FIVE-LIST-COLORING GRAPHS ON SURFACES I. TWO LISTS OF SIZE TWO IN PLANAR GRAPHS

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ABSTRACT

Let G be a plane graph with outer cycle C, let $v_1, v_2 \in V(C)$ and let $(L(v) : v \in V(G))$ be a family of sets such that $|L(v_1)| = |L(v_2)| = 2$, $|L(v)| \geq 3$ for every $v \in V(C) \setminus \{v_1, v_2\}$ and $|L(v)| \geq 5$ for every $v \in V(G) \setminus V(C)$. We prove a conjecture of Hutchinson that G has a (proper) coloring ϕ such that $\phi(v) \in L(v)$ for every $v \in V(G)$. We will use this as a lemma in subsequent papers.

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1 Introduction

All graphs in this paper are finite and simple. A *list-assignment* for a graph G is a family of non-empty sets $L = (L(v) : v \in V(G))$. An *L-coloring* of G is a (proper) coloring ϕ such that $\phi(v) \in L(v)$ for all $v \in V(G)$. A graph is *L-colorable* if it has at least one L-coloring. A graph G is *k-choosable*, also called *k-list-colorable*, if G has an L-coloring for every list-assignment L for G such that $|L(v)| \geq k$ for every $v \in V(G)$. List coloring was introduced and first studied by Vizing [10] and Erdős, Rubin and Taylor [4].

Clearly every k-choosable graph is k-colorable, but the converse is false. One notable example of this is that the Four-Color Theorem does not generalize to list-coloring. Indeed Voigt [11] constructed a planar graph that is not 4-choosable. On the other hand Thomassen [8] proved the following remarkable theorem with an outstandingly short proof.

Theorem 1.1 (Thomassen) *Every planar graph is 5-choosable.*

Actually, Thomassen [8] proved a stronger theorem.

Theorem 1.2 (Thomassen) *If G is a plane graph with outer cycle C, $P = p_1p_2$ is a subpath of C of length one, and L is a list assignment for G such that $|L(v)| \geq 5$ for all $v \in V(G) \setminus V(C)$, $|L(v)| \geq 3$ for all $v \in V(C) \setminus V(P)$, $|L(p_1)| = |L(p_2)| = 1$ and $L(p_1) \neq L(p_2)$, then G is L-colorable.*

Hutchinson [5] conjectured the following variation of Theorem 1.2, which is the main result of this paper:

Theorem 1.3 *If G is a plane graph with outer cycle C, $v_1, v_2 \in V(C)$ and L is a list assignment for G with $|L(v)| \geq 5$ for all $v \in V(G) \setminus V(C)$, $|L(v)| \geq 3$ for all $v \in V(C) \setminus \{v_1, v_2\}$, and $|L(v_1)| = |L(v_2)| = 2$, then G is L-colorable.*

Hutchinson [5] proved Theorem 1.3 for outerplanar graphs. In fact, Theorem 1.3 implies Theorem 1.2, as we now show.

Proof of Theorem 1.2, assuming Theorem 1.3. Let us assume for a contradiction that G, L and $P = p_1p_2$ give a counterexample to Theorem 1.2, and the triple is chosen so that $|V(G)|$ is minimum and subject to that $|E(G)|$ maximum. It follows from the minimality of G that the outer cycle C of G has no chords and that G is 2-connected. Since C has no chords it follows that for $i = 1, 2$ the vertex p_i has a unique neighbor in $C \setminus V(P)$, say v_i. Let $G' := G \setminus V(P)$. The graph G' is 2-connected, for otherwise it has a cutvertex, say v; but then v is adjacent to v_1 by the maximality of $|E(G)|$, and hence vv_1 is a chord of C, a contradiction. We deduce that $v_1 \neq v_2$, for otherwise we could color v_1 using a color $c \notin L(p_1) \cup L(p_2)$, delete v_1, remove c from the list of neighbors of v_1, and extend the coloring of v_1 to an L-coloring of G by the minimality of G, a contradiction. Thus $v_1 \neq v_2$.

Let C' be the outer cycle of G', and for $v \in V(G) \setminus V(P)$ let $L'(v)$ be obtained from $L(v)$ by deleting $L(v_i)$ for all $i \in \{1,2\}$ such that v_i is a neighbor of v. Then $|L'(v)| \geq 3$ for all $v \in V(C') \setminus \{v_1, v_2\}$ and $|L'(v_1)|, |L'(v_2)| \geq 2$. By Theorem 1.3 applied to G', C', v_1, v_2 and L' the graph G' has an L'-coloring. It follows that G has an L-coloring, a contradiction. $\square$

We will use Theorem 1.3 in subsequent papers to deduce various extensions of Theorem 1.2 for paths P of length greater than two. In particular, we will prove the following theorem.

Theorem 1.4 *If G is a plane graph with outer cycle C, P is a subpath of C and L is a list assignment for G with $|L(v)| \geq 5$ for all $v \in V(G) \setminus V(C)$, and $|L(v)| \geq 3$ for all $v \in V(C) \setminus V(P)$, then there exists a subgraph H of G with $|V(H)| = O(|V(P)|)$ such that for every L-coloring ϕ of P, either ϕ extends to an L-coloring of G or ϕ does not extend to an L-coloring of H.*

We need Theorem 1.4 and several similar results to show that graphs on a fixed surface that are minimally not 5-list-colorable satisfy certain isoperimetric inequalities. Those inequalities, in turn, imply several new and old results about 5-list-coloring graphs on surfaces [6, 7].

2 Preliminaries

Definition 2.1 (Canvas) We say that (G, S, L) is a *canvas* if G is a plane graph, S is a subgraph of the boundary of the outer face of G, and L is a list assignment for the vertices of G such that $|L(v)| \geq 5$ for all $v \in V(G)$ not incident with the outer face, $|L(v)| \geq 3$ for all $v \in V(G) \setminus V(S)$, and there exists a proper L-coloring of S.

Thus Theorem 1.2 can be restated in the following slightly stronger form, which follows easily from Theorem 1.2.

Theorem 2.2 *If (G, P, L) is a canvas, where P is a path of length one, then G is L-colorable.*

It should be noted that Theorem 1.3 is not true when one allows three vertices with list of size two. Indeed, Thomassen [9, Theorem 3] characterized the canvases (G, S, L) such that S is a path of length two and some L-coloring of S does not extended to an L-coloring of G. One of Thomassen's obstructions does not extend even when the three vertices in S are given lists of size two. To prove Theorem 1.3 we will need the following lemma, a consequence of [9, Lemma 1] and [9, Theorem 3].

Lemma 2.3 *Let (G, P, L) be a canvas, where G has outer cycle C, $P = p_1p_2p_3$ is a path on three vertices and G has no path Q with ends p_1 and p_3 such that every vertex of Q belongs to C and is adjacent to p_2. Then there exists at most one L-coloring of P that does not extend to an L-coloring of G.*

Definition 2.4 Let (G, S, L) be a canvas and let C be the outer walk of G. We say a cutvertex v of G is *essential* if whenever G can be written as $G = G_1 \cup G_2$, where $V(G_1), V(G_2) \neq V(G)$ and $V(G_1) \cap V(G_2) = \{v\}$, then $V(S) \not\subseteq V(G_i)$ for $i = 1, 2$. Similarly, we say a chord uv of C is *essential* if whenever G can be written as $G = G_1 \cup G_2$, where $V(G_1), V(G_2) \neq V(G)$ and $V(G_1) \cap V(G_2) = \{u, v\}$, then $V(S) \not\subseteq V(G_i)$ for $i = 1, 2$.

Definition 2.5 We say that a canvas (G, S, L) is *critical* if there does not an exist an L-coloring of G but for every edge $e \in E(G) \setminus E(S)$ there exists an L-coloring of $G \setminus e$.

Lemma 2.6 *If (G, S, L) is a critical canvas, then*

(1) *every cutvertex of G and every chord of the outer walk of G is essential, and*

(2) *every cycle of G of length at most four bounds an open disk containing no vertex of G.*

Proof. To prove (1) suppose for a contradiction that the graphs G_1, G_2 satisfy the requirements in the definition of essential cutvertex or essential chord, except that $V(S) \subseteq V(G_1)$. Since (G, S, L) is critical, there exists an L-coloring ϕ of G_1. By Theorem 2.2, ϕ can be extended to G_2. Thus G has an L-coloring, a contradiction. This proves (1).

Statement (2) is a special case of [3, Theorem 6]. It can also be deduced from Theorem 2.2. □

3 Proof of the Two with Lists of Size Two Theorem

In this section, we prove Theorem 1.3 in the following stronger form. We say that an edge uv *separates* vertices x and y if x and y belong to different components of $G \setminus \{u, v\}$.

Theorem 3.1 *Let (G, S, L) be a canvas, where S has two components: a path P and an isolated vertex u with $|L(u)| \geq 2$. Assume that if $|V(P)| \geq 2$, then G is 2-connected, u is not adjacent to an internal vertex of P and there does not exist a chord of the outer walk of G with an end in P which separates a vertex of P from u. Let L_0 be a set of size two. If $L(v) = L_0$ for all $v \in V(P)$, then G has an L-coloring, unless $L(u) = L_0$ and $V(S)$ induces an odd cycle in G.*

Proof. Let us assume for a contradiction that (G, S, L) is a counterexample with $|V(G)|$ minimum and subject to that with $|V(P)|$ maximum. Hence G is connected and (G, S, L) is critical. Let C be the outer walk of G. By the first statement of Lemma 2.6 all cutvertices of G and all chords of C are essential. Thus we have proved:

Claim 3.2 *There is no chord with an end in P.*

Claim 3.3 *G is 2-connected.*

Proof. Suppose there is a cutvertex v of G. By assumption then, $|V(P)| = 1$. Since v is a cutvertex the graph G can be expressed as $G = G_1 \cup G_2$, where $V(G_1) \cap V(G_2) = \{v\}$ and $V(G_1)\setminus V(G_2)$ and $V(G_2)\setminus V(G_1)$ are both non-empty. As v is an essential cutvertex of G, we may suppose without loss of generality that $u \in V(G_2) \setminus V(G_1)$ and $V(P) \subseteq V(G_1) \setminus V(G_2)$.

Consider the canvas (G_1, S_1, L), where $S_1 = P + v$, the graph obtained from P by adding v as an isolated vertex.. As $|V(G_1)| < |V(G)|$, there exists an L-coloring ϕ_1 of G_1. Let $L_1 = (L_1(x) : x \in V(G))$, where $L_1(v) = L(v) \setminus \{\phi_1(v)\}$ and $L_1(x) = L(x)$ for all $x \in V(G_1) \setminus \{v\}$. Similarly, there exists an L_1-coloring ϕ_2 of G_1 by the minimality of G. Note that $\phi_1(v) \neq \phi_2(v)$. Let $L_2 = (L_2(x) : x \in V(G_2))$, where $L_2(v) = \{\phi_1(v), \phi_2(v)\}$ and $L_2(x) = L(x)$ for all $x \in V(G_2)\setminus\{v\}$, and consider the canvas (G_2, S_2, L_2), where S_2 consists of the isolated vertices v and u. As $|V(G_2)| < |V(G)|$, there exists an L_2-coloring ϕ of G_2. Let i be such that $\phi_i(v) = \phi(v)$. Therefore, $\phi \cup \phi_i$ is an L-coloring of G, contrary to the fact that (G, S, L) is a counterexample. □

Let v_1 and v_2 be the two (not necessarily distinct) vertices of C adjacent to a vertex of P. There are at most two such vertices by Claim 3.2.

Claim 3.4 $v_1 \neq v_2$.

Proof. Suppose not; then $v_1 = v_2 = u$ and $V(S) = V(P) \cup \{u\}$ by Claim 3.3. By Claim 3.2 the graph $G[V(S)]$ is a cycle, and if it is odd, then $L(u) \setminus L_0 \neq \emptyset$ by hypothesis. In either case the graph $G[V(S)]$ has an L-coloring ϕ. Let $G' := G \setminus V(P)$ and let $L' = (L'(x) : x \in V(G'))$ be defined by $L'(x) := L(x) \setminus L_0$ if x has a neighbor in P and $L'(x) := L(x)$ otherwise. By Theorem 2.2 the graph G' has an L'-coloring ϕ' with $\phi'(u) = \phi(u)$, and thus G has an L-coloring, a contradiction. □

Claim 3.5 *For all $i \in \{1, 2\}$, if $v_i \neq u$, then v_i is the end of an essential chord of C.*

Proof. As v_1 and v_2 are symmetric, it suffices to prove the claim for v_1. So suppose $v_1 \neq u$ and v_1 is not an end of an essential chord of C. First suppose that $|L(v_1) \setminus L_0| \geq 2$. Let $G' = G \setminus V(P)$ and let S' consist of the isolated vertices v_1 and u. Furthermore, let $L'(v_1)$ be a subset of size two of $L(v_1)\setminus L_0$, let $L'(x) := L(x)\setminus L_0$ for all vertices $x \in V(G')\setminus\{v_1, v_2\}$ with a neighbor in P and let $L'(x) := L(x)$ otherwise. Note that the canvas (G', L', S') satisfies the hypotheses of Theorem 3.1. Hence as $|V(G')| < |V(G)|$ and (G, S, L) is a minimum counterexample, it follows that G' has an L'-coloring ϕ'. Since ϕ' can be extended to P, G has an L-coloring, a contradiction.

So we may assume that $L_0 \subseteq L(v_1)$ and $|L(v_1)| = 3$. Let P' be the path obtained from P by adding v_1. Let $S' = P' + u$, and let $L' = (L'(x); x \in V(G))$, where $L'(v_1) = L_0$ and $L'(x) = L(x)$ for all $x \in V(G) \setminus \{v_1\}$. Consider the canvas (G, S', L'). As v_1 is not the end of an essential chord of C and (G, S, L) was chosen so that $|V(P)|$ was maximized, we find that $G[V(S')]$ is an odd cycle and $L(u) = L_0$.

Now color G as follows. Let $\phi(v_1) \in L(v_1) \setminus L_0$; then we can extend ϕ to a coloring of $G[V(S')]$. Let $L'(v_1) = \{\phi(v_1)\}$, and for $x \in V(G) \setminus V(S')$ let $L'(x) = L(x) \setminus L_0$ if x has a neighbor in S and let $L'(x) = L(x)$ otherwise. By Theorem 2.2, there exists an L'-coloring of $G \setminus V(S)$ and hence ϕ can be extended to an L-coloring of G, a contradiction. □

By Claim 3.4 we may assume without loss of generality that $v_1 \neq u$. By Claim 3.5, v_1 is an end of an essential chord of C. But this and Claim 3.2 imply that $v_2 \neq u$. By Claim 3.5, v_2 is an end of an essential chord of C. As G is planar, it follows from Claim 3.3 that v_1v_2 is a chord of C.

Claim 3.6 $|V(P)| = 1$

Proof. Suppose not. Let G_1, G_2 be subgraphs of G such that $G = G_1 \cup G_2$, $V(G_1) \cap V(G_2) = \{v_1, v_2\}$, $V(P) \subseteq V(G_1)$ and $u \in V(G_2)$. Let $v \notin V(G)$ be a new vertex and construct a new graph G' with $V(G') = V(G_2) \cup \{v\}$ and $E(G') = E(G_2) \cup \{vv_1, vv_2\}$. Let $L(v) = L_0$. Consider the canvas (G', S', L), where S' consists of the isolated vertices v and u. As $|V(P)| \geq 2$, $|V(G')| < |V(G)|$. By the minimality of (G, S, L), there exists an L-coloring ϕ of G'. Hence there exists an L-coloring ϕ of G_2, where $\{\phi(v_1), \phi(v_2)\} \neq L_0$. We extend ϕ to an L-coloring of $P \cup G_2$. Let $L'(v_1) = \{\phi(v_1)\}$ and $L'(v_2) = \{\phi(v_2)\}$, and for $x \in V(G_1) \setminus (V(P) \cup \{v_1, v_2\})$ let $L'(x) = L(x) \setminus L_0$ if x has a neighbor in P and let $L'(x) = L(x)$ otherwise. By Theorem 2.2, there exists an L'-coloring ϕ' of $G_1 \setminus V(P)$. As ϕ' can be extended to P, G has an L-coloring, a contradiction. □

Let v be such that that $V(P) = \{v\}$.

Claim 3.7 *For* $i \in \{1, 2\}$, $L_0 \subseteq L(v_i)$ *and* $|L(v_i)| = 3$.

Proof. By symmetry it suffices to prove the claim for v_1. If $|L(v_1)| \geq 4$, then let $c \in L_0$. If $|L(v_1)| = 3$, then we may assume for a contradiction that $L_0 \setminus L(v_1) \neq \emptyset$. In that case let $c \in L_0 \setminus L(v_1)$.

In either case, let $L'(v_1) = L(v_1) \setminus \{c\}$, $L'(v_2) = L(v_2) \setminus \{c\}$ and $L'(x) = L(x)$ otherwise. Consider the canvas (G', S', L'), where $G' = G \setminus \{v\}$ and S' consists of the isolated vertices v_2 and u. As $|V(G')| < |V(G)|$, there exists an L'-coloring ϕ' of G' by the minimality of (G, S, L). Now ϕ' can be extended to an L-coloring of G by letting $\phi'(v) = c$, a contradiction. □

Claim 3.8 $L(v_1) = L(v_2)$

Proof. Suppose not. As G is planar, either v_1 is not an end of a chord of C separating v_2 from u, or v_2 is an the end of a chord separating v_1 from u. Assume without loss of generality that v_1 is not in a chord of C separating v_2 from u. This implies that v_1 is not an end of a chord in C other than v_1v_2. Let v' be the vertex in C distinct from v_2 and v that is adjacent to v_1.

Let $c \in L(v_1) \setminus L_0$. Let $G' = G \setminus \{v, v_1\}$, $L'(x) = L(x) \setminus \{c\}$ if x is adjacent to v_1 and $L'(x) = L(x)$ otherwise. Note that $|L'(v_2)| \geq 3$ as $L(v_1) \neq L(v_2)$ and $L_0 \subseteq L(v_1) \cap L(v_2)$. Let S' consist of isolated vertices v' and u. By considering the canvas (G', S', L') we deduce that G' has an L'-coloring ϕ'; if $u \neq v'$, then it follows by the minimality of G, because in that case $|L(u)|, |L(v')| \geq 2$; and if $u = v'$, then it follows from Theorem 2.2. As ϕ' can be extended to $\{v, v_1\}$, there exists an L-coloring of G, a contradiction. □

Claim 3.9 *One of v_1, v_2 is the end of an essential chord of C distinct from v_1v_2.*

Proof. Suppose for a contradiction that there is no such essential chord. Let $c \in L(v_1) \setminus L_0 = L(v_2) \setminus L_0$, and let L_1 be a set of size two such that $c \in L_1 \subseteq L(v_1)$ and $L_0 \neq L_1$. Let $L_1(v_1) = L_1(v_2) = L_1$ and $L_1(x) = L(x)$ for all $x \in V(G) \setminus \{v, v_1, v_2\}$. Let P' denote the path with vertex-set $\{v_1, v_2\}$ and consider the canvas $(G \setminus v, P' + u, L_1)$. Note that $G \setminus v$ is 2-connected, since G is 2-connected and there are no vertices in the open disk bounded by the triangle vv_1v_2 by the second assertion of Lemma 2.6. Since P' has no internal vertex, the canvas $(G \setminus v, P' + u, L_1)$ satisfies the hypotheses of Theorem 3.1. As $|V(G')| < |V(G)|$, there exists an L_1-coloring ϕ' of $G \setminus v$. But then ϕ' can be extended to an L-coloring of G, a contradiction. □

Suppose without loss of generality that v_2 is the end of an essential chord of C distinct from v_1v_2. Choose such a chord v_2u_1 such that u_1 is closest to v_1 measured by the distance in $C \setminus v_2$. Let G_1 and G_2 be connected subgraphs of G such that $V(G_1) \cap V(G_2) = \{v_2, u_1\}$, $G_1 \cup G_2 = G$, $v \in V(G_1)$ and $u \in V(G_2)$.

We now select an element c as follows. If v_1 is adjacent to u_1, then let $c \in L(v_1) \setminus L_0 = L(v_2) \setminus L_0$. Note that in this case $V(G_1) = \{v, v_1, v_2, u_1\}$ by the second assertion of Lemma 2.6. If v_1 is not adjacent to u_1, then we consider the canvas (G_1, P'', L), where $P'' = vv_2u_1$. As u_1 is not adjacent to v_1, there does not exist a path Q in G_1 as in Lemma 2.3. By Lemma 2.3, there is at most one coloring of P'' which does not extend to G_1. If such a coloring exists, then let c be the color of u_1 in that coloring; otherwise let c be arbitrary.

Consider the canvas (G_2, S', L'), where S' consists of the isolated vertices u_1 and u, $L'(u_1) = L(u_1) \setminus \{c\}$ and $L'(x) = L(x)$ otherwise. As $|V(G_2)| < |V(G)|$, there exists an L'-coloring ϕ of G_2 by the minimality of (G, S, L). But then we may extend ϕ to G_1 by the choice of c to obtain an L-coloring of G, a contradiction. □

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