TUTTE'S EDGE-COLOURING CONJECTURE

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ABSTRACT

Tutte made the conjecture in 1966 that every 2-connected cubic graph not containing the Petersen graph as a minor is 3-edge-colourable. The conjecture is still open, but we show that it is true in general provided it is true for two special kinds of cubic graphs that are almost planar.

1. INTRODUCTION

The following well-known conjecture is due to Tutte [9]:

(1.1) (*Conjecture*) *Every* 2*-connected cubic graph with no Petersen minor is* 3*-edge-colourable.*

(All graphs in this paper are finite and loopless; H is a *minor* of G if H can be obtained from a subgraph of G by contracting edges; and *Petersen* means the Petersen graph.)

This extends the four-colour theorem (Tait [8] showed that the four-colour-theorem is equivalent to the statement that every planar 2-connected cubic graph is 3-edge-colourable.) It also implies that certain non-planar graphs are 3-edge-colourable. Let us say G is *apex* if $G\backslash v$ is planar for some v (we use $\backslash$ to denote deletion); and G is *doublecross* if it can be drawn in the plane with crossings, but with at most two crossings, and with all the crossings on the boundary of the infinite region. Both apex and doublecross graphs have no Petersen minor, so (1.1) implies:

(1.2) (*Conjecture*)

(i) *Every* 2*-connected apex cubic graph is* 3*-edge-colourable, and*

(ii) *Every* 2*-connected doublecross cubic graph is* 3*-edge-colourable.*

Both the conjectures of (1.2) are still open, but since both kinds of graphs are almost planar, there is hope of modifying a proof of the four-colour theorem to prove (1.2). Indeed, preliminary work by Dan Sanders and Robin Thomas appears to indicate that this is feasible.

It is the objective of this paper to prove the equivalence of (1.1) and (1.2). That follows immediately from the following. (A *minimum counterexample* means a 2-connected cubic graph G with no Petersen minor which is not 3-edge-colourable, with $|V(G)|$ minimum.)

(1.3) *Every minimum counterexample is either apex or doublecross.*

If $X \subseteq V(G)$ we denote by $\delta(X)$ or $\delta_G(X)$ the set of edges of G with exactly one end in X. We say a cubic graph G is *cyclically* 5*-connected* if $|V(G)| \geq 8$ (to avoid some trivialities) and $|\delta(X)| \geq 5$ for every $X \subseteq V(G)$ with $|X|, |V(G) - X| \geq 3$. We say G is *theta-connected* if it is cubic and cyclically 5-connected, and $|\delta_G(X)| \geq 6$ for every $X \subseteq V(G)$ with $|X|, |V(G) - X| \geq 7$. The main theorem of [5] asserts:

(1.4) *Let G be theta-connected, with no Petersen minor. Then either G is apex, or G is doublecross, or G is isomorphic to Starfish.*

(*Starfish* is one particular cubic graph with twenty vertices, described in [5]. Here, all we need about Starfish is that it is 3-edge-colourable, which can easily be verified.) Consequently, (1.3) follows from (1.4) and the following.

(1.5) *Every minimum counterexample that is not theta-connected is apex.*

Proving (1.5) is therefore the objective of the paper.

2. PRELIMINARIES

Let G be a cubic graph. A *shore* of G is a subset $X \subseteq V(G)$ with $\emptyset \neq X \neq V(G)$ such that no two edges in $\delta(X)$ have a common end. Provided that $|V(G)| \geq 8$, it is easy to see that G is cyclically 5-connected if and only if $|\delta(X)| \geq 5$ for every shore X.

A *matching* of G means a set $F \subseteq E(G)$ so that no two members of F have a common end. Let X be a shore of a cubic graph G. An X-*colouring* of G means a set $\{F_1, F_2, F_3\}$ of three matchings of G, pairwise disjoint, so that $F_1 \cup F_2 \cup F_3$ is the set of edges of G with at least one end in X. A $\delta(X)$-*colouring* means a multiset $\{M_1, M_2, M_3\}$ of three matchings of G, pairwise disjoint, with union $\delta(X)$. (This is a multiset rather than a set, because two of the M_i's may be equal, but only if they are both null.) Now if $\{F_1, F_2, F_3\}$ is an X-colouring, the multiset

$$\{F_1 \cap \delta(X), F_2 \cap \delta(X), F_3 \cap \delta(X)\}$$

is a $\delta(X)$-colouring, and the set of all multisets that arise in this way is denoted by $\mathcal{C}(X)$. We need the following folklore result, whose proof we omit.

(2.1) *Let X be a shore of a cubic graph G, and let $\{M_1, M_2, M_3\} \in \mathcal{C}(X)$. Then*

$$|M_1| \equiv |M_2| \equiv |M_3| \equiv |\delta(X)| \, (mod\, 2).$$

We also shall sometimes need the following strengthening of (2.1), essentially due to Tait [8] (again, we omit its proof).

(2.2) *Let X be a shore of a cubic graph G, and let $\{M_1, M_2, M_3\} \in \mathcal{C}(X)$. Then there is a partition of $M_1 \cup M_2$ into sets $B_1, \dots, B_k$ each of cardinality 2, such that*

(i) *there are k paths $P_1, \dots, P_k$ of $G|X$, pairwise disjoint, so that for $1 \le i \le k$ both ends of P_i are incident with edges in B_i, and*

(ii) *for any $I \subseteq M_1 \cup M_2$ expressible as a union of some of $B_1, \dots, B_k$, $\mathcal{C}(X)$ contains*

$$\{(M_1 - I) \cup (M_2 \cap I), (M_1 \cap I) \cup (M_2 - I), M_3\}.$$

3. CYCLIC 5-CONNECTIVITY

A minor H of G is *proper* if H is not isomorphic to G. By a *minimal counterexample* we mean a 2-connected cubic graph G that is not 3-edge-colourable and has no Petersen minor, such that every 2-connected cubic proper minor of G is 3-edge-colourable. (Later we shall need the stronger hypothesis that G is a minimum counterexample, but we shall avoid this as long as possible.) The next theorem is related to results of Goldberg [2] and Isaacs [3].

(3.1) *Every minimal counterexample is cyclically 5-connected.*

Proof. Let G be a minimal counterexample, and suppose it is not cyclically 5-connected. Certainly $|V(G)| \geq 8$, and so there is a shore X with $|\delta(X)| \leq 4$. Choose X with $|\delta(X)|$ minimum. Let $\delta(X) = \{e_1, \ldots, e_k\}$, let $Y = V(G) - X$, and let e_i have ends $x_i \in X$ and $y_i \in Y$ $(1 \leq i \leq k)$. Since G is not 3-edge-colourable, we obviously have

(1) $\mathcal{C}(X) \cap \mathcal{C}(Y) = \emptyset$.

Now certainly $k \geq 2$ since G is 2-edge-connected. Suppose that $k = 2$. Let H be obtained from $G|X$ by adding the edge x_1x_2. Then H is 2-connected, cubic and isomorphic to a proper minor of G, and so H is 3-edge-colourable. Consequently,

$$\{\{e_1, e_2\}, \emptyset, \emptyset\} \in \mathcal{C}(X).$$

Similarly it belongs to $\mathcal{C}(Y)$, contradicting (1). Thus, $k \geq 3$. In particular, G is simple and 3-connected.

Suppose that $k = 3$. Let H be obtained from $G|X$ by adding a new vertex v and three new edges vx_1, vx_2, vx_3. Then H is 2-connected, cubic and isomorphic to a proper minor of G, and so H is 3-edge-colourable. Consequently,

$$\{\{e_1\},\{e_2\},\{e_3\}\} \in \mathcal{C}(X).$$

Similarly it belongs to $\mathcal{C}(Y)$, contradicting (1). Thus $k \geq 4$.

To complete the proof, we suppose for a contradiction that $k = 4$. Let $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ be the $\delta(X)$-colourings

$$\begin{aligned}
\alpha_0 &= \{\{e_1, e_2, e_3, e_4\}, \emptyset, \emptyset\} \\
\alpha_1 &= \{\{e_1, e_2\}, \{e_3, e_4\}, \emptyset\} \\
\alpha_2 &= \{\{e_1, e_3\}, \{e_2, e_4\}, \emptyset\} \\
\alpha_3 &= \{\{e_1, e_4\}, \{e_2, e_3\}, \emptyset\}.
\end{aligned}$$

If $\{a, b, c, d\} = \{x_1, x_2, x_3, x_4\}$ and there are two disjoint paths P, Q of $G|X$ such that P has ends a and b, and Q has ends c and d, we say that (a, b, c, d) is *feasible* in $G|X$.

(2) *If (x_1, x_2, x_3, x_4) is feasible in $G|X$ then*

$$\mathcal{C}(Y) \cap \{\alpha_0, \alpha_1\} \neq \emptyset \neq \mathcal{C}(Y) \cap \{\alpha_2, \alpha_3\}$$

If (x_1, x_3, x_2, x_4) is feasible in $G|X$ then

$$\mathcal{C}(Y) \cap \{\alpha_0, \alpha_2\} \neq \emptyset \neq \mathcal{C}(Y) \cap \{\alpha_1, \alpha_3\}.$$

If (x_1, x_4, x_2, x_3) is feasible in $G|X$ then

$$\mathcal{C}(Y) \cap \{\alpha_0, \alpha_3\} \neq \emptyset \neq \mathcal{C}(Y) \cap \{\alpha_1, \alpha_2\}.$$

Subproof. Suppose that (x_1, x_2, x_3, x_4) is feasible in $G|X$. Let H_1 be obtained from $G|Y$ by adding the edges y_1y_2 and y_3y_4. Then H_1 is 2-connected, cubic, and isomorphic to a proper minor of G, and so H_1 is 3-edge-colourable. By (2.1), $\mathcal{C}(Y) \cap \{\alpha_0, \alpha_1\} \neq \emptyset$.

Let H_2 be obtained from $G|Y$ by adding two new vertices u, v and five edges $uv, uy_1, uy_2, vy_3, vy_4$. Then H_2 is isomorphic to a proper minor of G since $G|X$ is connected; and so as usual, H_2 is 3-edge-colourable, and hence by (2.1), $\mathcal{C}(Y) \cap \{\alpha_2, \alpha_3\} \neq \emptyset$. The other claims follow by symmetry. This proves (2).

Let $\pi_1 = (x_1, x_2, x_3, x_4)$, $\pi_2 = (x_1, x_3, x_2, x_4)$, $\pi_3 = (x_1, x_4, x_2, x_3)$. By Menger's theorem (since $k \geq 4$), there are two disjoint paths of $G|X$ from $\{x_1, x_2\}$ to $\{x_3, x_4\}$, and so either π_2 or π_3 is feasible in $G|X$. Similarly one of π_1, π_3 and one of π_1, π_2 is feasible. Consequently, at least two of π_1, π_2, π_3 are feasible in $G|X$. From (2), $\mathcal{C}(Y)$ contains at least two of $\alpha_0, \alpha_1, \alpha_2, \alpha_3$, and similarly so does $\mathcal{C}(X)$. By (1), we may assume that

$$\begin{aligned} \mathcal{C}(X) &= \{\alpha_0, \alpha_1\} \\ \mathcal{C}(Y) &= \{\alpha_2, \alpha_3\}. \end{aligned}$$

From (2.2), π_1 is feasible in $G|X$, contrary to (2).

Hence $k \geq 5$, and the result follows. ■

4. COLOURINGS OF A 5-CUT

In view of (3.1), to complete the proof of (1.5) we need to examine the case of shores X with $|\delta(X)| = 5$. In this section we examine the possibilities for $\mathcal{C}(X)$ and $\mathcal{C}(V(G) - X)$ with such a shore X. Our approach is similar to that of [1,2,4].

Thus, let G be a minimal counterexample. Let $X \subseteq V(G)$ be a shore with $|\delta(X)| = 5$, and let $Y = V(G) - X$. Let K be the complete graph with vertex set $\delta(X)$. It is helpful to associate edges of K with members of $\mathcal{C}(X)$. Let $L(X)$ be the subgraph of K in which

(i) an edge ef of K is an edge of $L(X)$ if and only if

$$\{\delta(X) - \{e, f\}, \{e\}, \{f\}\} \in \mathcal{C}(X)$$

(ii) $L(X)$ has no vertices of degree 0.

By (2.1) there is a one-to-one correspondence between the members of $\mathcal{C}(X)$ and the edges of $L(X)$. Define $L(Y)$ similarly.

(4.1) *$E(L(X) \cap L(Y)) = \emptyset$, and $L(X)$ and $L(Y)$ both have minimum degree ≥ 2.*

Proof. The first claim follows since $\mathcal{C}(X) \cap \mathcal{C}(Y) = \emptyset$, and the second follows from (2.2). ■

(4.2) *$L(X)$ and $L(Y)$ both have ≥ 4 vertices.*

Proof. Since X is a shore, it follows that $G|X$ has a circuit. Since G is cyclically 5-connected, we may number $\delta(X) = \{e_1, \ldots, e_5\}$ so that (letting each e_i be incident with $x_i \in X$ and $y_i \in Y$), H_1 is isomorphic to a minor of G, where H_1 is obtained from $G|Y$ by adding five new vertices $v_1, \ldots, v_5$ and edges $v_1y_1, v_2y_2, v_3y_3, v_4y_4, v_5y_5, v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_1v_5$. Suppose that some two of $e_1, \ldots, e_5$ are not vertices of $L(Y)$, say e_i, e_j. From the rotational symmetry of $e_1, \ldots, e_5$ we may assume that $i, j \in \{1, 2, 3\}$. Let H_2 be obtained from $G|Y$ by adding a new vertex v and edges vy_1, vy_2, vy_3, y_4y_5. Then H_2 is 2-connected and cubic, and isomorphic to a proper minor of H_1 and hence of G. Hence H_2 is 3-edge-colourable, and so one of $e_1e_2, e_1e_3, e_2e_3 \in E(L(Y))$, contradicting that two of e_1, e_2, e_3 are not vertices of $L(Y)$. The result follows. ■

We deduce

(4.3) *Either*

(i) *$L(X)$ and $L(Y)$ are complementary circuits of K, both of length 5, or*

(ii) *one of $L(X), L(Y)$ consists of a circuit of length 4, and the other is its complement in K.*

Proof. Suppose that $L(X)$ has a circuit of length 3, with vertex set $\{e_1, e_2, e_3\}$ say, where $\delta(X) = \{e_1, \ldots, e_5\}$. By (4.2) $L(X)$ has at least one more edge, and by (4.1) we may assume that $e_1e_4 \in E(L(X))$. Hence e_1 has degree ≥ 3 in $L(X)$, and therefore does not belong to $L(Y)$ by (4.1). By (4.1), e_2 and e_3 both have degree ≥ 2 in $L(Y)$, and since $e_2e_3 \notin E(L(Y))$ it follows that $e_2e_4, e_2e_5, e_3e_4, e_3e_5 \in E(L(Y))$. Hence

$$\{e_1e_2, e_1e_3, e_2e_3, e_1e_4\} \subseteq E(L(X)) \subseteq \{e_1e_2, e_1e_3, e_2e_3, e_1e_4, e_1e_5, e_4e_5\}$$

and so by (4.1), the second inclusion is an equality; and therefore (ii) holds.

Consequently we may assume that neither $L(X)$ nor $L(Y)$ has a circuit of length 3. Since $L(X)$ and $L(Y)$ both have circuits by (4.1) and (4.2), and these circuits are edge-disjoint, it follows that (i) holds. ■

5. THETA-CONNECTIVITY

Theorem (4.3) was proved under the assumption that G is a minimal counterexample, but now we need to strengthen that; in this section it will be a minimum counterexample.

We need the following theorem of Seymour and Truemper [7].

(5.1) *Let C be a circuit of length 5 of a cubic graph H. Suppose that H has a Petersen minor, and there is no $X \subseteq V(H) - V(C)$ with $|X| \geq 3$ and $|\delta(X)| \leq 4$. Then H has a subgraph P with $C \subseteq P$, such that P is a subdivision of Petersen.*

We deduce from (5.1) that

(5.2) *Let G be a minimum counterexample, and let X, Y, K etc. be as in section 4.*

If $|X|, |Y| \geq 7$ *then* $L(X)$ *is not a circuit of length* 5.

Proof. Suppose that $L(X)$ is a circuit of length 5, with vertices $e_1, \dots, e_5$ in order say. By (4.3), $L(Y)$ is also a circuit of length 5, with vertices e_1, e_3, e_5, e_2, e_4 in order.

Let H_1 be obtained from $G|X$ by adding five new vertices $u_1, \dots, u_5$ and edges

$$u_1x_1, u_2x_2, u_3x_3, u_4x_4, u_5x_5, u_1u_3, u_3u_5, u_2u_5, u_2u_4, u_1u_4.$$

Let H_2 be obtained from $G|Y$ by adding vertices $v_1, \dots, v_5$ and edges

$$v_1y_1, y_2y_2, v_3y_3, v_4y_4, v_5y_5, v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_1v_5\ .$$

Since the only edges of $L(X)$ are e_1e_2, e_2e_3, e_3e_4, e_4e_5 and e_1e_5, it follows that H_1 is not 3-edge-colourable, and similarly neither is H_2. Since $|V(H_1)| < |V(G)|$ and H_1 is 2-connected, it follows that H_1 has a Petersen minor, and similarly so does H_2. Let C be the circuit of H_1 with vertex set $\{u_1, \dots, u_5\}$. Since G is cyclically 5-connected, it follows from (5.1) applied to H_1 and C that there is a subgraph P of H_1 with $C \subseteq P$ which is a subdivision of Petersen. But in Petersen, the subgraph obtained by deleting the vertex set of any 5-circuit is another 5-circuit in "opposite" order. Consequently H_2 is isomorphic to a minor of G, a contradiction since H_2 has a Petersen minor and G does not. The result follows. ■

A *candidate* $(G, x_1, \dots, x_5)$ consists of a graph G and five distinct vertices $x_1, \dots, x_5$ of G, such that $x_1, \dots, x_5$ have degree 2 in G, and every other vertex has degree 3.

Let $(G, x_1, \dots, x_5)$ be a candidate. A *policy* of $(G, x_1, \dots, x_5)$ is a tree of G expressible in the form $P \cup Q \cup R \cup S$, where P is a path from x_1 to x_3, Q is a path from x_2 to x_4, $P \cap Q$ is null, R is a path from some vertex of P to some vertex of Q with no other vertices in $P \cup Q$, and S is a path from an internal vertex of R to x_5 with no other vertex in $P \cup Q \cup R$.

A *left wing* of $(G, x_1, \ldots, x_5)$ is a subgraph of G expressible in the form $P \cup Q \cup R \cup S$, where P, Q, R, S are paths of G, pairwise disjoint except for their ends, and for some vertex t, P is from x_1 to t, Q is from x_3 to t, R is from x_5 to t, and S is from x_2 to x_4.

A *right wing* is defined similarly, except that P is from x_2 to t, Q is from x_4 to t, R is from x_5 to t, and S is from x_1 to x_3. Thus both left and right wings are forests with two components. See Figure 1.

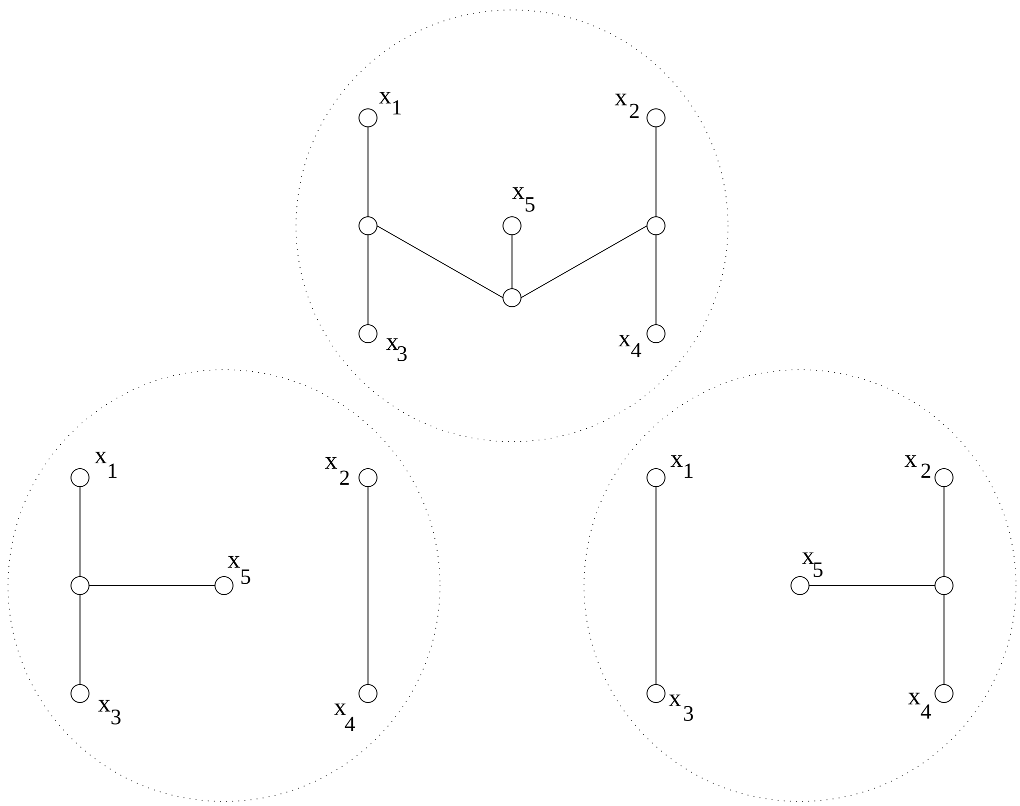

Figure 1: a policy, a left wing, and a right wing (the lines represent paths, not edges).

To avoid repetition, let us extract the following hypothesis, common to several statements that follow.

Hypothesis J. *Let G be a minimum counterexample, and let $X \subseteq V(G)$ be a shore with $|\delta(X)| = 5$, $\delta(X) = \{e_1, \ldots, e_5\}$ say. Let $Y = V(G) - X$, and for $1 \leq i \leq 5$ let e_i have ends $x_i \in X$ and $y_i \in Y$. Let $K, L(X)$ and $L(Y)$ be defined as section 4, and let $L(X)$ be a circuit of K with vertices e_1, e_2, e_3, e_4 in order.*

(5.3) *Under hypothesis* **J**, $(G|X, x_1, x_2, x_3, x_4, x_5)$ *is a candidate with no policy.*

Proof. Clearly it is a candidate, since G is cubic and X is a shore. Suppose it has a policy. Let H be obtained from $G|Y$ by adding three new vertices u, v, w and edges

$$uv, vw, uy_1, uy_3, wy_2, wy_4, vy_5 .$$

Then H is 2-connected, cubic and isomorphic to a proper minor of G, and hence is 3-edge-colourable. Consequently, $\mathcal{C}(Y)$ contains a $\delta(X)$-colouring $\{\{a\}, \{b\}, \{e_1, \ldots, e_5\} - \{a, b\}\}$ where $a \in \{e_1, e_3\}$ and $b \in \{e_2, e_4\}$; that is, one of the edges $e_1e_2, e_2e_3, e_3e_4, e_1e_4$ is an edge of $L(Y)$. But all such edges belong to $L(X)$, contrary to (4.1). ■

(5.4) *Under hypothesis* **J**, $|Y| \leq 7$.

Proof. Let H be obtained from $G|X$ by adding seven new vertices $v_1, \ldots, v_7$ and edges

$$v_1x_1, v_2x_2, v_3x_3, v_4x_4, v_5x_5, v_1v_2, v_2v_6, v_3v_6, v_3v_4, v_4v_7, v_1v_7, v_5v_6, v_5v_7 .$$

Let C be the circuit of H with vertex set $\{v_1, v_2, v_6, v_5, v_7\}$.

Suppose that H has a Petersen minor. By (5.1), there is a subgraph P of H with $C \subseteq P$ which is a subdivision of Petersen. But this contradicts (5.3).

Hence H has no Petersen minor. But it is 2-connected and cubic, and not 3-edge-colourable (we leave checking this to the reader). Since G is a minimum counterexample, $|V(H)| \geq |V(G)|$, and so $|Y| \leq 7$, as required. ∎

In passing, we note:

(5.5) *Let G be a minimum counterexample, and let $X \subseteq V(G)$ be a shore with $|\delta(X)| \leq 5$. Then $|\delta(X)| = 5$, and either $|X| \leq 7$ or $|V(G) - X| \leq 7$.*

Proof. From (3.1), $|\delta(X)| = 5$. From (4.3) we may assume that $L(X)$ is a 4-circuit or a 5-circuit, and $L(Y)$ is its complement where $Y = V(G) - X$. By (5.2) if $L(X)$ is a 5-circuit then either $|X| = 5$ or $|Y| = 5$, as required. By (5.4), if $L(X)$ is a 4-circuit then $|Y| \leq 7$, as required. ∎

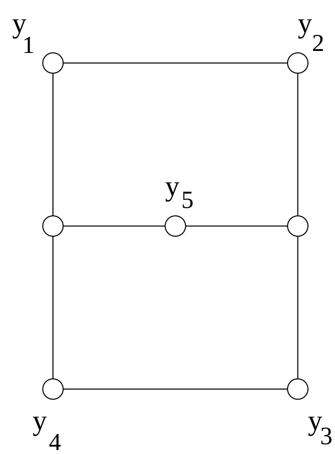

Figure 2 : a domino

We say a candidate $(G, y_1, y_2, y_3, y_4, y_5)$ is a *domino* if $|V(G)| = 7$, $V(G) = \{y_1, \ldots, y_7\}$

say, and

$$y_1y_2, y_2y_6, y_3y_6, y_3y_4, y_4y_7, y_1y_7, y_5y_6, y_5y_7$$

are edges. (See Figure 2.) It is a *turned domino* if $(G, y_1, y_4, y_3, y_2, y_5)$ is a domino.

(5.6) *Under hypothesis* **J***,* $(G|Y, y_1, y_2, y_3, y_4, y_5)$ *is either a domino or a turned domino.*

Proof. Since Y is a shore and $|\delta(Y)| = 5$, it follows that $|Y| \geq 5$ and Y includes the vertex set of a circuit of G. Also, $|\delta(Y)|$ is odd and so $|Y|$ is odd. By (5.4), either $|Y| = 5$ or $|Y| = 7$. If $|Y| = 5$ then $L(Y)$ is a circuit of length 5, contrary to (4.3). Thus $|Y| = 7$. Since $|\delta(Y)| = 5$ and every circuit of G has length ≥ 5 (by (3.1)), it follows that $G|Y$ is a subdivision of $K_{2,3}$, and indeed $\{y_1, \ldots, y_5\} = \{y_{i_1}, \ldots, y_{i_5}\}$ where $(G|Y, y_{i_1}, \ldots, y_{i_5})$ is a domino. Consequently $L(Y)$ is the complement of a 4-circuit with vertex set $e_{i_1}, e_{i_2}, e_{i_3}, e_{i_4}$ in order. But $L(Y)$ is the complement of a 4-circuit with vertex set e_1, e_2, e_3, e_4 in order, by (4.3). Hence $i_5 = 5$, and we may therefore assume that $i_1 = 1$; and consequently (i_2, i_3, i_4) is either (2, 3, 4) or (4, 3, 2). The result follows. ■

(5.7) *Under hypothesis* **J***, the candidate* $(G|X, x_1, x_2, x_3, x_4, x_5)$ *has a left wing and a right wing.*

Proof. We shall prove it has a left wing; then by symmetry it also has a right wing.

(1) *Let* $\{y'_1, \ldots y'_5\} = \{y_1, \ldots, y_5\}$*; then either* $(G|Y, y'_1, y'_2, y'_3, y'_4, y'_5)$ *has a left wing or* $(G|Y, y'_1, y'_2, y'_3, y'_5, y'_4)$ *has a left wing.*

Subproof. From (5.6), $(G|Y, y_1, y_2, y_3, y_4, y_5)$ is either a domino or a turned domino, and the result follows by checking all possibilities for $y'_1, \ldots, y'_5$. This proves (1).

Now $|X| \geq 5$ since X is a shore, and $|X| \neq 5$ since $L(X)$ is not a 5-circuit, and $|X| \neq 7$ since $L(X)$ is not the complement of a 4-circuit. Thus $|X| \geq 9$. Suppose that x_1, x_3 are adjacent. Then $X - \{x_1, x_3\}$ is a shore, and so by (5.5), $|X| = 9$, which easy case analysis shows to be impossible. Thus x_1, x_3 are not adjacent.

Let H be obtained from $G|X$ by adding a new vertex v and four edges vx_2, vx_4, vx_5, x_1x_3. Then H is 3-connected (since x_1, x_3 are not adjacent in G). Moreover, since $e_1e_2, e_2e_3, e_3e_4, e_1e_4$ are the only edges of $L(X)$ it follows that H is not 3-edge-colourable. Since $|V(H)| < |V(G)|$ and G is a minimum counterexample, it follows that H has a Petersen minor. By [7, theorem (2.2)] there is a subgraph P of G which is a subdivision of Petersen, so that $v \in V(P)$ and v has degree 3 in P. Since G has no Petersen minor it follows that $x_1x_3 \in E(P)$.

An *arc* of P means a path of P with at least one edge so that its ends have degree 3 in P and all its internal vertices have degree 2 in P. Thus, P has precisely fifteen arcs.

Now certainly vx_2, vx_4, vx_5 belong to different arcs of P. Suppose that Q is an arc of P containing x_1x_3 and one of vx_2, vx_4, vx_5, say vx_e where $\{x_2, x_4, x_5\} = \{x_a, x_b, x_e\}$. Let

$$\{1, 2, 3, 4, 5\} - \{a, b, e\} = \{c, d\}$$

where x_d lies in Q between v and x_c (we recall that the edge $x_cx_d = x_1x_3$ belongs to Q). Let T be a connected subgraph of $G|Y$ with $y_a, y_b, y_c \in V(T)$, minimal with the property; then $T \cup (P \cap (G|X))$, together with the edges e_a, e_b, e_c, form a subgraph of G which is a subdivision of Petersen, a contradiction.

Thus, vx_2, vx_4, vx_5, x_1x_3 all belong to different arcs of P. Let x_1x_3 belong to the arc Q say; and suppose that some arc R has a common end u with Q, and contains one of vx_2, vx_4, vx_5, say vx_e where $\{x_2, x_4, x_5\} = \{x_a, x_b, x_e\}$. Let

$$\{1, \ldots, 5\} - \{a, b, e\} = \{c, d\}$$

where x_d lies in Q between u and x_c. By (1), one of

$$(G|Y, y_a, y_c, y_b, y_d, y_e)$$

$$(G|Y, y_a, y_c, y_b, y_e, y_d)$$

has a left wing T say; and then $T \cup (P \cap (G|X))$, together with the five edges in $\delta(X)$, form a subgraph of G which is a subdivision of Petersen, a contradiction.

Thus, the arc containing x_1x_3 has no common end with any of the arcs containing v. There remain six possibilities for the arc containing x_1x_3 (in fact fewer, with use of symmetry); and we leave the reader to check that in each case, there is a subgraph of $P \cap (G|X)$ which is a left wing of $(G|X, x_1, x_2, x_3, x_4, x_5)$, as required. ■

A candidate $(G, x_1, x_2, x_3, x_4, x_5)$ is *strong* if for every $Z \subseteq V(G)$ which includes the vertex set of a circuit,

$$|\delta(Z)| + |Z \cap \{x_1, \ldots, x_5\}| \geq 5\,,$$

and if equality holds then either $|Z| \leq 7$ or $Z = V(G)$.

(5.8) *Under hypothesis* **J**, $(G|X, x_1, x_2, x_3, x_4, x_5)$ *is strong.*

Proof. Let $Z \subseteq V(G|X) = X$, including the vertex set of a circuit. By (3.1), $|\delta_G(Z)| \geq 5$, and so

$$|\delta_{G'}(Z)| + |Z \cap \{x_1, x_2, x_3, x_4, x_5\}| \geq 5\,,$$

where $G' = G|X$. Suppose equality holds. Then Z is a shore of G with $|\delta_G(Z)| = 5$, and so either $|Z| \leq 7$ or $|V(G) - Z| \leq 7$, by (5.5). If $|Z| \leq 7$ we are done, so we assume that $|V(G) - Z| \leq 7$. Then since $|Y| = 7$ by (5.6), and $Y \subseteq V(G) - Z$, it follows that $Z = X$ as required. ■

To complete the proof of (1.5) we need one more lemma, which is proved in the next section. But to motivate it, let us deduce (1.5).

Proof of (1.5), *assuming* (6.1).

Let G be a minimum counterexample that is not theta-connected. By (3.1), (4.3) and (5.2) we may assume that hypothesis **J** holds. By (5.3), (5.7) and (5.8), $(G|X, x_1, x_2, x_3, x_4, x_5)$ is a strong candidate with a left wing and a right wing but with no policy. By (6.1), $(G|X)\backslash x_5$ can be drawn in a disc with x_1, x_2, x_3, x_4 on the boundary in order. By (5.6), $G|Y$ can be drawn in a disc with y_1, y_2, y_3, y_4 on the boundary in order. Consequently, $G\backslash x_5$ is planar, and so G is apex. ■

6. CANDIDATES WITH NO POLICY

In this section we prove the following.

(6.1) *Let* $(G, x_1, x_2, x_3, x_4, x_5)$ *be a strong candidate with a left wing and a right wing but with no policy. Then* $G\backslash x_5$ *can be drawn in a disc with* x_1, x_2, x_3, x_4 *on the boundary in order.*

Before the proof, let us see that this can really happen. Let $(G, x_1, \ldots, x_5)$ be a candidate such that $G\backslash x_5$ can be drawn in a disc with x_1, x_2, x_3, x_4 on the boundary in order. Then it is easy to see that $(G, x_1, \ldots, x_5)$ has no policy; but in general it has a left wing and a right wing, and it can be arranged to be strong. Thus, (6.1) has a sort of converse.

Our proof of (6.1) is in three steps, and for the first we need another definition. Let $(G, x_1, \ldots, x_5)$ be a candidate. A *backer* for it is a subgraph

$$R_1 \cup \ldots \cup R_7 \cup S_1 \cup \ldots \cup S_4 \cup T_1 \cup \ldots \cup T_5$$

of G, where for some distinct vertices $a_1, \ldots, a_5, b_1, \ldots, b_4$ of G,

(i) $R_1, \dots, R_7, S_1, \dots, S_4$, $T_1, \dots, T_5$ are paths of G, pairwise disjoint except for their ends

(ii) $T_1, \dots, T_4$ may have no edges, but $R_1, \dots, R_7, S_1, \dots, S_4, T_5$ all have at least one edge

(iii) the paths join the pairs of vertices indicated in one of the two graphs of Figure 3.

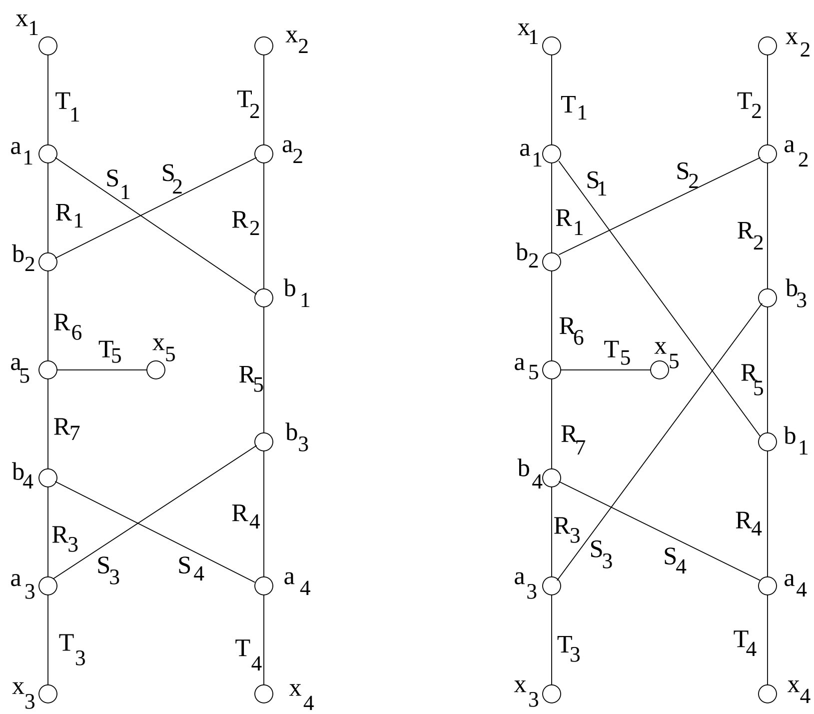

Figure 3 : the two posssibilities for a backer

We first prove:

(6.2) *Let $(G, x_1, \dots, x_5)$ be a strong candidate with a backer. Then $(G, x_1, \dots, x_5)$ has a* *policy.*

Proof. Choose a backer $H = R_1 \cup \ldots \cup R_7 \cup S_1 \cup \ldots \cup S_4 \cup T_1 \cup \ldots \cup T_5$ and vertices $a_1, \ldots, a_5, b_1, \ldots, b_4$ labelled as above, with $R_5 \cup R_6 \cup R_7 \cup T_5$ minimal. Let

$$X = V(R_5 \cup R_6 \cup R_7 \cup T_5) - \{b_1, b_2, b_3, b_4\}$$

$$Y = V(R_1 \cup R_2 \cup R_3 \cup R_4 \cup S_1 \cup S_2 \cup S_3 \cup S_4 \cup T_1 \cup T_2 \cup T_3 \cup T_4) - \{b_1, b_2, b_3, b_4\}.$$

We suppose, for a contradiction, that $(G, x_1, \ldots, x_5)$ has no policy.

(1) *There is no path in $G \backslash \{b_1, b_2, b_3, b_4\}$ from X to Y.*

Subproof. Suppose that P is such a path, with ends $x \in X$ and $y \in Y$ say; we may assume $V(P) \cap (X \cup Y) = \{x, y\}$. We examine the possible positions of x and y. Since x is incident with an edge of G not in H and all vertices of G have degree ≤ 3, it follows that $x \neq a_5$ and $y \neq a_1, \ldots, a_4$. Also, $x, y \neq b_1, \ldots, b_4$ since P is a path of $G \backslash \{b_1, b_2, b_3, b_4\}$.

From the symmetry we may assume that

$$y \in V(R_1 \cup R_2 \cup S_1 \cup S_2 \cup T_1 \cup T_2)\,.$$

But then $x \notin V(R_5 \cup R_6)$ from the minimality of $R_5 \cup R_6 \cup R_7 \cup T_5$, and $x \notin V(R_7 \cup T_5)$ since $(G, x_1, \ldots, x_5)$ has no policy, a contradiction. This proves (1).

From (1), there exists $Z \subseteq V(G) - \{b_1, \ldots, b_4\}$ with $X \subseteq Z$ and $Z \cap Y = \emptyset$, such that there is no edge of G from Z to $V(G) - (Z \cup \{b_1, \ldots, b_4\})$. Since $b_1, \ldots, b_4$ each have at least two neighbours in $Y \subseteq V(G) - Z$ and hence at most one in Z, it follows that $|\delta(Z)| \leq 4$. Let $z_0 = x_5$, and choose a sequence $z_0, z_1, \ldots, z_k$ of vertices of G, with k maximal such that

(i) $z_0, \ldots, z_k$ are all distinct

(ii) $z_1, \ldots, z_k \notin V(H)$

(iii) for $1 \leq i \leq k$, z_i is adjacent in G to z_{i-1}.

Since $z_1, \ldots, z_k \neq b_1, \ldots, b_4$ it follows that $z_1, \ldots, z_k \in Z$. Let z_{k+1} be a neighbour of z_k different from z_{k-1} (or, if $k = 0$, different from the vertex adjacent to z_0 in T_5). From the maximality of k, either $z_{k+1} \in \{z_1, \ldots, z_k\}$ or $z_{k+1} \in V(H)$. Now $z_{k+1} \notin V(R_5)$ since $(G, x_1, \ldots, x_5)$ has no policy, and so

$$z_{k+1} \in V(R_6 \cup R_7 \cup T_5) \cup \{z_1, \ldots, z_{k-1}\}.$$

Consequently there is a circuit C of G with $V(C) \subseteq Z - V(R_5)$. Since $|\delta(Z)| \leq 4$ and hence

$$|\delta(Z)| + |Z \cap \{x_1, \ldots, x_5\}| \leq 5\,,$$

and $(G, x_1, \ldots, x_5)$ is strong, it follows that $|\delta(Z)| = 4$ and $|Z| \leq 7$. But $|V(C)| \geq 5$ and so $|Z \cap V(R_5)| \leq 2$. Also, since $(G, x_1, \ldots, x_5)$ is strong, it follows that $|\delta(Z - V(R_5))| \geq 4$, and so $|Z \cap V(R_5)| = 2$. Hence

$$|Z \cap V(R_6 \cup R_7 \cup T_5)| \leq 5\,,$$

and since $|V(C)| \geq 5$ it follows that

$$Z \cap V(R_6 \cup R_7 \cup T_5) = V(C)\,.$$

Hence C is unique, and so one of R_6, R_7 has no internal vertices, say R_7; and

$$V(C) = \{z_1, \ldots, z_k\} \cup V(T_5) \cup (V(R_6) - \{b_2\})\,.$$

Now the two vertices in $V(R_5) \cap Z$ both have neighbours in $Z - V(R_5) = V(C)$, say c_1 and c_2. But c_1, c_2 are not adjacent since every circuit of G has length ≥ 5; and

$$c_1, c_2 \notin (V(T_5) - \{a_5\}) \cup \{z_1, \ldots, z_k\}$$

since $(G, x_1, \ldots, x_5)$ has no policy. Consequently,

$$c_1, c_2 \in V(R_6) - \{b_2, a_5\},$$

and since they are non-adjacent it follows that $k = 0$ and one of c_1, c_2 is z_{k+1}, which is impossible since it has degree 3. The result follows. ■

If x, y are vertices of a path P, we denote the subpath from x to y by $P[x, y]$. Next, we prove:

(6.3) *Let $(G, x_1, \ldots, x_5)$ be a strong candidate. If there is a left wing H_1 and a right wing H_2 so that x_5 has degree 1 in at least one of H_1, H_2, then $(G, x_1, \ldots, x_5)$ has a policy.*

Proof. We suppose for a contradiction that $(G, x_1, \ldots, x_5)$ has no policy. Consequently neither does $(G, x_3, x_2, x_1, x_4, x_5)$. From (6.2) we deduce

(1) *$(G, x_1, \ldots, x_5)$ and $(G, x_3, x_2, x_1, x_4, x_5)$ both have no backer.*

Choose H_1, H_2 as in the theorem with $H_1 \cup H_2$ minimal. Let $H_1 = P_1 \cup Q_1 \cup L_1$ where P_1 is a path from x_1 to x_3, Q_1 is a path from x_2 to x_4, $P_1 \cap Q_1$ is null, and L_1 is a path from x_5 to some vertex t_1 of P_1, and L_1 has no vertex in $P_1 \cup Q_1$ except t_1. Similarly, let $H_2 = P_2 \cup Q_2 \cup L_2$, where P_2 has ends x_1, x_3, Q_2 has ends x_2, x_4, and L_2 has ends x_5 and some $t_2 \in V(Q_2)$. At least one of L_1, L_2 has at least one edge.

An H_1-*arc* means a path in H_2 with distinct ends both in $V(H_1)$ and with no edge or internal vertex in H_1. An H_2-*arc* is defined similarly.

(2) *Every H_1-arc P with ends x, y has either*

(i) *one of x, y in $V(Q_1)$ and the other in $V(P_1)$, or*

(ii) *one of x, y equal to x_5 and the other in $V(P_1)$, and $L_1 \subseteq H_2$.*

Subproof. If $x, y \in V(Q_1)$ then $Q_1[x, y] \nsubseteq H_2$ (because H_2 has no circuit), and so by replacing $Q_1[x, y]$ by P we contradict the minimality of $H_1 \cup H_2$. Thus we may assume that $x \in V(P_1 \cup L_1)$. If $y \in V(Q_1)$ then since $(G, x_1, \dots, x_5)$ has no policy, it follows that $x \in V(P_1)$ and (i) holds, so we may assume that $x, y \in V(P_1 \cup L_1)$. If $x, y \in V(L_1)$ then by replacing $L_1[x, y]$ by P we contradict the minimality of $H_1 \cup H_2$. Thus we may assume that $x \in V(P_1)$, say $x \in V(P_1[x_1, t_1])$. From the minimality of $H_1 \cup H_2$ it follows that $y \notin V(P_1)$, and so $y \in V(L_1)$. If $L_1[y, t_1] \nsubseteq H_2$, then replacing it with P contradicts the minimality of $H_1 \cup H_2$. Thus $L_1[y, t_1] \subseteq H_2$, and so $P_1[x, t_1] \nsubseteq H_2$ since H_2 has no circuit. If $y \neq x_5$ then we may replace $P_1[x, t_1]$ by P, a contradiction to the minimality of $H_1 \cup H_2$; and so $y = x_5$, and $L_1 = L_1[y, t_1] \subseteq H_2$, and (ii) holds. This proves (2).

(3) $L_1 \subseteq H_2$ *and* $L_2 \subseteq H_1$.

Subproof. Suppose that $L_1 \nsubseteq H_2$. Then by (2), no H_1-arc has an end in $V(L_1) - \{t_1\}$. Since H_2 contains a path Q from x_5 to x_2, it follows that $L_1 \subseteq Q \subseteq H_2$, a contradiction. Thus $L_1 \subseteq H_2$, and $L_2 \subseteq H_1$ by symmetry. This proves (3).

From (3) it follows that either $L_1 \subseteq L_2$ or $L_2 \subseteq L_1$, and from the symmetry we may assume that $L_2 \subseteq L_1$. Hence $t_2 \in V(L_1)$, and so Q_2 includes at least two H_1-arcs. Let S_2, S_4 be the first and last H_1-arcs in Q_2, with ends a_2, b_2 and a_4, b_4, so that $x_2, a_2, b_2, b_4, a_4, x_4$ are in order in Q_2.

Now $L_1 \subseteq H_2$ by (3), and x_5 and x_1 belong to different components of H_2, so $P_1[x_1, t_1] \nsubseteq H_2$, and similarly $P_1[x_3, t_1] \nsubseteq H_2$. Thus by (2) there are at least two H_1-arcs in P_2, one with an end in $P_1[x_1, t_1]$ and another with an end in $P_1[x_3, t_1]$. Let S_1, S_3 be the first and last H_1-arcs in P_2, with ends a_1, b_1 and a_3, b_3, so that $x_1, a_1, b_1, b_3, a_3, x_3$ are in order in P_2. It follows that $a_1 \in V(P_1[x_1, t_1])$ and $a_3 \in V(P_1[x_3, t_1])$. Since $x_5 \notin V(P_2)$

it follows from (2) that $b_1, b_3 \in V(Q_1)$. Since

$$Q_1[x_2, a_2] \cup Q_1[x_4, a_4] \subseteq Q_2$$

we deduce that

$$b_1, b_3 \in V(Q_1[a_2, a_4]) - \{a_2, a_4\} .$$

Also, from (2), b_2 and b_4 belong to $V(P_1)$. Not both of them belong to $V(P_1[x_1, t_1])$ since $(G, x_1, \ldots, x_5)$ has no policy, and similarly not both belong to $V(P_1[x_3, t_1])$. Since

$$P_1[x_1, a_1] \cup P_1[x_3, a_3] \subseteq P_2$$

it follows that one of b_2, b_4 belongs to $V(P_1[a_1, t_1]) - \{a_1, t_1\}$ and the other to $V(P_1[a_3, t_1]) - \{a_3, t_1\}$. Hence H is a backer of one of $(G, x_1, \ldots, x_5)$, $(G, x_3, x_2, x_1, x_4, x_5)$, contrary to (1). The result follows. ■

Proof of (6.1).

Let $(G, x_1, \ldots, x_5)$ be a strong candidate with a right wing and a left wing, and with no policy. By (6.3), x_5 has degree ≥ 2 in every right wing and in every left wing.

Suppose that there are two disjoint paths P, Q of $G \backslash x_5$ from x_1 to x_3 and x_2 to x_4. Since G is connected (because $G, x_1, \ldots, x_5)$ is strong) there is a path R from x_5 to $V(P \cup Q)$. Choose a minimal such path R. Then $P \cup Q \cup R$ is either a left or right wing, and x_5 has degree 1 in it, a contradiction.

Hence such paths P, Q do not exist. By [6, theorem (2.4)], $G \backslash x_5$ can be drawn in a disc with $x_1, \ldots, x_4$ on the boundary in order. ■

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