $M_i \subseteq \mathcal{I}_0$. It follows that $T^i_j \in \mathcal{I}'_j$ for $j = 1, 2, \ldots, m$, $\{T^i_{m+1}, T^i_{m+2}, \ldots, T^i_t\} \cup M_i \subseteq \mathcal{I}'_0$ and $k + n - t \leq k'$. We deduce that $g_i \in \mathcal{G}_{l'}$, as desired. This proves (3).

(4) $\mathcal{I} \subseteq I(\mathcal{B}_0)$.

We prove (4) by induction on height. Let $T \in \mathcal{I}$, and assume that every tree in $\mathcal{I}$ of strictly smaller height belongs to $I(\mathcal{B}_0)$. Let $T =$Tree$(T_1, T_2, \ldots, T_t)$. Then $g = (T_1, T_2, \ldots, T_t; 0; \emptyset)$ is a germ of $\mathcal{I}$, and so $g \in \mathcal{G}_l$ for some $l = 1, 2, \ldots, p$. Let $g_1, g_2, \ldots$ be a fundamental sequence of $\mathcal{G}_l$. Then $g \leq g_i$ for some integer $i \geq 1$. Thus there exist a set $S \subseteq \{1, 2, \ldots, t\}$ and a mapping $f : \{1, 2, \ldots, t\} - S \to \{0, 1, \ldots, n\}$ satisfying (i)-(iv). Since $T^i_j \in \mathcal{I}_j$ for $j = 1, 2, \ldots, n$, $M_i \subseteq \mathcal{I}_0$ and $T_j \in I(\mathcal{B}_0)$ for $j \in S$ by the choice of T, we deduce that $T \in I(\mathcal{B}_0)$, as desired. This proves (4).

(5) $I(\mathcal{B}_0) \subseteq \mathcal{I}$.

Again, we prove (5) by induction on height. Let $T \in I(\mathcal{B}_0)$, and assume that every tree in $I(\mathcal{B}_0)$ of strictly smaller height belongs to $\mathcal{I}$. Let $l \in \{1, 2, \ldots, p\}$ be such that T conforms to B_l in $\mathcal{B}_0$, and let $B_l = (\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n; k; \mathcal{I}_0)$. Then either $T \in \mathcal{I}_0 \cup \mathcal{I}_1 \cup \ldots \cup \mathcal{I}_n$, or T has the form Tree$(T_1, T_2, \ldots$ $\ldots, T_{n+k+m})$, where $m \geq 0$, $T_i \in \mathcal{I}_i \cup \{\Gamma\}$ for $i = 1, 2, \ldots, n$, $T_{n+i} \in I(\mathcal{B}_0) \cup \{\Gamma\}$ for $i = 1, 2, \ldots, k$ and $T_{n+k+i} \in \mathcal{I}_0$ for $i = 1, 2, \ldots, m$. In the former case $T \in \mathcal{I}$, and so we assume the latter. Let $g_1, g_2, \ldots$ be a fundamental sequence of $\mathcal{G}_l$. Then there exists an integer $i \geq 1$ such that T_j is topologically contained in T^i_j for $j = 1, 2, \ldots, n$ and T_{n+k+j} is topologically contained in some member of M_i for $j = 1, 2, \ldots, m$. By the induction hypothesis $T_{n+j} \in \mathcal{I} \cup \{\Gamma\}$ for $j = 1, 2, \ldots, k$. Thus $T \in H(g_i, \mathcal{I}) \subseteq \mathcal{I}$, as desired. This proves (5).

It follows from (2), (3), (4), (5) that $\mathcal{B}_0$ satisfies (N1), (N2) and (N3). Let us call a bit B *proper* if B has width zero and is not of the form $(\mathcal{I}_1; 0; \emptyset)$, where $\mathcal{I}_1 = I(\{B\})$. We choose a finite set of bits $\mathcal{B}$ such that

(i) $\mathcal{B}$ satisfies (N1), (N2) and (N3),

(ii) subject to (i), $|\mathcal{B}|$ is minimum, and

(iii) subject to (i) and (ii), the number of proper bits in $\mathcal{B}$ is minimum.

Such a choice is possible, because $\mathcal{B}_0$ satisfies (i). We claim that $\mathcal{B}$ satisfies (N4). Indeed, let $B \in \mathcal{B}$ have width zero, and let $I(\{B\}) \subseteq \mathcal{I}'$ for some ideal $\mathcal{I}'$ of a bit $B' \in \mathcal{B}$. We must show that $B = B'$ and that B is not proper. If $B \neq B'$ we put $\mathcal{B}' = \mathcal{B} - \{B\}$; then $I(\mathcal{B}') = \mathcal{I}$ by (3.1), and hence $\mathcal{B}'$ satisfies (N1), (N2) and (N3), contrary to (ii). Thus $B = B'$. Now $\mathcal{I}' \subseteq I(\{(\mathcal{I}'; 0; \emptyset)\}) \subseteq I(\{B'\}) = I(\{B\}) \subseteq \mathcal{I}'$; hence equality holds throughout and thus $(\mathcal{I}'; 0; \emptyset)$ is not proper. If B is is proper let $\mathcal{B}' = (\mathcal{B} - \{B\}) \cup \{(\mathcal{I}'; 0; \emptyset)\}$, and again, using (3.1) we see that $\mathcal{B}'$ satisfies (i), contrary to (iii). This proves that $B = B'$ and that B is not proper, and hence $\mathcal{B}$ satisfies (N4), as desired. $\square$

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