EXCLUDED MINOR THEOREMS

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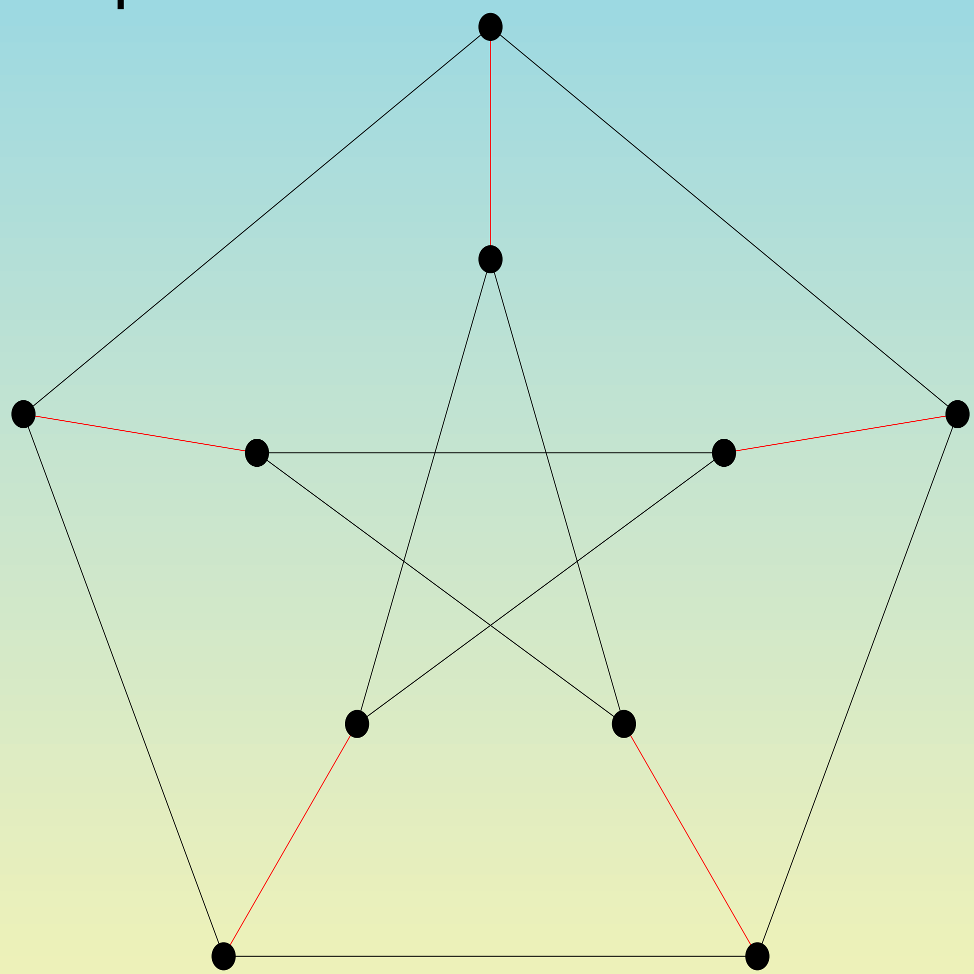
A graph H is a minor of a graph G if H can be obtained from a subgraph of G by contracting edges. An H minor is a minor isomorphic to H.



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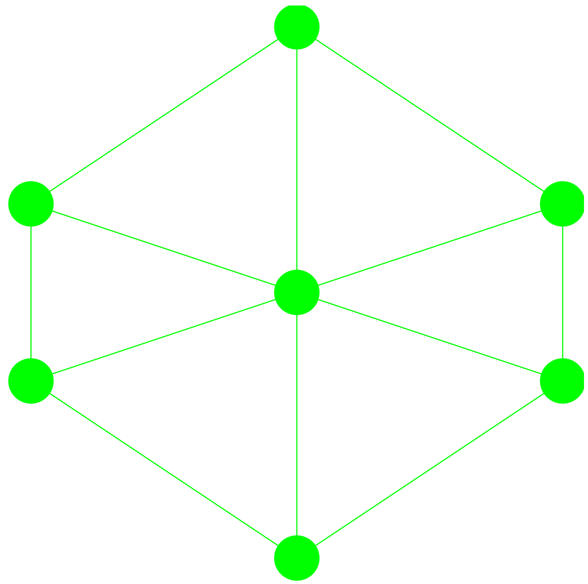

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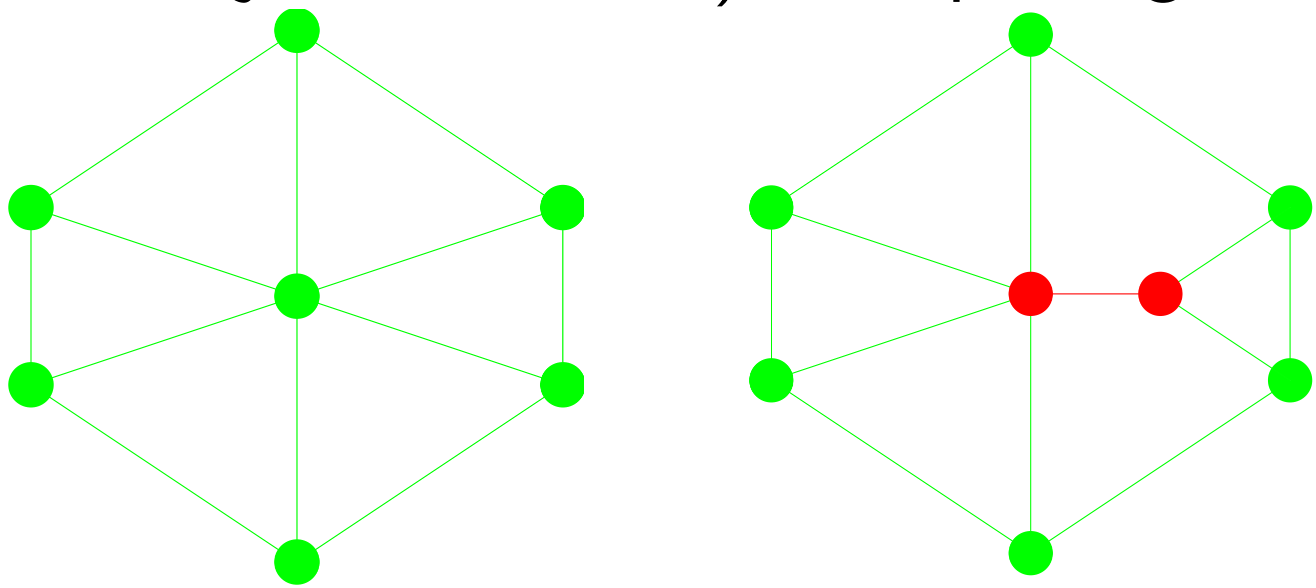

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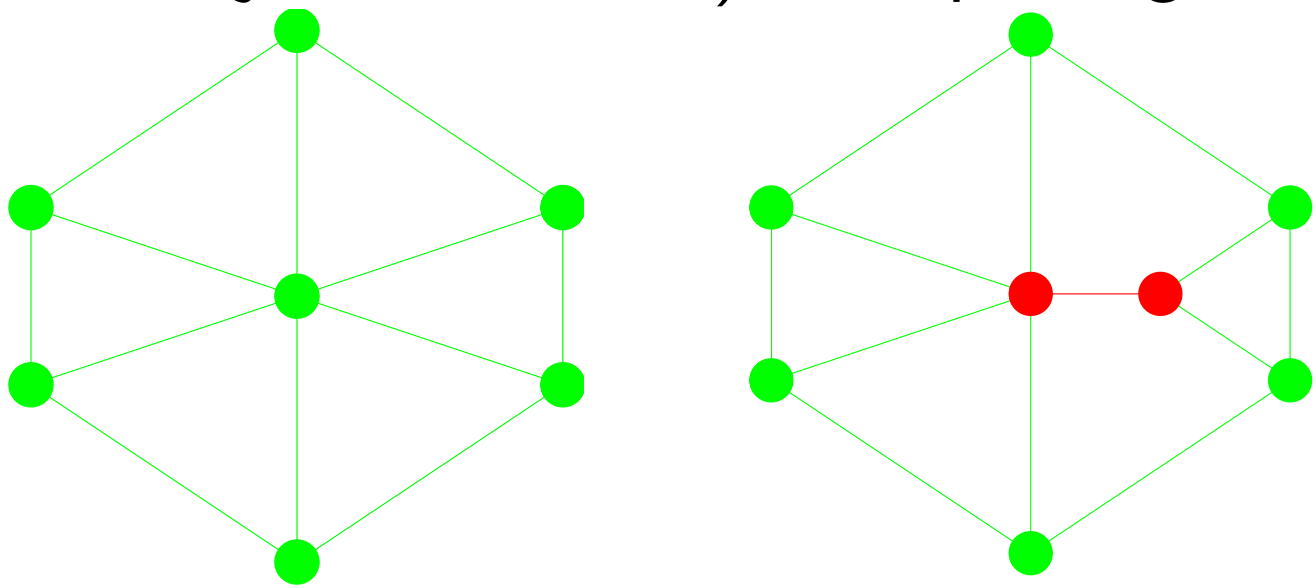

SEYMOUR'S SPLITTER THM Let $H \neq K_4$ and $G \neq$wheel be simple 3-connected, $H \leq_m G$. Then G can be obtained from H by repeatedly adding edges (between nonadjacent vertices) and splitting vertices.

KURATOWSKI'S THEOREM. A graph is planar $\Leftrightarrow$ it has no K_5 or $K_{3,3}$ minor.

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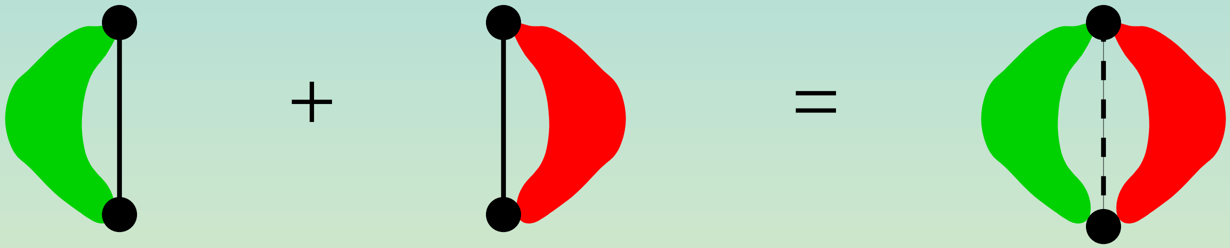

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PROOF of $\Rightarrow$. We may assume G is nonplanar. By Kuratowski's theorem G has a K_5 minor. By Seymour's theorem G can be obtained from K_5 as stated. Now $G \cong K_5$, for otherwise G has a $K_{3,3}$ minor.

THEOREM (Wagner) A graph has no K_5 minor $\Leftrightarrow$ it can be obtained by means of 0-, 1-, 2-, and 3-sums from planar graphs and V_8.

A graph G is internally 4-connected (I4C) if it is simple, 3-connected, has at least five vertices and for every separation (A, B) of order 3, one of A, B has at most 3 edges.

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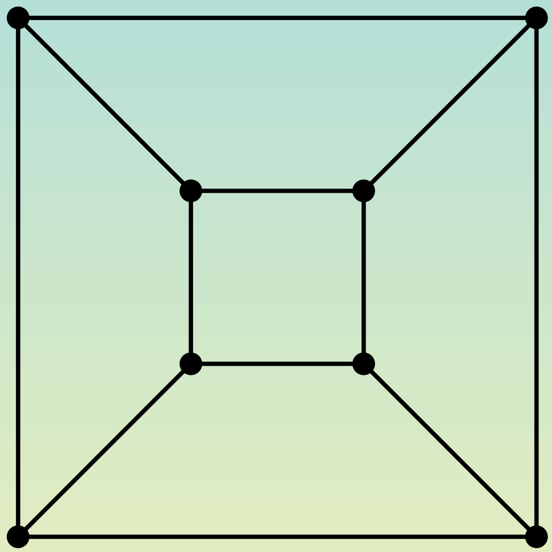

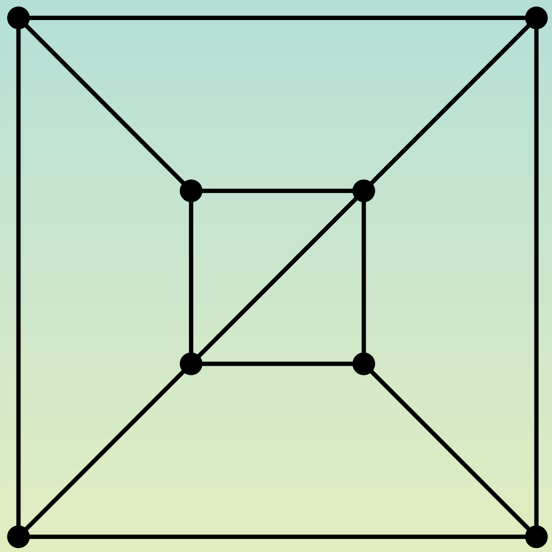

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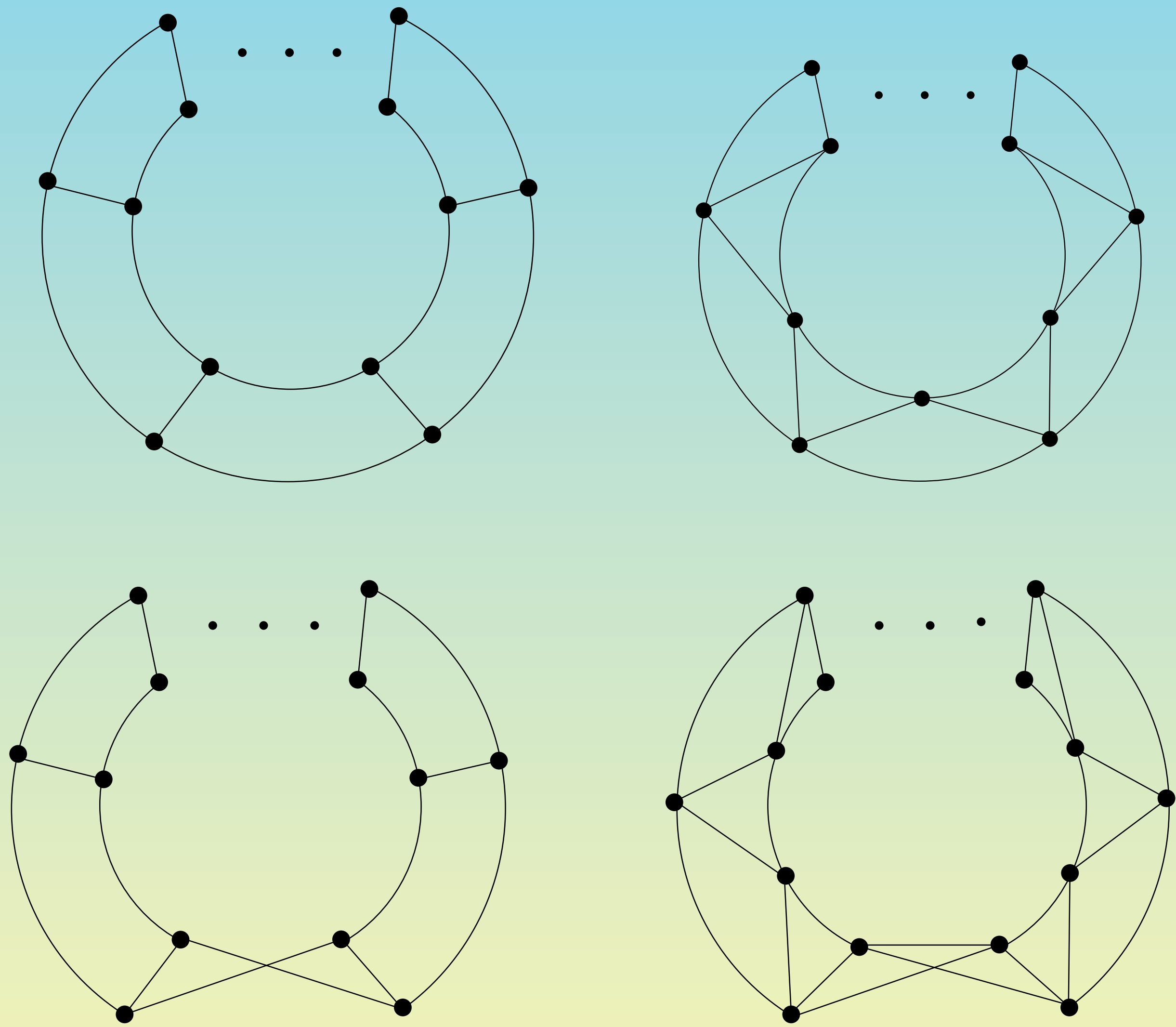
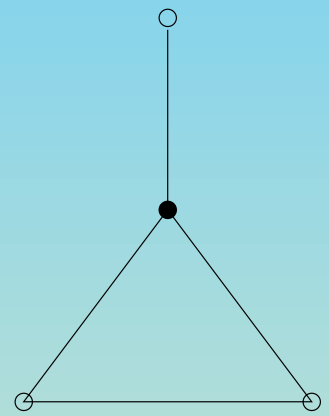
THM (Johnson, RT) Except for eight well-defined families, an I4C graph G can be "built" from an I4C minor of itself similarly as in Seymour's theorem. The intermediate graphs are allowed to have one "violation" of I4C, but the next graph in the sequence "repairs" this violation.

LADDERS



Violating vertex, edge, pair



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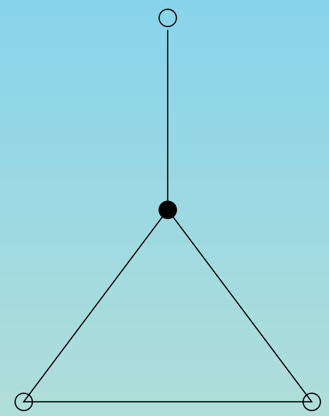

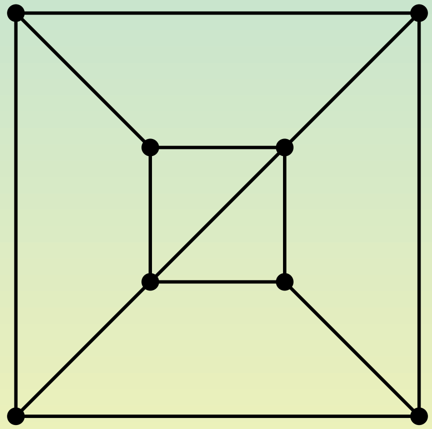

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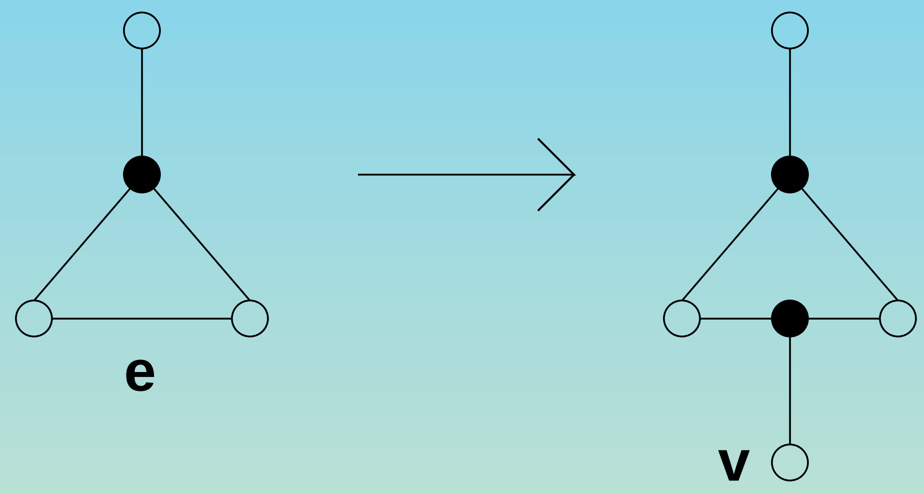

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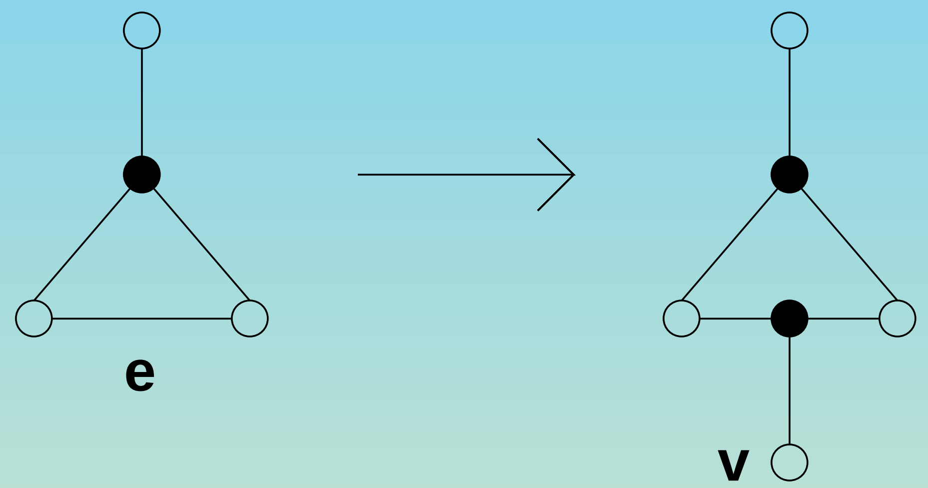
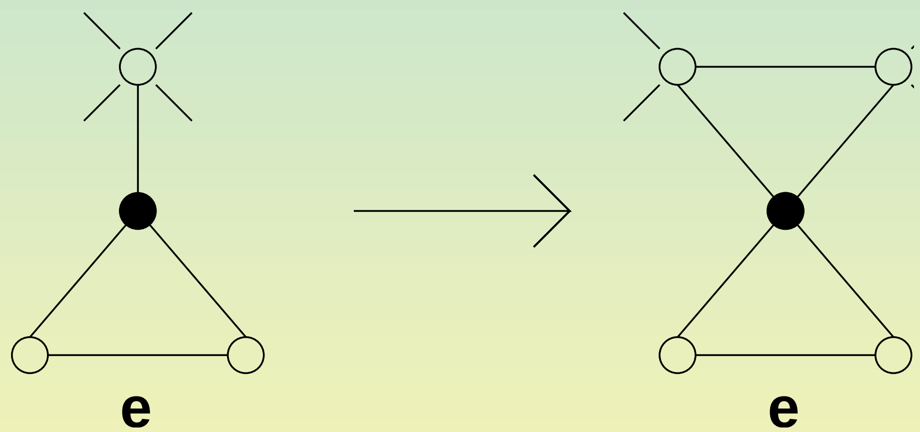


SPECIAL SPLIT



THM Johnson, RT If $H \leq_m G$, H is not $K_{3,3}, K_5$, cube or octahedron, G is not a ladder or biwheel, then $\exists$ sequence $J_0 = H, J_1, \ldots, J_k = G$

- each J_i is I4C except possibly for one violating edge
- no edge is violating in J_i and J_{i+1}
- J_i is obtained from J_{i-1} by (special) addition or (special) split

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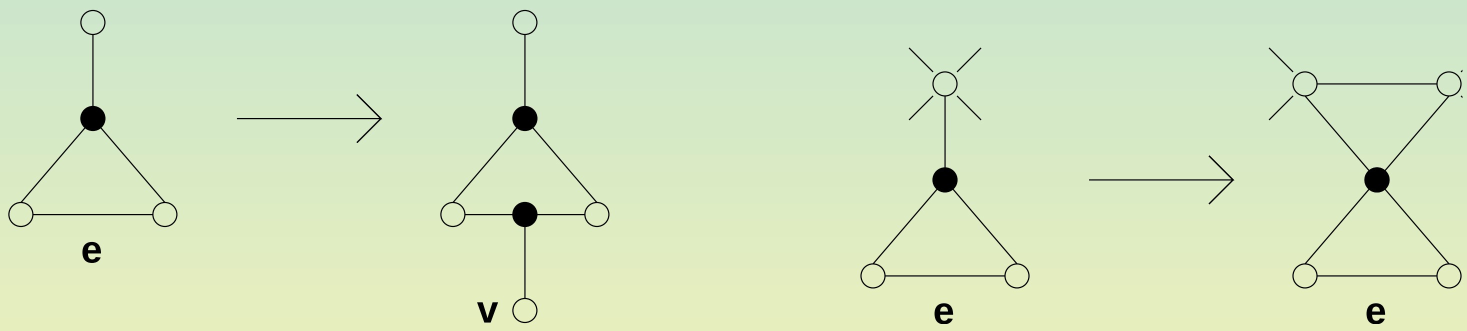

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THM Johnson, RT The minimal nonplanar I4C graphs other than $K_{3,3}, K_5$ are: $K_6^{=}$, $\overline{C}_7$, $K_{3,3}$+deg 4 vertex, V_8, cube+diagonal.

Application to Negami's conjecture.

A graph K is a cover of a graph H if there exists an onto mapping $p : V(K) \to V(H)$ such that for every $v \in V(K)$ the neighbors of v in K are mapped bijectively onto the neighbors of $p(v)$ in H.

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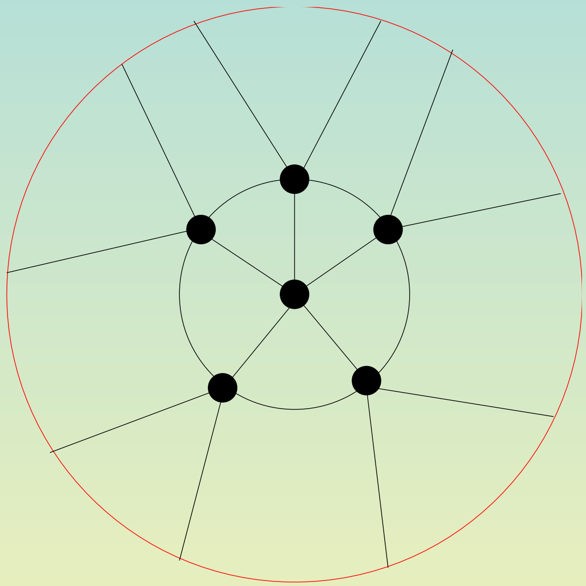

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REMARK. It suffices to show that $K_{1,2,2,2}$ has no planar cover.

ROBERTSON'S THEOREM

THM An I4C graph G has no V_8 minor $\Leftrightarrow$

(1) G is planar, or

(2) $G \backslash X$ is edgeless for some $X \subseteq V(G)$, $|X| \leq 4$, or

(3) $G \backslash u \backslash v$ is a cycle for some $u, v \in V(G)$, or

(4) $G \cong L(K_{3,3})$, or

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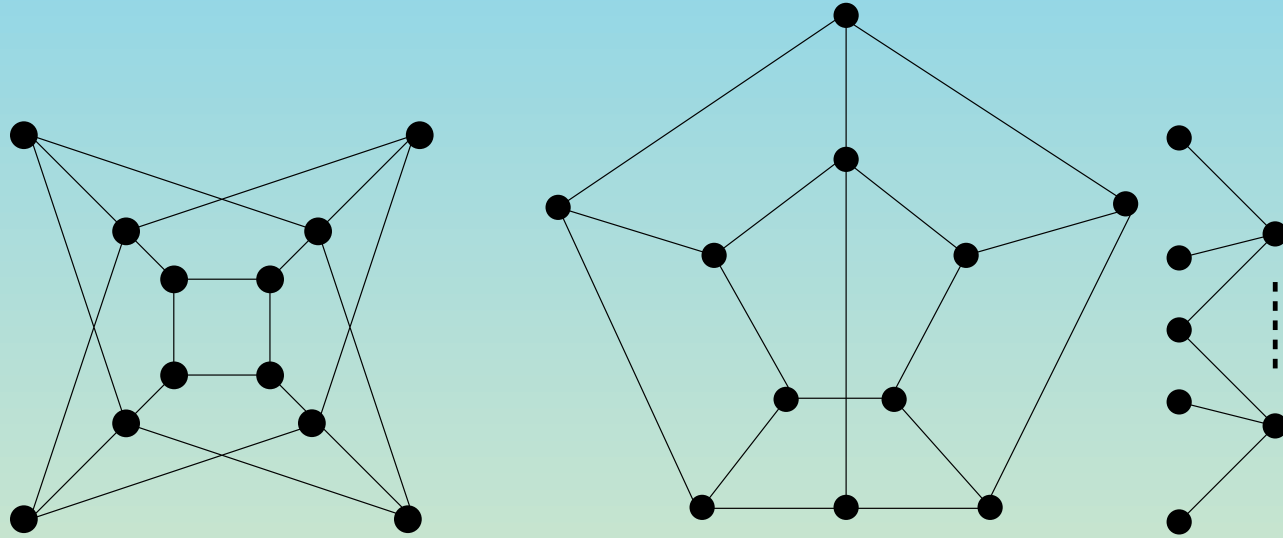
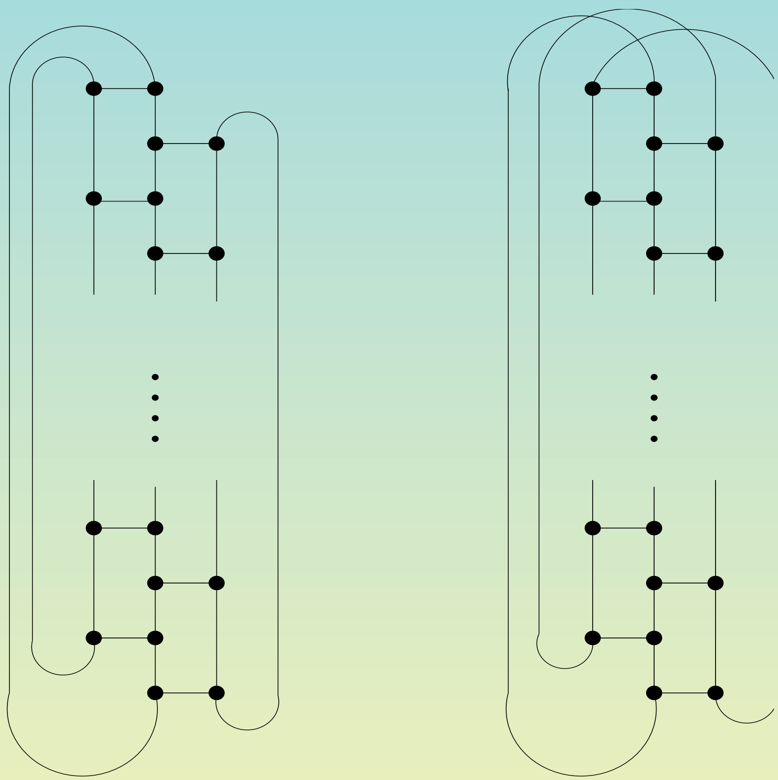
PROOF Let G be nonplanar, I4C, no V_8 minor. We know $G \geq_m K_6^{=}$, $\overline{C}_7$, $K_{3,3}$+deg 4 vertex, V_8, or cube+diag.

THEOREM An I4C graph has no octahedron minor $\Leftrightarrow$
(1) G is a Möbius ladder, or
(2) G is isomorphic to a minor of Petersen,

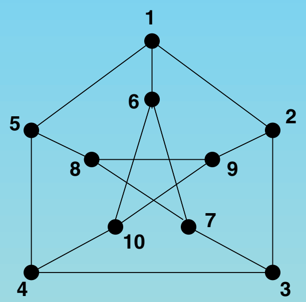


The last graph has all possible triads with feet in the 5-element independent set.

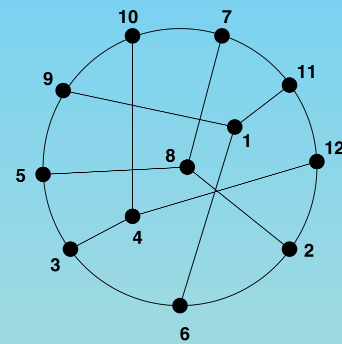
A cubic graph is cyclically 5-connected (C5C) if it is simple, 3-connected, $\neq K_4$, and for every set $F \subseteq E(G)$ of size at most 4, at most 1 component of $G \backslash F$ has cycles.



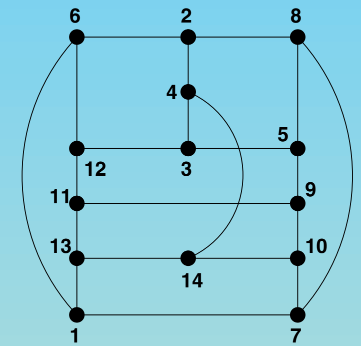
Biladders



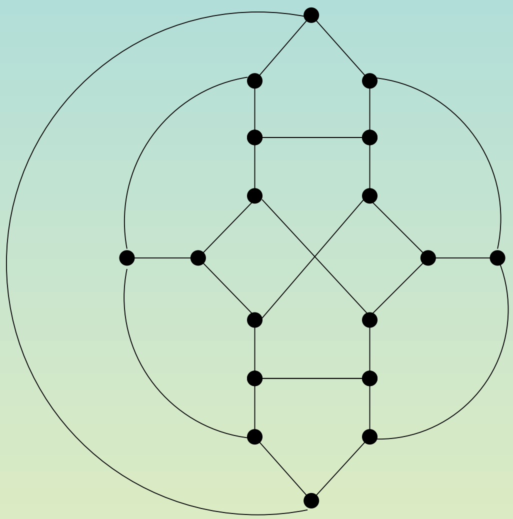
Petersen



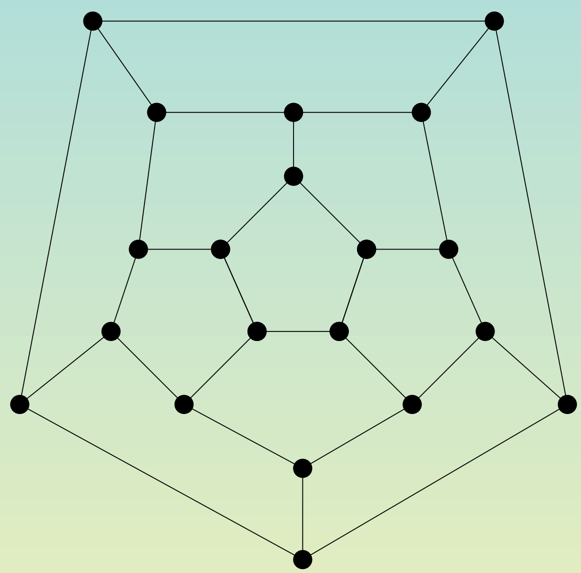
T r i p l e x



Rects



Ruby



Dodecahedron

Cyclically 5-connected graphs

THEOREM (Robertson, Seymour, RT) Let G be a C5C cubic graph that is not a biladder, and let H be a C5C minor of G. Then G can be obtained from H by repeatedly applying the operations of

(i) adding a handle

(ii) adding a pentagon.

THEOREM (Robertson, Seymour, RT) A C5C cubic graph G has no Petersen minor if and only if it is

(i) apex ($G\backslash v$ planar for some v), or

(ii) doublecross (2 crossings on the same region), or

(iii) has a "hamburger structure", or

(iv) has a "hose structure".

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THEOREM (Mader) If G has n vertices and no K_6-minor, then G has at most $4n-10$ edges.

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JORGENSEN'S CONJECTURE Every 6-connected graph with no K_6-minor is apex (=planar $+$ one vertex).

EXTREMAL PROBLEMS

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THM Thomason

No K_t minor $\Rightarrow$ at most $(0.319 + o(1))t\sqrt{\log t}n$ edges

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CONJECTURE $\forall t\ \exists N$ if G is $(t-2)$-connected and $|G| > N$, then $|E(G)| \le (t-2)n - \binom{t-1}{2}$.

THM Jorgensen No $K_{4,4}$ minor $\Rightarrow \leq 4n - 8$ edges.

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THEOREM (Mader, conjectured by Dirac)
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Implies Mader's theorem (Kezdy, McGuiness)