EIDMA MINICOURSE ON STRUCTUAL GRAPH THEORY

Robin Thomas

LECTURE 2: Tree-decompositons of graphs I.

Topics: Path-width, tree-width and applications to structure theory and the design of algorithms. The concept of a haven (bramble) as an obstacle to low tree-width and its use. Excluding a planar graph. The Erdös-Pósa property.

Recommended reading: [D, Chapter 12]. See the first edition of [D] or [1] for path-width.

Exercises

1. A graph is series-parallel if and only if it has tree-width at most two.

2. The $k \times k$ grid has tree-width exactly k.

3. A graph has tree-width at most t if and only if it is a subgraph of a chordal graph with no K_{t+2} subgraph.

4. Prove that there exist trees of arbitrarily large path-width.

Let G be a graph and let $X \subseteq V(G)$. By $\delta(X)$ we denote the set of edges with one end in X and the other in $V(G) - X$. A set of edges of the form $\delta(X)$ for some $X \subseteq V(G)$ is called a *cut*. Two cuts $\delta(X)$ and $\delta(Y)$ *do not cross* if either $X \subseteq Y$, or $Y \subseteq X$, or $X \cap Y = \emptyset$ or $X \cup Y = V(G)$.

5. Let G be a graph, let T be a tree and $(W_t : t \in V(T))$ be a partition of $V(G)$ into disjoint (possibly empty) sets. For each edge $e \in E(T)$ let S_e be the vertex-set of one of the components of $T \backslash e$, and let $X_e = \bigcup_{t \in S_e} W_t$. Then the cuts $\delta(X_e)$ pairwise do not cross. Prove that, conversely, every family of pairwise non-crossing cuts arises in this way.

Reference

1. R. Diestel, Graph Minors I: a short proof of the path-width theorem, *Combinatorics, Probability and Computing* **4** (1995), 27–30.