BEYOND GROTZSCH'S THEOREM

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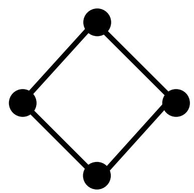
joint work with
Zdenek Dvorak and K. Kawarabayashi
and
Zdenek Dvorak and Daniel Kral

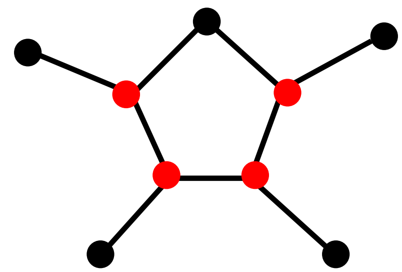
THM (Grotzsch) ∀ triangle-free planar graph is 3-colorable

PROOF Reducible configurations:

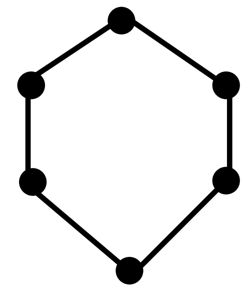
monogram



tetragram

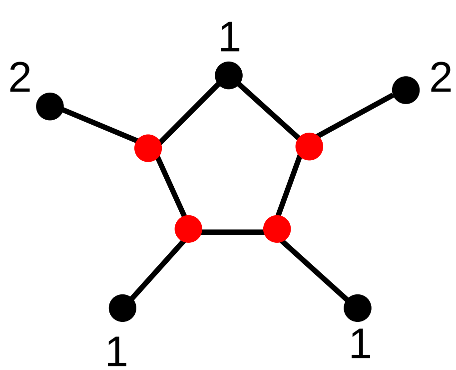


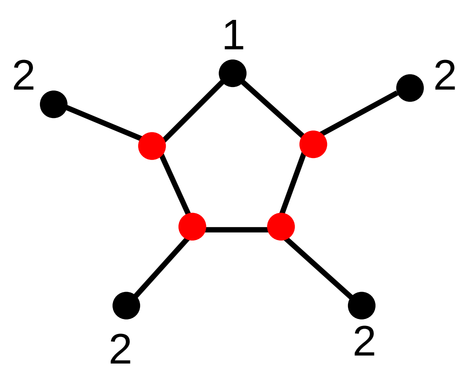
pentagram

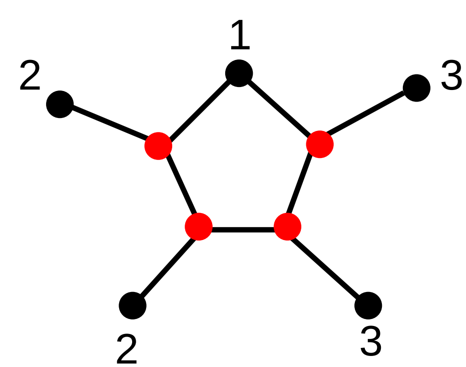


hexagram

Non-extendable pentagram colorings:







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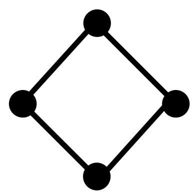
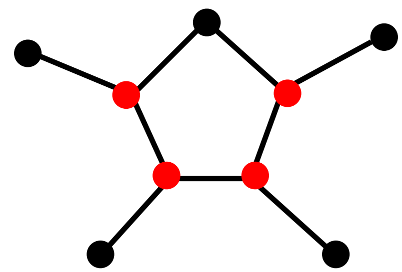
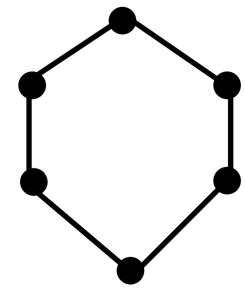


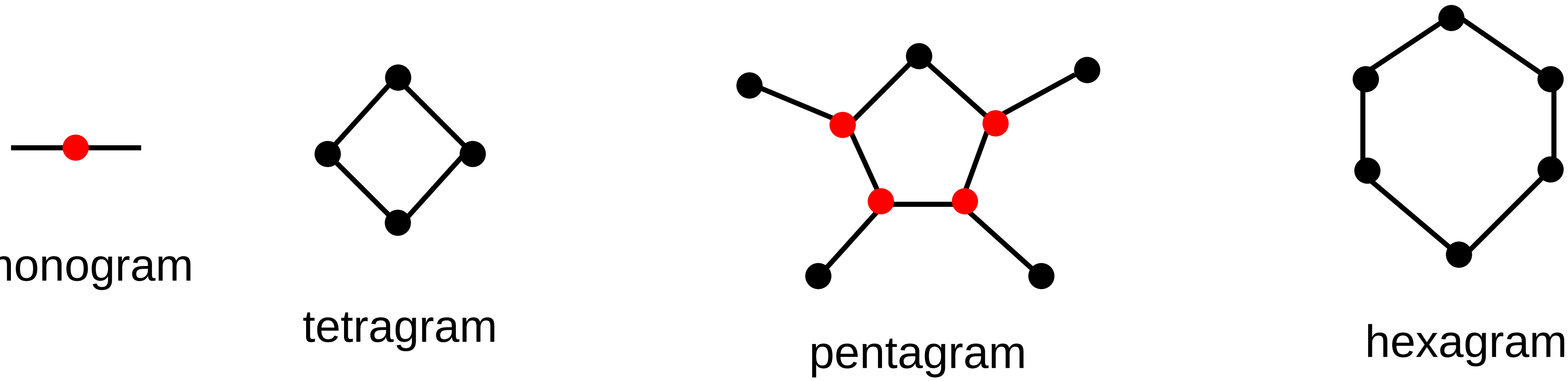
tetragram



pentagram



hexagram



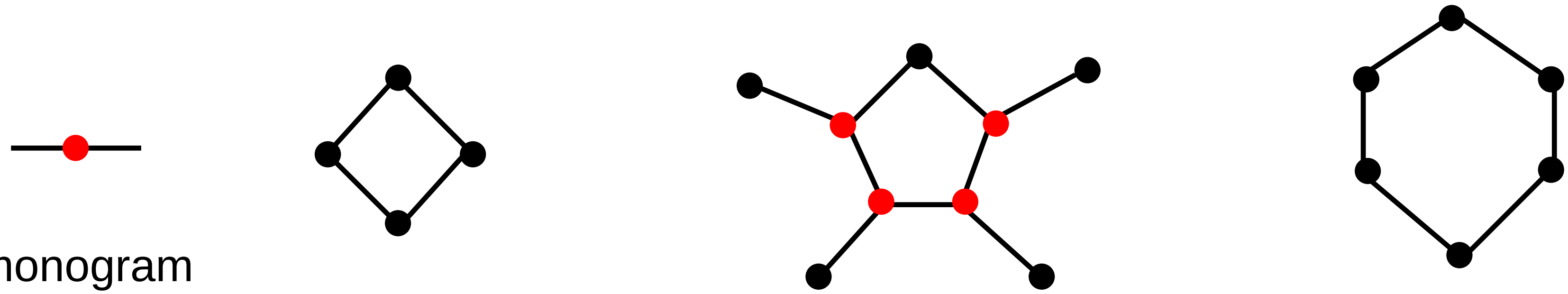
LEMMA 1 G connected triangle-free planar, C outer cycle, $|C|\leq 6$, $G\neq C \Rightarrow \exists$ multigram s.t. red vertices are not in C

PF

$$ch(v)=\begin{cases} \deg(v)-4 & v \text{ not in } C \\ -1/3 & v\in C, \deg 2 \\ 0 & \text{o.w.} \end{cases} \qquad ch(f)=\begin{cases} |f|-4 & f \text{ internal} \\ 0 & f \text{ outer} \end{cases}$$

$$\sum\text{charges} = \sum(d(v)-4) + \sum(|f|-4) - t_2/3 + \sum_{v\in C}(4-d(v)) - (l-4)$$

$$\leq 2E-4V+2E-4F-t_2/3+2t_2+t_3-l+4 \leq 2t_2/3-4 < 0.$$



monogram tetragram pentagram hexagram

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Each face will send 1/3 to every incident vertex v s.t. either $v \in G-C$ has degree 3, or $v \in C$ has degree 2.

After that there is a face of charge $<0 \Rightarrow$ tetragram or pentagram. Q.E.D.

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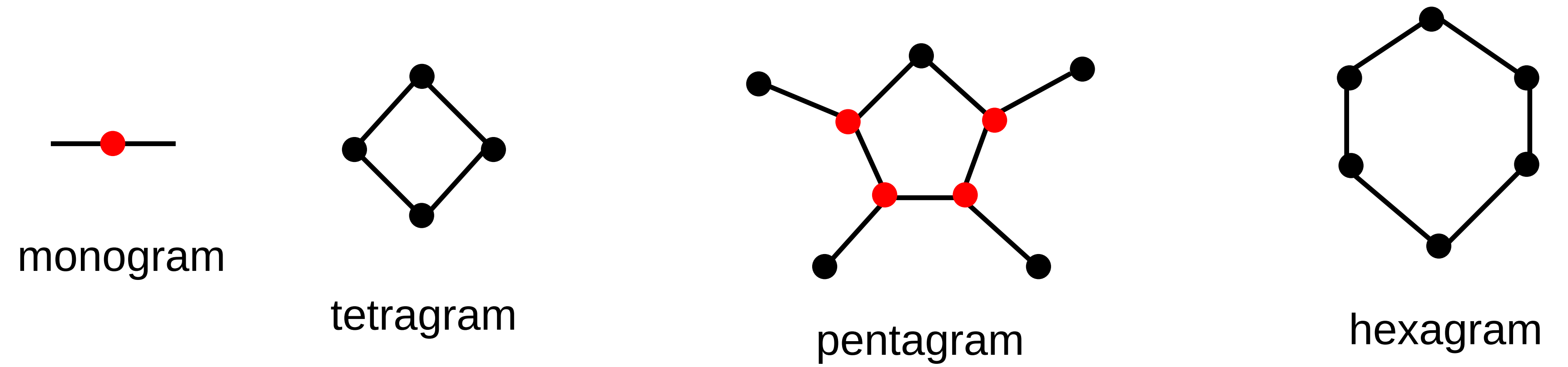


LEMMA 2 Every triangle-free planar graph has a safe multigram (reduction does not create triangles)

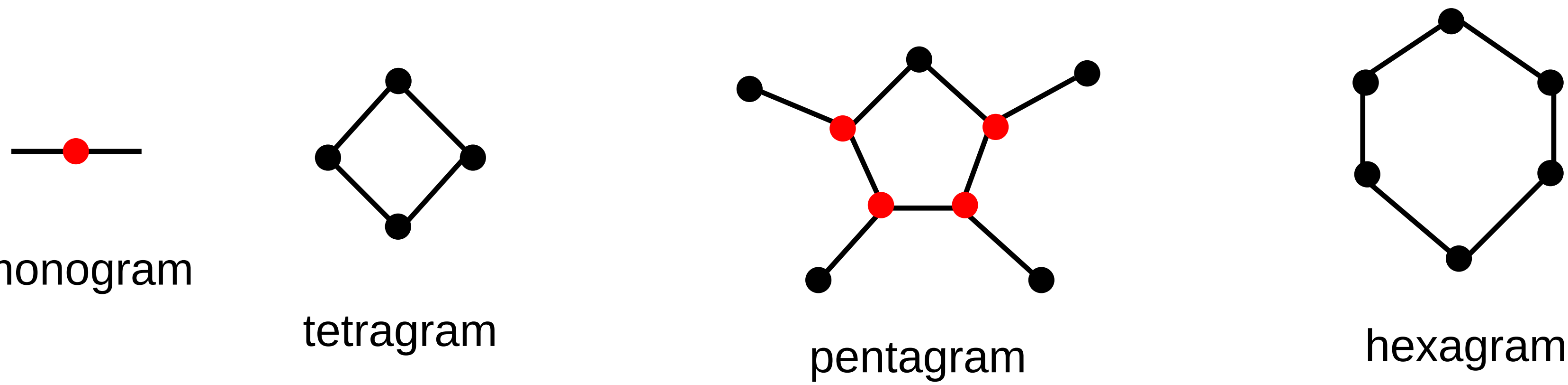
PF Pick a separating ≤ 6-cycle C with smallest inside, or facial ≤ 6-cycle if no separating cycle. Apply Lemma 1 to C and its interior. Q.E.D.

PF of Grotzsch: multigram $\Rightarrow$ smaller graph $\Rightarrow$ induction

LINEAR-TIME ALGORITHM

A problem: which vertices of a C_4 to identify?

Kowalik: An O(n log n) algorithm



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LINEAR-TIME ALGORITHM

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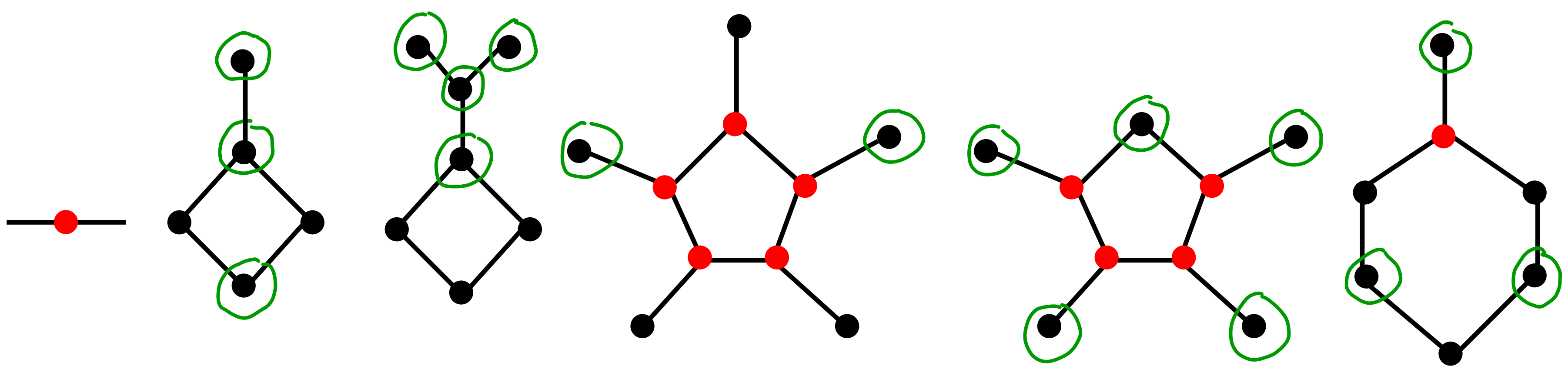
LEMMA 3 Every triangle-free planar graph has a secure multigram (safe+marked vertices have bounded degree)



ALGORITHM Find secure multigram and reduce in constant time to a smaller graph; recurse

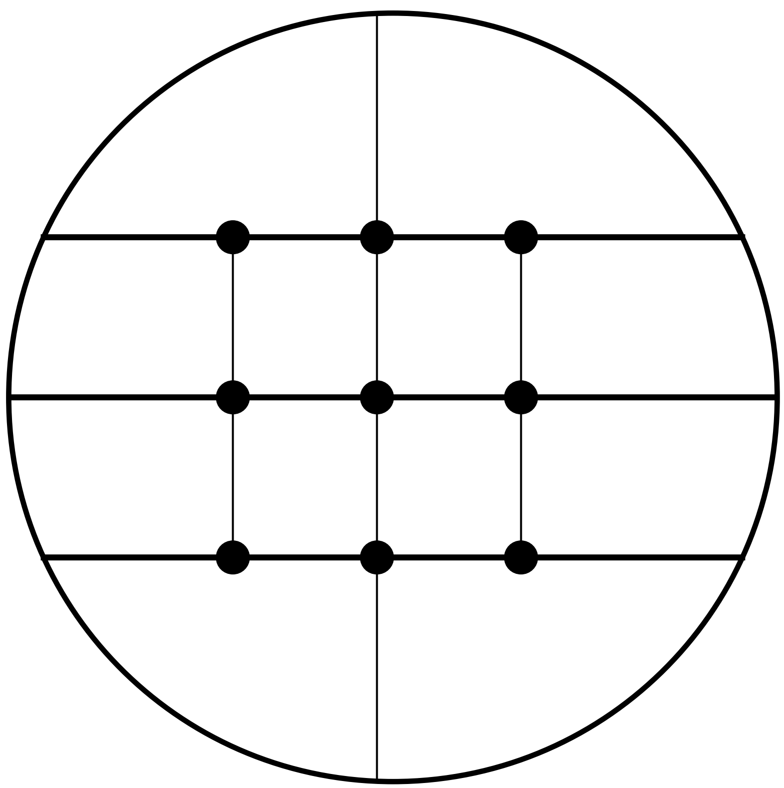
PART II

Triangle-free graphs on surfaces

THM (Thomassen) Every 4-critical graph of girth ≥ 5 in Σ has at most $f(g(\Sigma))$ vertices.

Not true for graphs of girth ≥ 4:

THM (Youngs) For every non-bipartite quadrangulation G of the projective plane $\chi(G)=4$.



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THM For every 4-critical triangle-free graph in Σ the sum of face-lengths of faces of length >4 is $\leq cg(\Sigma)$.

COR Every 4-critical graph of girth ≥ 5 in Σ has at most $cg(\Sigma)$ vertices.

APPLICATION For every surface Σ there is a linear-time algorithm to test whether a given triangle-free graph in Σ is 3-colorable.

COR (Kawarabayashi, Thomassen) $\forall\ \Sigma\ \forall$ triangle-free $G \hookrightarrow \Sigma$ all but $cg(\Sigma)$ vertices of G can be 3-colored.

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MAIN LEMMA G plane, girth ≥ 5, outer cycle C.
If G is C-critical and $|C| \geq 12$, then

$$\sum (|f|-8)^+ \leq |C|-12$$

(sum over all finite faces)

G is C-critical if for every proper subgraph H containing C some 3-coloring of C extends to H but not to G

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Open problems

Structure of 3-colorable triangle-free toroidal graphs?

Fix a proper minor-closed class F. Can $\chi(G)$ be approximated to within an additive error of 10 in poly-time for $G \in F$?

HAPPY BIRTHDAY, CARSTEN