The Tree-Width Compactness Theorem for Hypergraphs

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Abstract

A hypergraph H has tree-width k (a notion introduced by Robertson and Seymour) if k is the least integer such that H admits a tree-decomposition of tree-width k. We prove a compactness theorem for this notion, that is, if every finite subhypergraph of H has tree-width $< k$, then H itself has tree-width $< k$. This result will be used in a later paper on well-quasi-ordering infinite graphs.

1. Introduction

In this paper hypergraphs may be infinite, but to avoid trivial problems each edge may be incident only with a finite number of vertices. For other definitions see the end of this section. The following notion is the key one.

Definition. (Robertson and Seymour [4]). A *tree-decomposition* of a hypergraph $(V,(W_e : e \in E))$ is a couple (T,X), where T is a tree and $X = (X^t : t \in V(T))$ is a family of sets such that

(W1) $\bigcup_{t\in V(T)} X^t \supseteq V$,

(W2) for each $e \in E$ there exists $t \in V(T)$ such that $W_e \subseteq X^t$, and

(W3) $X^t \cap X^{t''} \subseteq X^{t'}$ whenever t' is on the path joining t and t'' in T.

The *tree-width* of a tree-decompositon (T,X) is

$$\sup_{t\in V(T)} (|X^t| - 1).$$

The *tree-width* of a hypergraph H is the least w such that H admits a tree-decomposition of tree-width w.

Originally, Robertson and Seymour introduced this notion for finite graphs only. The extension to hypergraphs (also used in more recent work of Robertson and Seymour) is obvious and the proofs are not more difficult for them. But the reader will lose nothing if he always assumes that our hypergraphs are in fact graphs. However, we have found the extension to infinite graphs (hypergraphs) important. In Section 2 we prove the following

Countable Tree-Width Compactness Theorem. Let H be an at most countable hypergraph and let every finite subhypergraph of H have tree-width $< w$. Then H has tree-width $< w$.

This is then used in Section 3 to prove the general

Tree-Width Compactness Theorem. If H is a hypergraph such that every finite subhypergraph of H has tree-width $< w$, then H itself has tree-width $< w$.

We were led to this by our aim to extend the results of Robertson and Seymour on well-quasi-ordering finite graphs to infinite ones. In [6] we prove, using the Tree-Width Compactness

Theorem, the following extension of a theorem from [5].

Theorem. *Let G_1 be a finite planar graph and let $G_2, G_3, \cdots$ be a sequence of graphs. Then there are indices $1 \leq i < j$ such that G_j contains a subgraph which can be contracted onto G_i.*

In the rest of this section we introduce some terminology. We shall indicate that k denotes a non-negative integer by writing $k < \omega$. If S is an arbitrary set, then the sets of finite, at most countable and non-empty subsets of S will be denoted by $[S]^{<\omega}, [S]^{\leq\omega}, [S]^{>0}$ respectively. We define a *division* in a set V to be a set of disjoint non-empty subsets of V. A *partition* of V is a division $\mathcal{D}$ in V such that V is the union of the members of $\mathcal{D}$. If $\mathcal{D}, \mathcal{D}'$ are divisions in V then $\mathcal{D}^*$ will denote $\cup\{[D]^{>0} : D \in \mathcal{D}\}$, and $\mathcal{D}' \prec \mathcal{D}$ will mean that $\mathcal{D}' \subseteq \mathcal{D}^*$, i.e. that every member of $\mathcal{D}'$ is a subset of a member of $\mathcal{D}$. For any set S and function $f : \mathcal{D} \to S$, we define a function $f^* : \mathcal{D}^* \to S$ by the rule that if $\emptyset \neq X \subseteq D \in \mathcal{D}$ then $f^*(X) = f(D)$.

A *hypergraph* is a pair $H = (V, (W_e : e \in E))$, where V, E are arbitrary (possibly infinite) sets and for each $e \in E$, W_e is a finite subset of V. The members of V and E are called the *vertices* and *edges* of H respectively. If $u, v \in W_e$ for some $e \in E$ then we say that u, v are *adjacent* in H. We define $V(H) = V, E(H) = E$. The hypergraph H is *finite* if the set V is finite and is *at most countable* if the set V is at most countable. The hypergraph H is a *graph* if each W_e has cardinality 1 or 2. If $V' \subseteq V, E' \subseteq E$, and $W_e \subseteq V'$ for every $e \in E'$, then the hypergraph $(V', (W_e : e \in E'))$ is called a *subhypergraph* of H. If $H = (V, (W_e : e \in E))$ is a hypergraph and $A \subseteq V$, then $H|A$ is the hypergraph

$$(A, (W_e : e \in E \text{ and } W_e \subseteq A))$$

and $N_H(A)$ (or simply $N(A)$ if no confusion is likely) is the set

$$\{v \in V : v \in W_e \text{ and } W_e \cap A \backslash \{v\} \neq \emptyset \text{ for some } e \in E\}.$$

Let T be a tree and $t_1, t_2 \in V(T)$. We denote by $[t_1, t_2]_T$ the path joining t_1 and t_2 in T, so that $\{t_1, t_2\} \subseteq [t_1, t_2]_T \subseteq V(T)$. (For the purposes of this paper, it is convenient to regard a path in T as being a subset of $V(T)$.) The subscript T will be omitted if no confusion may arise. A *path-sequence* in a tree T is a finite sequence $t_1, \cdots, t_n$ of distinct vertices of T such that $t_1, \cdots, t_n$

are the vertices of the path $[t_1, t_n]_T$ in the order of their occurrence as this path is described from t_1 to t_n.

Let (T, X) be a tree-decomposition of a subhypergraph of H and let $u, v \in V(H)$ and $t \in V(T)$. We shall say that u is *t-tied* to v in (T, X) if there do not exist vertices r, s of T such that $u \in X^r, v \in X^s$ and $t \in [r, s]_T$. If $t, t' \in V(T)$ and S is a subset of $V(H)$ and $t' \in [t, r]_T$ for every $r \in V(T)$ such that $X^r \cap S \neq \emptyset$, then we shall say that t' *separates* t *from* S *in* (T, X). If $L = (T, X)$ is a tree-decomposition of H and A is a set, then $L|A$ will denote the tree-decomposition $(T, X|A)$ of $H|A \cap V(H)$, where $X|A = (X^t \cap A : t \in V(T))$. We shall say that L is *finite* if the tree T is finite.

If T and S are trees, then a mapping $\varphi : V(T) \to V(S)$ is called an *embedding* of T into S if

(i) φ is 1-1, and

(ii) if $t' \in [t, t'']_T$ then $\varphi(t') \in [\varphi(t), \varphi(t'')]_S$.

If φ is an embedding of a tree T into a tree S then $f(T, S, \varphi)$ will denote the number of vertices of the smallest subtree of S which contains all the vertices $\varphi(t), t \in V(T)$.

If (T, X) and (S, Y) are tree-decompositions we write

$$(T, X) \leq (S, Y)$$

to mean that there is an embedding φ of T into S which satisfies (i), (ii) and

(iii) $X^t \subseteq Y^{\varphi(t)}$ for every $t \in V(T)$.

If φ satisfies (i), (ii) and (iii) then it is called an *embedding* of (T, X) into (S, Y). By $m((T, X), (S, Y))$ we denote the minimum of $f(T, S, \varphi)$ over all embeddings φ of (T, X) into (S, Y).

In Section 2 we shall use the following immediate corollary of a well-known theorem of Kruskal.

Theorem (Kruskal [2]). *If $(T_1, X_1), (T_2, X_2), \cdots$ is an infinite sequence of finite tree-decompositions of some hypergraphs such that $\bigcup_{i=1}^{\infty} \bigcup_{t \in V(T_i)} X_i^t$ is finite, then there exists a sequence $i_1 < i_2 < \cdots$ such that*

$$(T_{i_1}, X_{i_1}) \leq (T_{i_2}, X_{i_2}) \leq \cdots.$$

For a simplified proof see [3].

Acknowledgments. I would like to express my thanks to Professor C. St. J. A. Nash-Williams whose suggestions considerably improved the presentation of this paper. After the results of this paper were announced, a simpler proof of the Tree-Width Compactness Theorem was found by Thomassen [7]. In [1] a yet different proof is given, relying on Thomassen's alternative definition of tree-width.

2. Proof of the Countable Tree-Width Compactness Theorem

Let $H = (V, (W_e : e \in E))$ be an at most countable hypergraph such that every finite subhypergraph of H has tree-width $< w$. Let $V_1 \subseteq V_2 \subseteq \cdots$ be finite sets such that $\bigcup_{n=1}^{\infty} V_n = V$.

(1) **Claim.** *There is a sequence*

(2) $(T_1, X_1), (T_2, X_2), \cdots$

of tree-decompositions such that for every positive integer n

(3) (T_n, X_n) *is a finite tree-decomposition of $H|V_n$ of tree-width $< w$ and*

(4) $(T_n, X_n) \leq (T_{n+1}, X_{n+1})$.

Proof. By assumption each $H|V_n$ has tree-width $< w$, and so it is easily seen that there exists a sequence $M_1, M_2, \cdots$ such that M_n is a finite tree-decomposition of $H|V_n$ of tree-width $< w$ for $n = 1, 2, \cdots$. By Kruskal's theorem $1, 2 \cdots$ has a subsequence $i(1,1), i(1,2), \cdots$ such that

$$M_{i(1,1)}|V_1 \leq M_{i(1,2)}|V_1 \leq \cdots$$

and $i(1,1), i(1,2), \cdots$ has a subsequence $i(2,1), i(2,2), \cdots$ such that

$$M_{i(2,1)}|V_2 \leq M_{i(2,2)}|V_2 \leq \cdots$$

and $i(2,1), i(2,2) \cdots$ has a subsequence $i(3,1), i(3,2), \cdots$ such that

$$M_{i(3,1)}|V_3 \leq M_{i(3,2)}|V_3 \leq \cdots$$

and so forth. The sequence

$$M_{i(1,1)}|V_1, M_{i(2,2)}|V_2, M_{i(3,3)}|V_3, \cdots$$

satisfies (4), since for any k there exits $\ell > k$ such that

$$M_{i(k,k)}|V_k \leq M_{i(k,\ell)}|V_k = M_{i(k+1,k+1)}|V_k \leq M_{i(k+1,k+1)}|V_{k+1},$$

which proves (1). $\square$

We choose a sequence (2) satisfying (3), (4) in a special way. First choose (T_1, X_1) such that it is a first term of a sequence (2) satisfying (3), (4). Then choose a tree-decomposition (T_2, X_2) such that $(T_1,X_1),(T_2,X_2)$ (in this order) are the first two terms of a sequence (2) satisfying (3), (4) and $m((T_1,X_1),,(T_2,X_2))$ is the least possible. Then choose a tree-decomposition (T_3,X_3) such that $(T_1,X_1),(T_2,X_2),(T_3,X_3)$ (in this order) are the first three terms of a sequence (2) satisfying (3), (4) and $m((T_2,X_2),(T_3,X_3))$ is the least possible. Then choose $(T_4,X_4)\cdots$ and so forth. Finally we arrive at a sequence (2) satisfying (3), (4) which is minimal in the above sense.

For any $n=1,2,\cdots$ let φ_n be an embedding of (T_n,X_n) into (T_{n+1},X_{n+1}) such that

$$f(T_n,T_{n+1},\varphi_n)=m((T_n,X_n),(T_{n+1},X_{n+1})).$$

(5) **Claim**. *For every n and for every edge $\{t,t'\}\in E(T_n)$ there is a constant c such that for every $i>n$ the length of the path joining*

$$\varphi_{i-1}(\varphi_{i-2}(\cdots\varphi_n(t)\cdots)) \text{ and } \varphi_{i-1}(\varphi_{i-2}(\cdots\varphi_n(t')\cdots))$$

in T_i is less than c.

Proof. Assume the contrary. Then starting with an edge $\{t_n,t'_n\}\in E(T_n)$ we can find edges $\{t_i,t'_i\}\in E(T_i)$ and paths P_i in T_i such that

(6) $P_{i+1}=[\varphi_i(t_i),\varphi_i(t'_i)]_{T_{i+1}} \qquad i=n,n+1,\cdots,$

(7) $t_i,t'_i\in P_i \qquad i=n+1,n+2,\cdots,$

(8) $|P_i|>2$ for infinitely many i.

From the definition of $\le$ and from (W3) follows

$$X_j^{t_j}\cap X_j^{t'_j}\subseteq X_{j+1}^{\varphi_j(t_j)}\cap X_{j+1}^{\varphi_j(t'_j)}\subseteq X_{j+1}^{t_{j+1}}\cap X_{j+1}^{t'_{j+1}}$$

and since each of the above sets contains at most w elements, there exists j_0 such that

(9) $X_j^{t_j}\cap X_j^{t'_j}=X_{j+1}^{t_{j+1}}\cap X_{j+1}^{t'_{j+1}} \qquad (j\ge j_0).$

By (8) there is $k>j_0$ such that

(10) $|P_k|>2$.

Let

$$r_{k-1}=t_{k-1},r'_{k-1}=t'_{k-1}$$

$$r_{j+1}=\varphi_j(r_j),r'_{j+1}=\varphi_j(r'_j) \qquad (j\ge k-1).$$

Using (W3), (9) and definition of $\leq$ we get

$$X_j^{r_j} \cap X_j^{r'_j} \subseteq X_j^{t_j} \cap X_j^{t'_j} = X_{k-1}^{t_{k-1}} \cap X_{k-1}^{t'_{k-1}} \subseteq X_j^{r_j} \cap X_j^{r'_j} \qquad (j \geq k-1)$$

and hence

(11) $X_j^{r_j} \cap X_j^{r'_j} = X_j^{t_j} \cap X_j^{t'_j} \qquad (j \geq k-1).$

Let us define, for $j \geq k, \tilde{T}_j$ to be the tree obtained from T_j by deleting the edge $\{t_j, t'_j\}$ and adding the edge $\{r_j, r'_j\}$.

(12) **Claim.** *The sequence*

(13) $(T_1, X_1), (T_2, X_2), \cdots, (T_{k-1}, X_{k-1}), (\tilde{T}_k, X_k), (\tilde{T}_{k+1}, X_{k+1}), \cdots$

satisfies (3), (4).

Proof. To prove (3) it suffices to verify that $(\tilde{T}_j, X_j)$ satisfies (W3) for $j \geq k$. To this end and let $t, t', t'' \in V(\tilde{T}_j) = V(T_j)$ be such that $t' \in [t, t'']_{\tilde{T}_j}$ but $t' \notin [t, t'']_{T_j}$. Then necessarily $t_j, t'_j \in [t, t'']_{T_j}$ and $t' \in [r_j, r'_j]_{T_j}$. Hence

$$X_j^t \cap X_j^{t''} \subseteq X_j^{t_j} \cap X_j^{t'_j} = X_j^{r_j} \cap X_j^{r'_j} \subseteq X_j^{t'}$$

by (W3) and (11), which proves (3). Condition (4) is clear, because the embeddings φ_n apply for $(\tilde{T}_n, X_n)$ as well. Thus (12) is proved. □

By (10),

$$m((T_{k-1}, X_{k-1}), (\tilde{T}_k, X_k)) \leq f(T_{k-1}, \tilde{T}_k, \varphi_{k-1}) < f(T_{k-1}, T_k, \varphi_{k-1})$$
$$= m((T_{k-1}, X_{k-1}), (T_k, X_k)),$$

which contradicts the choice of (T_k, X_k). This proves (5). □

We may assume that the trees $T_1, T_2, \cdots$ are chosen in such a way that $\varphi_i(t) = t$ for $t \in V(T_i)$ and $i = 1, 2, \cdots$. Put

$$V(T) = \bigcup_{n=1}^{\infty} V(T_n),$$
$$E(T) = \{\{t, t'\} : \text{there is } n_0 \text{ such that } \{t, t'\} \in E(T_n) \text{ for } n \geq n_0\},$$
$$X^t = \bigcup_{n=n_1}^{\infty} X_n^t \quad \text{for } t \in V(T), \text{ where } n_1 \text{ is such that } t \in V(T_{n_1}).$$

By (5) T is a tree. Hence (T, X) is clearly a tree-decomposition of H of tree-width $< w$.

3. Proof of the Tree-Width Compactness Theorem

Let $H = (V, (W_e : e \in E))$ be a hypergraph such that every finite (and hence by the result of Section 2 every countable) subhypergraph of H has tree-width $< w$. We start with the following lemmas.

(14) **Lemma**. *Let S be a subset of V and q be any specified object. For each $A \in [V]^{\leq\omega}$ let $\mathcal{M}(A)$ be a class of tree-decompositions of $H|A$ such that $q \in V(T)$ and $X^q \cap S = \emptyset$ for every $(T, X) \in \mathcal{M}(A)$. Suppose that $\mathcal{M}(A') \subseteq \mathcal{M}(A)$ whenever $A \subseteq A' \in [V]^{\leq\omega}$. Let a subset F of $S \times S$ be defined by the rule that an element (u, v) of $S \times S$ belongs to F iff there exists $B \in [V]^{\leq\omega}$ such that u is q-tied tov in every tree-decomposition belonging to $\mathcal{M}(B)$. Then F is an equivalence relation on S.*

Proof. Let $u \in S$. If $A \in [V]^{\leq\omega}$ and $(T, X) \in \mathcal{M}(A)$ and $t', t'' \in V(T)$ and $u \in X^{t'}, u \in X^{t''}$ then $u \in X^t$ for every $t \in [t', t'']_T$ by (W3), and therefore $q \notin [t', t'']_T$ since $u \in S$ and $X^q \cap S = \emptyset$. Therefore u is q-tied to u in all tree-decompositions belonging to $\mathcal{M}(A)$, and so $(u, u) \in F$. It is obvious that if $(u, v) \in F$ then $(v, u) \in F$. Finally, suppose that $(u, v) \in F$ and $(v, w) \in F$. Then there exist $C, D \in [V]^{\leq\omega}$ such that u is q-tied to v in every tree-decompositon belonging to $\mathcal{M}(C)$ and v is q-tied to w in every tree-decomposition belonging to $\mathcal{M}(D)$. Let (T, X) be a tree-decomposition belonging to $\mathcal{M}(C \cup D \cup \{v\})$. Then $(T, X) \in \mathcal{M}(C)$ and $(T, X) \in \mathcal{M}(D)$ and so u is q-tied to v and v is q-tied to w in (T, X). Suppose that $t', t'' \in V(T)$ and $u \in X^{t'}, w \in X^{t''}$. Since (T, X) is a tree-decomposition of $H|C \cup D \cup \{v\}$, it follows that $v \in X^r$ for some $r \in V(T)$. Since u is q-tied to v and v is q-tied to w in (T, X), it follows that $q \notin [t', r]_T \cup [r, t'']_T$ and therefore $q \notin [t', t'']_T$. This proves that u is q-tied to w in every tree-decomposition belonging to $\mathcal{M}(C \cup D \cup \{v\})$, and so $(u, w) \in F$. □

(15) **Lemma**. *Let the notation be as in* (14). *If $u, v \in S$ are adjacent in H, then $(u, v) \in F$.*

Proof. Let $e \in E$ be such that $u, v \in W_e$. We claim that u is q-tied to v in every tree-decompositon belonging to $\mathcal{M}(W_e)$. Indeed, let $(T, X) \in \mathcal{M}(W_e)$ be such that u is not q-tied to v in (T, X). Then $u \in X^s, v \in X^r$ for some $s, r \in V(T)$ such that $q \in [s, r]$. By (W2) there is $t \in V(T)$ such that $W_e \subseteq X^t$,and so $q \in [t, s]$ or $q \in [r, t]$. Assume without loss of generality that the former case holds: then $u \in X^t \cap X^s \subseteq X^q$ by (W3), contradicting $X^q \cap S = \emptyset$.

Definition. If the hypotheses of (14) are satisfied, and if F is the equivalence relation described in (14), then the set of equivalence classes of F is a partition of S. This partition will be called the *partition of S determined by q and the classes $\mathcal{M}(A)(A \in [V]^{\leq \omega})$.*

We shall now construct a sequence $\mathcal{D}_0 \succ \mathcal{D}_1 \succ \mathcal{D}_2 \succ \cdots$ of divisions in V and functions $F_k : \mathcal{D}_k \to [V]^{<\omega}(k = 0, 1, 2, \cdots)$. The division $\mathcal{D}_k$ and function F_k will be defined by induction on k.

First let $\mathcal{D}_0 = \{V\}$. Choose a subset $F_0(V)$ of V maximal subject to the requirement that for every $A \in [V]^{\leq \omega}$ there exists a tree-decomposition (T, X) of $H|A$ of tree-width $< w$ such that $F_0(V) \subseteq X^t$ for some $t \in V(T)$.

Define $D(S, 0)$ to be V and $t(S, 0)$ to be the ordered pair $(V, 0)$ for every $S \in [V]^{>0} = \mathcal{D}_0^*$. If $A \in [V]^{\leq \omega}$ let $\mathcal{M}_0(A, V)$ be the class of all tree-decompositions of $H|A$ of tree-width $< w$ such that $(V, 0) \in V(T)$ and $X^{(V,0)} = F_0(V)$. Observe that if $A \in [V]^{\leq \omega}$ then, by the definition of $F_0(V)$, there exist a tree-decomposition (S, Y) of $H|A$ of tree-width $< w$ and a vertex s of S such that $F_0(V) \subseteq Y^s$. We can clearly choose (S, Y) so that $(V, 0) \notin V(S)$. If we now construct a tree T from S by adjoining a new vertex $(V, 0)$ and an edge joining $(V, 0)$ to s, and let $X^s = Y^s$ for $s \in V(S), X^{(V,0)} = F_0(V)$, then this yields a tree-decomposition $(T, X) \in \mathcal{M}_0(A, V)$, thus showing that there exists a $(T, X) \in \mathcal{M}_0(A, V)$ with $|V(T)| \geq 2$. Let $\mathcal{D}_1$ be the partition of $V \backslash F_0(V)$ determined by $(V, 0)$ and the classes $\mathcal{M}_0(A, V)$ $(A \in [V]^{\leq \omega})$. For each $R \in \mathcal{D}_1$ choose a subset $F_1(R)$ of $R \cup (N(R) \cap F_0(V))$ maximal subject to the requirement that for every $A \in [V]^{\leq \omega}$ there exists a tree-decomposition $(T, X) \in \mathcal{M}_0(A, V)$ such that $F_1(R) \subseteq X^t$ for some $t \in N_T((V, 0))$. Such a choice is possible because we have seen that $\mathcal{M}_0(A, V)$ always has at least one element (T, X) with $|V(T)| \geq 2$ and so $F_1(R)$ can at least be $\emptyset$ if no other subset of $R \cup (N(R) \cap F_0(V))$ meets our requirements. In fact $F_1(R)$ is always nonempty but we do not need this fact.

To continue our construction, for $S \in \mathcal{D}_1^*$ let $D(S, 1)$ be the member of $\mathcal{D}_1$ which contains S and $t(S, 1)$ be the ordered pair $(D(S, 1), 1)$. If $A \in [V]^{\leq \omega}$ and $R \in \mathcal{D}_1$, let $\mathcal{M}_1(A, R)$ be the class of all tree-decompositions (T, X) of $H|A$ of tree-width $< w$ such that $t(R, 0) = (V, 0), t(R, 1) = (R, 1)$

are adjacent vertices of T and $t(R,1)$ separates $t(R,0)$ from $A\cap R$ in (T,X) and $X^{t(R,0)}=F_0(V)$ and

$$X^{t(R,1)}\cap[R\cup(N(R)\cap F_0^*(R))]=F_1(R).$$

For each $R\in\mathcal{D}_1$, let $\mathcal{D}_2^R$ be the partition of $R\backslash F_1(R)$ determined by $t(R,1)$ and the classes $\mathcal{M}_1(A,R)$ $(A\in[V]^{\leq\omega})$. Let $\mathcal{D}_2=\cup\{\mathcal{D}_2^R : R\in\mathcal{D}_1\}$. For each $R\in\mathcal{D}_2$ choose a subset $F_2(R)$ of $R\cup(N(R)\cap(F_0^*(R)\cup F_1^*(R)))$ maximal subject to the requirement that for every $A\in[V]^{\leq\omega}$ there exists a tree-decomposition $(T,X)\in\mathcal{M}_1(A,D(R,1))$ such that $F_2(R)\subseteq X^t$ for some $t\in N_T(t(R,1))\backslash\{(V,0)\}$. [Such a choice is clearly possible unless there exists an $A\in[V]^{\leq\omega}$ such that $N_T(t(R,1))=\{(V,0)\}$ for every $(T,X)\in\mathcal{M}_1(A,D(R,1))$, in which case let $F_2(R)=\emptyset$.]

In general, let k be a positive integer and suppose that we have defined divisions $\mathcal{D}_0\succ\mathcal{D}_1\succ\cdots\succ\mathcal{D}_k$ in V and functions $F_i:\mathcal{D}_i\rightarrow[V]^{<\omega}$ $(i=0,1,\cdots,k)$. For each $S\in\mathcal{D}_k^*$ and each $i\in\{0,1,\cdots,k\}$ let $D(S,i)$ be the member of $\mathcal{D}_i$ which contains S and let $t(S,i)=(D(S,i),i)$. For $R\in\mathcal{D}_k^*$ we denote by $P(R,i)$ the set

$$D(R,i)\cup(N(D(R,i))\cap\bigcup_{j=0}^{i-1}F_j^*(R))\qquad(i=1,\cdots,k)$$

and put $P(R,0)=V$.

If $A\in[V]^{\leq\omega}$ and $R\in\mathcal{D}_k$ let $\mathcal{M}_k(A,R)$ be the class of all tree-decompositions (T,X) of $H|A$ of tree-width $<w$ such that the sequence

$$t(R,0)=(V,0),t(R,1),\cdots,t(R,k-1),t(R,k)=(R,k)$$

is a path-sequence in T, and

(16) $t(R,i)$ separates $(V,0)$ from $A\cap D(R,i)$ in (T,X) for $i=1,\cdots,k$, and

(17) $X^{t(R,i)}\cap P(R,i)=F_i^*(R)$ for $i=0,1,\cdots,k$.

Then it is easy to see that

(18) $\mathcal{M}_k(A',R)\subseteq\mathcal{M}_k(A,R)$ whenever $A\subseteq A'\in[V]^{\leq\omega}$.

For each $R\in\mathcal{D}_k$ let $\mathcal{D}_{k+1}^R$ be the partition of $R\backslash F_k(R)$ determined by $t(R,k)$ and the classes $\mathcal{M}_k(A,R)$ $(A\in[V]^{\leq\omega})$. Let $\mathcal{D}_{k+1}=\cup\{\mathcal{D}_{k+1}^R : R\in\mathcal{D}_k\}$. For each $R\in\mathcal{D}_{k+1}$, choose a subset

$F_{k+1}(R)$ of $P(R,k+1) := R \cup (N(R) \cap \bigcup_{j=0}^{k} F_j^*(R))$ maximal subject to the requirement that for every $A \in [V]^{\leq\omega}$ there exists a tree-decomposition $(T,X) \in \mathcal{M}_k(A, D(R,k))$ such that $F_{k+1}(R) \subseteq X^t$ for some $t \in N_T(t(R,k)) \backslash \{t(R,k-1)\}$. [Such a choice is clearly possible unless there exists an $A \in [V]^{\leq\omega}$ such that $N_T(t(R,k)) = \{t(R,k-1)\}$ for every $(T,X) \in \mathcal{M}_k(A, D(R,k))$, in which case let $F_{k+1}(R) = \emptyset$.] Having thus defined $\mathcal{D}_k, F_k$ for $k = 0, 1, \cdots$, we let

$$V(\hat{T}) = (\mathcal{D}_0 \times \{0\}) \cup (\mathcal{D}_1 \times \{1\}) \cup (\mathcal{D}_2 \times \{2\}) \cup \cdots$$

$$E(\hat{T}) = \{\{(R,k),(S,k+1)\} : k < \omega, R \in \mathcal{D}_k, S \in \mathcal{D}_{k+1}, R \supseteq S\}$$

$$\hat{X}^{(R,k)} = F_k(R) \; for \; (R,k) \in V(\hat{T})$$

$$\hat{X} = (\hat{X}^{(R,k)} : (R,k) \in V(\hat{T})).$$

We shall show that $(\hat{T}, \hat{X})$ is a tree-decomposition of H of tree-width $< w$. Clearly its tree-width is $< w$, and so it remains to show that it satisfies (W1)-(W3). We proceed in a series of claims.

If u, v are distinct vertices of a tree T then $V_{T-u}(v)$ will denote the set of vertices of the component of $T - u$ which includes v.

(19) **Claim**. *If $k < \omega, R \in \mathcal{D}_k$ and $A \in [V]^{\leq\omega}$ then $\mathcal{M}_k(A,R) \neq \emptyset$.*

Proof. We proceed by induction on k. For $k = 0$, we have $\mathcal{M}_0(A,V) \neq \emptyset$ since we showed above that there exists $(T,X) \in \mathcal{M}_0(A,V)$ with $|V(T)| \geq 2$. Now assume the inductive hypothesis that $\mathcal{M}_k(A', R')$ is non-empty for all $A' \in [V]^{\leq\omega}, R' \in \mathcal{D}_k$; and let $A \in [V]^{\leq\omega}, R \in \mathcal{D}_{k+1}$. We will show that $\mathcal{M}_{k+1}(A,R) \neq \emptyset$. To this end, choose a set $B \in [V]^{\leq\omega}$ such that

(20) $A \cup F_0^*(R) \cup F_1^*(R) \cup \cdots \cup F_k^*(R) \cup F_{k+1}(R) \subseteq B$,

(21) for every $v \in F_{k+1}(R) \cap N(R)$ there is an $e \in E$ such that $v \in W_e \subseteq B$ and $W_e \cap R \neq \emptyset$,

(22) $B \cap R \neq \emptyset$.

(Of course, (22) is an automatic consequence of (21) if $F_{k+1}(R) \cap N(R) \neq \emptyset$.) Let $Q = D(R,k)$. We note that $t(R,k) = t(Q,k) = (Q,k)$: denote this by q. For any $(S,Y) \in \mathcal{M}_k(G,Q)$, where $G \in [V]^{\leq\omega}$, let N^S denote $N_S(q) \backslash \{t(Q,k-1)\}$ if $k > 0$ and $N_S(q)$ if $k = 0$. Since $R \in \mathcal{D}_{k+1}$ it follows that R belongs to the partition $\mathcal{D}_{k+1}^Q$ of $Q \backslash F_k(Q)$ determined by q and the classes $\mathcal{M}_k(G,Q)$ $(G \in [V]^{\leq\omega})$. Therefore for any $u, v \in R$ there exists $C(u,v) \in [V]^{\leq\omega}$ such that u is

q-tied to v in every tree-decomposition belonging to $\mathcal{M}_k(C(u,v),Q)$. By the inductive hypothesis, $\mathcal{M}_k(B,Q) \neq \emptyset$ and so we can choose an $(I,\Lambda) \in \mathcal{M}_k(B,Q)$. If we construct a tree J from I by adjoining a new vertex a and an edge joining a to q and if we let $\Gamma^v = \Lambda^v$ for $v \in V(I), \Gamma^a = \emptyset$ then this shows that $\mathcal{M}_k(B,Q)$ has an element (J,Γ) such that $N^J \neq \emptyset$. This and the definition of $F_{k+1}(R)$ imply that

(i) for every $G \in [V]^{\leq\omega}$ there exist $(S,Y) \in \mathcal{M}_k(G,Q)$ and $s \in N^S$ such that $F_{k+1}(R) \subseteq Y^s$,

(ii) for every $v \in P(R,k+1) \backslash F_{k+1}(R)$ there exists $A(v) \in [V]^{\leq\omega}$ such that for every $(S,Y) \in \mathcal{M}_k(A(v),Q)$ and every $s \in N^S$ we have $\{v\} \cup F_{k+1}(R) \not\subseteq Y^s$.

Let

$$C = B \cup \bigcup\{A(v) : v \in B \cap P(R,k+1)\backslash F_{k+1}(R)\} \cup \bigcup\{C(u,v) : u,v \in B \cap R\}.$$

By (i), we can choose $(T,X) \in \mathcal{M}_k(C,Q)$ and $t_1 \in N^T$ such that $F_{k+1}(R) \subseteq X^{t_1}$.

Since $Q = D(R,k)$, it follows that $D(Q,i) = D(R,i)$ therefore $t(Q,i) = t(R,i)$ for $i = 0, \cdots, k$. Since $(T,X) \in \mathcal{M}_k(B,Q)$ by (18), we infer from (17) that

(23) $F_i^*(R) = F_i^*(Q) \subseteq X^{t(Q,i)} = X^{t(R,i)} (i = 0, \cdots, k)$.

By (18), $(T,X) \in \mathcal{M}_k(C(u,v),Q)$ for all $u,v \in B \cap R$ and so u is q-tied to v in (T,X) for all $u,v \in B \cap R$. Therefore there exists $t_2 \in N_T(q)$ such that all vertices t of T for which $X^t \cap B \cap R \neq \emptyset$ belong to $V_{T-q}(t_2)$ (and there is at least one such t by (W1) since $\emptyset \neq B \cap R \subseteq C = V(H|C)$). Since $(T,X) \in \mathcal{M}_k(B,Q)$ by (18), it follows that q separates $(V,0)$ from $B \cap Q$ in (T,X) and so, if $k > 0$, then $X^t \cap B \cap R = \emptyset$ for every $t \in V_{T-q}(t(Q,k-1))$ and therefore $t_2 \neq t(Q,k-1)$. Hence $t_2 \in N^T$. Let $Z = (Z^t : t \in V(T))$, where $Z^t = B \cap X^t$ for every $t \in V(T)$. If $R \cap F_{k+1}(R) \neq \emptyset$ then (since $F_{k+1}(R) \subseteq B$ and $F_{k+1}(R) \subseteq X^{t_1}$) we have $X^{t_1} \cap B \cap R \neq \emptyset$ and so $t_1 = t_2$ and therefore $F_{k+1}(R) \subseteq X^{t_2}$. If $R \cap F_{k+1}(R) = \emptyset$ then $F_{k+1}(R) \subseteq N(R) \cap \bigcup_{i=0}^{k} F_i^*(R)$. Therefore if $v \in F_{k+1}(R)$ then by (21) $v \in W_e \subseteq B$ and $W_e \cap R \neq \emptyset$ for some $e \in E$ and $W_e \subseteq X^r$ for some $r \in V(T)$ by (W2), so that $X^r \cap B \cap R \neq \emptyset$ and therefore $r \in V_{T-q}(t_2)$ and so

$$v \in X^r \cap \bigcup_{i=0}^{k} F_i^*(R) \subseteq X^r \cap \bigcup_{i=0}^{k} X^{t(R,i)} \subseteq X^{t_2}$$

by (23) and (W3). Hence, whether $R \cap F_{k+1}(R) = \emptyset$ or not, we have $F_{k+1}(R) \subseteq X^{t_2}$. Moreover, if $v \in B \cap P(R, k+1) \backslash F_{k+1}(R)$ then $(T, X) \in \mathcal{M}_k(A(v), Q)$ by (18) and so $\{v\} \cup F_{k+1}(R) \not\subseteq X^{t_2}$ by (ii) and therefore $v \notin X^{t_2}$. It follows that $B \cap P(R, k+1) \cap X^{t_2} = F_{k+1}(R)$, i.e. $Z^{t_2} \cap P(R, k+1) = F_{k+1}(R)$. It can now be checked that the tree-decomposition (T, Z) of $H|B$ becomes an element of $\mathcal{M}_{k+1}(B, R)$ if we replace the vertex t_2 of T by $t(R, k+1)$ (or interchange $t_2, t(R, k+1)$ if $t(R, k+1)$ is already a vertex of T). Therefore $\mathcal{M}_{k+1}(B, R) \neq \emptyset$ and so $\mathcal{M}_{k+1}(A, R) \neq \emptyset$ by (18).

□

Now we shall assume that $(\hat{T}, \hat{X})$ does not satisfy (W1) and obtain a contradiction. So let

$$V_0 = \bigcup_{k=0}^{\infty} \bigcup_{R \in \mathcal{D}_k} F_k(R).$$

Then $V_0 = \bigcup_{(R,k) \in V(\hat{T})} \hat{X}^{(R,k)}$ and so our assumption that $(\hat{T}, \hat{X})$ does not satisfy (W1) implies that $V_0 \neq V$. Let V_1 be a maximal subset of $V \backslash V_0$ satisfying

(24) for any two vertices $u, v \in V_1$ there are vertices $u_0 = u, u_1, \cdots, u_n = v \in V_1$ and edges $e_1, \cdots, e_n \in E$ such that $u_{i-1}, u_i \in W_{e_i} (i = 1, \cdots, u)$.

Then $V_1 \neq \emptyset$ since $V_0 \neq V$. It follows from the maximality of V_1 that

(25) *If v_1, v are adjacent in H and $v_1 \in V_1$ then $v \in V_0 \cup V_1$.*

(26) **Claim**. *For every $k < w$ the set V_1 belongs to $\mathcal{D}^*_k$.*

Proof. We proceed by induction. For $k = 0$ the statement is true since $V_1 \neq \emptyset$: so assume that $V_1 \subseteq R$ for some $R \in \mathcal{D}_k$ and consider the partition $\mathcal{D}^R_{k+1}$ of $R \backslash F_k(R)$ determined by $t(R, k)$ and the classes $\mathcal{M}_k(A, R) (A \in [V]^{\leq \omega})$. If $u, v \in V_1$ are adjacent in H, then they belong to the same element of $\mathcal{D}^R_{k+1}$ by (15), and, since V_1 is assumed to satisfy (24), the same is true for any two (not necessarily adjacent) elements of V_1. □

It follows from (26) that the expressions $D(V_1, k)$ and $F^*_k(V_1)$ make sense for every $k < \omega$.

(27) **Claim**. *If $v_0 \in V_0$ is adjacent to some vertex of V_1, then there exists $k(v_0) < \omega$ such that $v_0 \in F^*_k(V_1)$ for every $k \geq k(v_0)$.*

Proof. Let v_0 be adjacent to $v_1 \in V_1$, say $\{v_0, v_1\} \subseteq W_e$ for $e \in E$. Let p be the least integer such that $v_0 \in F_p(R)$ for some $R \in \mathcal{D}_p$, then

$$v_0 \in F_p(R) \setminus \bigcup_{i=0}^{p-1} F_i^*(R) \subseteq P(R,p) \setminus \bigcup_{i=0}^{p-1} F_i^*(R) \subseteq R.$$

We claim that $R = D(V_1, p)$. Indeed, $D(R,0) = V = D(V_1,0)$ and assuming that $D(R,\ell) = D(V_1,\ell)$ for some $\ell < p$ we have $v_0 \in R \subseteq D(R,\ell+1)$ and $v_1 \in V_1 \subseteq D(V_1,\ell+1)$ and hence $D(R,\ell+1) = D(V_1,\ell+1)$ by (15). Thus we have $R = D(R,p) = D(V_1,p)$.

We put $k(v_0) = p+1$. Now let $k \geq k(v_0)$ be given. In order to show that $v_0 \in F_k^*(V_1)$ we can by (19) choose a tree-decomposition $(T,X) \in \mathcal{M}_k(W_e, D(V_1,k))$. By (W2) there is a $t \in V(T)$ such that $\{v_0, v_1\} \subseteq W_e \subseteq X^t$.

Since $(T,X) \in \mathcal{M}_k(W_e, D(V_1,k))$ it follows that

(i) $t(V_1,0) = (V,0), t(V_1,1), t(V_1,2), \cdots, t(V_1,p) = (R,p), t(V_1,p{+}1), \cdots, t(V_1,k)$ is a path-sequence in T,

(ii) $t(V_1,k)$ separates $(V,0)$ from $W_e \cap D(V_1,k)$ in (T,X),

(iii)
$$X^{(R,p)} \cap P(D(V_1,k),p) = X^{t(D(V_1,k),p)} \cap P(D(V_1,k),p) = F_p^*(D(V_1,k)) = F_p(R),$$

(iv)
$$X^{t(V_1,k)} \cap P(D(V_1,k),k) = X^{t(D(V_1,k),k)} \cap P(D(V_1,k),k) = F_k^*(D(V_1,k),k) = F_k^*(V_1).$$

Since $v_1 \in W_e \subseteq X^t$ and $v_1 \in V_1 \subseteq D(V_1,k)$, it follows from (ii) that $t(V_1,k) \in [(V,0),t]$ and so, by (i), $t(V_1,k) \in [(R,p),t]$ and therefore $X^{(R,p)} \cap X^t \subseteq X^{t(V_1,k)}$ by (W3). Moreover $F_p(R) = F_p(D(V_1,k)) = F_p^*(V_1)$ and $F_p(R) \subseteq X^{(R,p)}$ by (iii). Furthermore

$$N(D(V_1,k)) \cap \bigcup_{j=1}^{k-1} F_j^*(V_1) = N(D(V_1,k)) \cap \bigcup_{j=1}^{k-1} F_j^*(D(V_1,k)) \subseteq P(D(V_1,k),k).$$

Therefore

$$v_0 \in F_p(R) \cap X^t \cap N(V_1) \subseteq X^{(R,p)} \cap X^t \cap N(D(V_1,k)) \cap \bigcup_{j=0}^{k-1} F_j^*(V_1)$$
$$\subseteq X^{t(V_1,k)} \cap P(D(V_1,k),k) = F_k^*(V_1)$$

by (iv). □

From (27) it follows that

(28) *there are at most w vertices $v_1, \cdots, v_m \in V_0$ which are adjacent to a vertex from V_1.*

We put

$$k_0 = max(k(v_1), \cdots k(v_m)) + 1.$$

(29) **Claim.** *For every $k \geq k_0$ the set V_1 belongs to $\mathcal{D}_k$.*

Proof. Let $k \geq k_0$. Suppose that $V_1 \notin \mathcal{D}_k$. From this and the fact that $V_1 \in \mathcal{D}_k^*$ by (26), it follows that we can choose $u \in V_1, v \in D(V_1,k) \backslash V_1$. Let $R = D(V_1, k-1), q = t(R, k-1)$. Let $A \in [V]^{\leq \omega}$. Let $A \cup \{u, v\} = C$. We shall construct a tree-decomposition $(S, Y) \in \mathcal{M}_{k-1}(C, R)$ such that u is not q-tied to v in (S, Y).

By (19), we can choose a tree-decomposition $(T, X) \in \mathcal{M}_k(C, D(V_1, k))$. Let B be the tree $T|V_{T-q}(t(V_1,k))$, let B' be an isomorphic copy of B such that $V(B') \cap V(T) = \emptyset$ and let $t \longmapsto t'$ be the corresponding isomorphism. Let S be the tree obtained from T by adjoining the tree B' and an edge joining q to $t(V_1,k)'$. We put

$$Y^t = \begin{cases} X^t \cap C & \text{for } t \in V(T) \backslash V(B) \\ X^t \cap C \cap (V_1 \cup \{v_1, \cdots, v_m\}) & \text{for } t \in V(B) \end{cases}$$
$$Y^{t'} = X^t \cap C \backslash V_1 \quad \text{for} \quad t \in V(B)$$

$$Y = (Y^s : s \in V(S)).$$

We claim that (S, Y) in the desired tree-decomposition. Clearly (S, Y) satisfies (W1). To prove that (S, Y) satisfies (W2) let e be an edge of $H|C$. Then $W_e \subseteq C$ and, since (X, T) satisfies (W2), $W_e \subseteq X^t$ for some $t \in V(T)$. Therefore $W_e \subseteq X^t \cap C$. If $t \notin V(B)$ then $W_e \subseteq X^t \cap C = Y^t$. If $t \in V(B)$ and $W_e \cap V_1 \neq \emptyset$ then $W_e \subseteq V_1 \cup \{v_1, \cdots, v_m\}$ by (25) and (28) and so $W_e \subseteq Y^t$. If $t \in V(B)$ and $W_e \cap V_1 = \emptyset$ then $W_e \subseteq Y^{t'}$.

To prove (W3) let $s_1, s_2, s_3 \in V(S)$ be such that s_2 is on the path between s_1 and s_3 in S. For $i = 1, 2, 3$ let $t_i \in V(T)$ be such that $t_i = s_i$ if $s_i \in V(T)$ and $t_i' = s_i$ otherwise.

If $\{s_1, s_3\} \subseteq V(T)$ or $\{s_1, s_3\} \subseteq (V(T)\backslash V(B)) \cup V(B')$ then $X^{t_1} \cap X^{t_3} \subseteq X^{t_2}$ because (T, X) satisfies (W3) and from this it easily follows that $Y^{t_1} \cap Y^{t_3} \subseteq Y^{t_2}$. We may therefore assume that $s_1 \in V(B), s_2 \in V(B) \cup V(B') \cup \{q\}$ and $s_3 \in V(B')$.

Since $(T, X) \in \mathcal{M}_k(C, D(V_1, k))$, it follows from (17) that $F^*_{k-1}(D(V_1, k)) \subseteq X^{t(D(V_1,k),k-1)}$, i.e. $F^*_{k-1}(V_1) \subseteq X^q$. Moreover for $i = 1, \cdots, m$ we have $k - 1 \geq k_0 - 1 \geq k(v_i)$ and so $v_i \in F^*_{k-1}(V_1)$. Therefore $\{v_1, \cdots, v_m\} \subseteq X^q$ and so, since (T, X) satisfies (W3), we have

$$\begin{aligned} Y^{s_1} \cap Y^{s_3} &\subseteq (\{v_1, \cdots, v_m\}\backslash V_1) \cap C \cap X^{t_1} \cap X^{t_3} \\ &\subseteq (\{v_1, \cdots, v_m\}\backslash V_1) \cap C \cap X^{t_1} \cap X^{t_3} \cap X^q \\ &\subseteq (\{v_1, \cdots, v_m\}\backslash V_1) \cap C \cap X^{t_2} \subseteq Y^{t_2}, \end{aligned}$$

which completes the proof that (S, Y) satisfies (W3).

Thus (S, Y) is a tree-decomposition of $H|A$ and since $(T, X) \in \mathcal{M}_k(C, D(V_1, k))$ it is not hard to see that $(S, Y) \in \mathcal{M}_{k-1}(C, R)$. By (W1) applied to (T, X) there exist vertices $t(u), t(v)$ of T such that $u \in X^{t(u)}, v \in X^{t(v)}$. Since $(T, X) \in \mathcal{M}_k(C, D(V_1, k))$ it follows that $t(V_1, k)$ separates $(V, 0)$ from $C \cap D(V_1, k)$ in (T, X). From this and the fact that $u \in X^{t(u)} \cap C \cap V_1 \subseteq X^{t(u)} \cap C \cap D(V_1, k)$ and $v \in X^{t(v)} \cap C \cap D(V_1, k)$ it follows that $t(u), t(v) \in V(B)$. Since $u \in V_1$ and $v \notin V_1$ it follows that $u \in Y^{t(u)}, v \in Y^{t(v)'}$ and so u is not q-tied to v in (S, Y). Since $(S, Y) \in \mathcal{M}_{k-1}(C, R) \subseteq \mathcal{M}_{k-1}(A, R)$ by (18), it is not true that u is q-tied to v in every tree-decomposition belonging to $\mathcal{M}_{k-1}(A, R)$. Since A was arbitrary, this shows that u, v cannot belong to the same member of the partition of $R\backslash F_{k-1}(R)$ determined by the classes $\mathcal{M}_{k-1}(A, R)(A \in [V]^{\leq\omega})$, which contradicts the fact that $u, v \in D(V_1, k)$. Thus the assumption that $V_1 \notin \mathcal{D}_k$ leads to a contradiction, and (29) is proved.

Since we have seen that $V_1 \neq \emptyset$, we can choose a particular $u_0 \in V_1$. For $u_0 \in A \in [V]^{\leq\omega}$ we let

$$d(A) = min\{card[(V, 0), t]_T : (T, X) \in \mathcal{M}_{k_0}(A, V_1), t \in V(T), u_0 \in X^t\}$$

(30) **Claim**. *There exists a constant $K < \omega$ such that $d(A) \leq K$ for every $A \in [V]^{\leq\omega}$ such that $u_0 \in A$.*

Proof. By (18),

(31) $d(A) \leq d(B)$ whenever $u_0 \in A \subseteq B \in [V]^{\leq\omega}$.

If (30) is false then for every $n < \omega$ there exists $A_n \in [V]^{\leq\omega}$ such that $u_0 \in A_n$ and $d(A_n) > n$. But then (31) implies that $d(A_0 \cup A_1 \cup A_2 \cup \cdots) > n$ for every $n < \omega$, which is impossible . □

For $k = k_0, k_0 + 1, \cdots$ and $A \in [V]^{\leq\omega}$ we let $\mathcal{N}_k(A) = \{(T, X) \in \mathcal{M}_k(A, V_1) :$ $card\ [(V, 0), t]_T \leq K$ for some $t \in V((T)$ such that $u_0 \in X^t\}$.

Clearly

(32) $\mathcal{N}_k(A') \subseteq \mathcal{N}_k(A)$ for $A \subseteq A' \in [V]^{\leq\omega}$.

(33) **Claim.** *If $k \geq k_0$ and $u_0 \in A \in [V]^{\leq\omega}$ then $\mathcal{N}_k(A) \neq \emptyset$.*

Proof. We proceed by induction on k. For $k = k_0$ the statement follows from (30). Now assume the inductive hypothesis that $\mathcal{N}_k(A')$ is non-empty for all $A' \in [V]^{\leq\omega}$ such that $u_0 \in A'$; and let $u_0 \in A \in [V]^{\leq\omega}$. We will show that $\mathcal{N}_{k+1}(A) \neq \emptyset$. To this end, choose a set $B \in [V]^{\leq\omega}$ such that

(34) $A \cup F_0^*(V_1) \cup F_1^*(V_1) \cup \cdots \cup F_k^*(V_1) \cup F_{k+1}(V_1) \subseteq B$,

(35) for every $i = 1, \cdots, m$ there exits an edge $e_i \in E$ such that $v_i \in W_{e_i} \subseteq B$ and $W_{e_i} \cap V_1 \neq \emptyset$,

(36) $B \cap V_1 \neq \emptyset$.

(Of couse, (36) is an automatic consequence of (35) if $m > 0$ but in any case B can be chosen so as to satisfy (36) since $V_1 \neq \emptyset$.) We note that $t(V_1, k) = (V_1, k)$: denote this by q. Since $V_1 \in \mathcal{D}_{k+1}$ it follows that V_1 belongs to the partition $\mathcal{D}_{k+1}^{V_1}$ of $V_1 = V_1 \backslash F_k(V_1)$ determined by q and the classes $\mathcal{M}_k(G, V_1)(G \in [V]^{\leq\omega})$. Therefore for any $u, v \in V_1$ there exists $C(u, v) \in [V]^{\leq\omega}$ such that u is q-tied to v in every tree-decomposition belonging to $\mathcal{M}_k(C(u, v), V_1)$. From the definiton of $F_{k+1}(V_1)$ it follows that

(37) for every $v \in P(V_1, k + 1)\backslash F_{k+1}(V_1)$ there exists $A(v) \in [V]^{\leq\omega}$ such that for every $(S, Y) \in \mathcal{M}_k(A(v), V_1)$ and every $s \in N_S(q)\backslash\{t(V_1, k - 1)\}$ we have $\{v\} \cup F_{k+1}(V_1) \not\subseteq Y^s$. Let

$$C = B \cup \bigcup\{A(v) : v \in B \cap P(V_1, k + 1)\backslash F_{k+1}(V_1)\} \cup \bigcup\{C(u, v) : u, v \in B \cap V_1\}.$$

By the inductive hypothesis, $\mathcal{N}_k(C) \neq \emptyset$, so we can choose $(T, X) \in \mathcal{N}_k(C)$. By (18), $(T, X) \in \mathcal{M}_k(C(u, v), V_1)$ for all $u, v \in B \cap V_1$ and so u is q-tied to v in (T, X) for all $u, v \in B \cap V_1$. Therefore there exists $t_2 \in N_T(q)$ such that all vertices t of T for which $X^t \cap B \cap V_1 \neq \emptyset$ belong to $V_{T-q}(t_2)$ (and there is at least one such t by (W1) since $\emptyset \neq B \cap V_1 \subseteq C = V(H|C)$). Since $(T, X) \in \mathcal{M}_k(B, V_1)$ by (18), it follows that q separates $(V, 0)$ from $B \cap V_1$ in (T, X) and so $X^t \cap B \cap V_1 = \emptyset$ for every $t \in V_{T-q}(t(V_1, k-1))$ and therefore $t_2 \neq t(V_1, k-1)$. Let $Z = (Z^t; t \in V(T))$, where $Z^t = B \cap X^t$ for every $t \in V(T)$. For $i = 1, \cdots, m$ we have $v_i \in F_k^*(V_1)$ since $k \geq k_0 > k(v_i)$ and there is by (35) and edge $e_i \in E$ such that $v_i \in W_{e_i} \subseteq B$ and $W_{e_i} \cap V_1 \neq \emptyset$. Then $W_{e_i} \subseteq X^{r_i}$ for some $r_i \in V(T)$ by (W2), so that $X^{r_i} \cap B \cap V_1 \neq \emptyset$ and therefore $r_i \in V_{T-q}(t_2)$ and so

$$v_i \in X^{r_i} \cap \bigcup_{j=0}^{k} F_j^*(V_1) \subseteq X^{r_i} \cap \bigcup_{j=0}^{k} X^{t(V_1, j)} \subseteq X^{t_2} \quad \text{by (W3)}.$$

Hence

$$F_{k+1}(V_1) \subseteq P(V_1, k+1) \cap V_0 \subseteq N(V_1) \cap V_0 = \{v_1, \cdots, v_m\} \subseteq X^{t_2}.$$

Moreover, if $v \in B \cap P(V_1, k+1) \backslash F_{k+1}(V_1)$ then $(T, X) \in \mathcal{M}_k(A(v), V_1)$ by (18) and so $\{v\} \cup F_{k+1}(V_1) \not\subseteq X^{t_2}$ by (37) and therefore $v \notin X^{t_2}$. It follows that $B \cap P(V_1, k+1) \cap X^{t_2} = F_{k+1}(V_1)$, i.e. $Z^{t_2} \cap P(V_1, k+1) = F_{k+1}(V_1)$. It can now be checked that the tree-decomposition (T, Z) of $H|B$ becomes an element of $\mathcal{N}_{k+1}(B)$ if we replace the vertex t_2 of T by $t(V_1, k+1)$ (or interchange $t_2, t(V_1, k+1)$ if $t(V_1, k+1)$ is already a vertex of T). Therefore $\mathcal{N}_{k+1}(B) \neq \emptyset$, and so $\mathcal{N}_{k+1}(A) \neq \emptyset$ by (32). □

Now we are ready to complete the proof of the fact that $(\hat{T}, \hat{X})$ satisfies (W1). Let us take $(T, X) \in \mathcal{N}_{k_0+K}(\{u_0\})$ and let $t \in V(T)$ be such that $u_0 \in X^t$ and *card* $[(V, 0), t] \leq K$. Since $t(V_1, k_0 + K)$ separates $(V, 0)$ from $\{u_0\}$ in (T, X) we have $t(V_1, k_0 + K) \in [t, (V, 0)]$, which implies *card*$[(V, 0), t] \geq$ *card* $[(V, 0), t(V_1, k_0 + K)] \geq K + 1$, a contradiction. Hence $(\hat{T}, \hat{X})$ satisfies (W1).

To prove that $(\hat{T}, \hat{X})$ satisfies (W2) let $e \in E$ and let $W_e = \{u_1, \cdots, u_n\}$. We may assume that $n \geq 1$, since otherwise $W_e = \emptyset \subseteq \hat{X}^{(V,0)}$. For $i = 1, \cdots, n$ let s_i be the least integer such that $u_i \in F_{s_i}(R_i)$ for some $R_i \in \mathcal{D}_{s_i}$ (there is at least one such integer since we have shown that $(\hat{T}, \hat{X})$ satisfies (W1)). We may assume without loss of generality that $s_1 \geq s_2 \geq \cdots \geq s_n$. Since $F_{s_i}(R_i) \subseteq R_i \cup \bigcup_{j=0}^{s_i-1} F_j^*(R_i)$, it follows from the minimality of s_i that $u_i \in R_i$. We will show that $W_e \subseteq \hat{X}^{(R_1, s_1)}$. By (19) we can choose $(T, X) \in \mathcal{M}_{s_1}(W_e, R_1)$ arbitrarily. By (W2) there is a $t \in V(T)$ such that $W_e \subseteq X^t$. Let $i \in \{2, \cdots, n\}$. Clearly $D(R_1, 0) = V = D(R_i, 0)$. If $D(R_1, p) = D(R_i, p)$ for some $p < s_i$, then, since $u_1 \in R_1 \subseteq D(R_1, p+1), u_i \in R_i \subseteq D(R_i, p+1)$, we have $D(R_1, p+1) = D(R_i, p+1)$ by (15). Hence $D(R_1, s_i) = D(R_i, s_i) = R_i$. Thus $t(R_i, s_i) = (R_i, s_i) \in V(T)$ and $u_i \in F_{s_i}(R_i) \subseteq X^{t(R_i, s_i)}$ and $u_i \in F_{s_i}(R_i) = F_{s_i}(D(R_1, s_i)) = F^*_{s_i}(R_1)$. Since (R_1, s_1) separates $(V, 0)$ from $W_e \cap R_1$ we have

$$(R_1, s_1) \in [t, t(V, 0)]_T,$$

hence

$$(R_1, s_1) \in [t, (R_i, s_i)]_T \qquad (i = 1, \cdots, n).$$

Now using (W3) we get for $s_i < s_1$

$$\begin{aligned} u_i \in X^t \cap X^{(R_i, s_i)} \cap N(R_1) \cap F^*_{s_i}(R_1) &\subseteq X^{(R_1, s_1)} \cap N(R_1) \cap \bigcup_{j=0}^{s_1 - 1} F^*_j(R_1) \\ &\subseteq X^{(R_1, s_1)} \cap P(R_1, s_1) = F_{s_1}(R_1) = \hat{X}^{(R_1, s_1)} \end{aligned}$$

and for $s_i = s_1$ clearly $u_i \in F_{s_1}(R_1) = \hat{X}^{(R_1, s_1)}$. Hence $W_e \subseteq \hat{X}^{(R_1, s_1)}$, as desired.

To prove that $(\hat{T}, \hat{X})$ satisfies (W3) let $u \in \hat{X}^{(R,r)} \cap \hat{X}^{(S,s)} = F_r(R) \cap F_s(S)$ for $R \in \mathcal{D}_r, S \in \mathcal{D}_s$. Let p be maximal such that $D(R, p) = D(S, p)$. We may assume without loss of generality that $r > p$, for if $r = p = s$ then there is nothing to prove. We have $u \in F_r(R) \subseteq (R \cup \bigcup_{j=0}^{r-1} F^*_j(R)) \subseteq (D(R, r-1) \cup \bigcup_{j=0}^{r-2} F^*_j(R)) \subseteq \cdots \subseteq (D(R, p+1) \cup \bigcup_{j=0}^{p} F^*_j(R))$.

If $s > p$ then a similar argument gives $u \in D(S, p+1) \cup \bigcup_{j=0}^{p} F^*_j(S) = D(S, p+1) \cup \bigcup_{j=0}^{p} F^*_j(R)$ and therefore $u \in \bigcup_{j=0}^{p} F^*_j(R)$ since $D(R, p+1) \cap D(S, p+1) = \emptyset$. If $s = p$ then $u \in F_s(S) = F_p(D(S, p)) = F_p(D(R, p)) = F^*_p(R) \subseteq \bigcup_{j=0}^{p} F^*_j(R)$.

Since $u \in \bigcup_{j=0}^{p} F^*_j(R), u \in F_r(R) \subseteq R \cup (N(R) \cap \bigcup_{j=0}^{r-1} F^*_j(R))$ and $R \cap \bigcup_{j=0}^{r-1} F^*_j(R) = \emptyset$ it follows that $u \in N(R)$. Let $i \in \{0, 1, \cdots, p\}$ be such that $u \in F^*_i(R)$. To conclude the proof it is sufficient to show that $u \in \hat{X}^{t(R,k)}$ for any $k \in \{i+1, \cdots, r\}$, the proof for S being similar. So let $k \in \{i+1, \cdots r\}$ and for an arbitrarily chosen $A \in [V]^{\leq \omega}$ we can by (19) take a tree-decomposition

$(T, X) \in \mathcal{M}_r(A, R)$. We have $u \in F_r(R) \cap F_i^*(R) \cap N(R) \subseteq X^{t(R,r)} \cap X^{t(R,i)} \cap N(R) \cap \bigcup_{j=0}^{i} F_j^*(R) \subseteq X^{t(R,k)} \cap N(D(R,k)) \cap \bigcup_{j=0}^{k-1} F_j^*(R) \subseteq F_k^*(R) = \hat{X}^{t(R,k)}$,which completes the proof of the theorem.

References

1. I. Kříž and R. Thomas, The menger-like property of the tree-width of infinite graphs and related compactness results, submitted.

2. J. B.Kruskal, Well-quasi-ordering, the Tree Theorem and Vaszonyi's Conjecture, Trans. Amer. Math. 95 (1960), 210-225.

3. C. St. J. A. Nash-Williams, On well-quasi-ordering finite trees, Proc. Cambridge Philos. Soc. 59 (1963), 833-835.

4. N. Robertson and P. D. Seymour, Graph Minors III. Planar tree-width, J. Combin. Theory Ser. B 36 (1984), 49-69.

5. N. Robertson and P. D. Seymour, Graph Minors IV. Tree-width and well-quasi-ordering, submitted.

6. R. Thomas, Well-quasi-ordering infinite graphs with forbidden finite planar minor, to appear in Trans. Amer. Math. Soc.

7. C. Thomassen, Configurations in graphs with large degree, connectivity or chromatic number, preprint.