TREE-DECOMPOSITIONS OF GRAPHS II.

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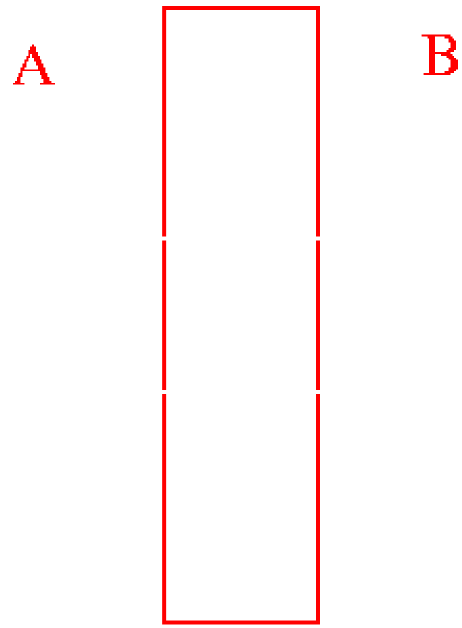

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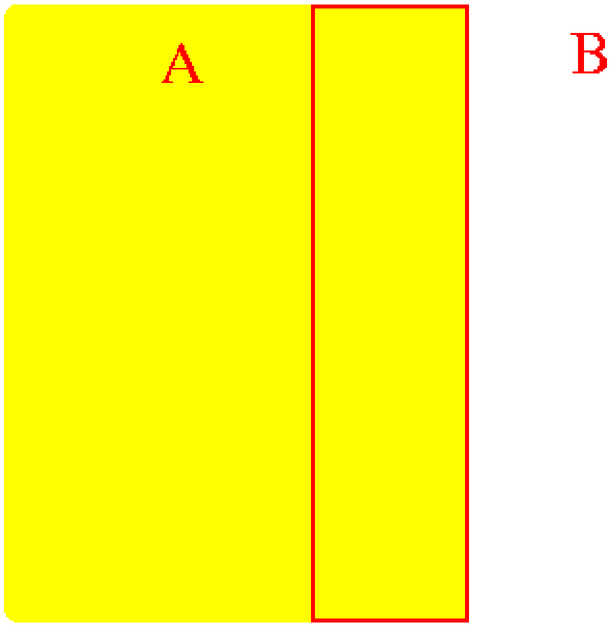

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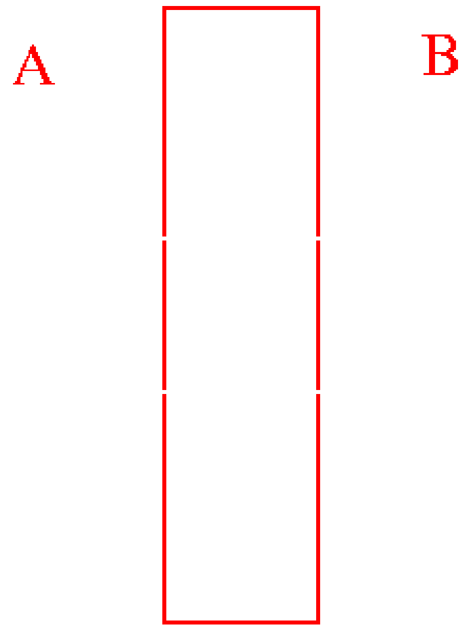

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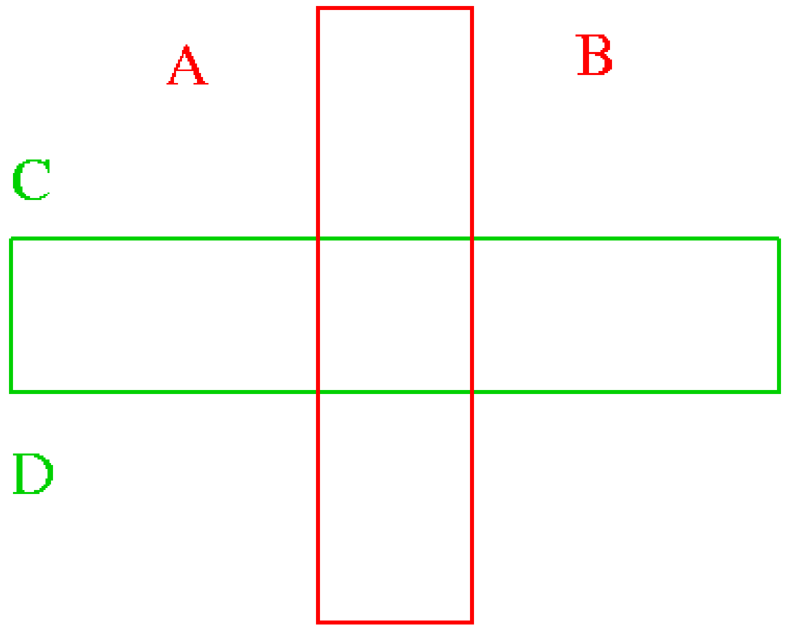

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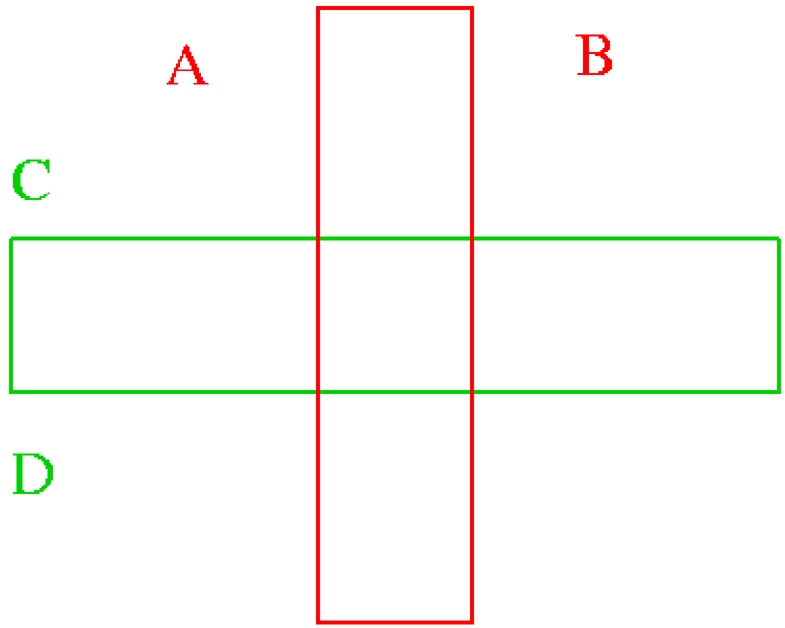

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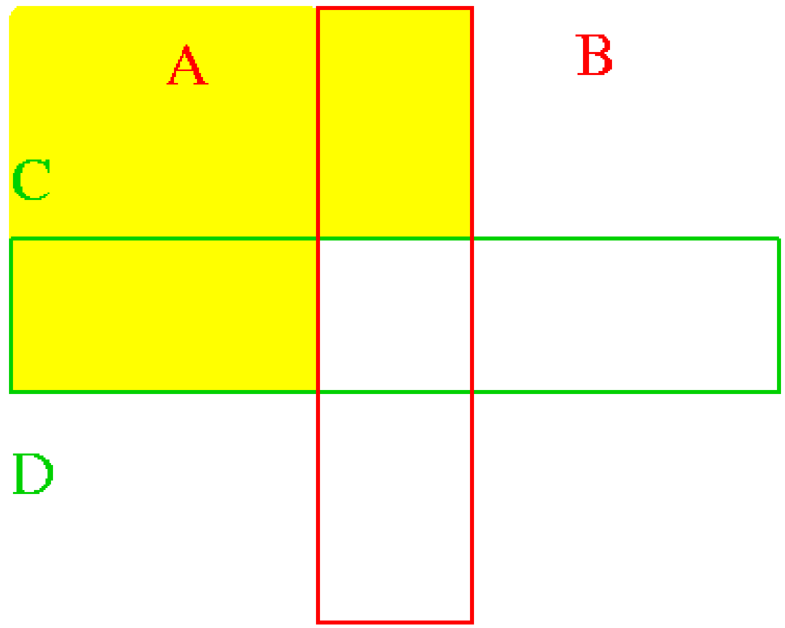

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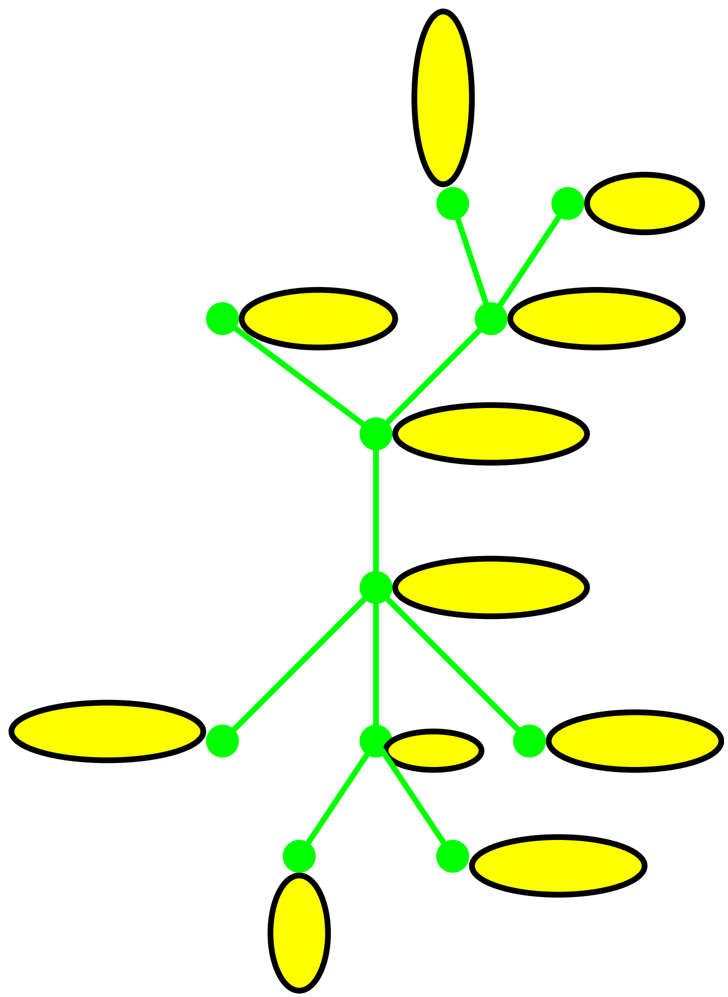
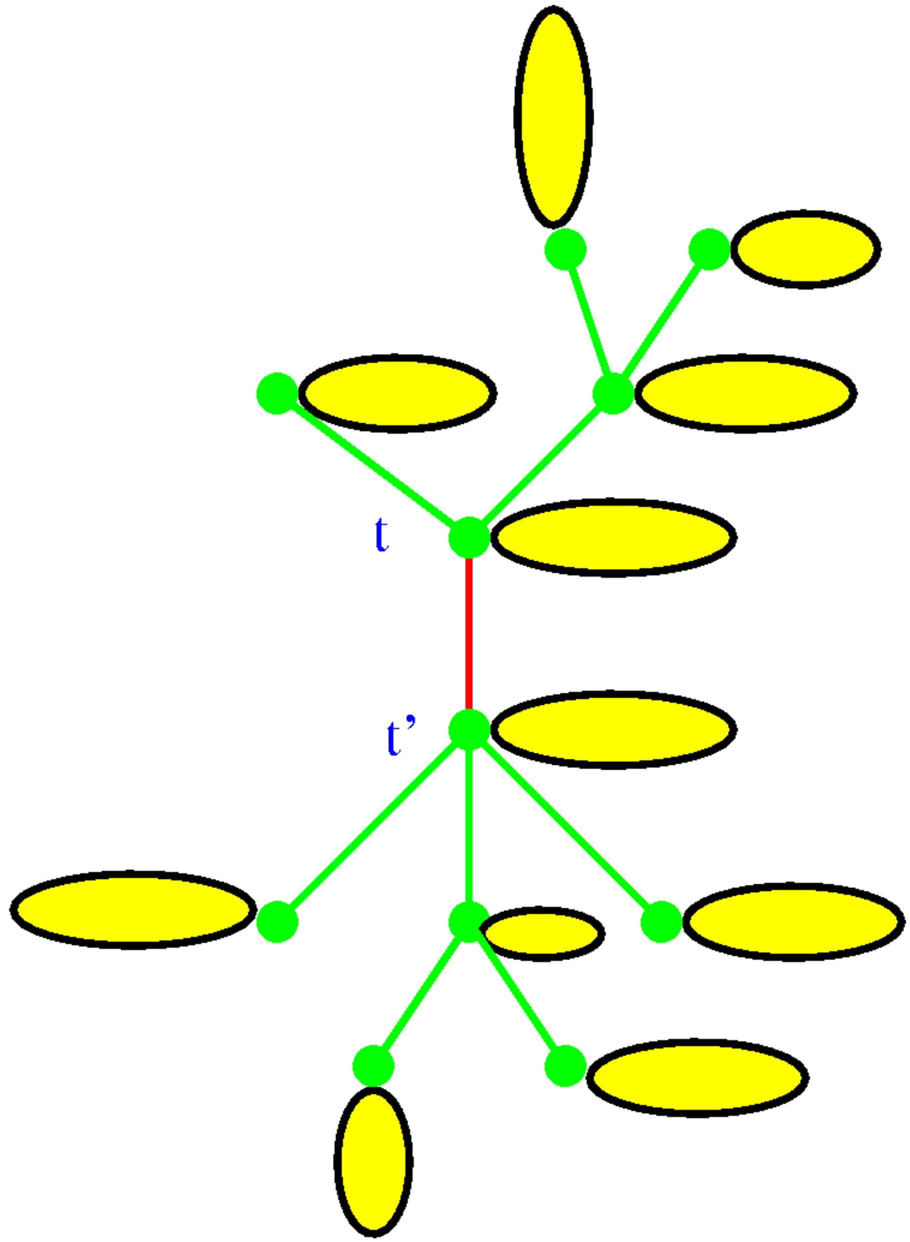
A family of cross-free separations gives rise to a tree-decompositon.



A tree-decomposition of a graph G is (T, W), where T is a tree and $W = (W_t : t \in V(T))$ satisfies
(T1) $\bigcup_{t \in V(T)} W_t = V(G)$,
(T2) if $t' \in T[t, t'']$, then $W_t \cap W_{t''} \subseteq W_{t'}$,
(T3) $\forall uv \in E(G) \exists t \in V(T)$ s.t. $u, v \in W_t$.

The width is $\max(|W_t| - 1 : t \in V(T))$.

The tree-width of G is the minimum width of a tree-decomposition of G.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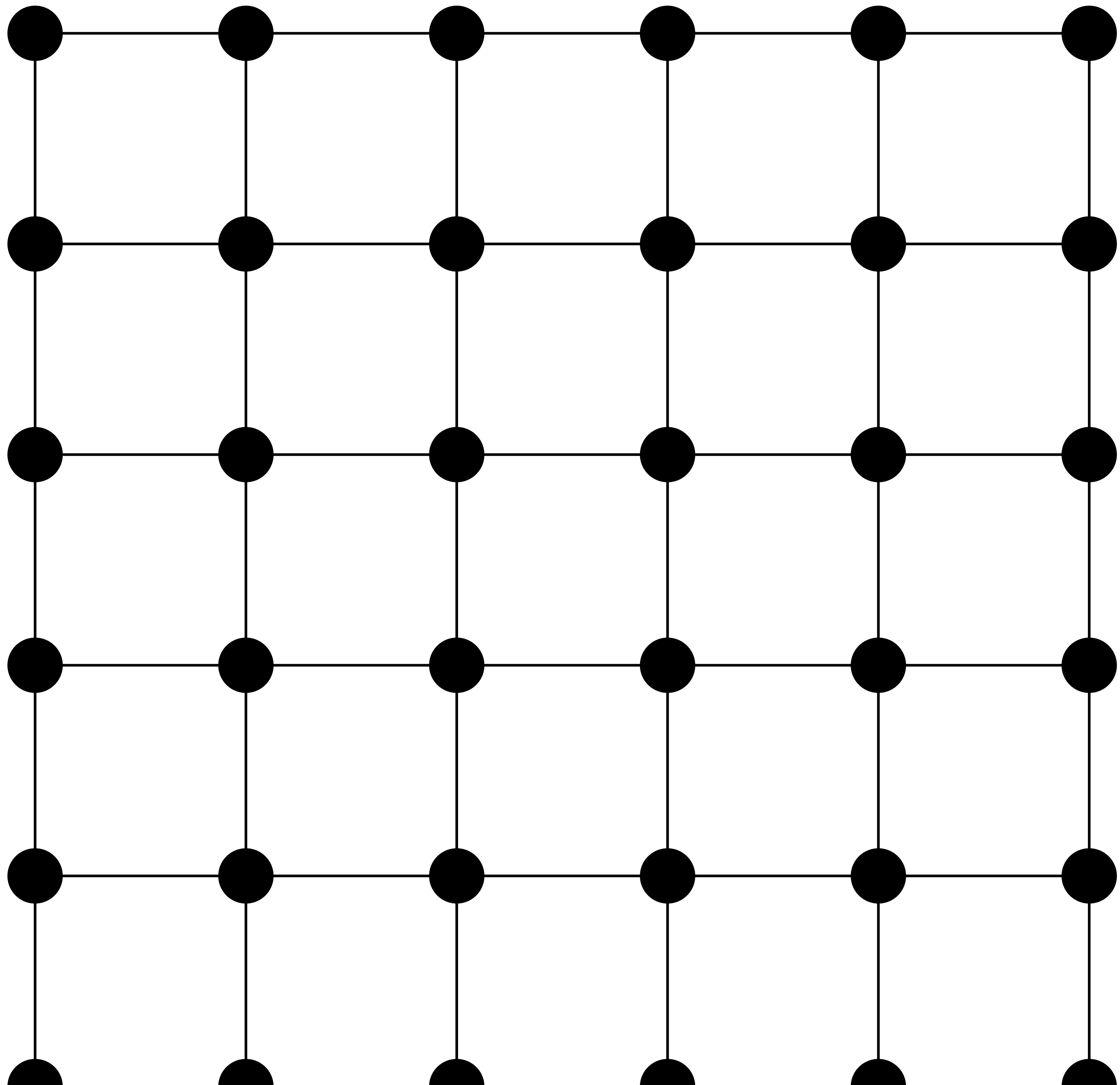

A haven β of order k in G assigns to every $X \in [V(G)]^{<k}$ the vertex-set of a component of $G \backslash X$ such that

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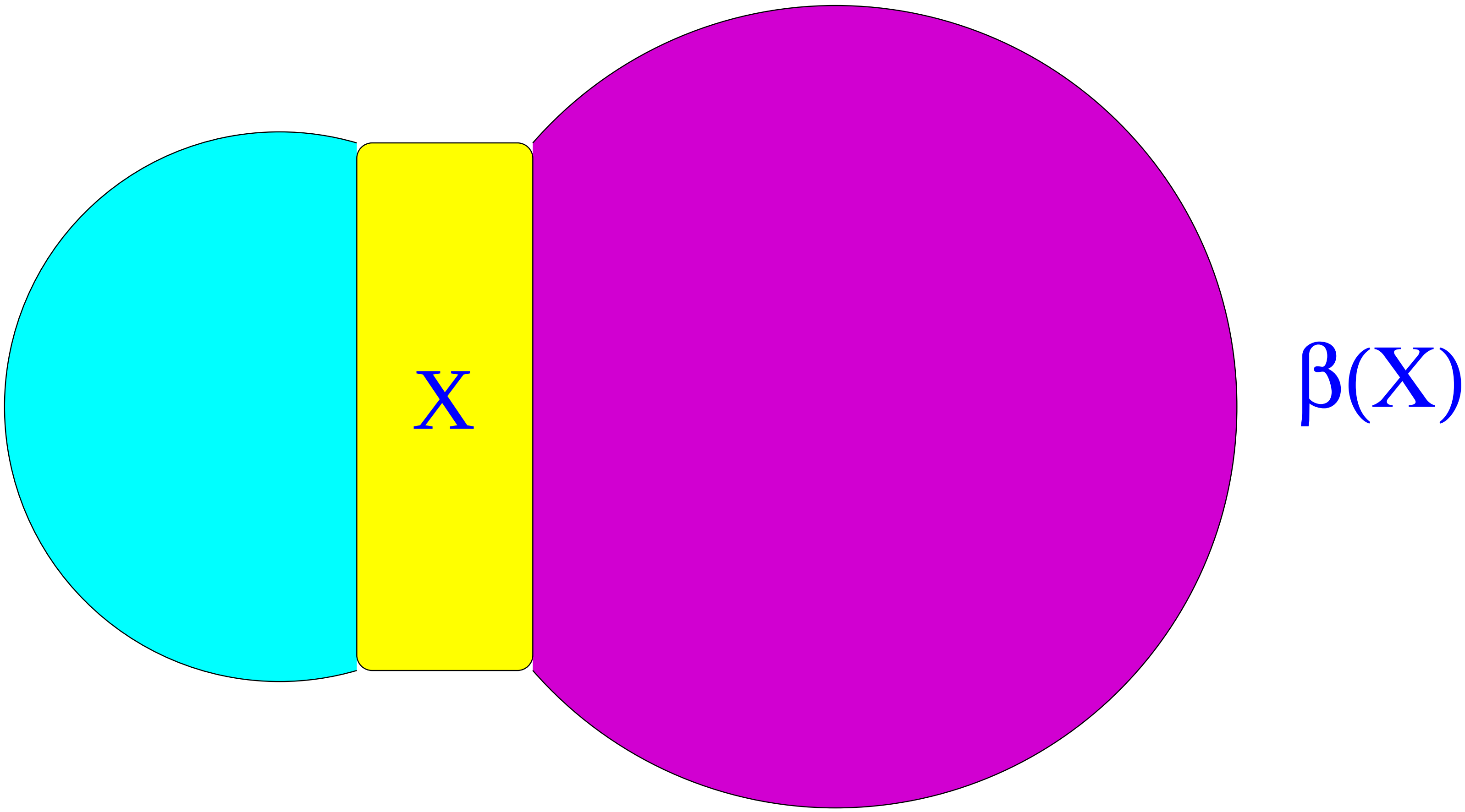

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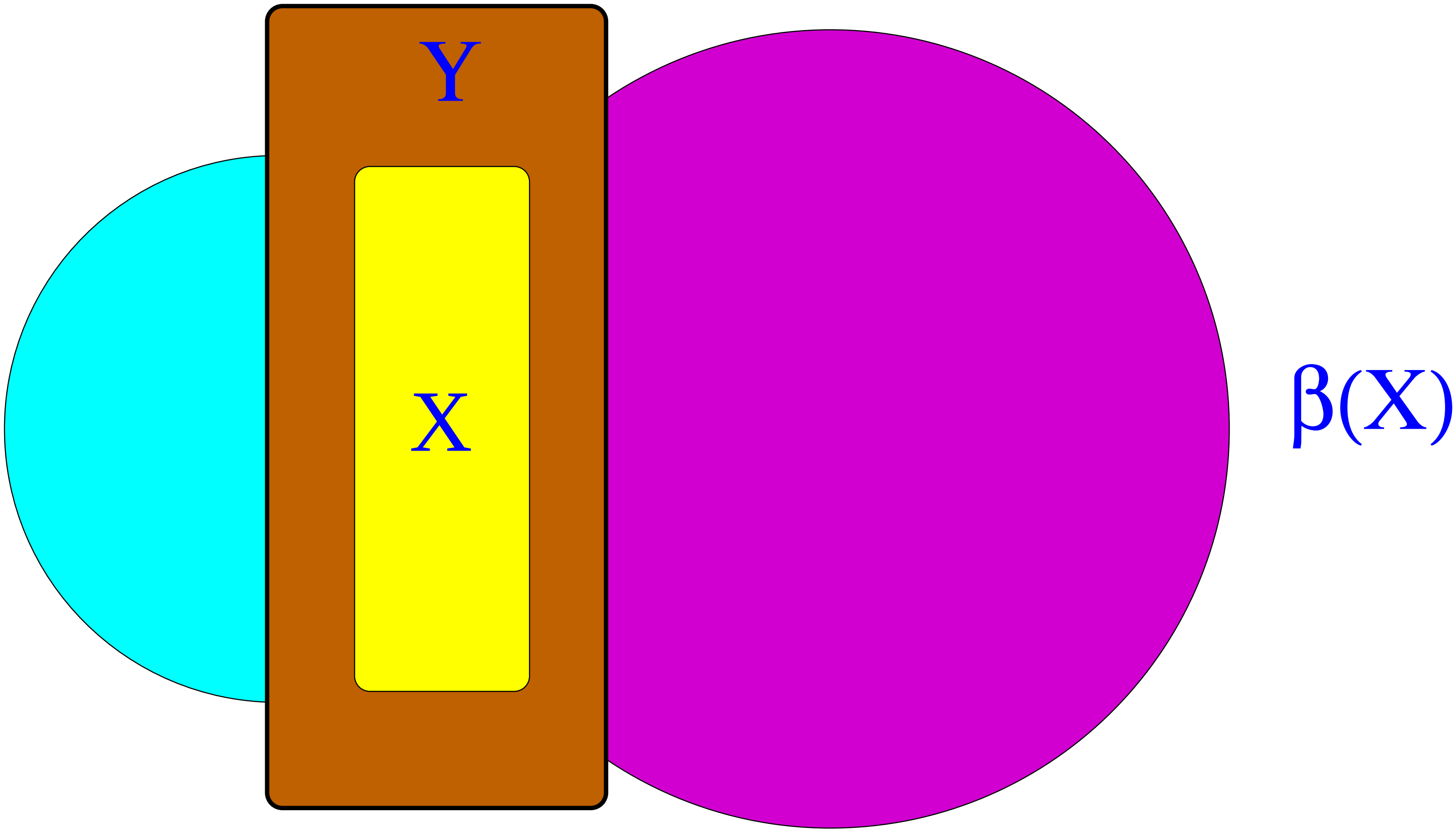

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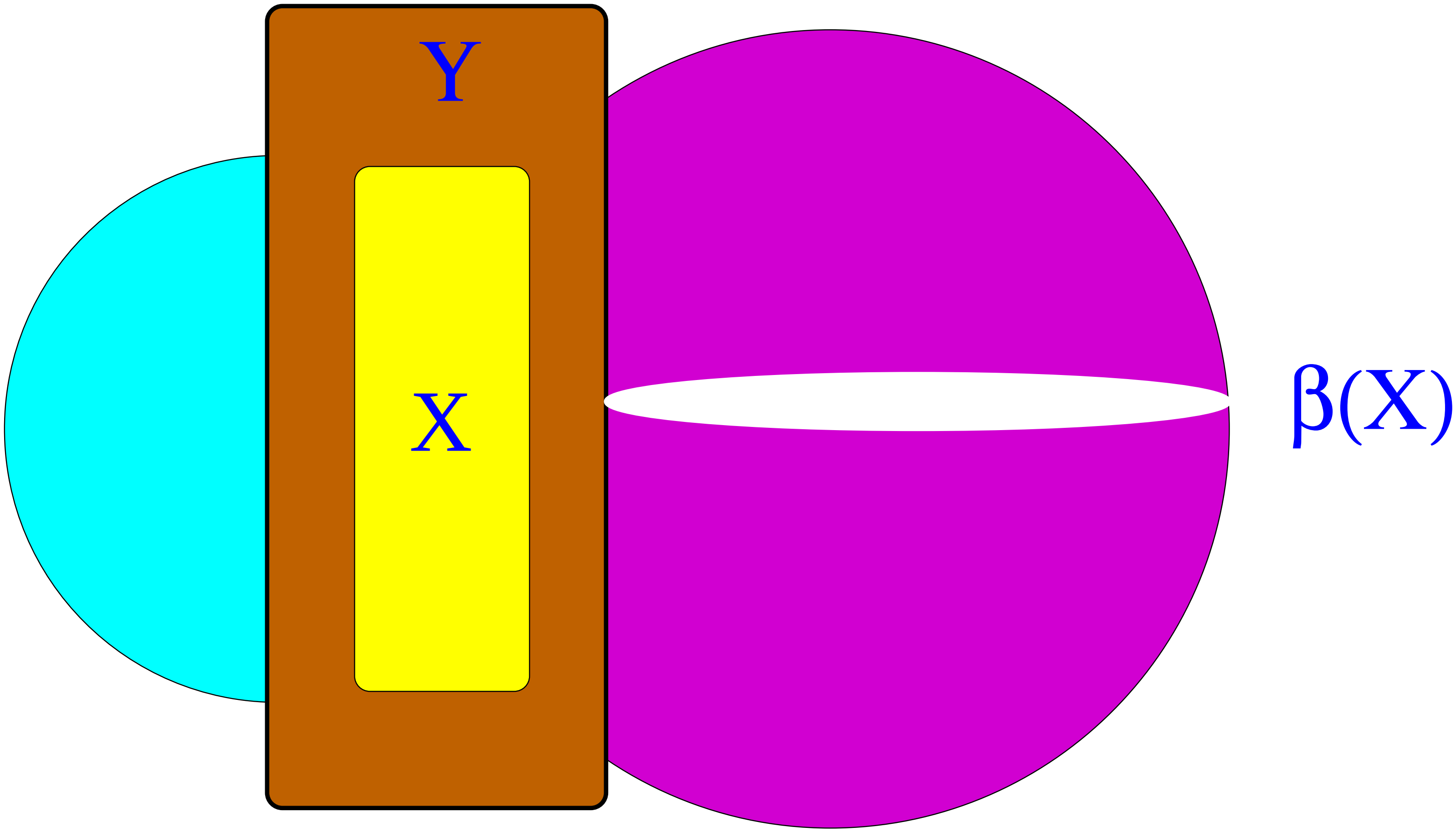

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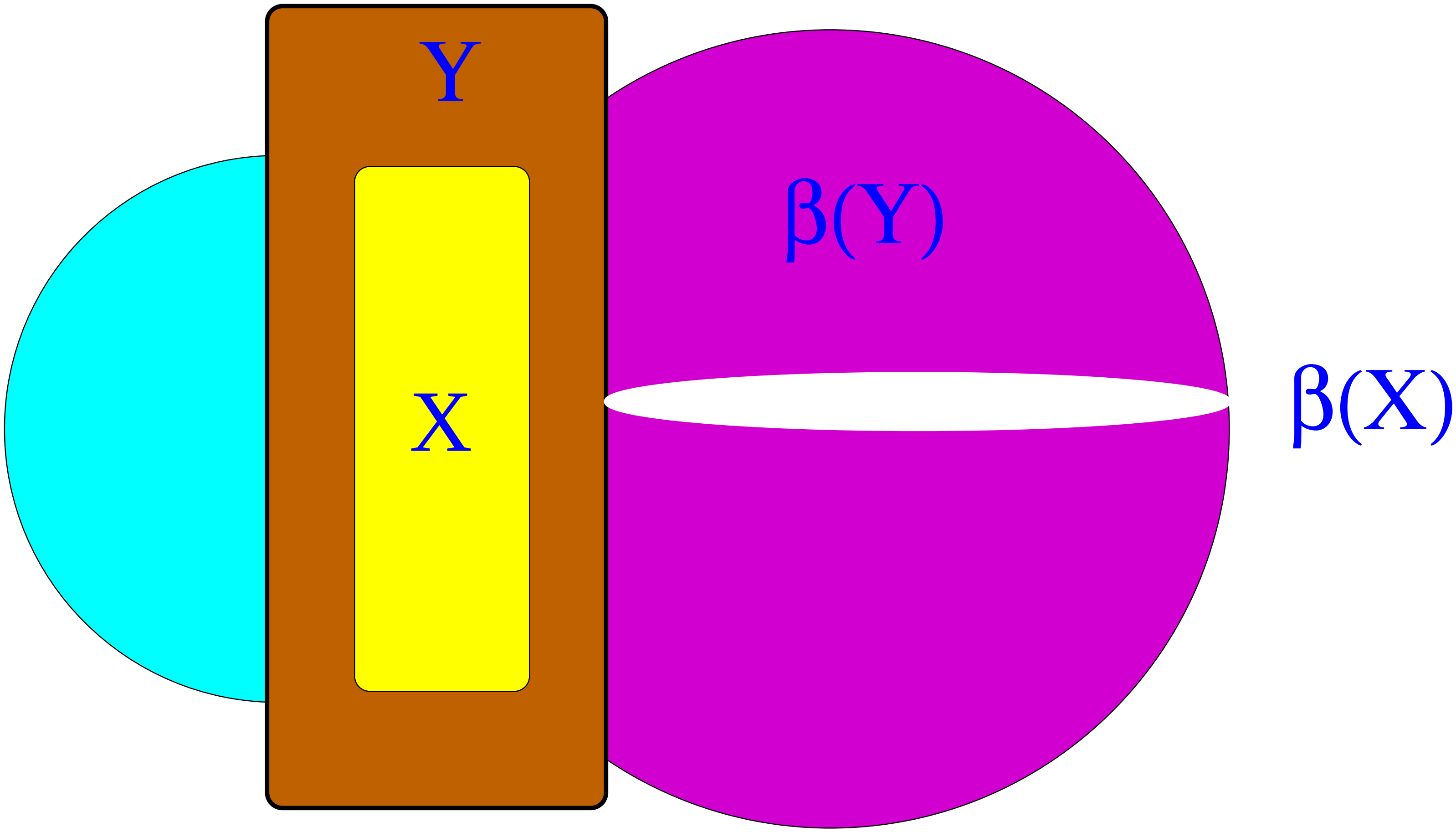

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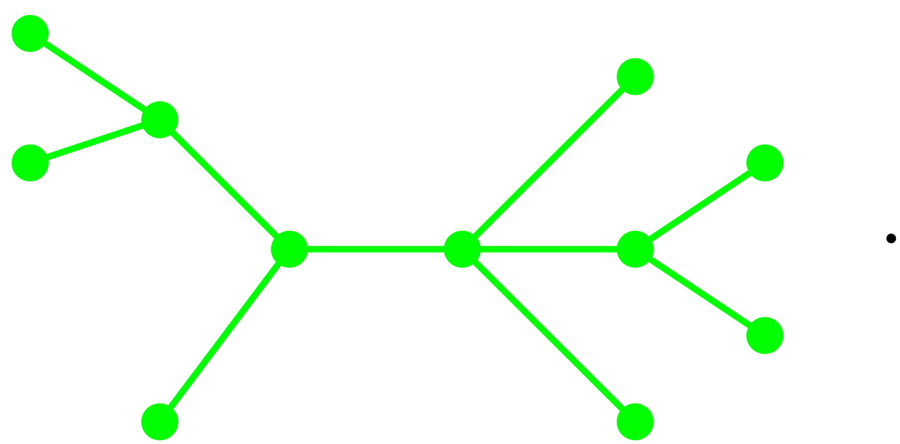

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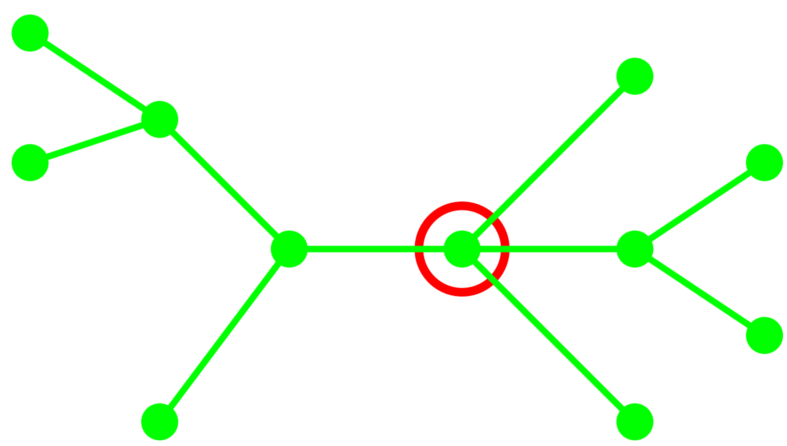

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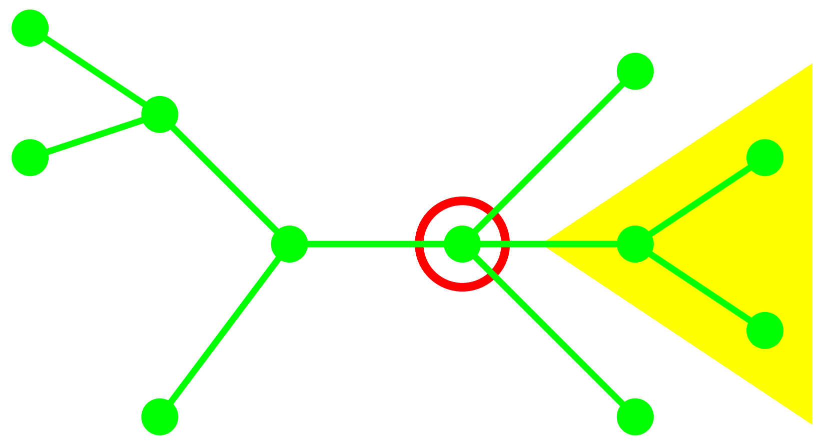

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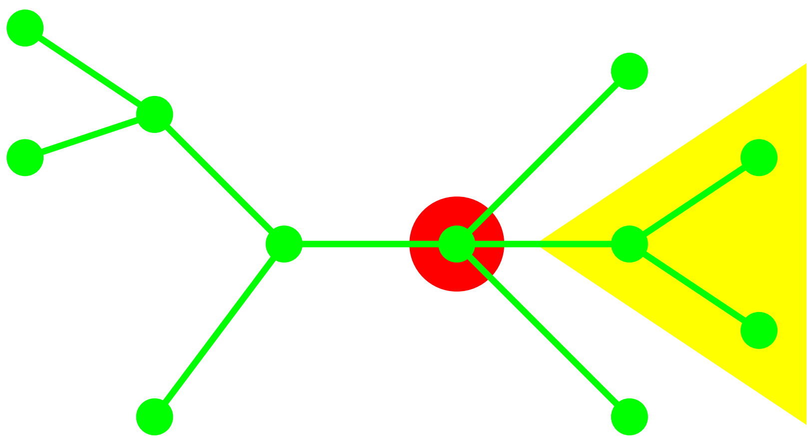

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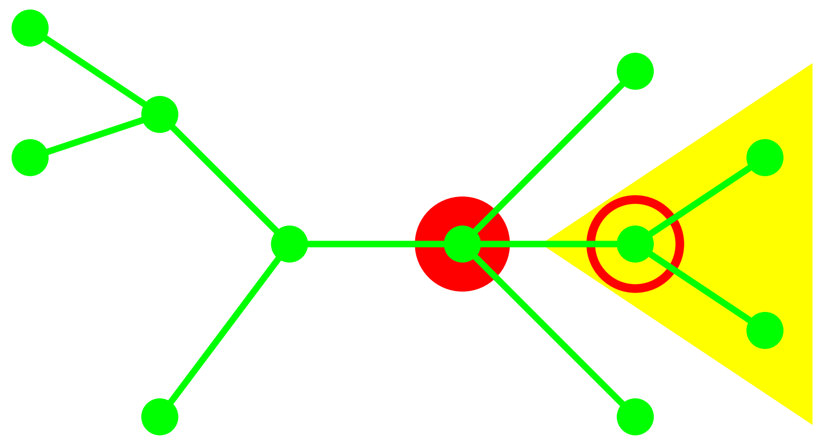

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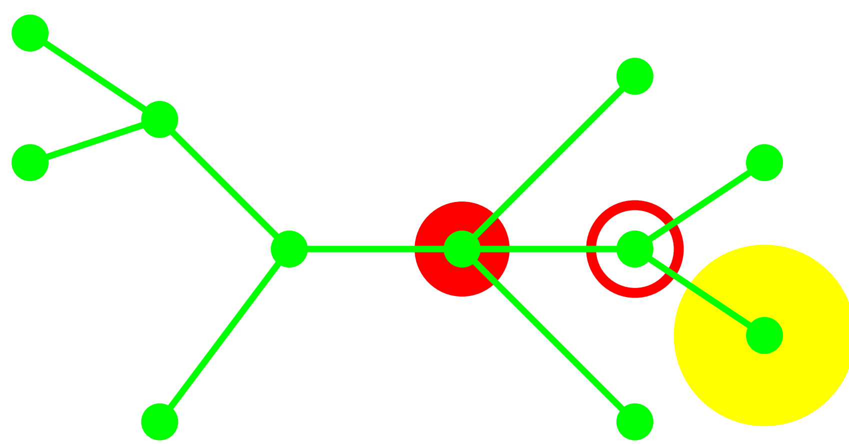

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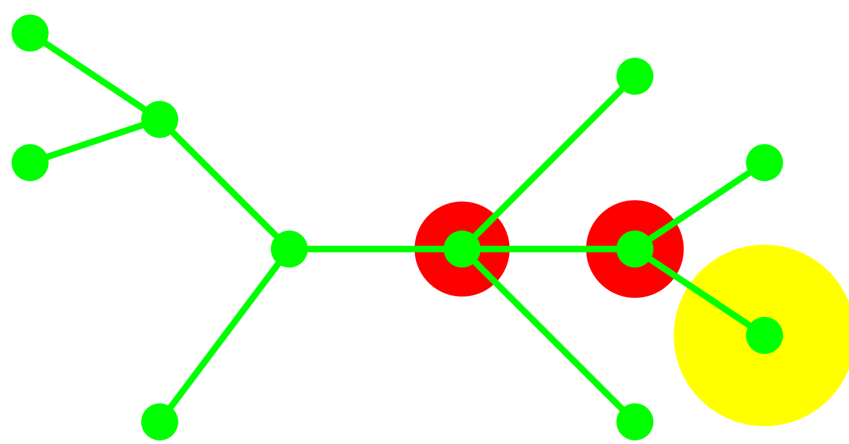

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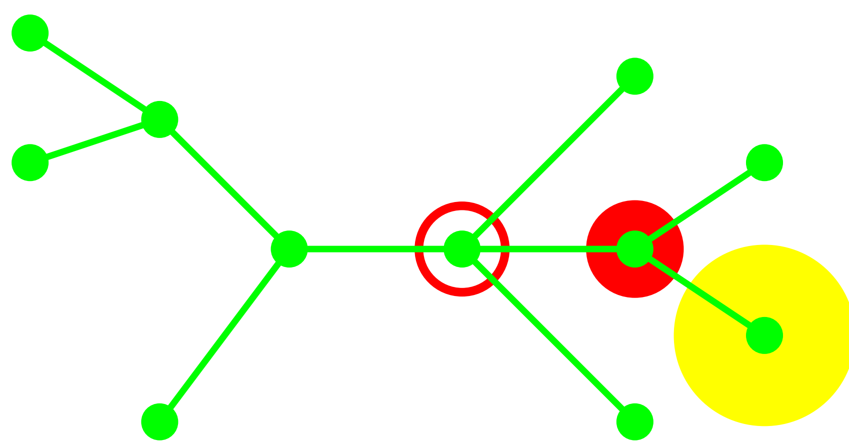

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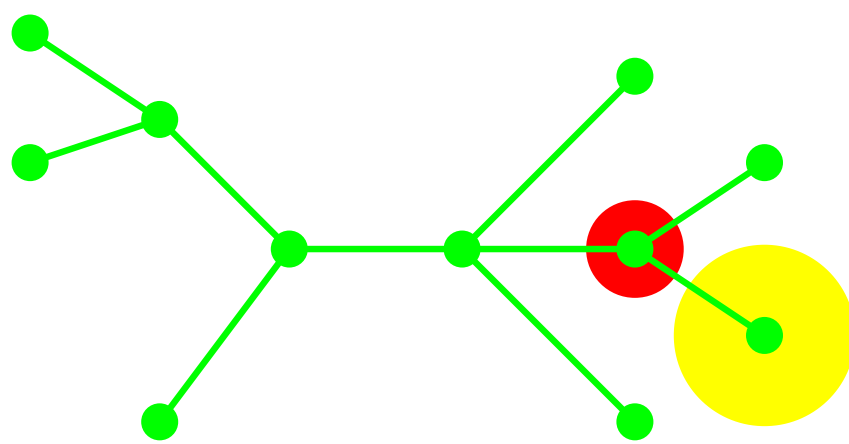

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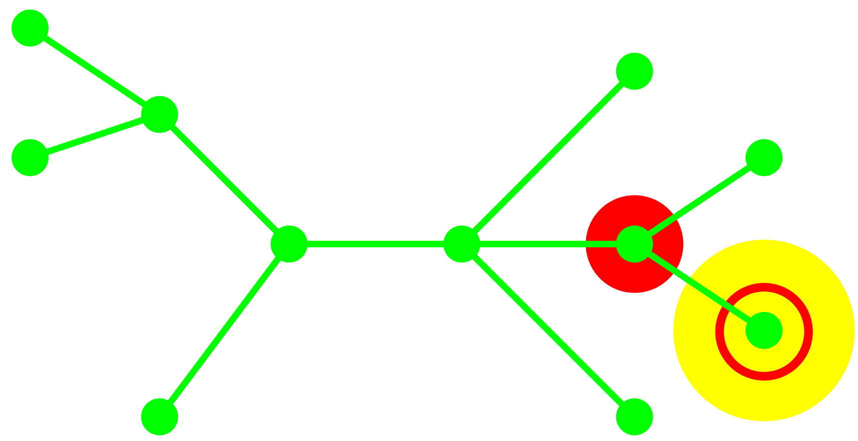

BACK TO MATHEMATICS

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Suppose $|W_t| \geq 3k$ for some t. Then t is a leaf; let t' be its neighbor. If for every $X \subseteq V(G)$ of size $< k$ there is a component containing $\geq k$ vertices of $W_t \cap W_{t'}$, then let $\beta(X)$ be that component. Then β is a haven.

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REMINDER: HOW TO USE A HAVEN

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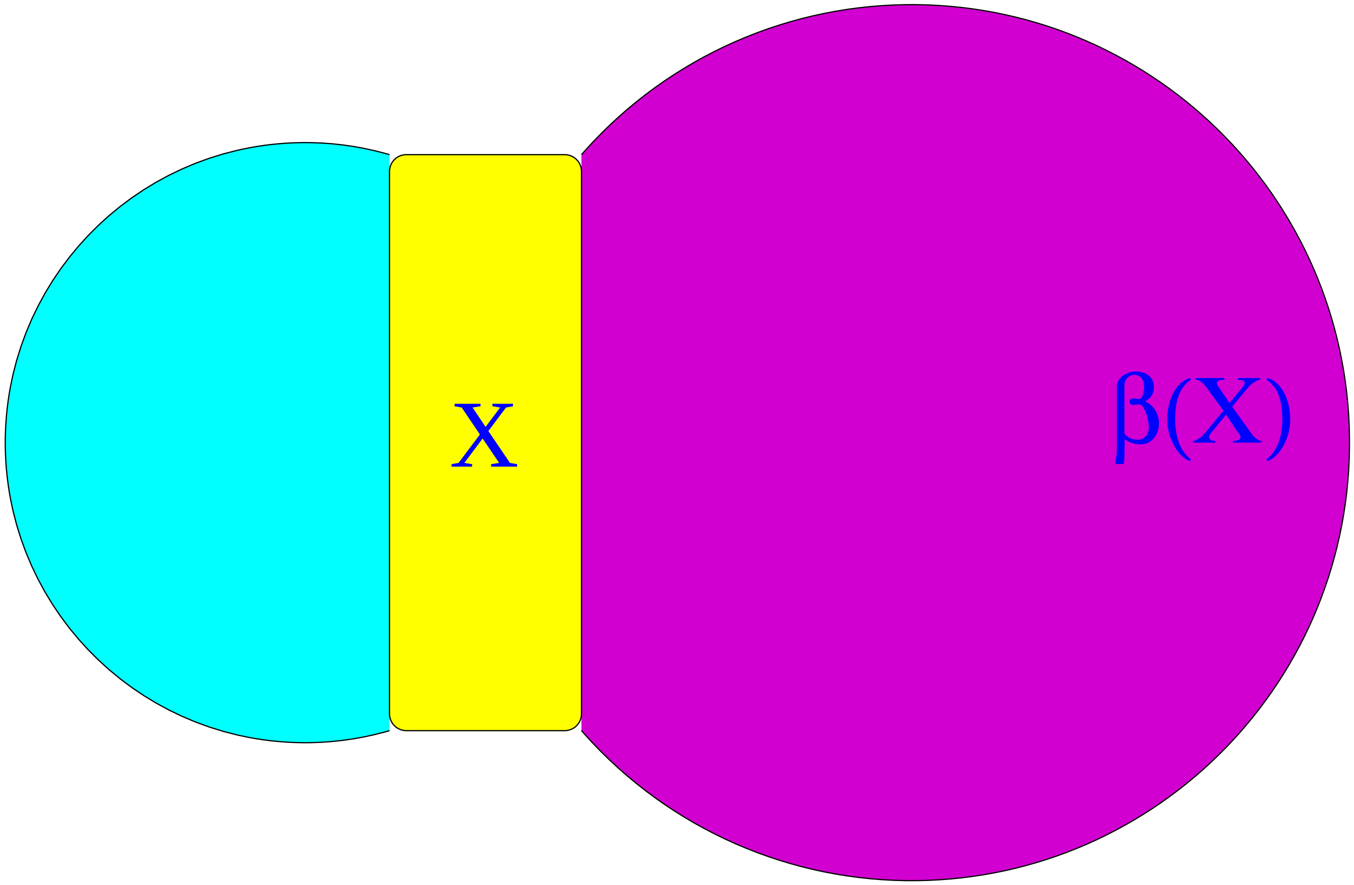

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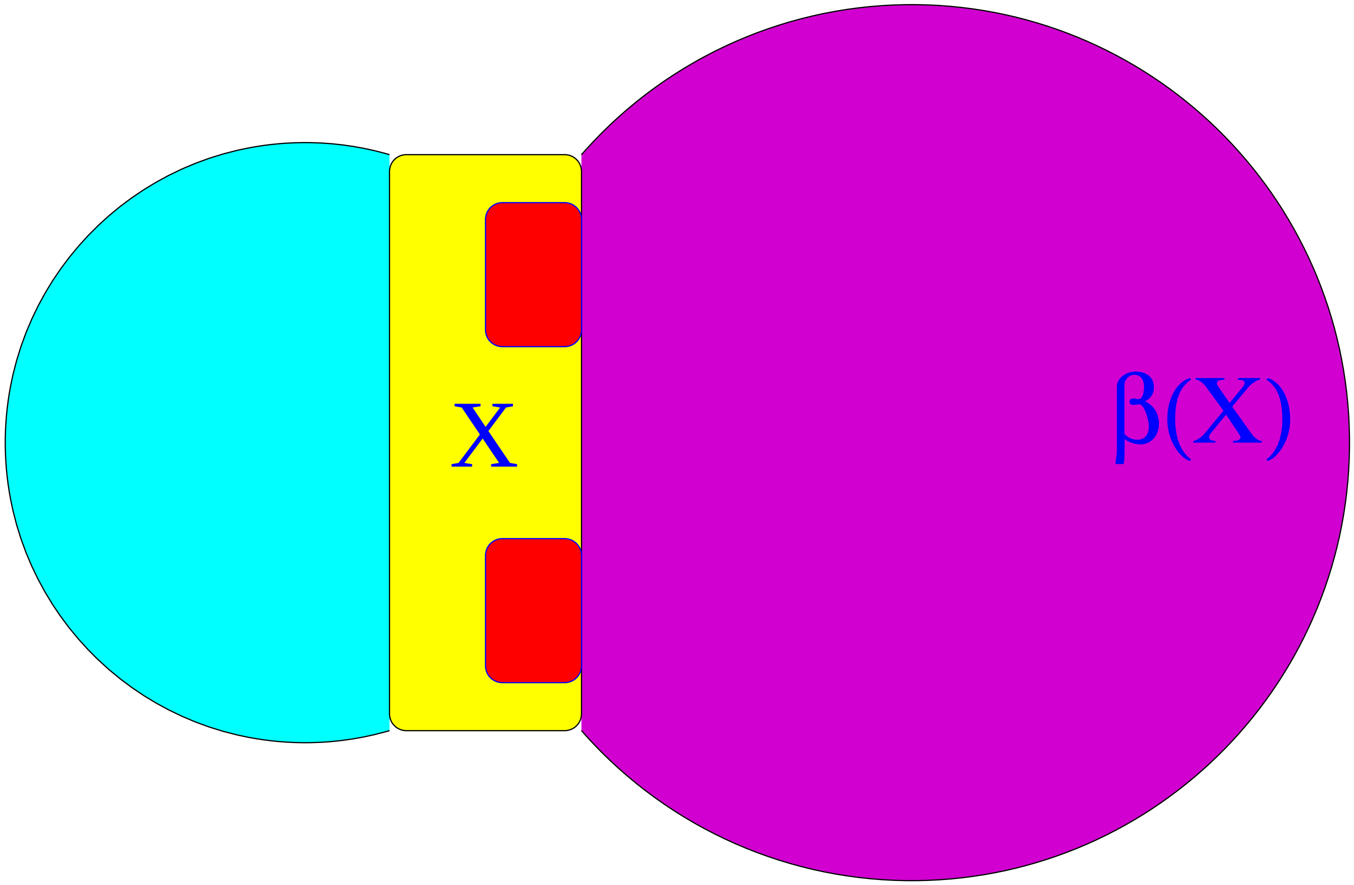

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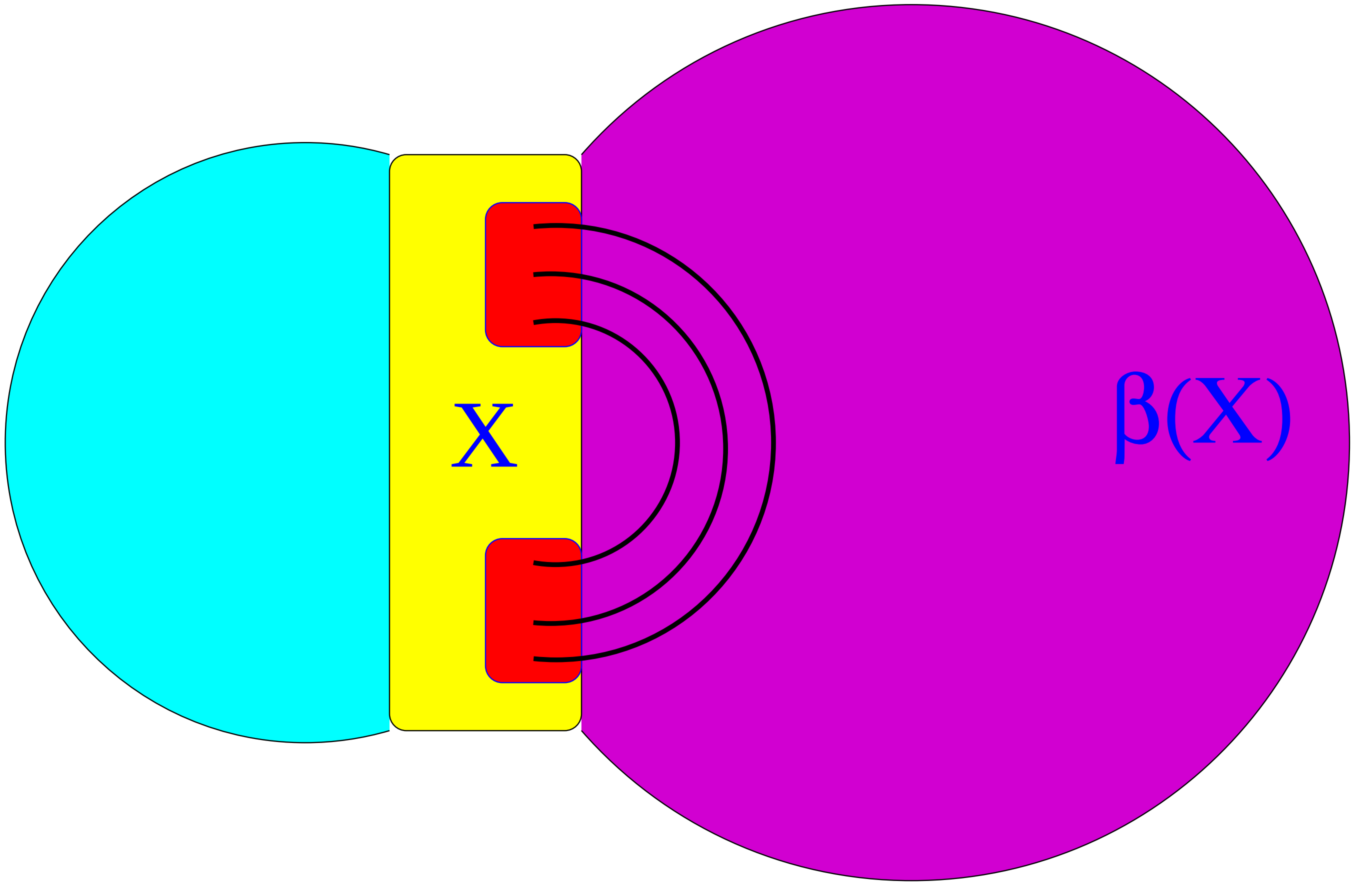

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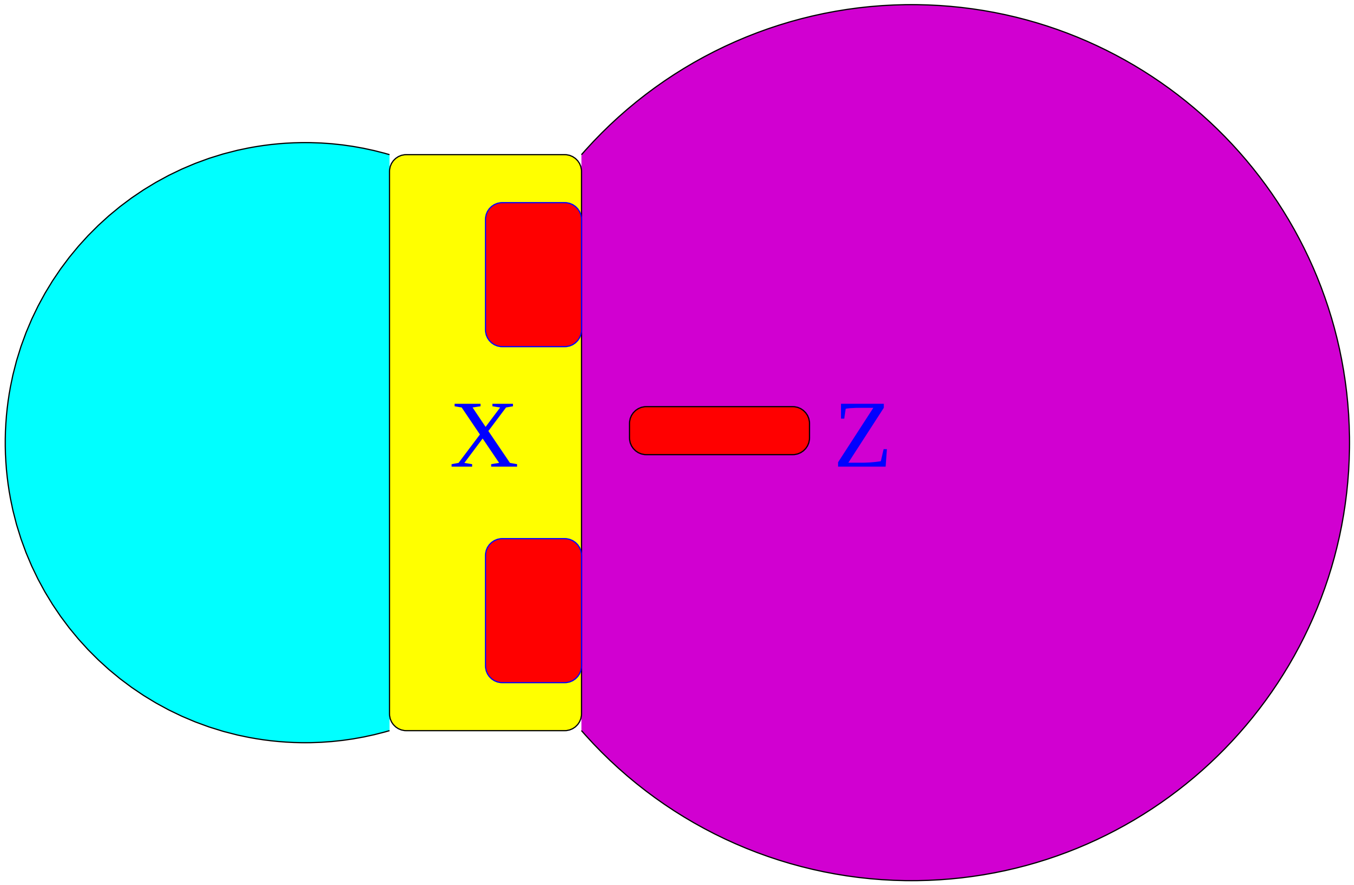

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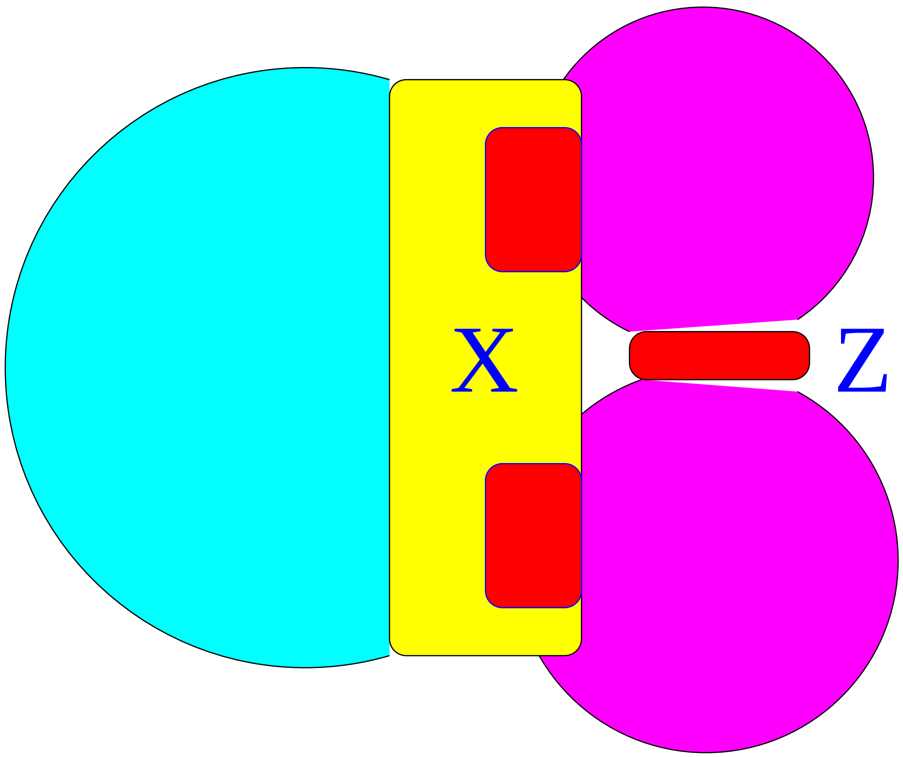

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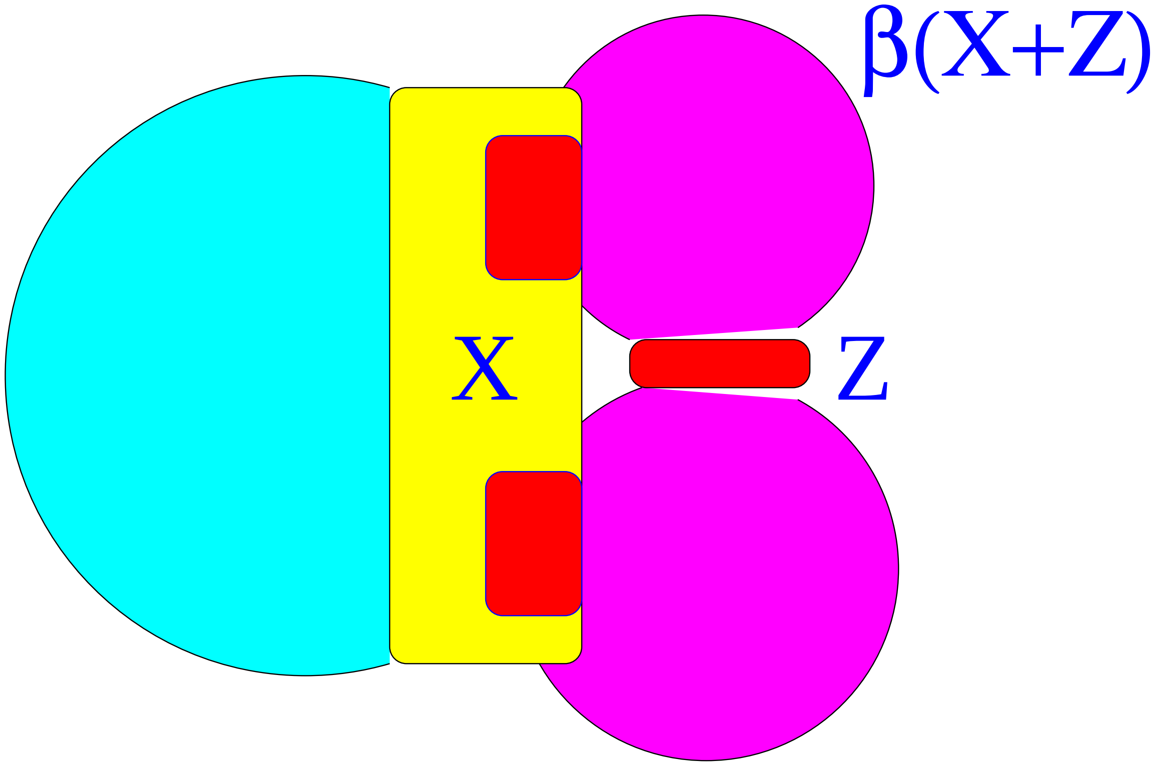

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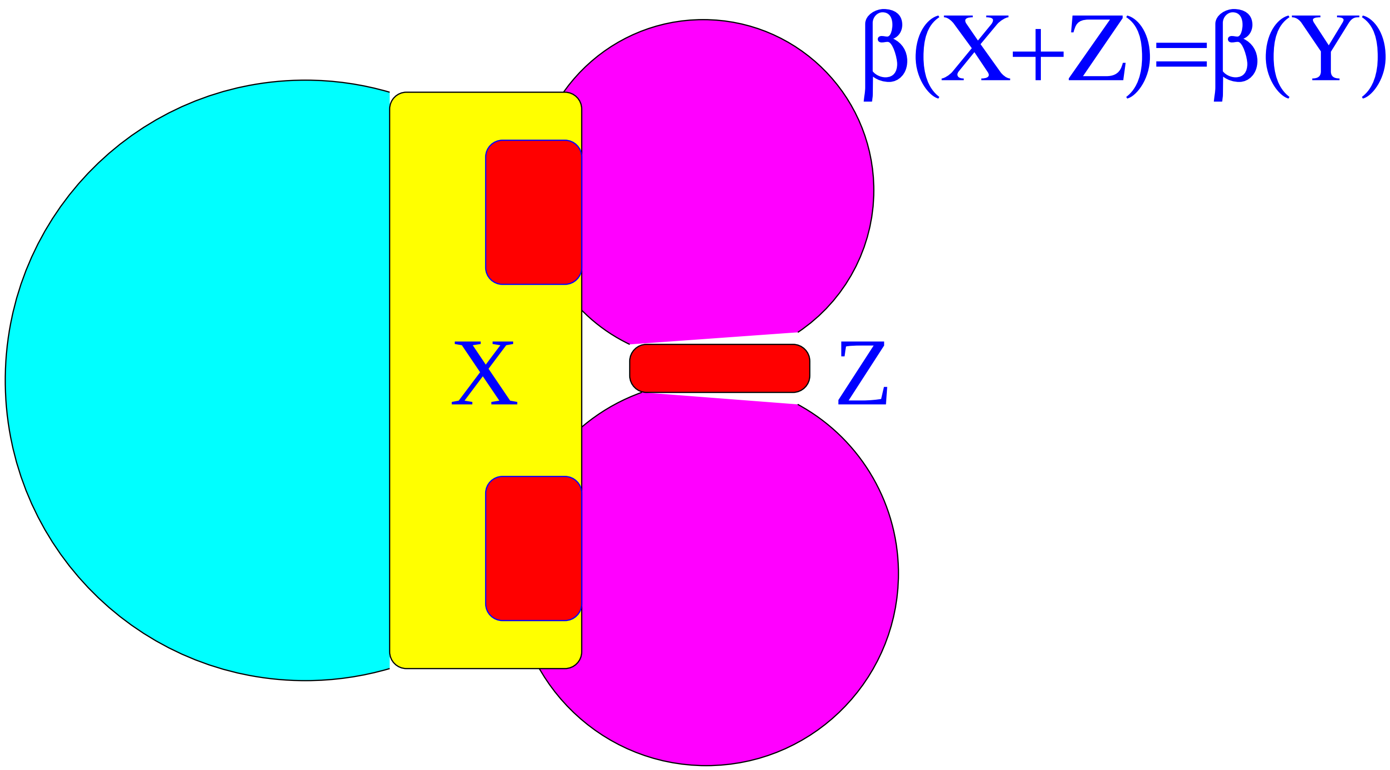

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MAIN IDEA Two sets of disjoint paths: $\mathcal{P}$ and $\mathcal{Q}$, where $\mathcal{P}$ joins A and B; $|\mathcal{Q}| >> |\mathcal{P}|$.

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- $V(P) \cap V(Q) = \emptyset \; \forall P \in \mathcal{P}' \; \forall Q \in \mathcal{Q}'$, or
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- $V(P) \cap V(Q) \neq \emptyset \ \forall P \in \mathcal{P}' \ \forall Q \in \mathcal{Q}'$.

In the latter case get a $k \times k$ grid, where $k = |\mathcal{P}'|$. Let $p = |\mathcal{P}|$.

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PROOF Take a haven β of large order. Pick X highly externally linked with a large binary tree T in $G \backslash \beta(X)$. In X find g^2 disjoint large sets $X_1, X_2, \ldots$, each connected by a disjoint subtree of T. Apply previous idea to X_i-X_j paths and X_p-X_q paths. Either we get a grid for some i, j, p, q, or we will make all the paths disjoint

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THEOREM (Ding, Oporowski, RT, Vertigan) Every internally 4-connected nonplanar graph on at least $f(t)$ vertices has a minor isomorphic to D'_t, M_t, or $K_{4,t}$.

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COROLLARY (Ding, Oporowski, RT, Vertigan) There exists a constant c such that every minimal graph of crossing number at least two on at least c vertices belongs to a well-defined family of graphs.

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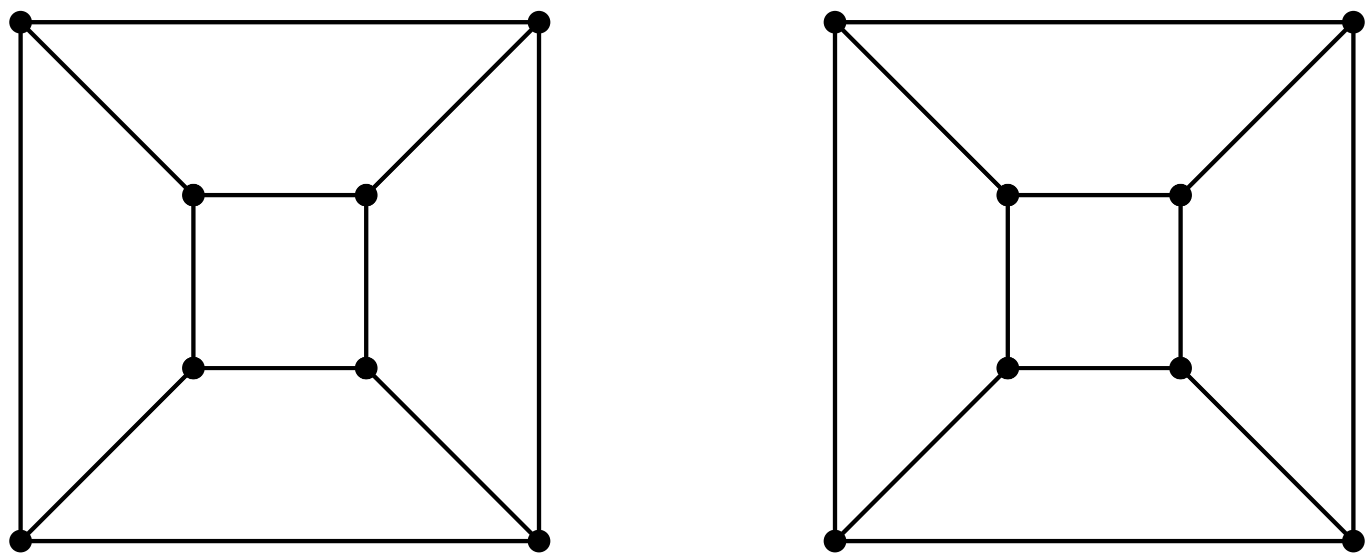

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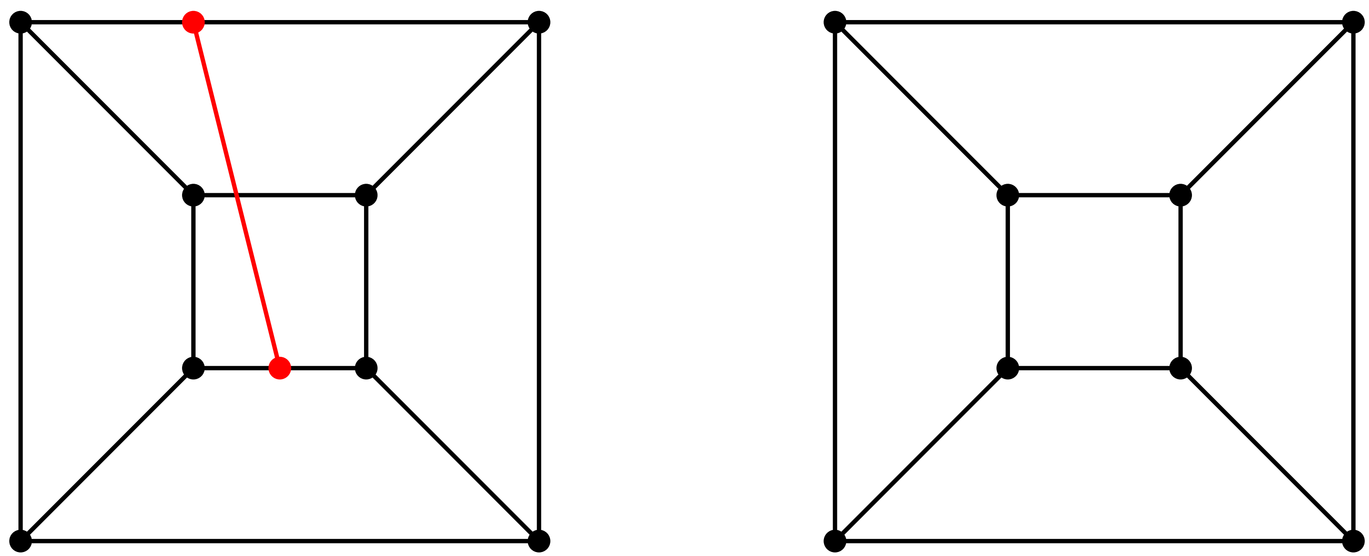

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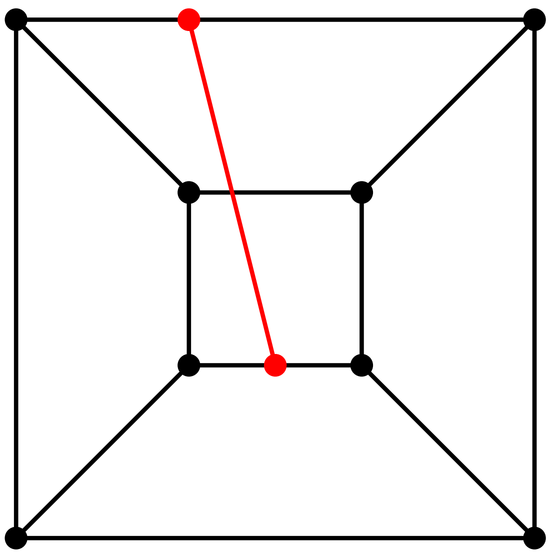

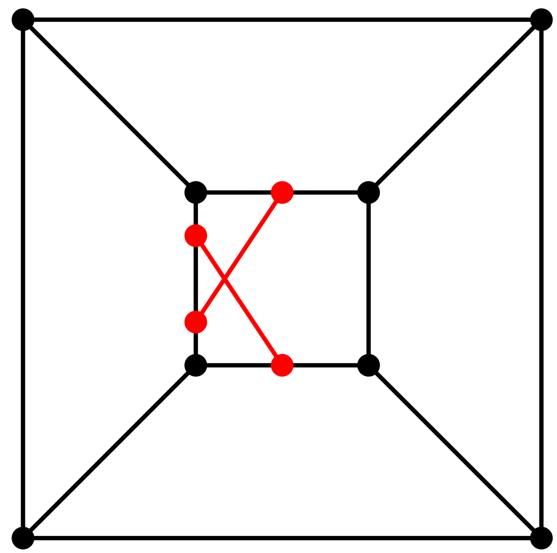

BRANCH-WIDTH

A branch-decomposition of G is a ternary tree T with leaves the edges of G. Every $\alpha \in E(T)$ defines a separation of G; the order of α is the order of this separation. The width of T is the maximum order of its edges. The branch-width of G, $bw(G)$, is the minimum width of a branch-decomposition

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- $bw(G) \leq 2 \Leftrightarrow G$ is series-parallel
- $bw(G) \leq 3 \Leftrightarrow$ no minor isomorphic to: K_5, cube, octahedron, V_8
- $bw(G^*) = bw(G)$
- $\frac{2}{3}\, tw(G) \leq bw(G) \leq tw(G) + 1$
- $bw(G)$ big $\Leftrightarrow G$ has a big grid minor

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A game Ratcatcher vs. rat. The ratcatcher carries a noisemaker of power k, and the rat will not move through any wall in which the noise level is too high.

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THM Geelen, Gerards, Whittle Every graph G has a linked branch-decomposition of width $bw(G)$:
If $\alpha, \beta \in E(T)$ and $|X_\alpha| = |X_\beta| =: k$, then either $|X_\gamma| < k$ for some γ between α and β, or there exist k disjoint paths between X_α and X_β.