EIDMA MINICOURSE ON STRUCTURAL GRAPH THEORY

Robin Thomas

LECTURE 4: Excluded minor theorems and well-quasi-ordering.

Topics: Seymour's splitter theorem and its applications. Analogues for higher connectivity and their applications. Separators. Typical subgraphs of large graphs. Excluded minor theorems for infinite graphs (time permitting). Well-quasi-ordering finite trees and extension to graphs of bounded tree-width. The finitness of obstruction sets for surface embeddings.

Recommended reading: [1, 2]. For introduction to well-quasi-ordering see [D, Chapter 12].

Exercises

1. Prove that a graph has no $K_{3,3}$ minor if and only if it has a tree-decomposition over planar graphs and K_5.

2. Prove that a graph has no K_5 minor if and only if it has a tree-decomposition over planar graphs and V_8 (= 8-cycle plus all the long diagonals)

3. Let k be an integer, and let $G_1, G_2, \ldots$ be a sequence of graphs such that each G_i has a set F_i of at most k edges such that $G_i \backslash F_i$ is a forest. Prove that there exist indices i and j such that $i < j$ and a subdivision of G_i is isomorphic to a subgraph of G_j.

4. Find a sequence of finite graphs $G_1, G_2, \ldots$ such that there are no indices i and j such that $i \neq j$ and a subdivision of G_i is isomorphic to a subgraph of G_j. In other words, the topological minor ordering is not a well-quasi-order.

5. Construct a well-quasi-ordered set Q such that Q^ω is not well-quasi-ordered, where Q^ω is quasi-ordered by the relation that $(q_1, q_2, \ldots) \leq (q'_1, q'_2, \ldots)$ if there exists a strictly increasing function $f : \{1, 2, \ldots\} \to \{1, 2, \ldots\}$ such that $q_i \leq q'_{f(i)}$ for all $i = 1, 2, \ldots$.

References

1. N. Robertson and P. D. Seymour, Graph Minors IV. Tree-width and well-quasi-ordering, *J. Combin. Theory Ser. B* **48** (1990), 227–254
2. R. Thomas, Recent excluded minor theorems, in Surveys in Combinatorics, 1999, (eds. J. D. Lamb and D. A. Preece), Cambridge University Press, Cambridge 1999.