TREE-DECOMPOSITIONS OF GRAPHS

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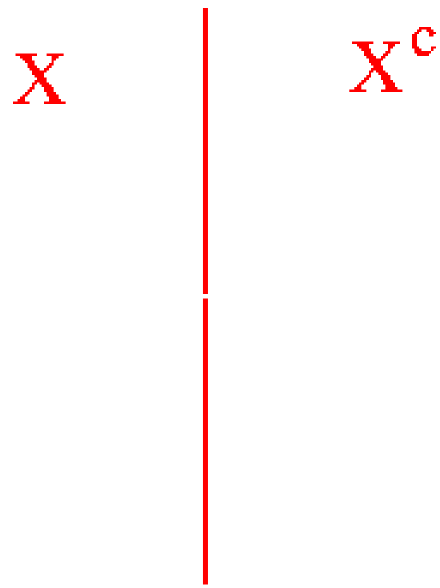
MOTIVATION

$\delta X =$ edges with one end in X, one in $V(G) - X$



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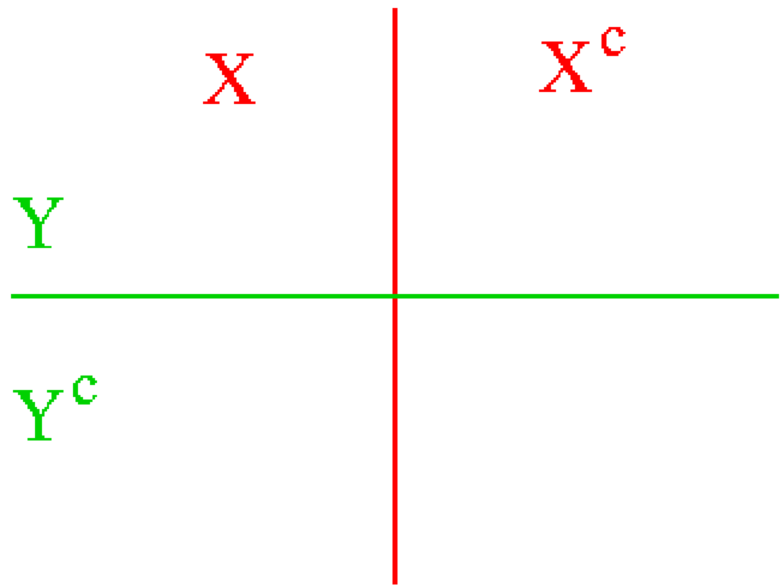

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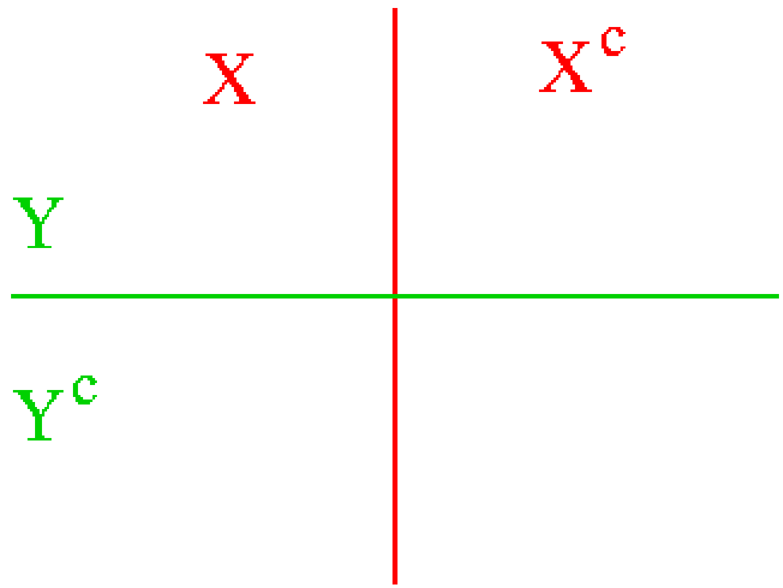
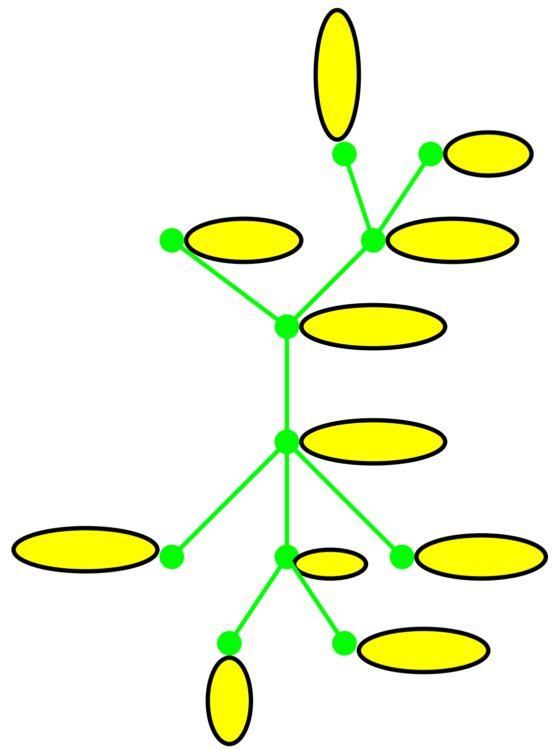


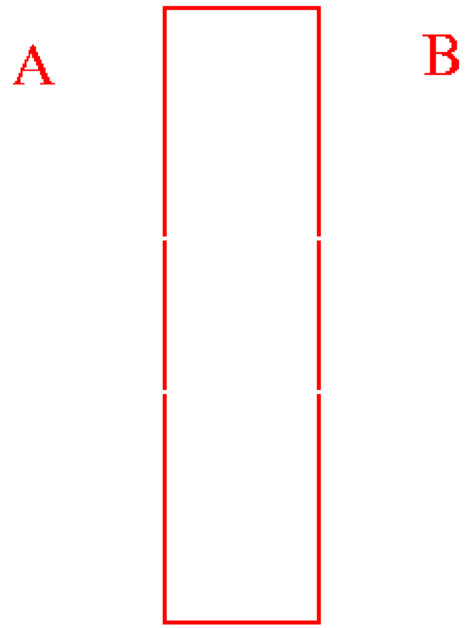
Two edge-cuts $\delta X, \delta Y$ do not cross if:
$X \subseteq Y$ or $X \subseteq Y^c$ or $X^c \subseteq Y$ or $X^c \subseteq Y^c$.

Example of a cross-free family of edge-cuts:

Let T be a tree, and $(W_t : t \in V(T))$ a partition of $V(G)$. Every edge of T defines a cut; the collection of cuts thus obtained is cross-free.



A separation of a graph G is a pair (A, B) such that $A \cup B = V(G)$ and there is no edge between $A - B$ and $B - A$.



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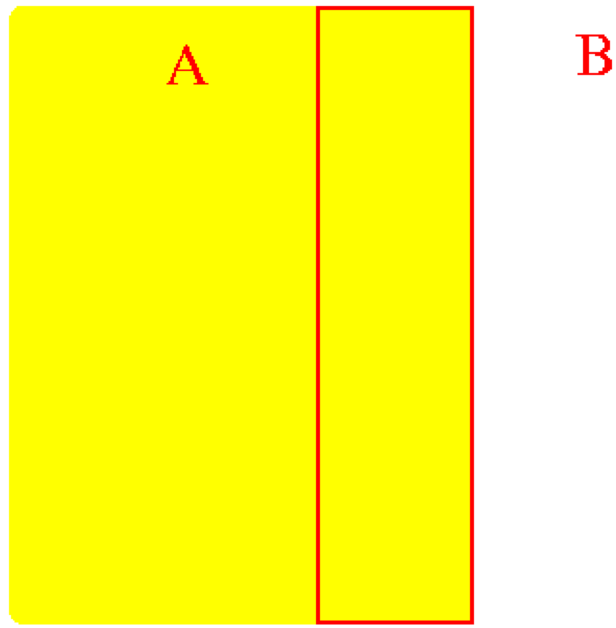

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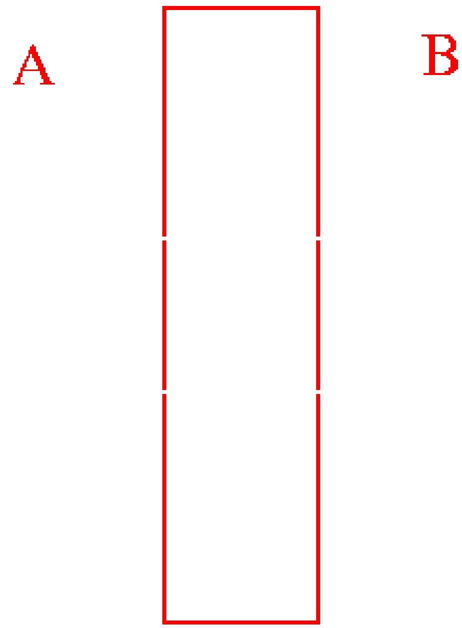

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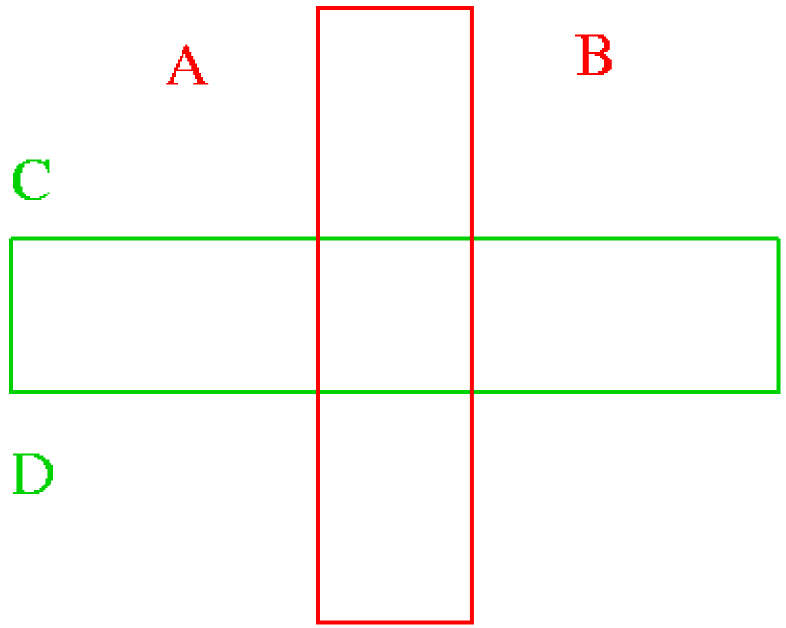

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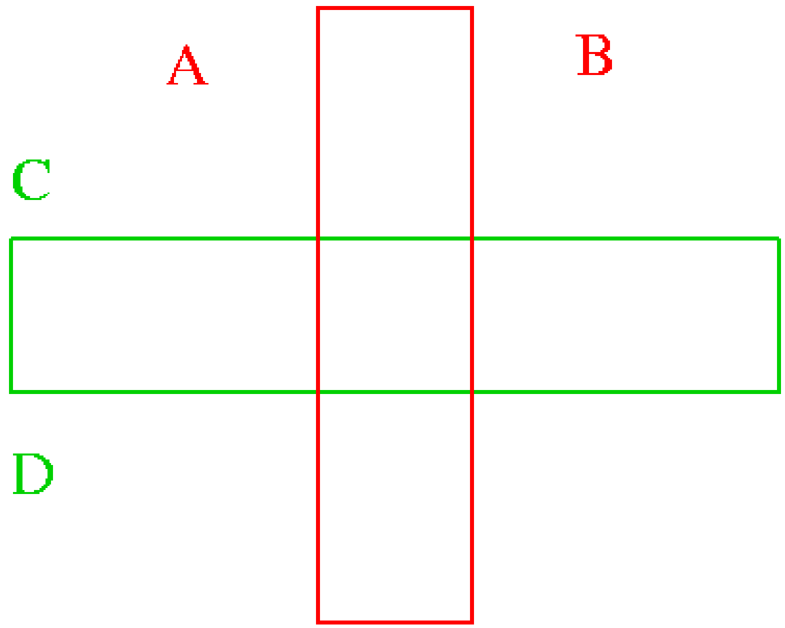

Two separations (A, B) and (C, D) do not cross if:
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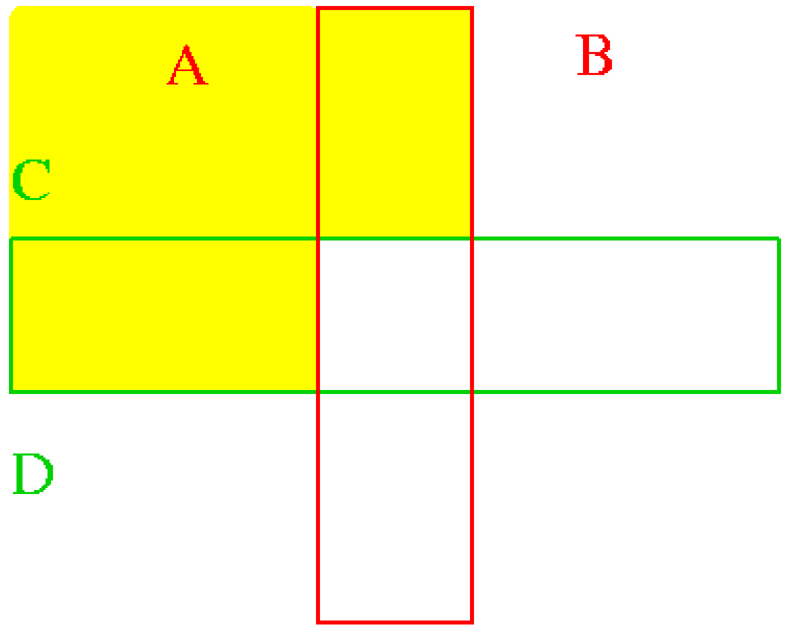

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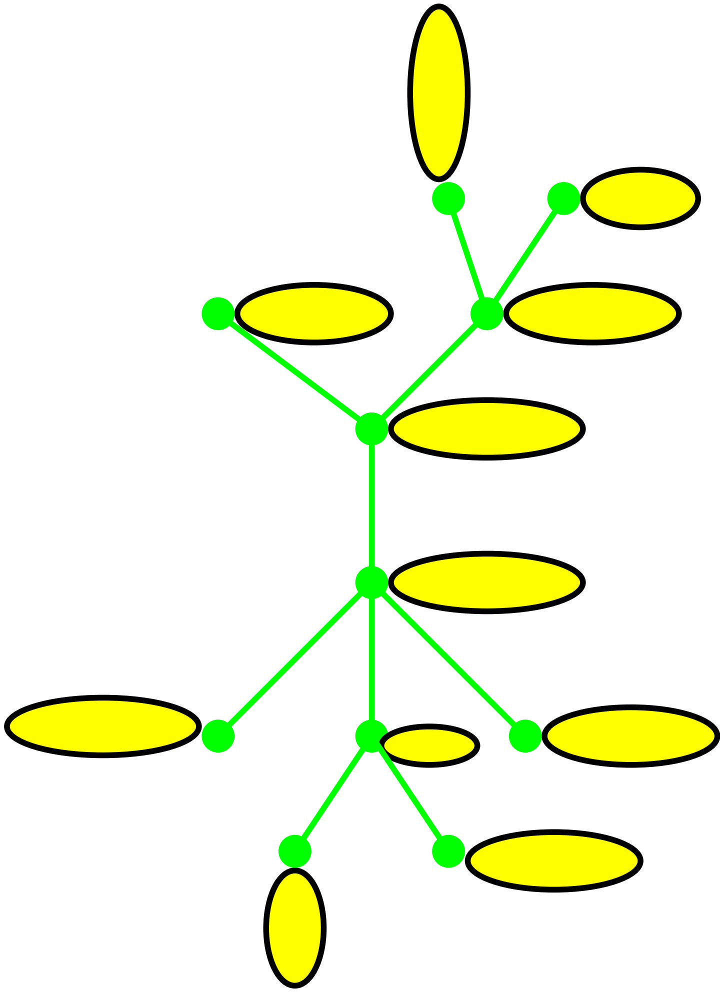
A family of cross-free separations gives rise to a tree-decompositon.



A tree-decomposition of a graph G is (T, W), where T is a tree and $W = (W_t : t \in V(T))$ satisfies

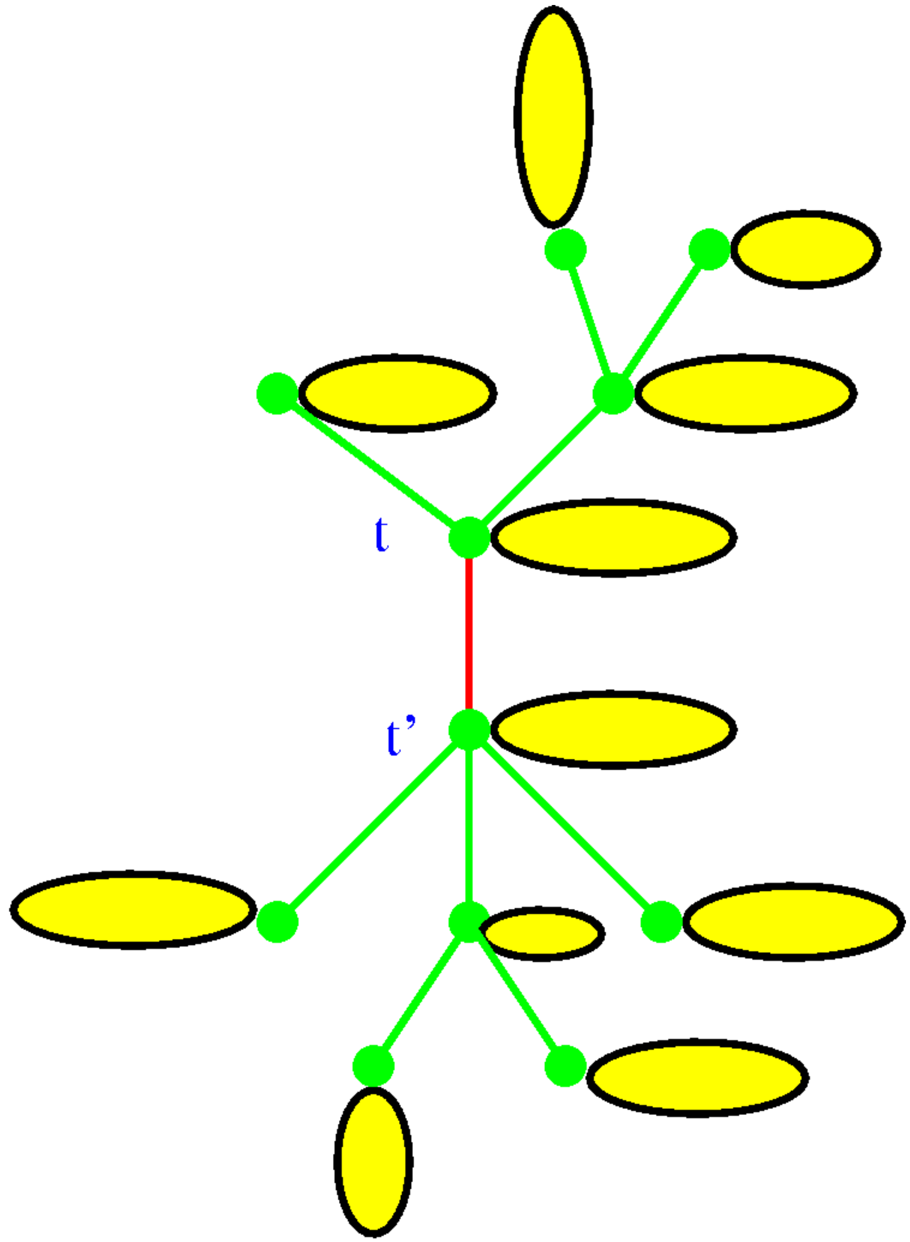
(T1) $\bigcup_{t \in V(T)} W_t = V(G)$,

(T2) if $t' \in T[t, t'']$, then $W_t \cap W_{t''} \subseteq W_{t'}$,

(T3) $\forall uv \in E(G) \exists t \in V(T)$ s.t. $u, v \in W_t$.

The width is $\max(|W_t| - 1 : t \in V(T))$.

The tree-width of G is the minimum width of a tree-decomposition of G.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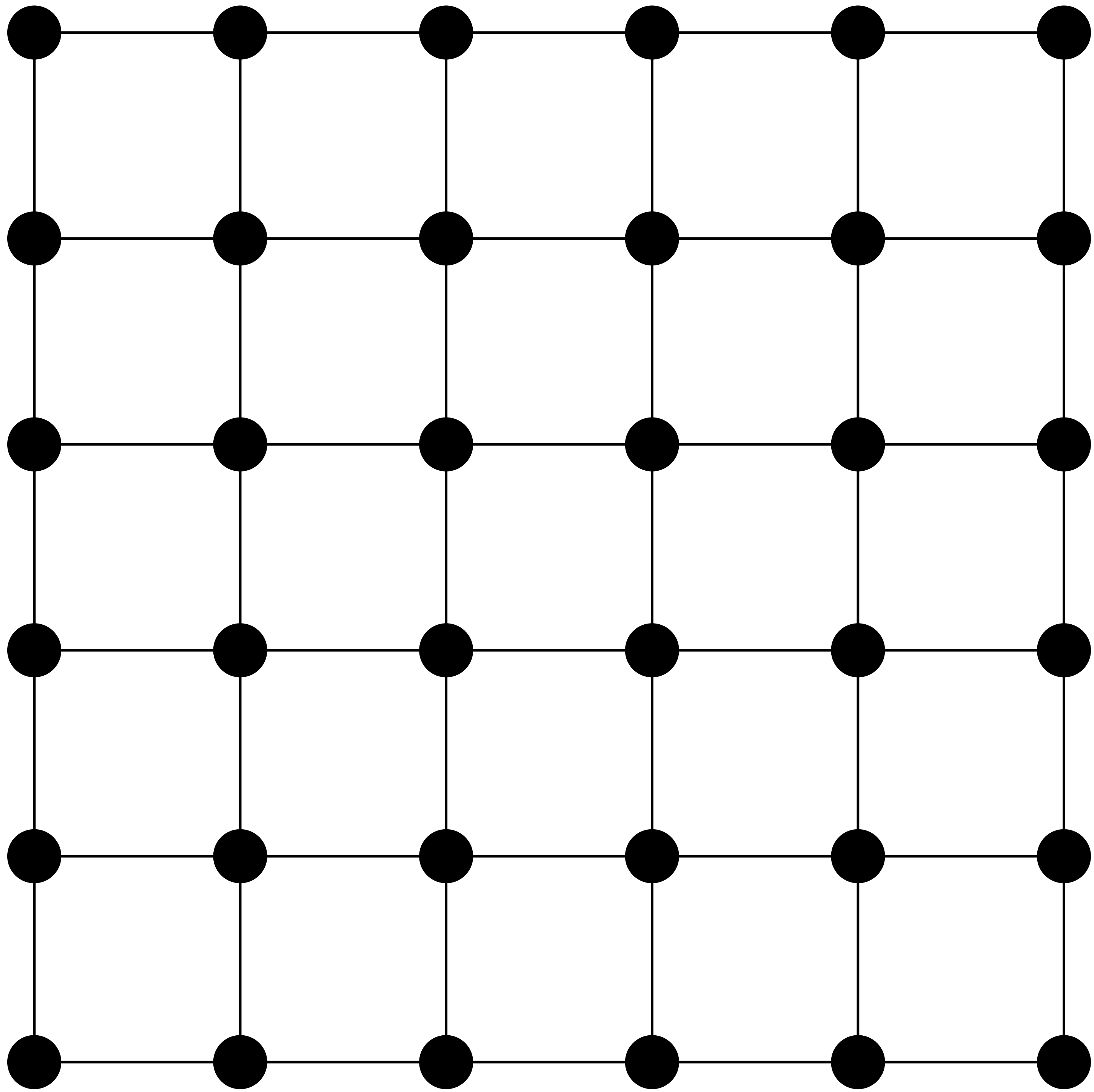
- The $k \times k$ grid has tree-width k



Consider all functions ϕ mapping graphs into integers such that

(1) $\phi(K_n) = n - 1$,

(2) G minor of $H \Rightarrow \phi(G) \leq \phi(H)$,

(3) If $G \cap H$ is a clique, then
$\phi(G \cup H) = \max\{\phi(G), \phi(H)\}$.

Order such functions by $\phi \leq \psi$ if $\phi(G) \leq \psi(G)$ for all G.

THEOREM (Halin) Tree-width is the maximum element in the above poset.

A haven β of order k in G assigns to every $X \in [V(G)]^{<k}$ the vertex-set of a component of $G\backslash X$ such that

(H) $X \subseteq Y \in [V(G)]^{<k} \Rightarrow \beta(Y) \subseteq \beta(X)$.

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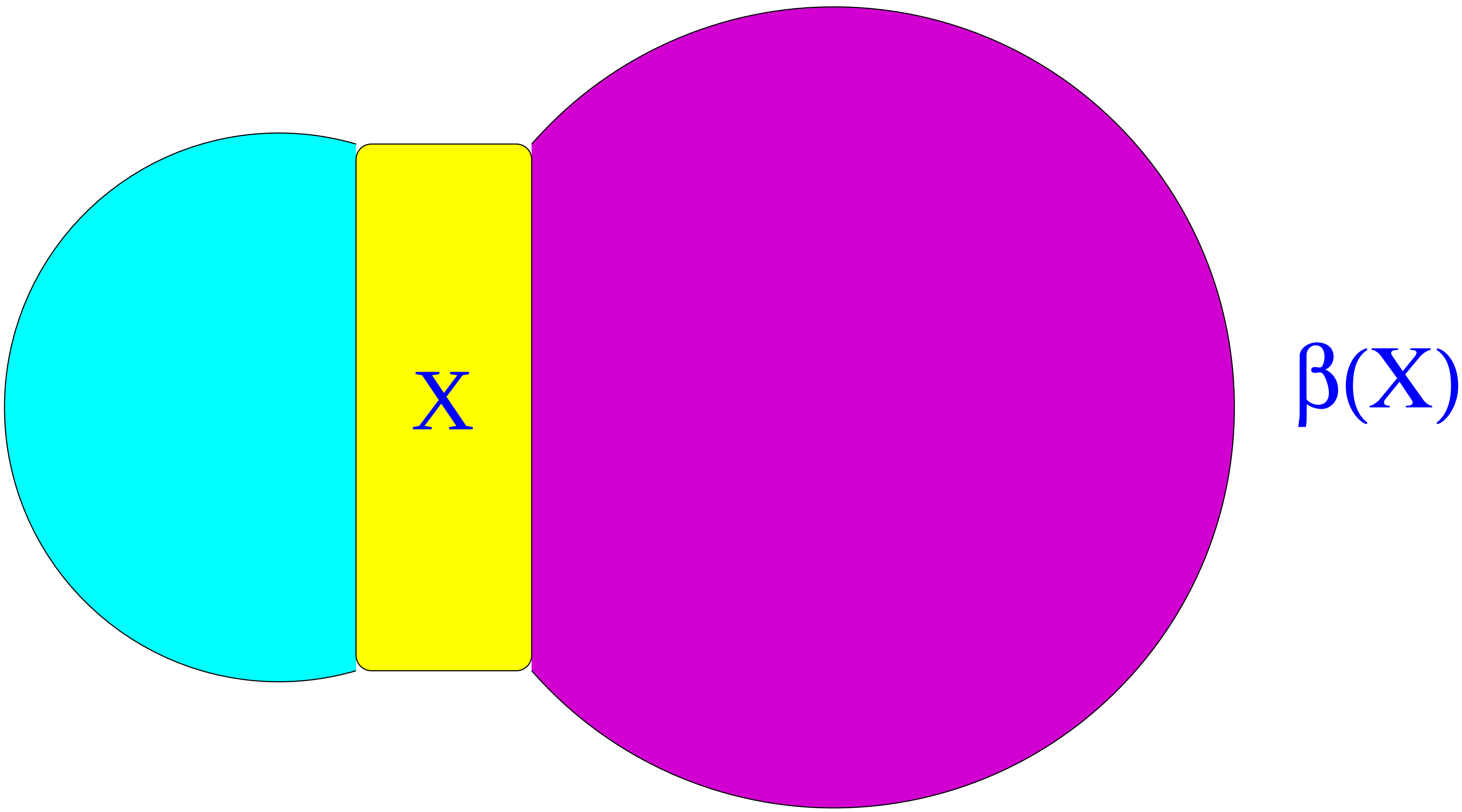

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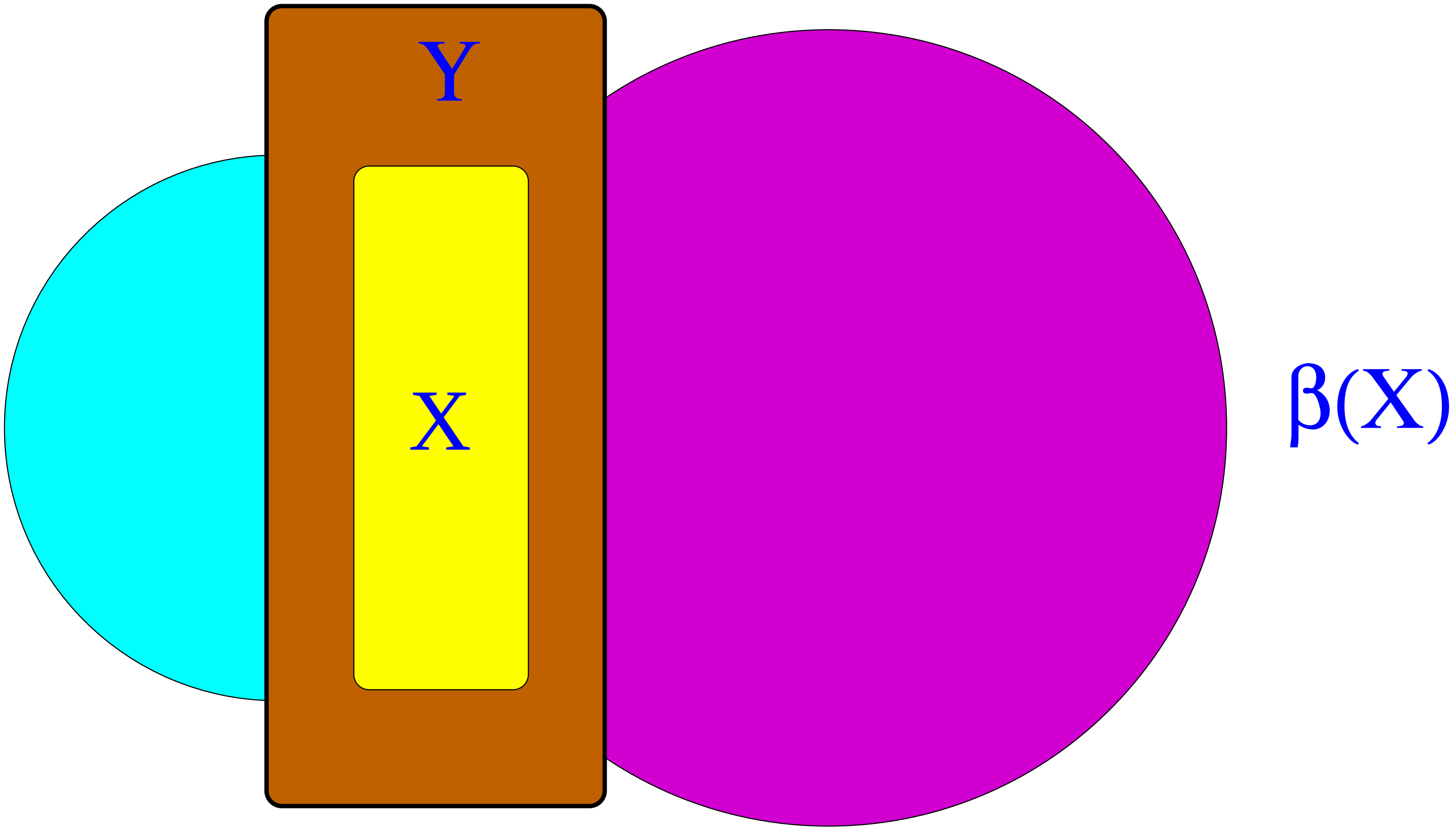

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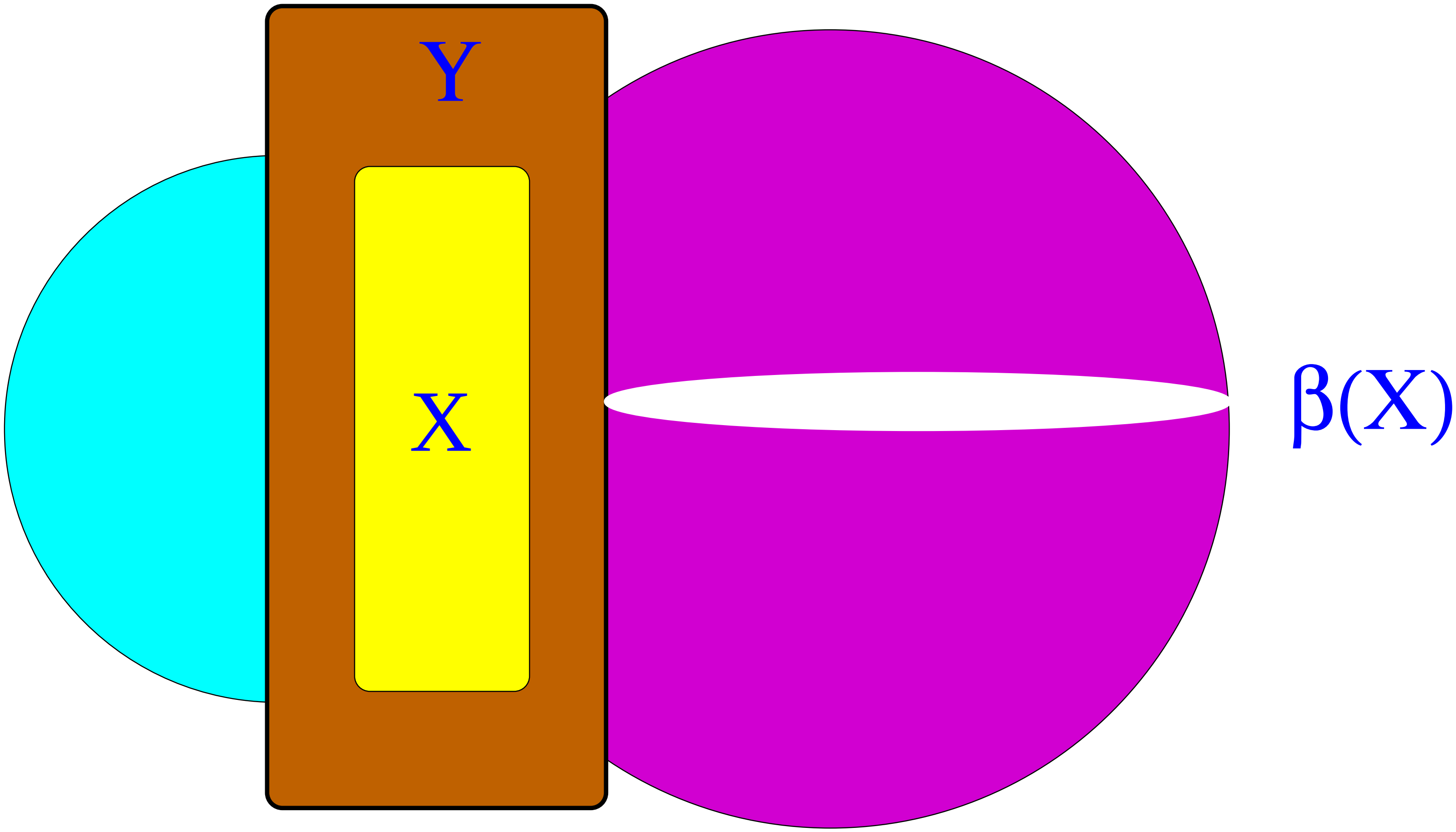

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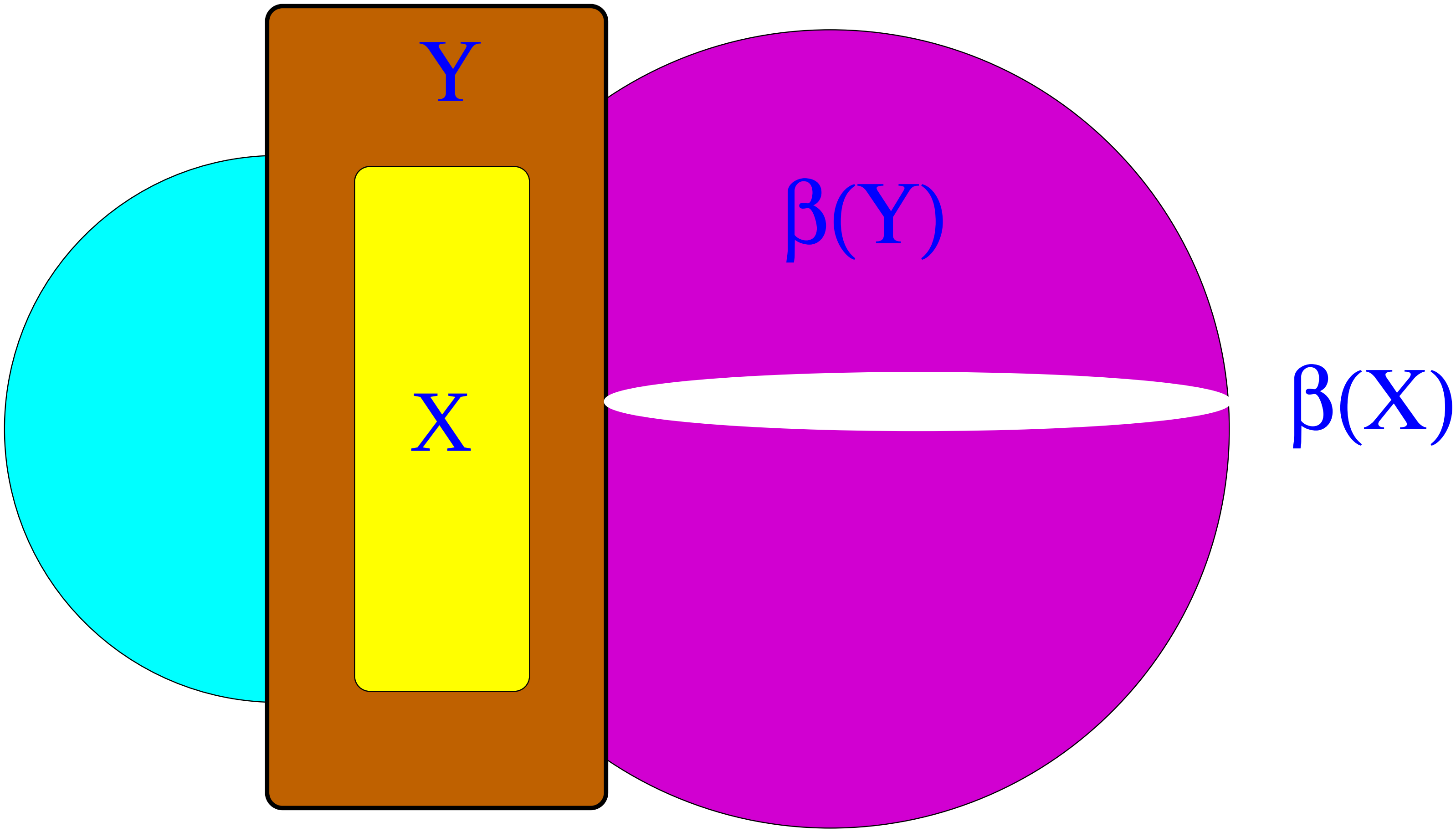

Cops and robbers. Fix a graph G and an integer k. There are k cops, they move slowly in helicopters. There is a robber, who moves infinitely fast along cop-free paths. He can see a helicopter landing, and can run to a safe place before the chopper lands.

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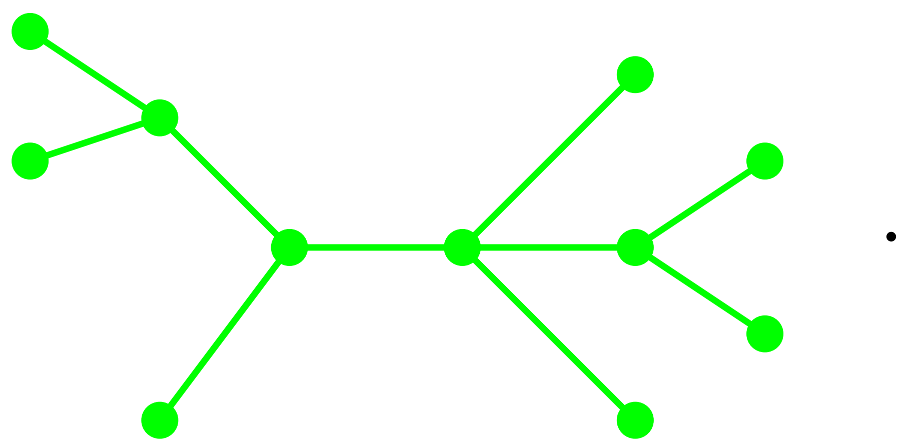

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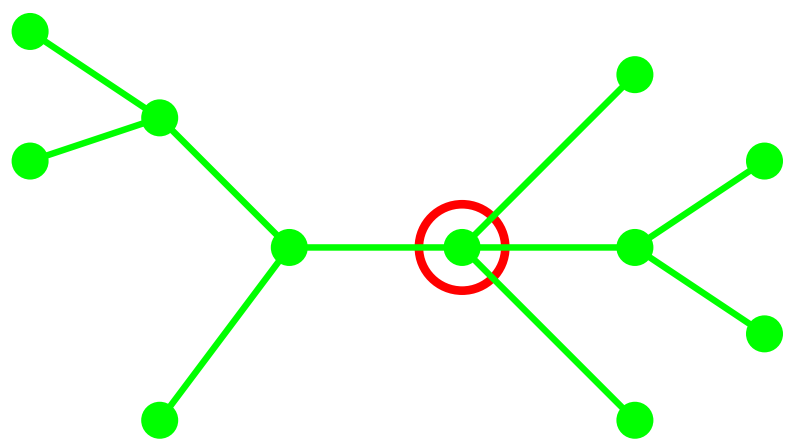

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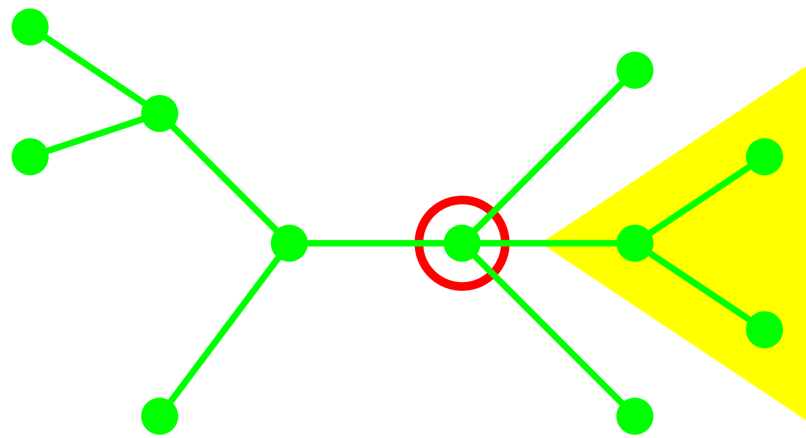

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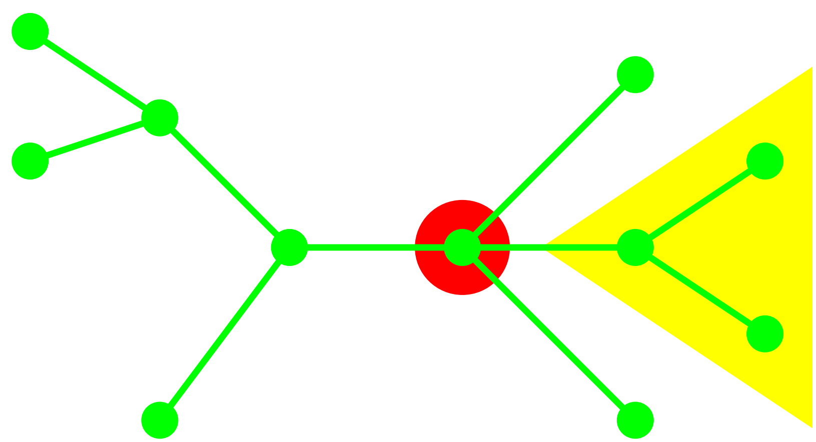

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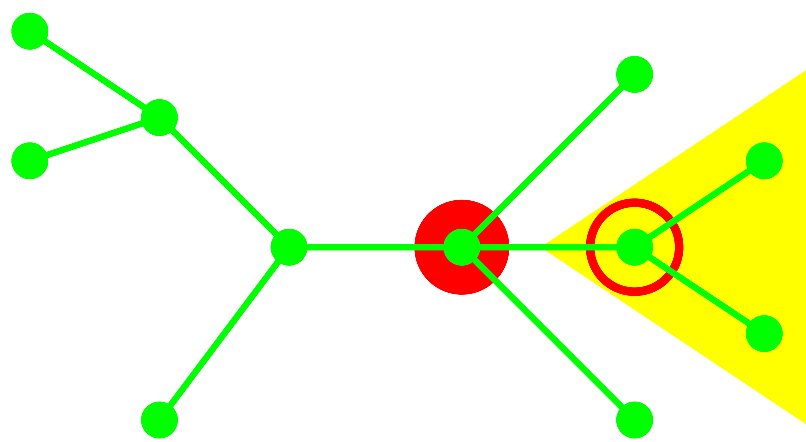

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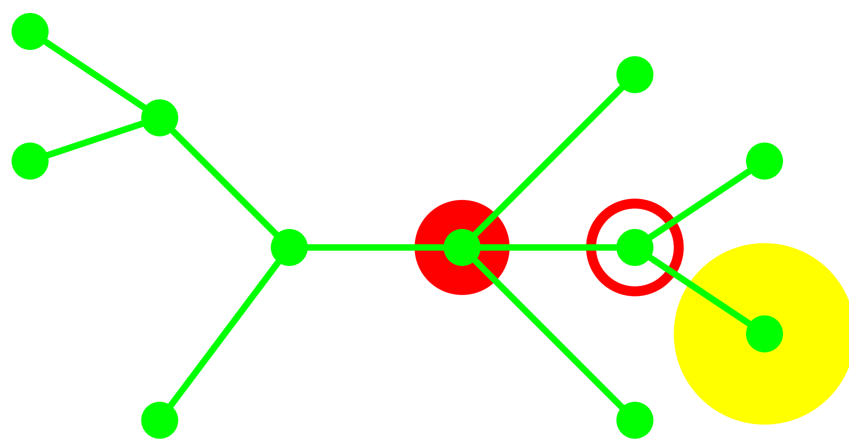

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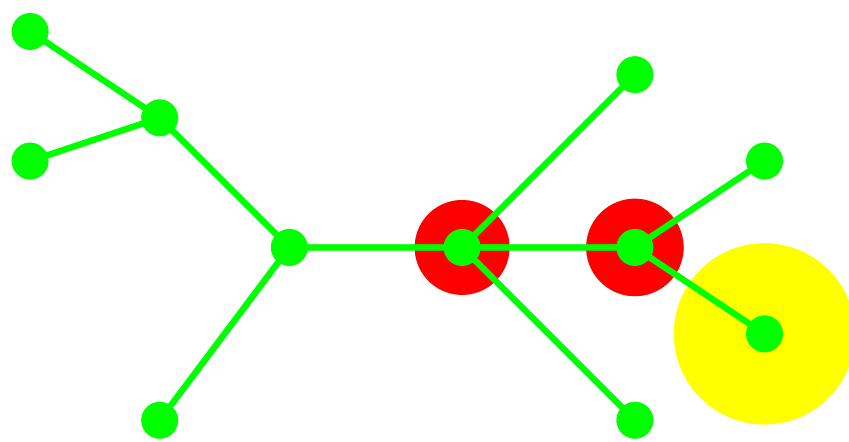

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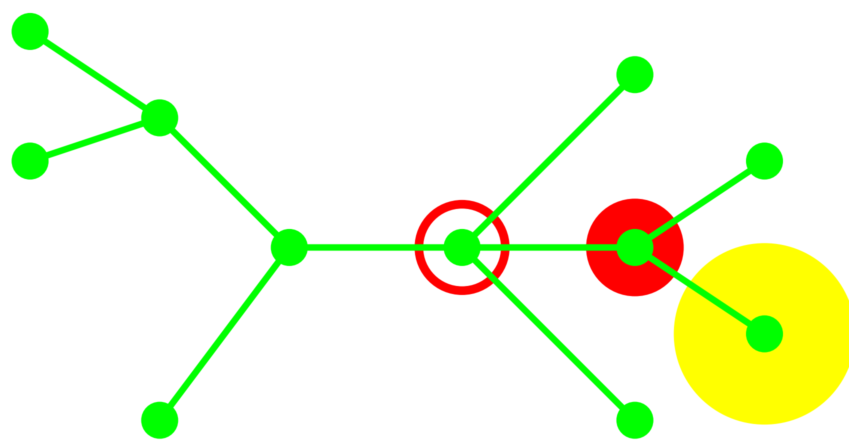

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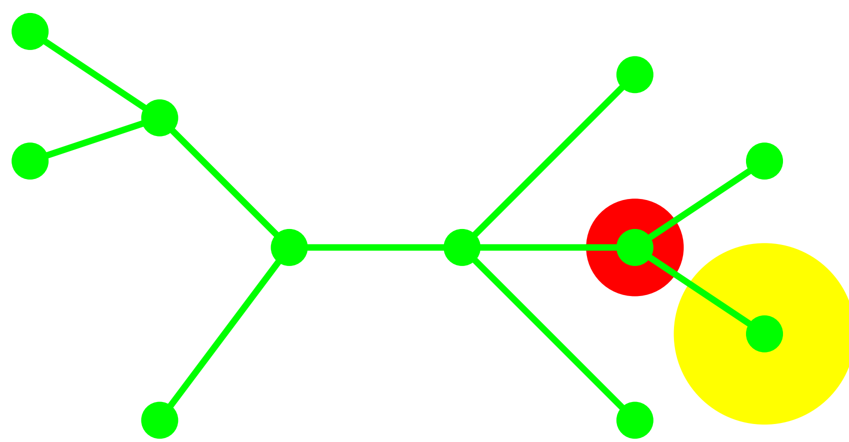

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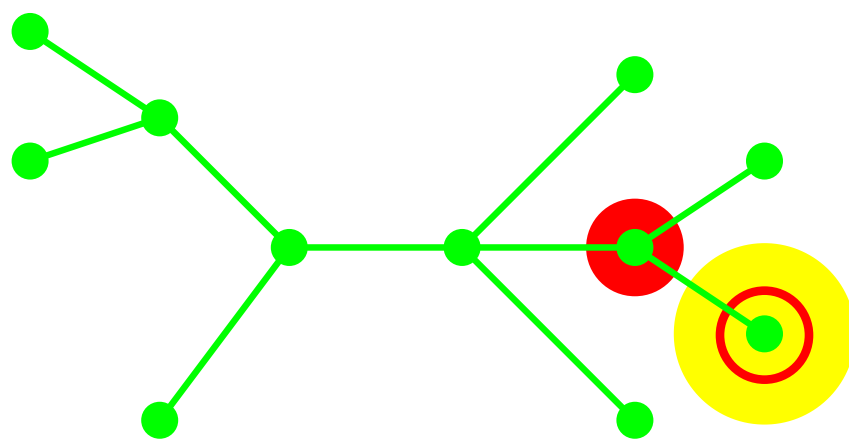

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Fact. A haven gives an escape strategy for the robber.

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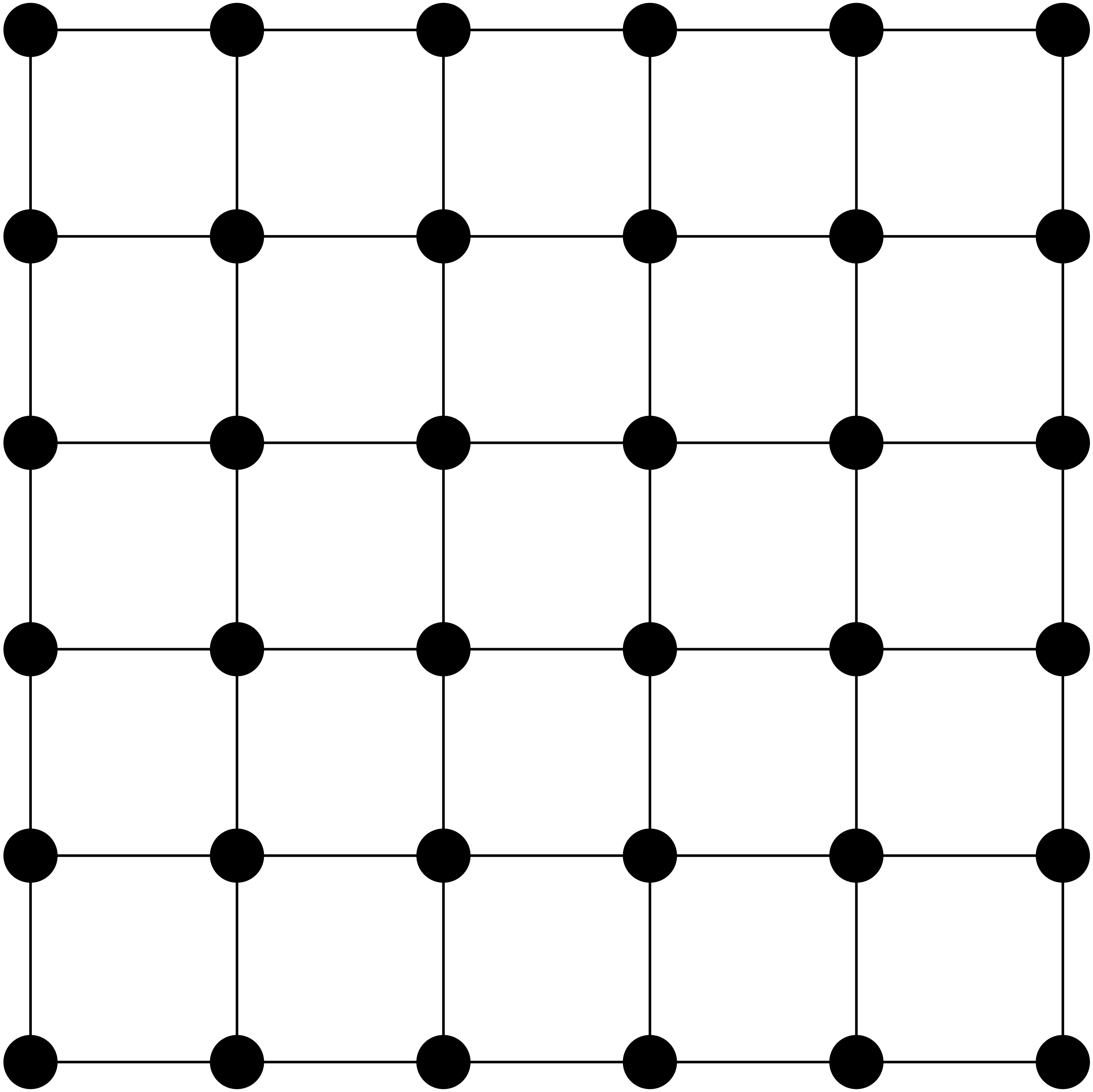
COR Search strategy $\Rightarrow$ monotone search strategy.



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THEOREM (Arnborg, Proskurowski, ...)
Many problems can be solved in linear time when restricted to graphs of bounded tree-width.

Tree-width is useful in

- theory
- design of theoretically fast algorithms
- practical computations

FEEDBACK VERTEX-SET FOR FIXED k

INSTANCE A graph G

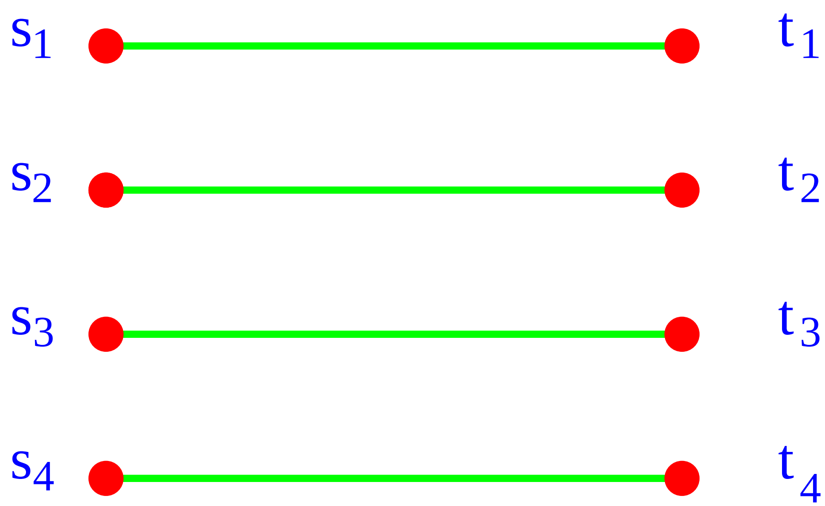
QUESTION Is there a set $X \subseteq V(G)$ such that $|X| \leq k$ and $G \backslash X$ is acyclic?

ALGORITHM If tw(G) is small use bounded tree-width methods. Otherwise answer “no”. That’s correct, because big tree-width $\Rightarrow$ big grid $\Rightarrow$ $k + 1$ disjoint circuits $\Rightarrow$ X does not exist.

k DISJOINT PATHS IN PLANAR GRAPHS

INSTANCE A planar graph G, vertices $s_1, s_2, \ldots, s_k, t_1, t_2, \ldots, t_k$ of G

QUESTION Are there disjoint paths $P_1, .., P_k$ such that P_i has ends s_i and t_i?



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ALGORITHM $\text{tw}(G)$ small $\Rightarrow$ bounded tree-width methods. Otherwise big grid minor $\Rightarrow$ big grid minor with the terminals outside. The middle vertex of this grid minor can be deleted, without affecting the feasibility of the problem.

APPLICATIONS

THEOREM (Erdös, Pósa) There exists a function f such that every graph has either k disjoint cycles, or a set X of at most $f(k)$ vertices such that $G \backslash X$ is acyclic.

THEOREM (Robertson, Seymour) For every planar graph H there exists a function f such that every graph has either k disjoint H minors, or a set X of at most $f(k)$ vertices such that $G \backslash X$ has no H minor.

False for every nonplanar graph H. Open for subdivisions.

THEOREM (Oporowski, Oxley, RT) There exists a function f such that every 3-connected graph on at least $f(t)$ vertices has a minor isomorphic to W_t or $K_{3,t}$.

THEOREM (Oporowski, Oxley, RT) There exists a function f such that every 4-connected graph on at least $f(t)$ vertices has a minor isomorphic to D_t, M_t, O_t, or $K_{4,t}$.

THEOREM (Ding, Oporowski, RT, Vertigan) There exists a function f such that every 4-connected nonplanar graph on at least $f(t)$ vertices has a minor isomorphic to D'_t, M_t, or $K_{4,t}$.

COROLLARY (Ding, Oporowski, RT, Vertigan) There exists a constant c such that every minimal graph of crossing number at least two on at least c vertices belongs to a well-defined family of graphs.

THEOREM (Arnborg, Proskurowski)

Let $P(G, Z)$ be some information about a graph G and set $Z \subseteq V(G)$ such that

(i) $P(G, Z)$ can be computed in constant time if $|V(G)| \leq k + 1$

(ii) if $Z' \subseteq Z$ then $P(G, Z')$ can be computed from $P(G, Z)$ in constant time

(iii) if (A, B) is a separation of G with $A \cap B \subseteq Z$, then $P(G, Z)$ can be computed from $P(G \upharpoonright A, A \cap Z), P(G \upharpoonright B, B \cap Z)$ in constant time.

Then $P(G, \emptyset)$ can be computed in linear time if a tree-decomposition of G of width $\leq k$ is given.

EXAMPLE. For $A \subseteq V(G)$, let α_A be the maximum cardinality of an independent set $I \subseteq V(G)$ with $I \cap Z = A$. Let $P(G, Z) = (\alpha_A : A \subseteq Z)$.

Discrete magnetic Schrödinger operators

$\mathcal{M}_G =$ all Hermitian matrices $A = (a_{ij})$ s.t. $a_{i,j} \neq 0$ if i, j are adjacent and $a_{i,j} = 0$ if $i \neq j$ and not adjacent.

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THEOREM (Colin de Verdière) $\nu(G) \leq tw'(G)$, where $tw'(G)$ is a slight variation of tree-width s.t. $tw(G) \leq tw'(G) \leq tw(G) + 1$.

A path decomposition of G is a sequence $W_1, W_2, \ldots, W_n$ such that

(i) $\bigcup W_i = V(G)$, and every edge has both ends in some W_i, and

(ii) if $i < i' < i''$ then $W_i \cap W_{i''} \subseteq W_{i'}$

The width of $W_1, \ldots, W_n$ is

$$\max\{|W_i| - 1 : 1 \leq i \leq n\}$$

The path-width of G is the minimum width of a path-decomposition.

THM F forest, $pw(G) \geq |V(F)| - 1 \Rightarrow F \leq_m G$.

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HISTORY Originally due to Robertson and Seymour.
Current bound by Bienstock, Robertson, Seymour, RT.
New proof by Diestel.

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Diestel's proof. Let $V(F) = \{v_1, v_2, \dots, v_k\}$ s.t. v_i is adjacent to ≤ 1 v_j for $j < i$. Let $\mathcal{L} = \{(A, B) : G \upharpoonright B$ has no path-decomposition $W_1, W_2, \dots, W_t$ of width $\leq k - 2$ with $A \cap B \subset W_1\}$.

Choose $i \in \{0, 1, \dots, k\}$ and $(A, B) \in \mathcal{L}$ such that

(i) $G \upharpoonright A$ has a minor isomorphic to $F \upharpoonright \{v_1, \dots, v_i\}$ s.t. each "node" intersects $A \cap B$ in precisely one vertex

(ii) $\nexists\, (A', B') \in \mathcal{L}$ with $A \subseteq A'$, $B \supseteq B'$, $|A' \cap B'| < |A \cap B|$

(iii) i is maximum subject to (i) and (ii)

(iv) $|B|$ is minimum subject to (i), (ii), (iii)

CLAIM $\nexists\ (A', B') \in \mathcal{L}$ with $A \subseteq A'$, $B \supseteq B'$, $|A' \cap B'| \leq |A \cap B|$.

PROOF OF CLAIM. Suppose not. By (ii) equality holds.

PROOF OF THM Let j be the only index $\leq i$ such that $v_{i+1} \sim v_j$. Pick any vertex of $B - A$ adjacent to the unique vertex of $A \cap B$ that belongs to the v_j-node.

Let (T, W) be a tree-decomposition of G and $t \in V(T)$. The torso at t is $G \restriction W_t$ plus all edges with both ends in $W_t \cap W_{t'}$ for some $t' \sim t$.

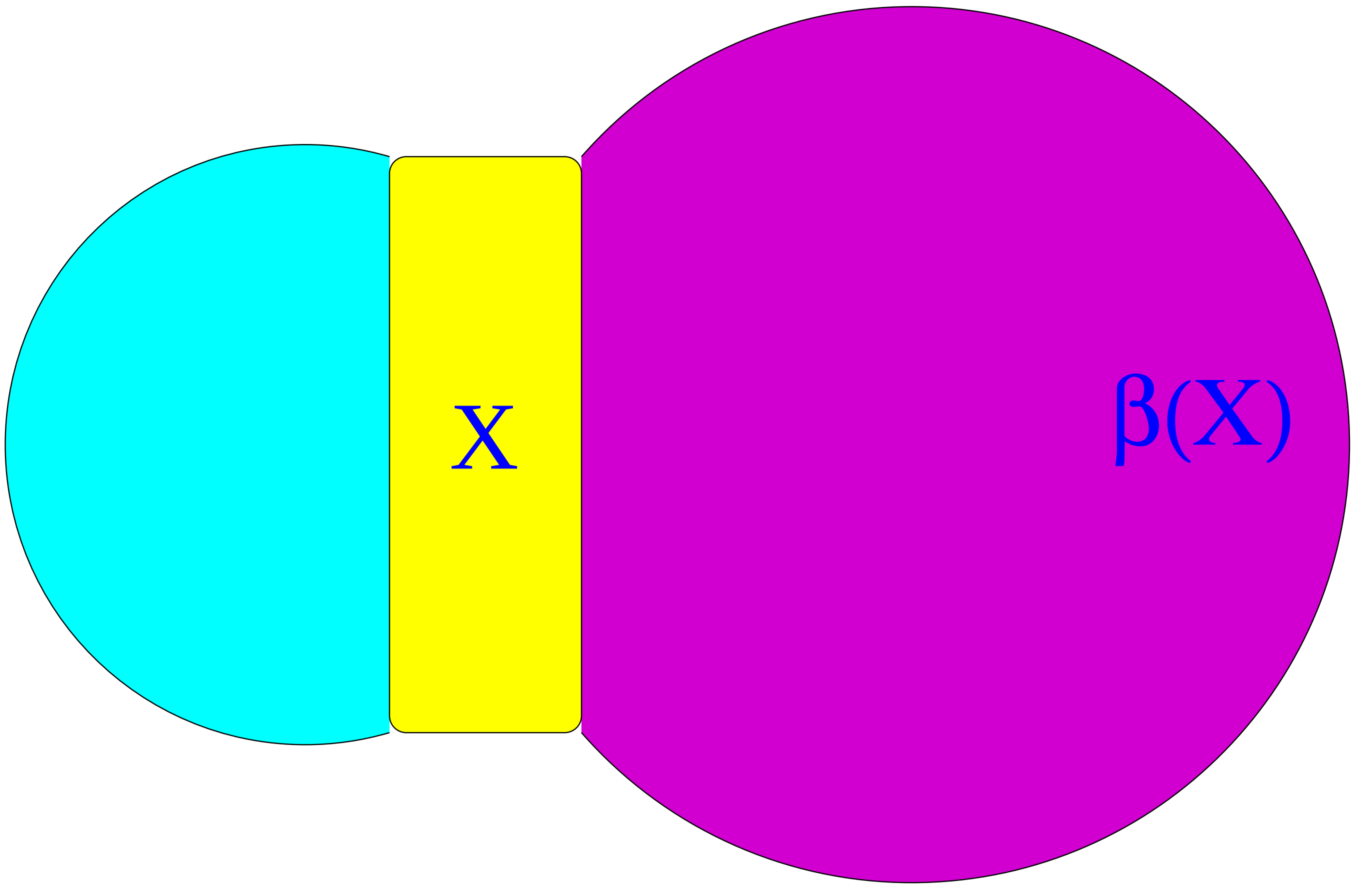
(T, W) is a tree-decomposion over $\mathcal{F}$ if every torso belongs to $\mathcal{F}$.

HOW TO USE A HAVEN?

REMINDER A haven β of order k in D assigns to every $X \in [V(D)]^{<k}$ the vertex-set of a strong component of $D \backslash X$ such that

(H) $X \subseteq Y \in [V(D)]^{<k} \Rightarrow \beta(Y) \subseteq \beta(X)$.

Let β be a haven of order k in G. Let $X \subseteq V(G)$ with $|X| \leq k/2$ and $\beta(X)$ minimum. Then X is "externally linked":



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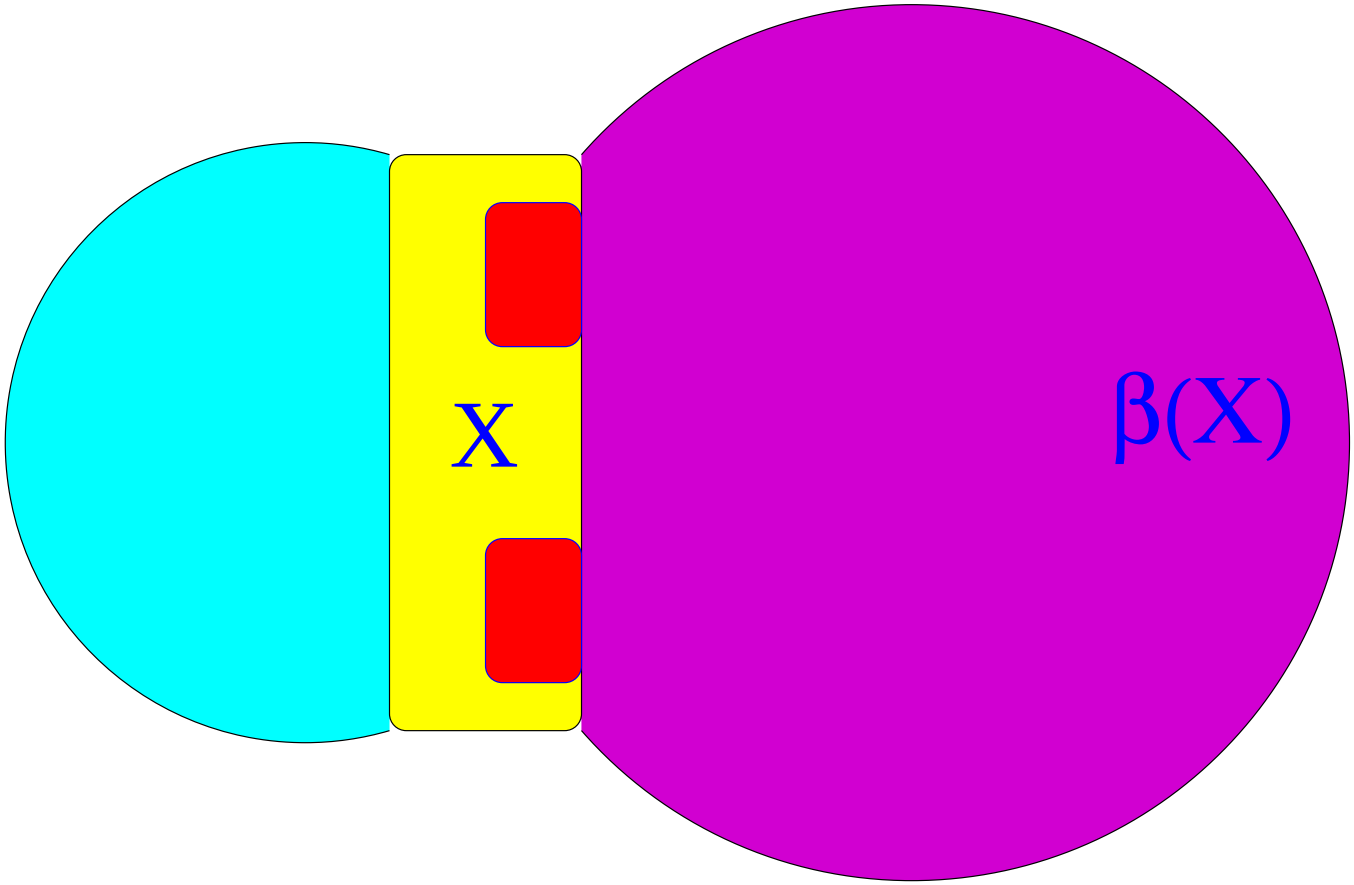

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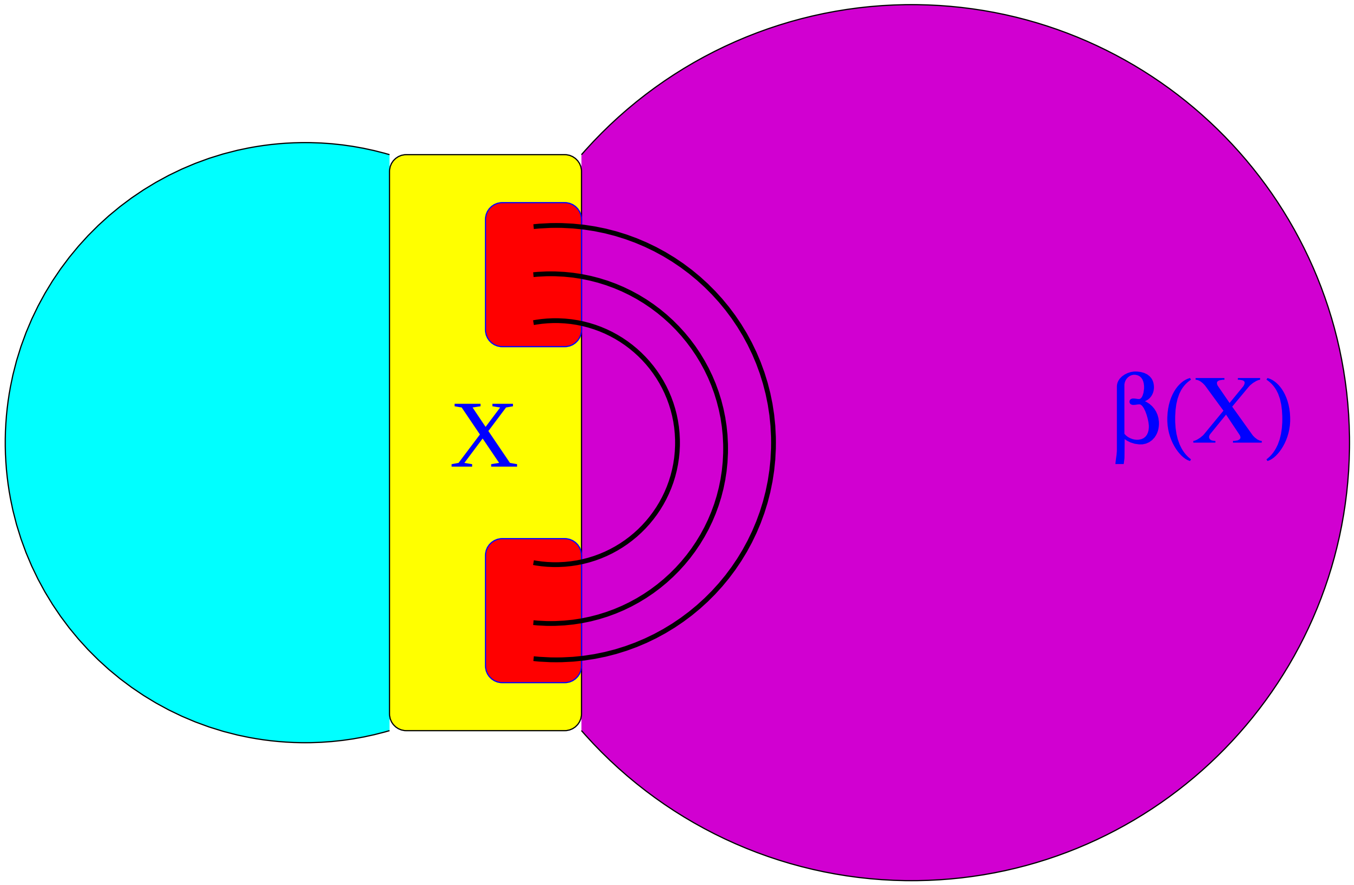

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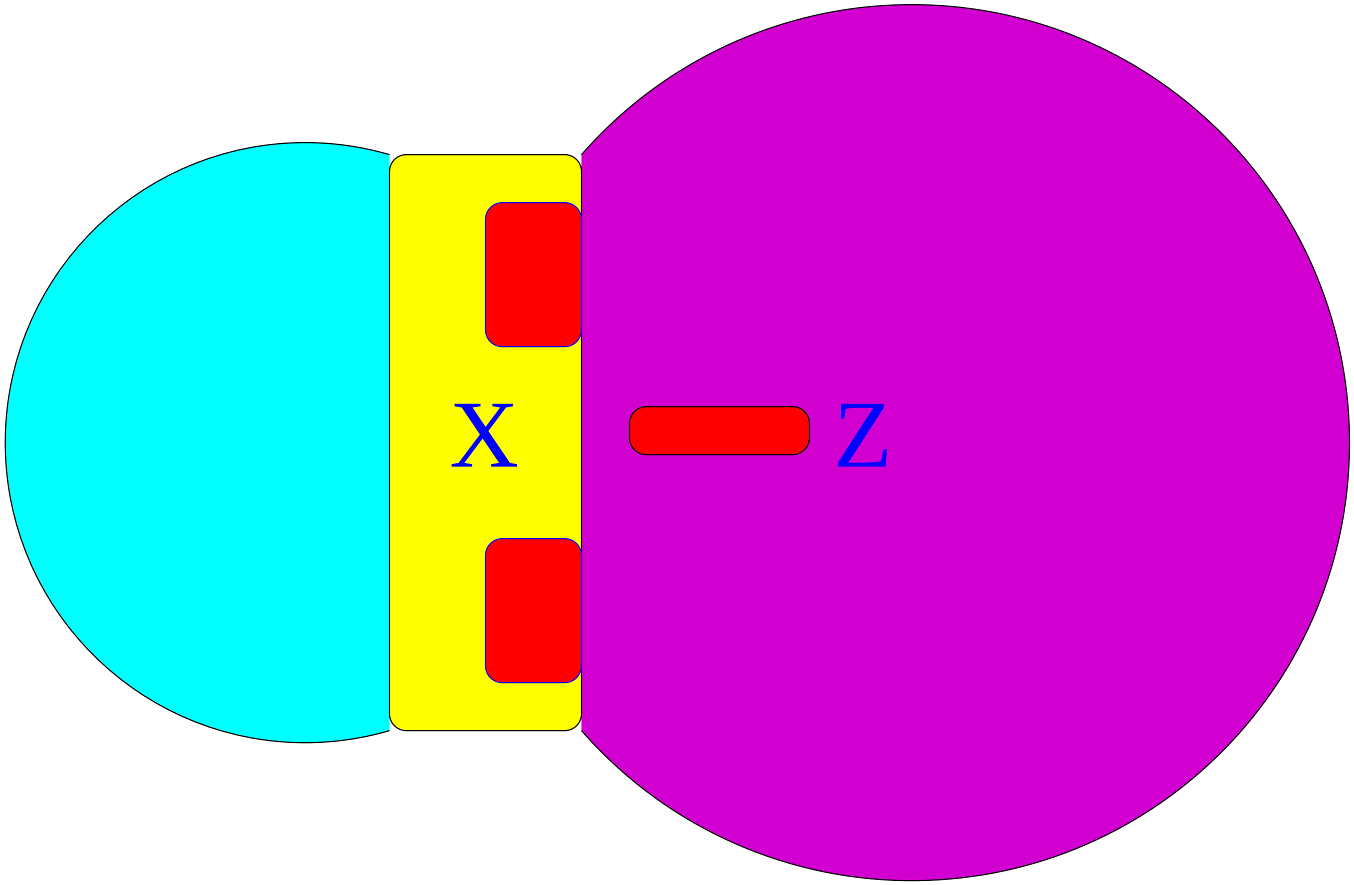

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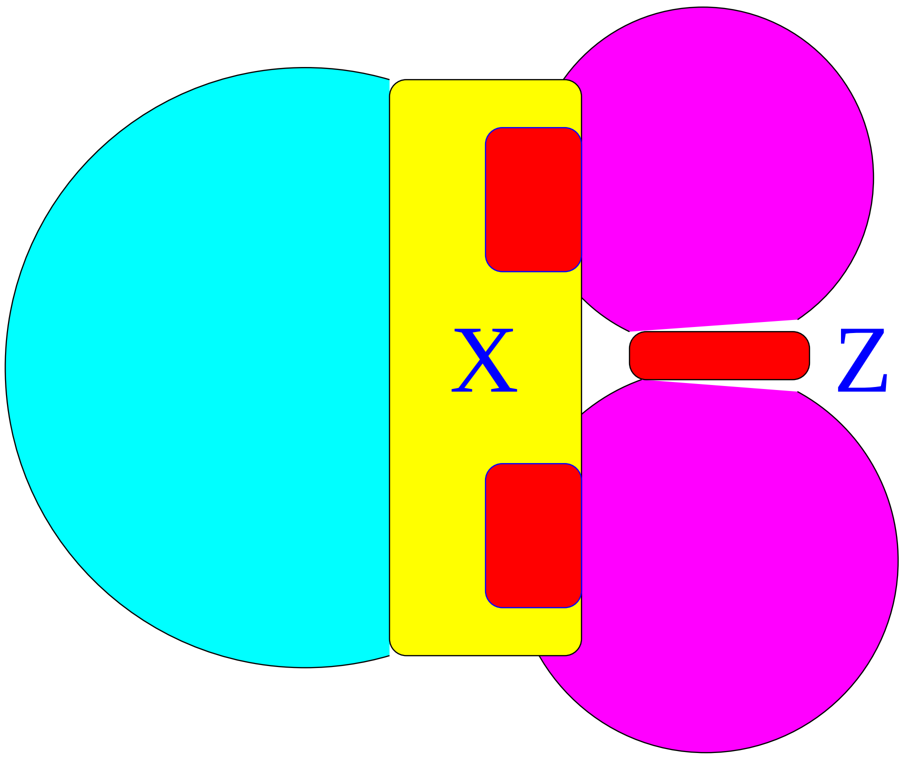

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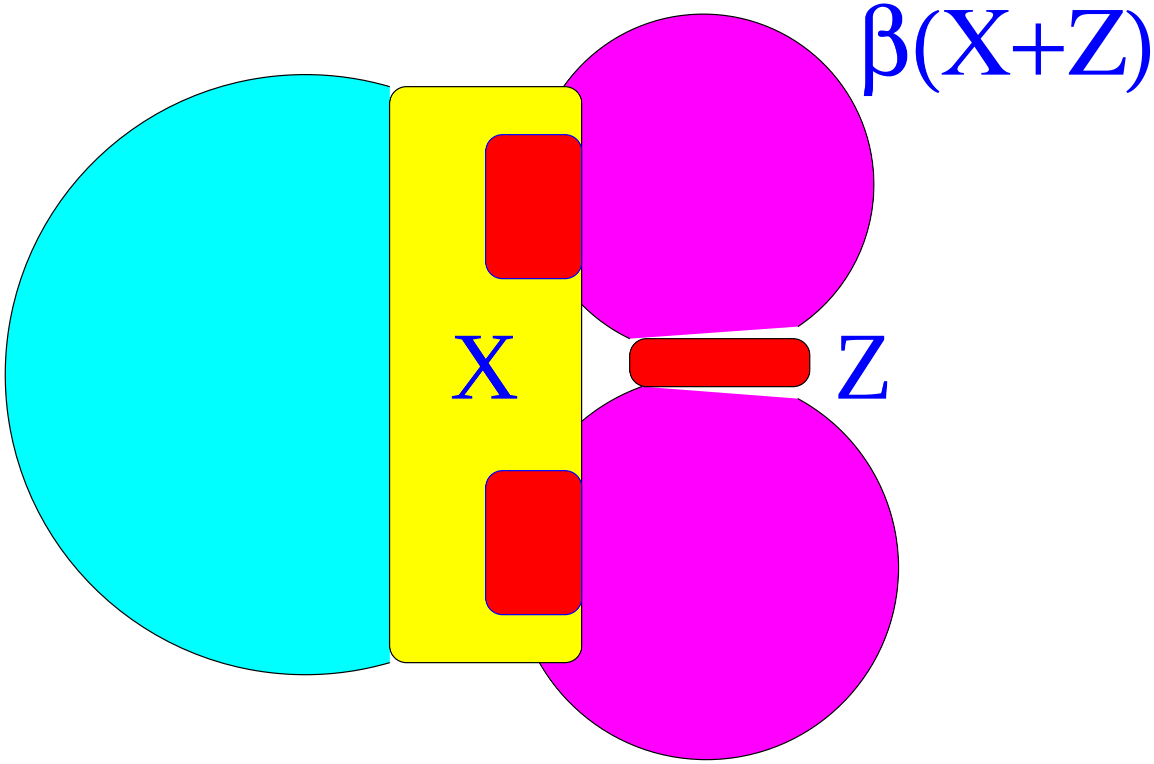

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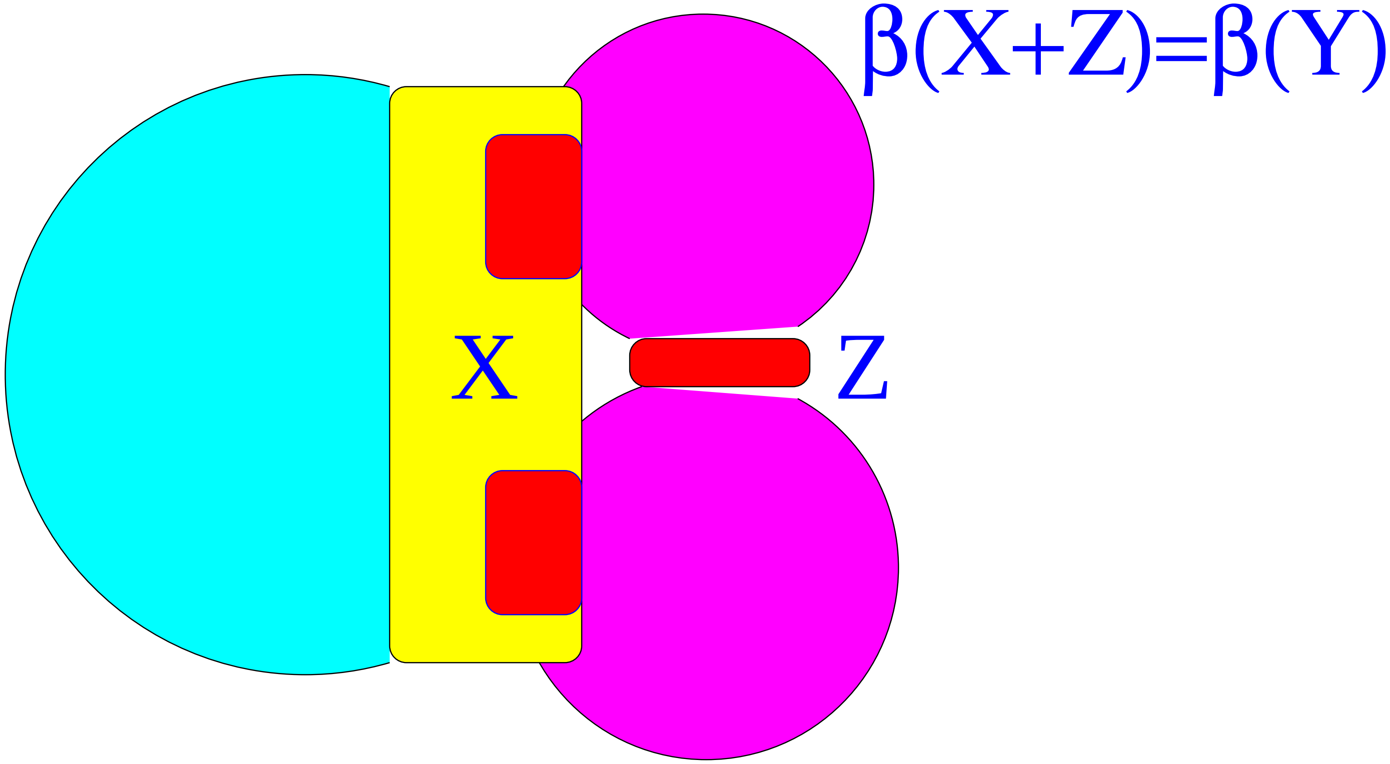

Let Σ be a surface with k holes, $C_1, \ldots, C_k$ their boundaries ("cuffs").

A graph G can be nearly embedded in Σ if G has a set X of at most k vertices such that $G \backslash X$ can be written as $G_0 \cup G_1 \cup \ldots \cup G_k$, where for $i > 0$:

1. G_0 has an embedding in Σ
2. G_i are pairwise disjoint
3. $U_i := V(G_0) \cap V(G_i) = V(G_0) \cap C_i$
4. G_i has a path decomposition $(X_u)_{u \in U_i}$ of width $< k$, s.t. $u \in X_u$ for all $u \in U_i$ (the order on U_i given by C_i)

NOTATION: $G \in \mathcal{F}(\Sigma)$

$\Sigma - k = \Sigma$ with k holes removed

$\Sigma_H =$ orientable surface of largest genus that does not embed H

Σ'_H same for non-orientable

THEOREM (Robertson, Seymour)
For every finite graph H there exists $k \geq 0$ such that every graph with no H minor has a tree-decomposition over

$$\mathcal{F}(\Sigma_H - k) \cup \mathcal{F}(\Sigma'_H - k).$$

INFINITE GRAPHS

THEOREM (Halin) A graph has no ray ($=1$-way infinite path) $\Leftrightarrow$ it has a tree-decomposition (T, W) such that T is rayless and each W_t is finite.

With Robertson and Seymour we characterize graphs with no K_κ minor, no T_κ subdivision, or no half-grid minor. Havens and searching play an important role.

SAMPLE RESULT. A graph has no $T_{\aleph_1}$-minor $\Leftrightarrow$ it has no tree-decomposition (T, W), where T is rayless and each W_t is at most countable.

MOTIVATION

THEOREM (RT) There exists a sequence $G_1, G_2, \ldots$ of (uncountable) graphs such that for $i < j$ G_j has no G_i minor.

CONJECTURE True for countable graphs.

THEOREM (RT) Known when G_1 is finite and planar.

FACT Not known even when every component is finite.

LEMMA (Kříž, RT) Let $\mathcal{F}$ be "compact" (if every finite subgraph of G belongs to $\mathcal{F}$, then $G \in \mathcal{F}$). If every finite subgraph of G has a tree-decomposition over $\mathcal{F}$, then so does G.

THEOREM (Diestel, Thomas) For every finite graph H there exists an integer k such that every (infinite) graph with no H minor has a tree-decomposition over

$$\mathcal{F}(\Sigma_H - k) \cup \mathcal{F}(\Sigma'_H - k).$$

A graph G is plane with one vortex if for some k it has a near-embedding $G_0, G_1, .., G_k$ in the sphere with k holes, where $G_2, \ldots, G_k$ are null.

A tree-dec. (T, W) has finite adhesion if

- for every t, $|W_t \cap W_{t'}|$ is bounded $(t' \sim t)$,
- for every ray $t_1, t_2, \ldots$ in T, $\liminf |W_{t_i} \cap W_{t_{i+1}}|$ is finite.

THEOREM (Diestel, Thomas) An infinite graph has no $K_{\aleph_0}$-minor if and only if it has a tree-decomposition of finite adhesion over plane graphs with at most one vortex.

THEOREM (Robertson, Seymour, RT)

Every planar graph with no minor isomorphic to a $g \times g$ grid has tree-width $< 5g$.

PROOF Suppose G has tree-width $\geq 5g$. Then G has a haven β of order $\geq 5g$. Take a planar drawing of G and a circular cutset X of order $\leq 4g$ with $\beta(X)$ inside X and with inside of X minimal.