EIDMA MINICOURSE ON STRUCTURAL GRAPH THEORY

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LECTURE 7: Matching structure, Pfaffian orientations and digraph structure I

Topics: Edmonds' matching theorem, the linear hull of perfect matchings, the matching structure: decomposition into bricks and braces, the matching lattice, Pfaffian orientations and their use, Pfaffian orientations of bipartite graphs, Polya's permanent problem, the even directed cycle problem, sign-nonsingular matrices, applications in economics.

Recommended reading: [LP] and [2] for Pfaffian orientations and their relation to other problems; [1] for directed tree-width.

Exercises

1. Let G be a bipartite graph with bipartition (A, B), and let M be a perfect matching in G. Let D be the directed graph obtained from G by directing every edge from A to B and contracting every edge of M, and let k be an integer. Prove that D is strongly k-connected if and only if G is k-extendable (that is, every matching of size at most k can be extended to a perfect matching).

2. Let G be a connected bipartite graph with bipartition (A, B), and let k be an integer. Prove that G is k-extendable if and only if for every set $X \subseteq A$, its neighborhood $N(X)$ in B either has size at least $|X| + k$ or is equal to B.

3. Let G be a bipartite graph with bipartition (A, B), and let M be a perfect matching in G. Let D be the directed graph obtained from G by directing every edge from A to B and contracting every edge of M. Prove that G has a Pfaffian orientation if and only if D has a subdivision with no even directed cycle.

4. Let G be a bipartite graph with bipartition (X, Y), and let A be the matrix with rows indexed by X and columns indexed by Y such that $a_{xy} = 1$ if $x \in X$ is adjacent to $y \in Y$ and $a_{xy} = 0$ otherwise. Prove that G has a Pfaffian orientation if and only if some of the entries of A can be changed to -1 in such a way that the determinant of the resulting matrix is equal to the permanent of A.

5. Prove that $K_{3,3}$ has no Pfaffian orientation.

References

1. T. Johnson, N. Robertson, P. D. Seymour and R. Thomas, Directed tree-width, *J. Combin. Theory Ser. B* **82** (2001), 138–154.
2. W. McCuaig, Pólya's permanent problem, *Electron. J. Combin.* **11** (2004), 83pp.