EIDMA MINICOURSE ON STRUCTURAL GRAPH THEORY

Robin Thomas

LECTURE 5: The Four-Color Theorem

Topics: History, equivalent formulations and an outline of a proof. What are the prospects for finding a computer-free proof?

Recommended reading: [1, 2, 3]

Exercises

1. Prove that the Four-Color Theorem is equivalent to the assertion that every 2-connected 3-regular planar graph is 3-edge-colorable.

2. Let $\times$ denote the vector cross product in $\mathbf{R}^3$, and let n be an integer. A *bracketing* is an expression obtained from the formal cross product $v_1 \times v_2 \times \cdots \times v_n$ by inserting $n-2$ pairs of brackets to make the order of evaluation well-defined. Prove that given two bracketings, there is an assignment of $(1,0,0),(0,1,0),(0,0,1)$ to $v_1, v_2, \ldots, v_n$ that makes the evaluations of the two bracketings equal and nonzero.

3. Prove that every simple triangulation of the sphere of minimum degree at least five has a triangle whose vertices have degrees $5,5,5$ or $5,5,6$ or $5,6,6$.

4. Prove that a minimal counterexample to the Four-Color Theorem is an internally 6-connected triangulation. (Internally 6-connected means that it is 5-connected, and for every cutset X of size five, the deletion of X creates exactly two components, one of which has exactly one vertex.)

5. Prove that every 2-connected 3-regular graph of crossing number one is 3-edge-colorable. Prove that the same is not true with “one” replaced by “two”.

References

1. T. L. Saaty, Thirteen colorful variations on Guthrie’s four-color conjecture, *Am. Math. Monthly* **79** (1972), 2–43.
2. T. L. Saaty and P C. Kainen, The Four Color Problem Assaults and Conquest, Dover Publications, New York 1986.
3. R. Thomas, An update on the Four-Color Theorem, *Notices Amer. Math. Soc.* **45** (1998), 848–859.