GRAPH PLANARITY

and

RELATED TOPICS

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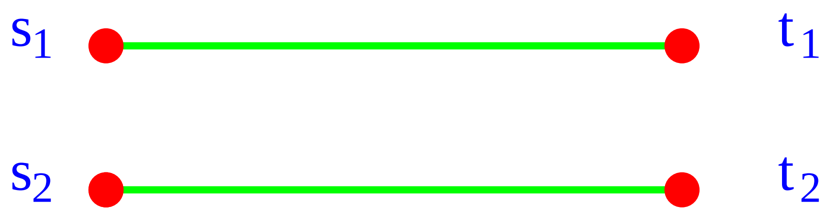
THE TWO PATHS PROBLEM

Given G and $s_1, s_2, t_1, t_2 \in V(G)$ do there exist disjoint paths joining s_i to t_i?



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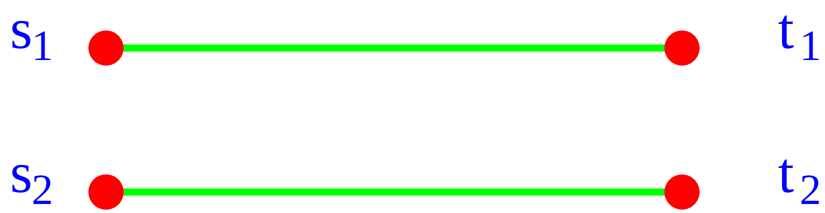

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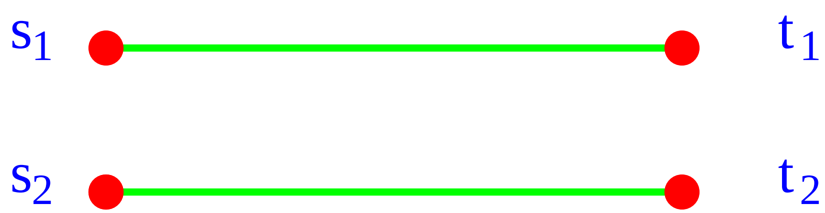
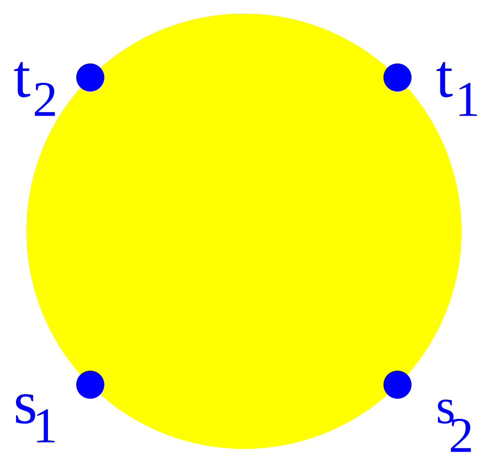


OBSTRUCTION: G drawn in a disk with s_1, s_2, t_1, t_2 drawn on the boundary in order



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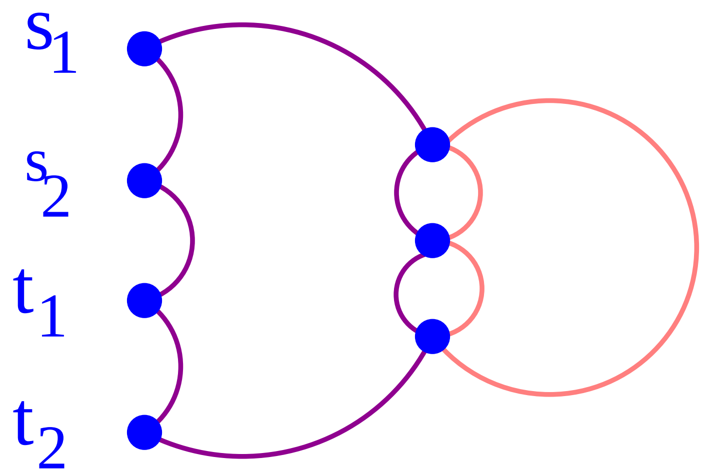

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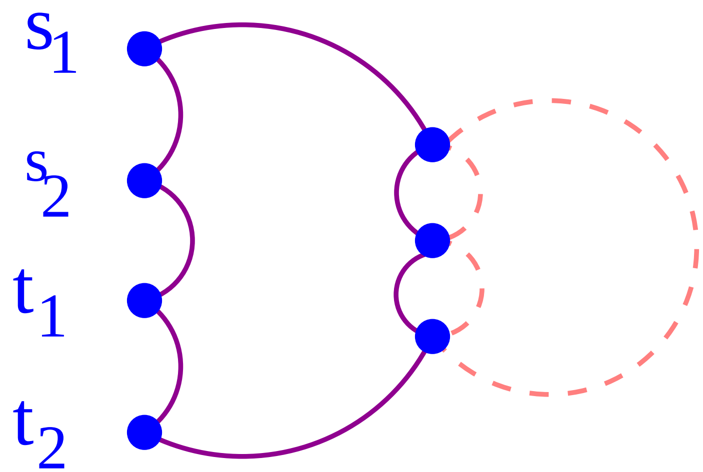

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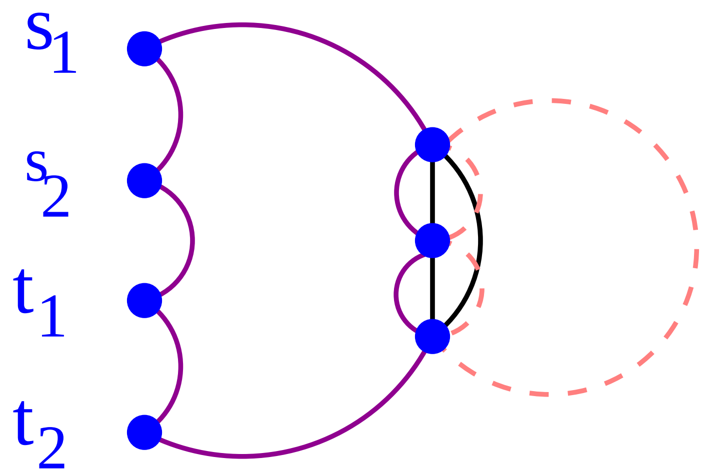

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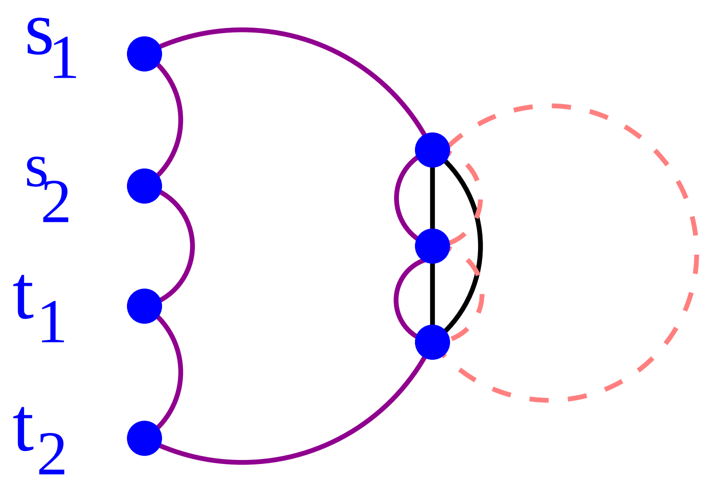

THEOREM Robertson, Seymour, Shiloach, Thomassen, Watkins If G is reduced, then the paths exist $\Leftrightarrow$ G cannot be drawn in a disk with s_1, s_2, t_1, t_2 drawn on the boundary of the disk in order.

PLANAR GRAPHS

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COROLLARY $|E(G)| \leq 3|V(G)| - 6$
$|E(G)| \leq 2|V(G)| - 4$ if G has no triangles

KURATOWSKI'S THEOREM. A graph is planar $\Leftrightarrow$ it has no subgraph isomorphic to a subdivision of K_5 or $K_{3,3}$.

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PF. $\Rightarrow$ K_5 and $K_{3,3}$ have too many edges.

$\Leftarrow$ (Thomassen) By induction. We may assume G is 3-connected, G/uv is 3-connected and planar.

TESTING PLANARITY IN LINEAR TIME

- 1974 Hopcroft and Tarjan
- 1967 Lempel, Even and Cederbaum, 1976 Booth and Lueker
- Shih and Hsu, Boyer and Myrvold, RT class notes

COLIN de VERDIERE'S PARAMETER

Let $\mu(G)$ be the maximum corank of a matrix M satisfying

(i) for $i \neq j$, $M_{ij} = 0$ if $ij \notin E$ and $M_{ij} < 0$ otherwise,

(ii) M has exactly one negative eigenvalue,

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THEOREM $\mu(G) \leq 3 \Leftrightarrow G$ is planar.

SEPARATORS

Let G have n vertices. A separator in G is a set S such that every component of $G \backslash S$ has at most $2n/3$ vertices.

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Alon, Seymour, RT improved to $\sqrt{4.5n}$, and proved that graphs not contractible to K_t have a separator of size at most $\sqrt{t^3n}$.

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THEOREM (Brightwell, Scheinerman) Every 3-connected plane graph has a primal-dual circle packing representation.

SCHNYDER'S THEOREM and DRAWING ON A GRID

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Given $\leq_1, \leq_2, \leq_3$ there exists a 1-1 map $v \in V(G) \rightarrow (v_1, v_2, v_3) \in \mathbf{R}^3$ s.t.

(i) $v_1 + v_2 + v_3 = 2n - 5$ for all $v \in V(G)$

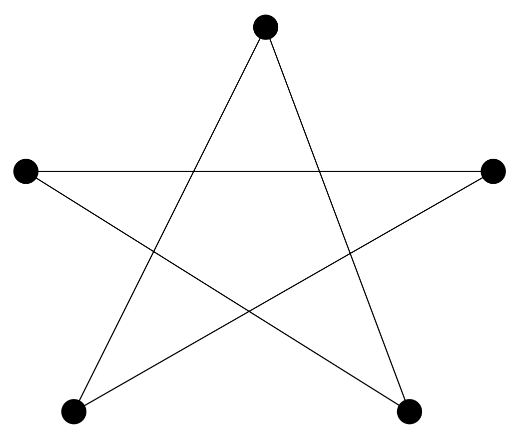
(ii) $v_i \in [0, 2n-5]$ is an integer

(iii) for $uv \in E(G)$: $u \leq_i v \Leftrightarrow u_i \leq v_i$

It follows that this gives a straight-line embedding.

THRACKLE CONJECTURE

CONJECTURE (Conway) If a graph G can be drawn in the plane such that every two distinct edges meet exactly once (cross or share an end), then $|E(G)| \leq |V(G)|$.



NOTE Enough to show that 1-sum of two even cycles cannot be drawn in such a way.

STRING GRAPHS

OPEN PROBLEM Can every planar graph be represented as an intersection graph of simple closed curve in the plane such that any two curves meet at most once?

$Y\Delta$-TRANSFORMATIONS

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APPLICATION Fast computation of spanning trees, ...

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- See August 1998 Notices of the AMS for a survey.

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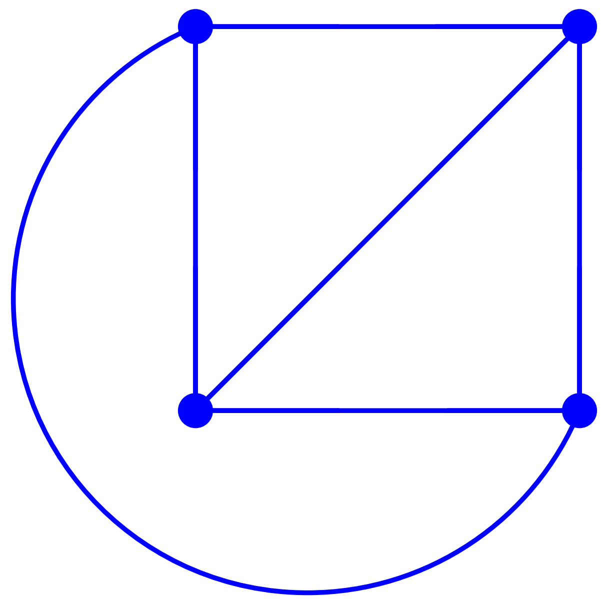
$$\prod \binom{A_k(m, c_1, .., c_{20}) + 7^n B_k(m, c_1, .., c_{20})}{C_k(m, c_1, .., c_{20}) + 7^n D_k(m, c_1, .., c_{20})}$$

is not divisible by 7.

Let G be a cubic planar graph.



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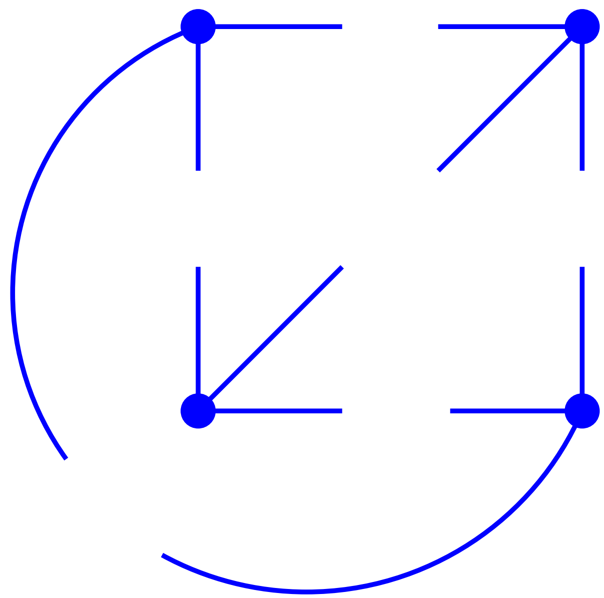

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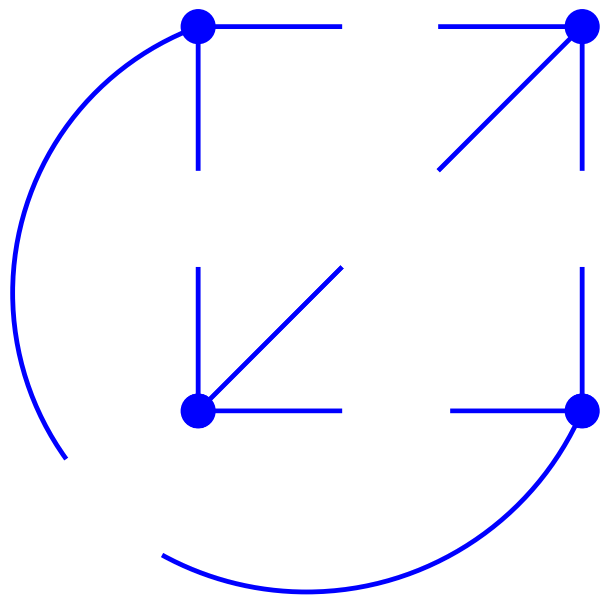

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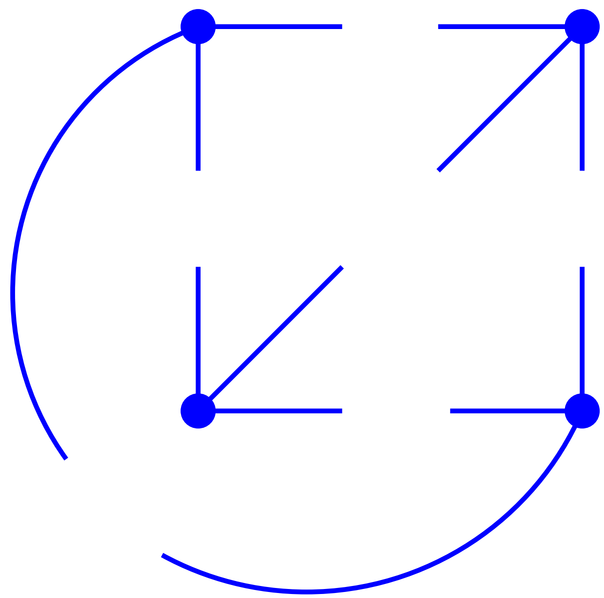

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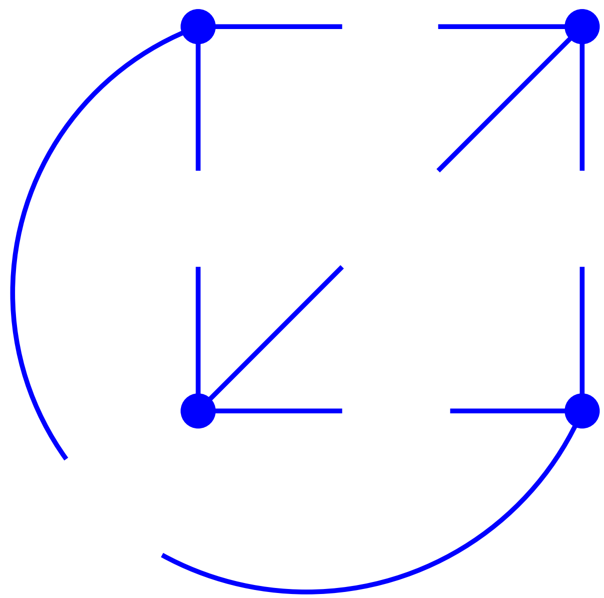

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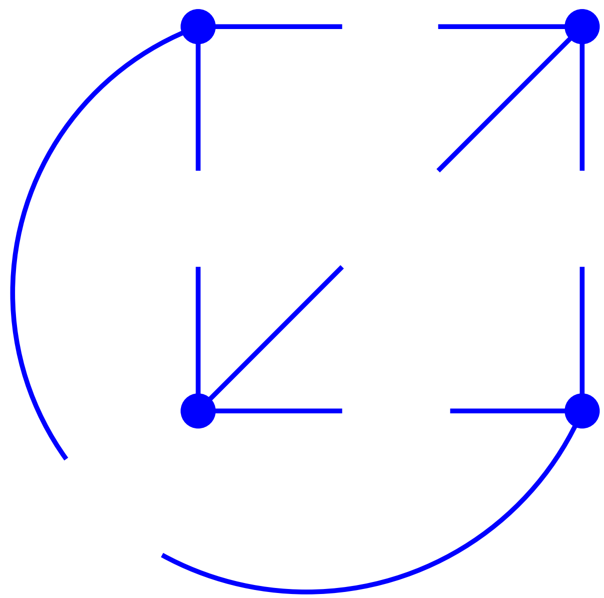

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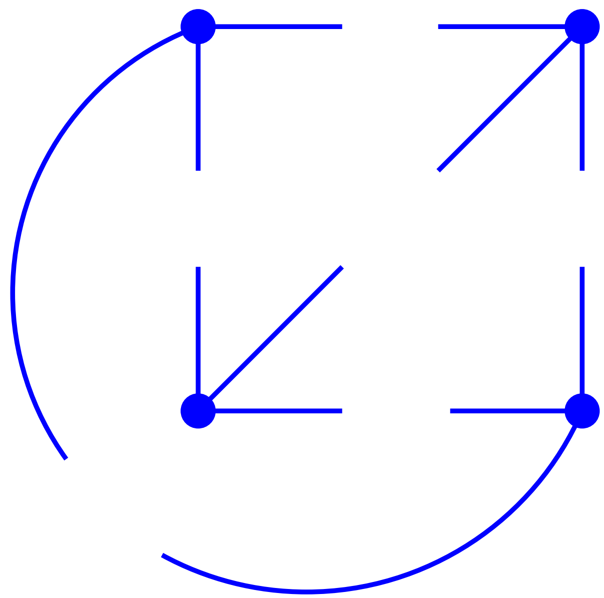

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$$P[B|A] - P[B] = 48^{-n} \cdot (\# \text{ edge 3-colorings of } G)$$

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MAIN STEPS IN THE PROOF

(1) Every minimal counterexample is apex or doublecross.
(2)True for apex graphs.
(3)True for doublecross graphs.

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Proved by Thomassen

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CONJECTURE (Tutte 1954) Every edge 2-connected graph has a 5-flow.

PROOF OF THE TWO PATHS THEOREM

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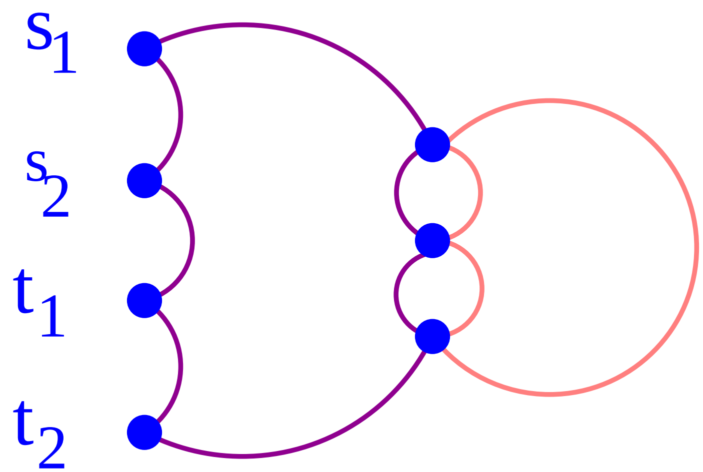

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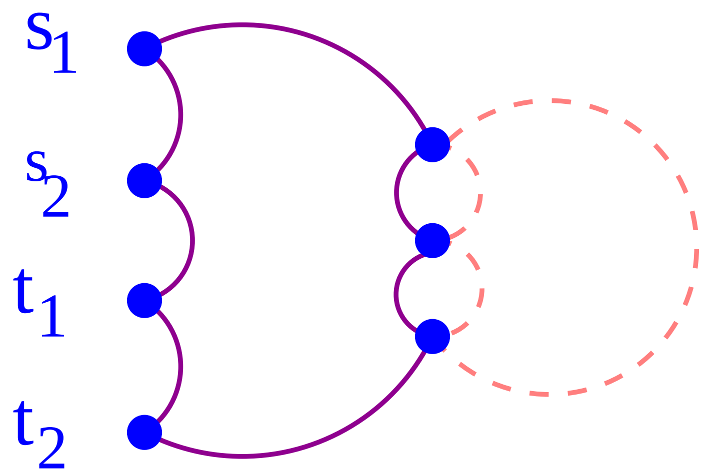

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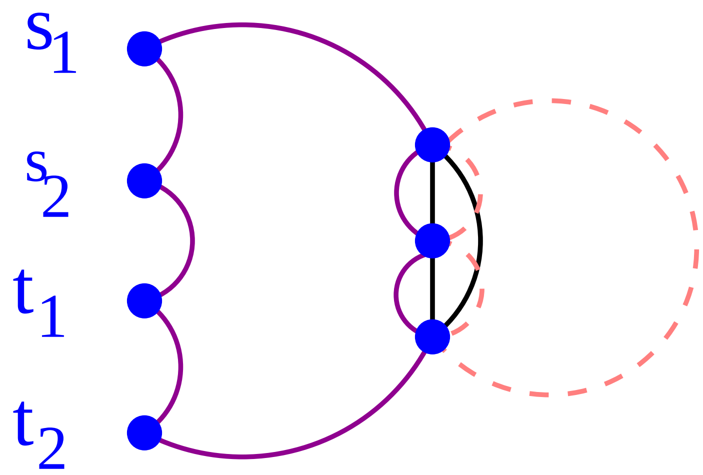

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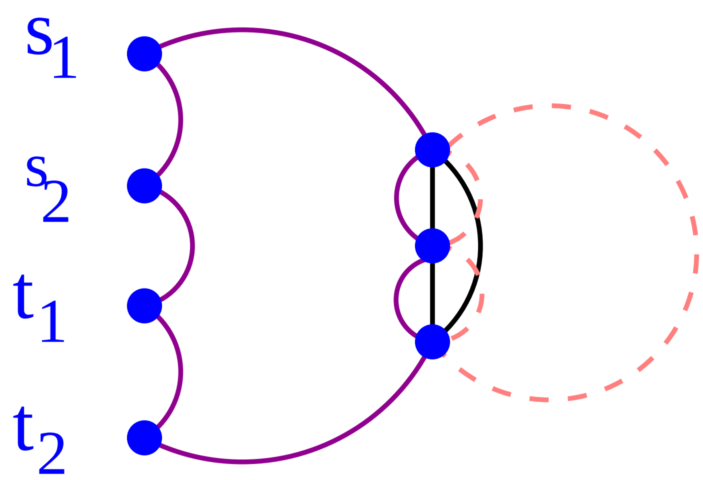

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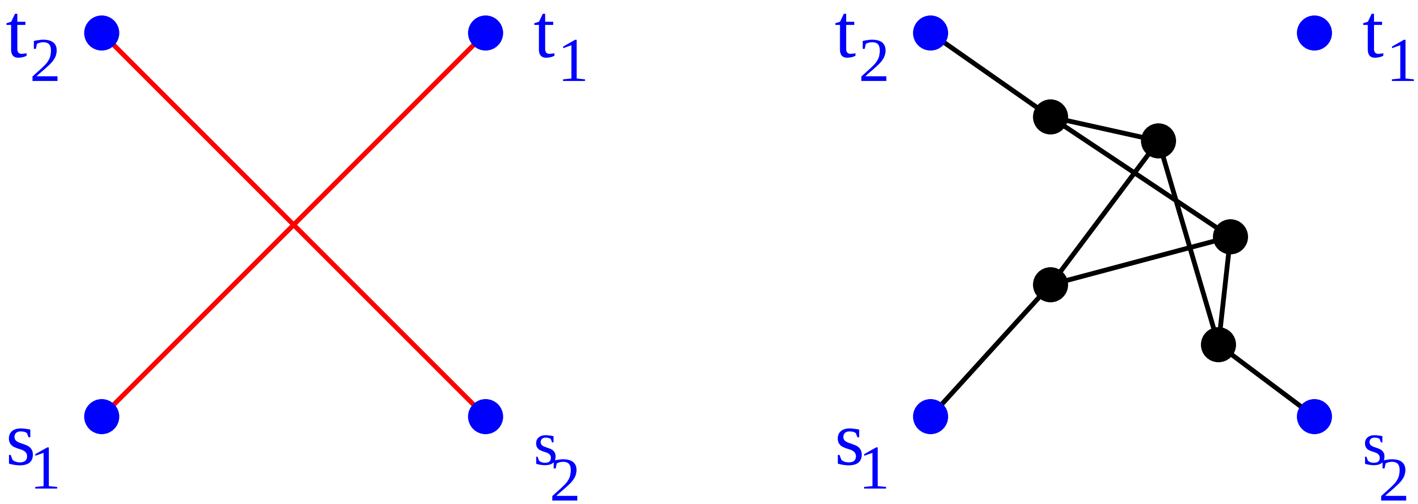

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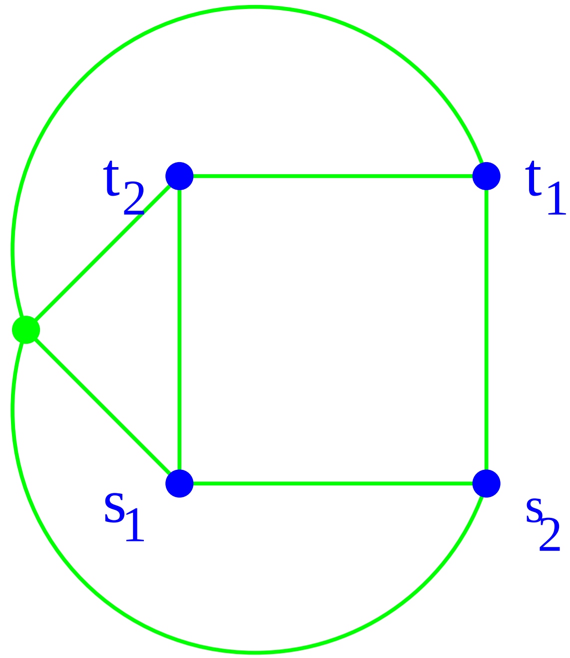

$\Rightarrow$ $K_{3,3}$ subdivision or K_5 subdivision

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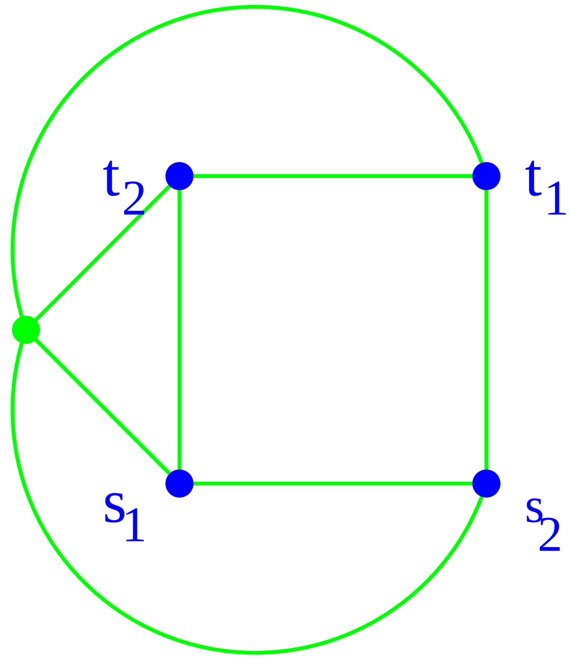

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PROOF OF THM By Lemma WMA double triad. Get a fourth path.

OPEN QUESTION

Can the TWO DISJOINT PATHS problem be solved in linear time?

Let G have n vertices. A separator in G is a set S such that every component of $G \setminus S$ has at most $2n/3$ vertices.

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PROOF WMA G is a triangulation. Let $k = \lfloor \sqrt{2n} \rfloor$. Choose a cycle C of length $\leq 2k$ with $\mathsf{out}(C) < 2n/3$ and $\mathsf{ins}(C) - \mathsf{out}(C)$ minimum. Then $\mathsf{ins}(C) < 2n/3$.