COLORING TRIANGLE-FREE GRAPHS ON SURFACES

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joint work with

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and

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Coloring graphs on surfaces

The classical problems:

The 4-color problem (Guthrie 1852)
Can every planar graph be 4-colored?

Coloring graphs on surfaces

The classical problems:

The map color problem: Find max $\chi(G)$ over all $G \hookrightarrow S_g$ for $g>0$.

Answer: $(7+\sqrt{48g+1})/2$

(Heawood 1890; Ringel, Youngs 1960s)

Modern point of view: Fix Σ,k. Can we test $\chi(G) \leq k$ in poly-time for G in Σ?

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Yes for $k \geq 6$ by Euler's formula

Yes for $k=5$ by Thomassen's theorem

Open for $k=4$ (think Four-Color Theorem)

No for $k=3$, unless P=NP

Triangle-free graphs:

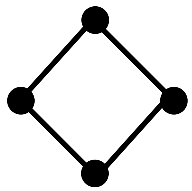
Yes for $k \geq 4$ by Euler's formula

Yes for $k=3$ (this talk)

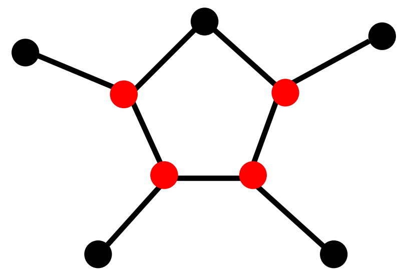
THM (Grotzsch) ∀ triangle-free planar graph is 3-colorable

PROOF Reducible configurations:

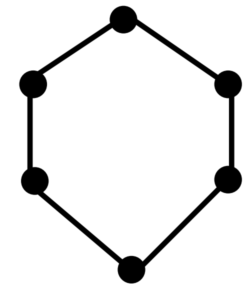
monogram



tetragram

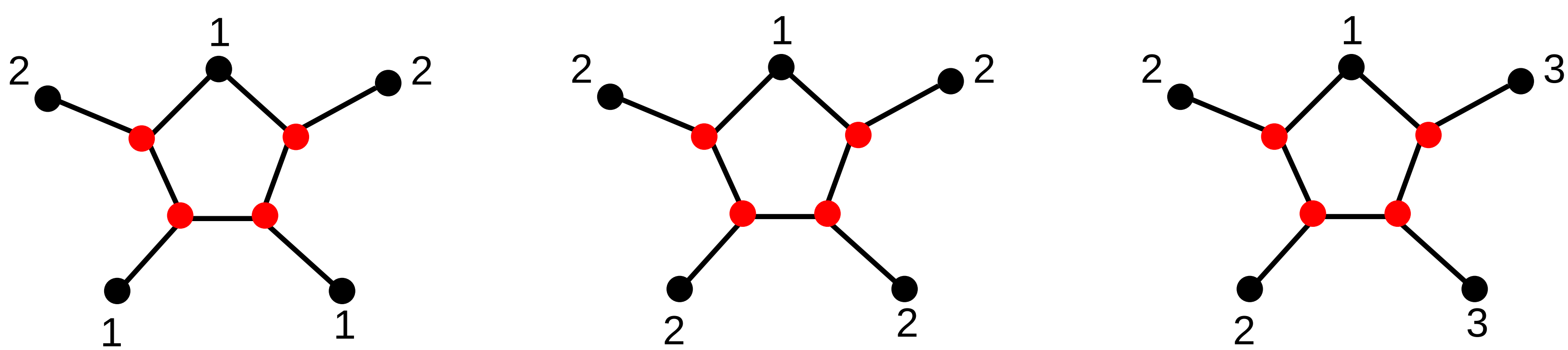


pentagram



hexagram

Non-extendable pentagram colorings:



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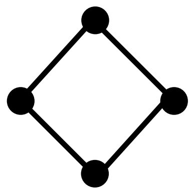

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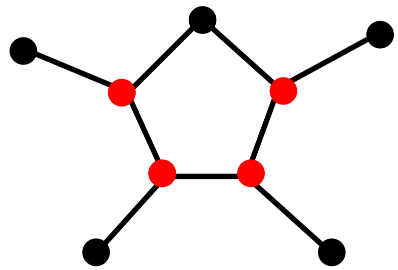
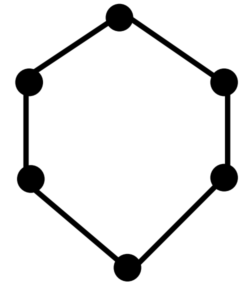


pentagram



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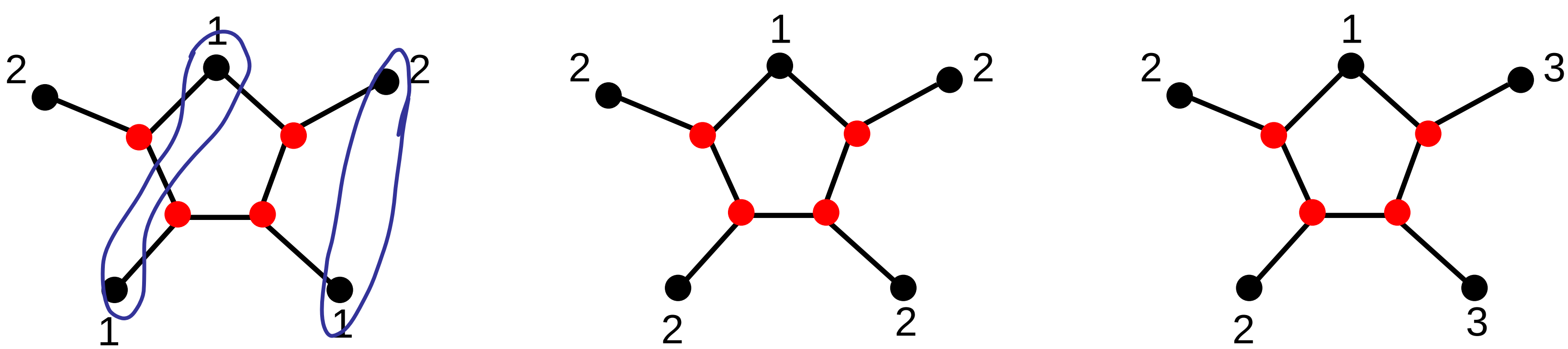
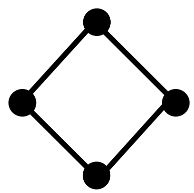


Reduction: delete red vertices, identify indicated pairs

THM (Grotzsch) ∀ triangle-free planar graph is 3-colorable

PROOF Reducible configurations:

monogram



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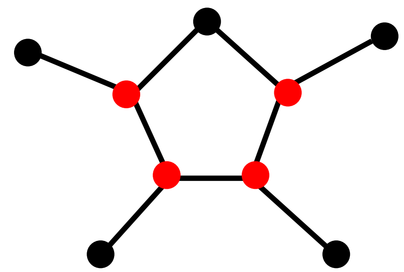

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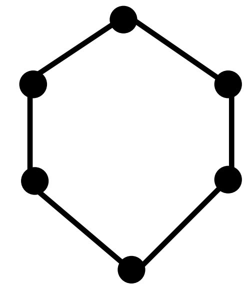

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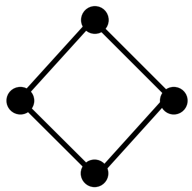

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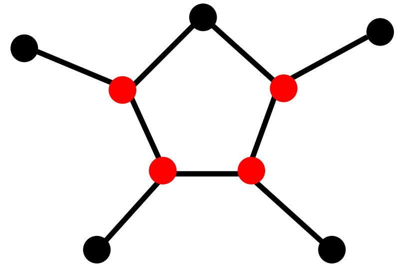

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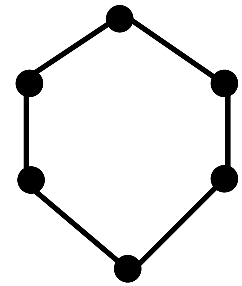


hexagram

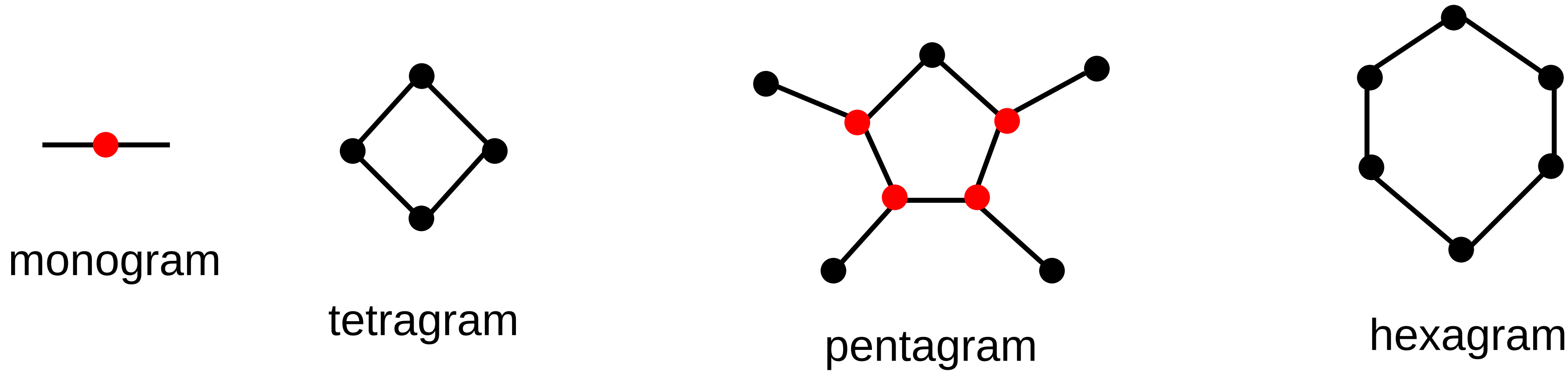
LEMMA 1 G connected triangle-free planar, C outer cycle, $|C| \leq 6$, $G \neq C \Rightarrow \exists$ multigram s.t. red vertices are not in C

PF

$$ch(v) = \begin{cases} \deg(v)-4 & v \text{ not in } C \\ -1/3 & v \in C, \deg 2 \\ 0 & \text{o.w.} \end{cases} \qquad ch(f) = \begin{cases} |f|-4 & f \text{ internal} \\ 0 & f \text{ outer} \end{cases}$$

$$\sum \text{charges} = \sum(d(v)-4) + \sum(|f|-4) - t_2/3 + \sum_{v \in C}(4-d(v)) - (|C|-4)$$

$$\leq 2E-4V+2E-4F-t_2/3+2t_2+t_3-|C|+4 \leq 2t_2/3-4 < 0.$$



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Each face will send 1/3 to every incident vertex v s.t. either $v\in G-C$ has degree 3, or $v\in C$ has degree 2.

After that there is a face of charge $<0 \Rightarrow$ tetragram or pentagram. Q.E.D.

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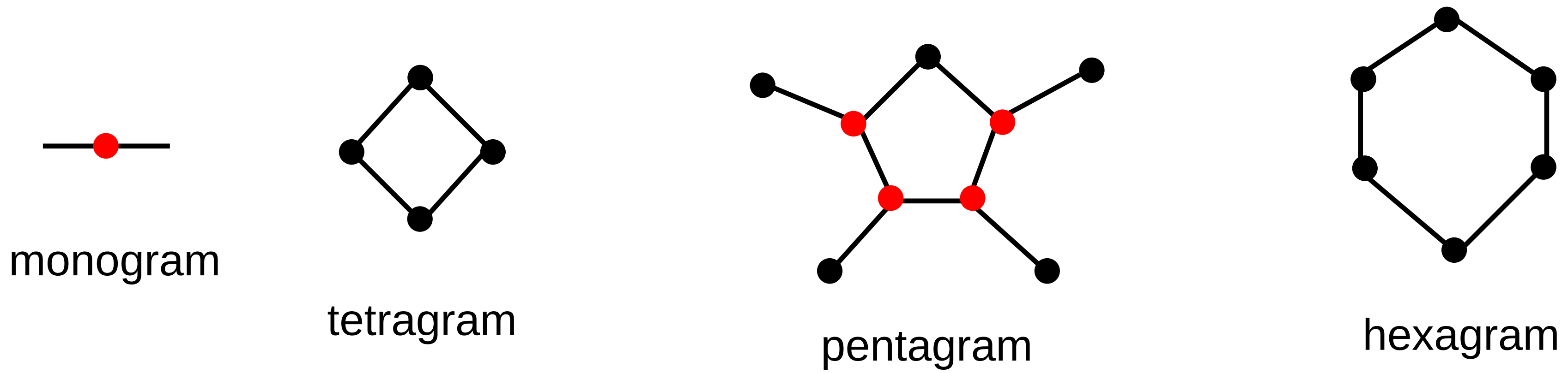


LEMMA 2 Every triangle-free planar graph has a safe multigram (reduction does not create triangles)

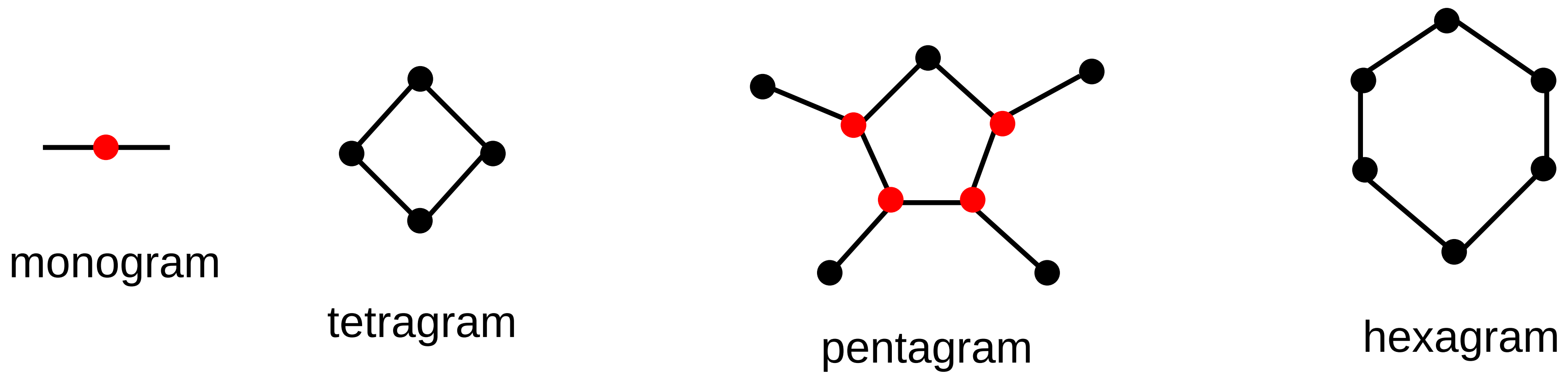
PF Pick a separating ≤ 6-cycle C with smallest inside, or facial ≤ 6-cycle if no separating cycle. Apply Lemma 1 to C and its interior. Q.E.D.

PF of Grotzsch: multigram $\Rightarrow$ smaller graph $\Rightarrow$ induction

LINEAR-TIME ALGORITHM

A problem: which vertices of a C_4 to identify?

Kowalik: An O(n log n) algorithm



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LINEAR-TIME ALGORITHM

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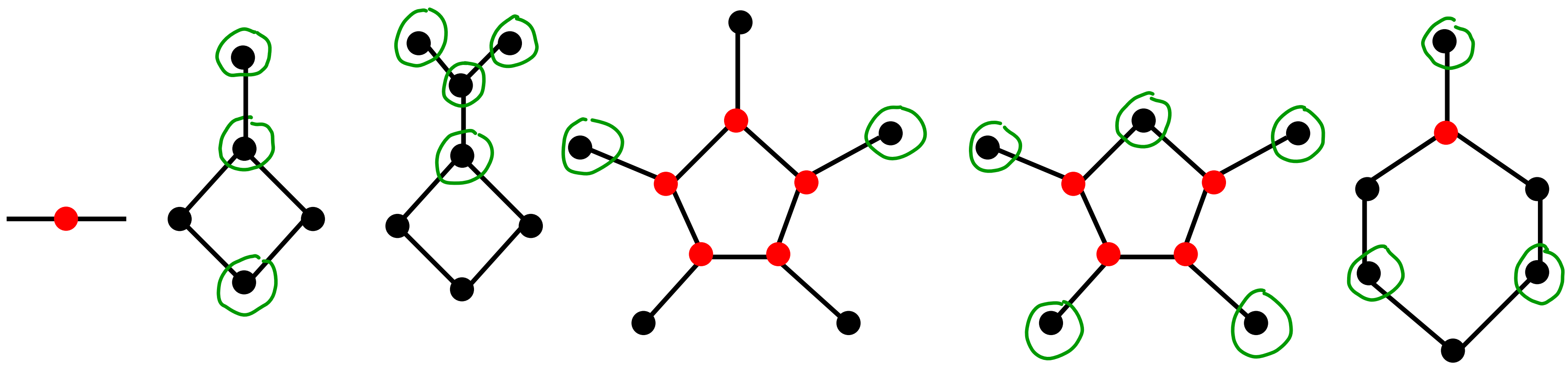
LEMMA 3 Every triangle-free planar graph has a secure multigram (safe+marked vertices have bounded degree)



ALGORITHM Find secure multigram and reduce in constant time to a smaller graph; recurse

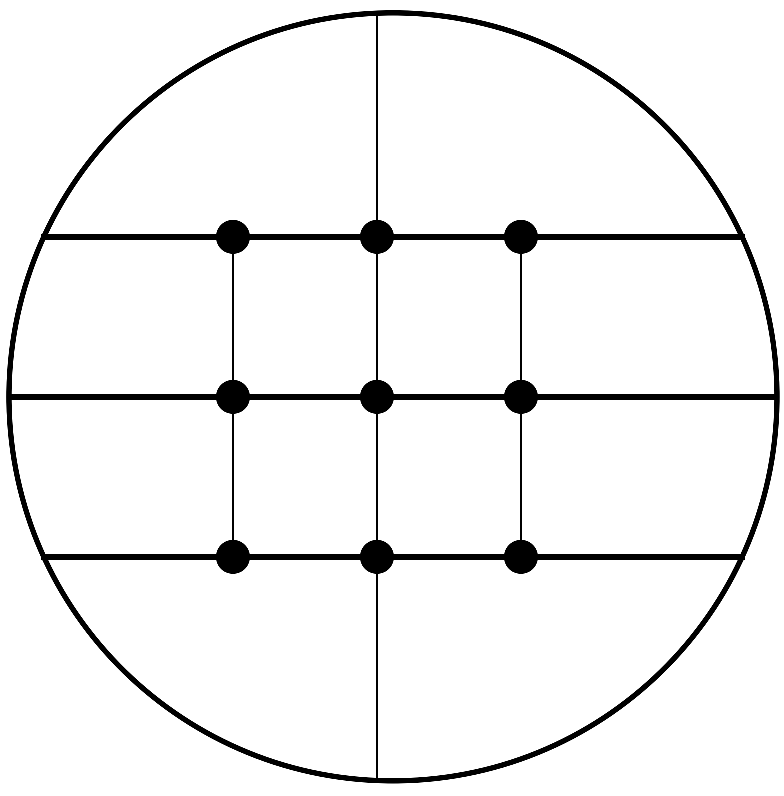
PART II

Triangle-free graphs on surfaces

THM (Thomassen) Every 4-critical graph of girth ≥ 5 in Σ has at most $f(\text{genus}(\Sigma))$ vertices.

Not true for graphs of girth ≥ 4:

THM (Youngs) For every non-bipartite quadrangulation G of the projective plane $\chi(G)=4$.



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THM For every 4-critical graph in a surface Σ
$\sum(|f|-4) \leq c(\text{genus}(\Sigma)+\text{no of triangles})$.

COR Every triangle-free 4-critical graph in Σ has at most $c \cdot \text{genus}(\Sigma)$ faces of size >4.

APPLICATION For every surface Σ there is a linear-time algorithm to test whether a given triangle-free graph in Σ is 3-colorable.

COR (Kawarabayashi, Thomassen) $\forall\ \Sigma\ \forall$ triangle-free G in Σ all but $cg(\Sigma)$ vertices of G can be 3-colored.

COR (Havel's conj) $\exists\ D$ s.t. if the triangles of a planar G are pairwise at distance $\geq D$, then 3-colorable.

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Testing 3-colorability of triangle-free G in Σ

Step 1: We either

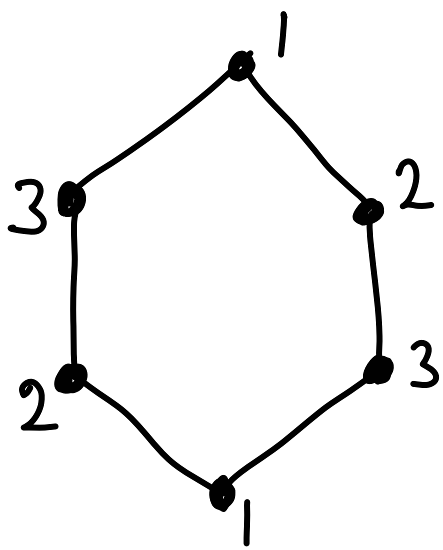
• determine that G is 3-colorable (has no 4-critical subgraph with few faces of length >4), or

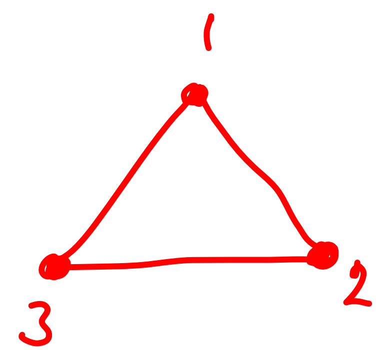
• find a subgraph H of bounded size s.t. each face of H is a locally planar quadrangulation

Step 2: We try all 3-colorings of H to see if one of them extends to G, using:

THM If G is in orientable Σ, G has k precolored faces, each of size $\leq k$, and G is a locally planar quadrangulation, then a precoloring extends iff sum of winding numbers is zero.

Let $G \hookrightarrow \Sigma$, Σ orientable, let $c: V(G) \to \{1,2,3\}$ be a 3-coloring. Orient edges $1 \to 2$, $2 \to 3$, $3 \to 1$. Let C be a cycle. Choose clockwise direction of C. Define winding number of C: $w(C) := (\#\text{forward edges} - \#\text{back edges})/3$.





FACT $w(C_4)=0$

FACT $\sum w(C)=0$, sum over all facial cycles

COR If some faces are precolored, and all other faces are quadrangles, then a necessary condition for the precoloring to extend is that the sum of winding #s be 0.

THM (Hutchinson) $\forall\ \Sigma$ orientable $\exists\ r$ s.t. every $G \hookrightarrow \Sigma$ with all faces even and edge-width $\geq r$ is 3-colorable.

THM B (Kral, RT) $\forall \Sigma$ orientable $\forall k\ \exists r\ \forall G \hookrightarrow \Sigma$ with precolored cycles $C_1,\ldots,C_k$ (“holes”) if

1. each hole has size $\leq k$
2. all other faces are C_4s
3. there is no “schism” of length $\leq r$
4. no C_i surrounded by cycle of length $<|C_i|$
5. sum of winding numbers is 0

$\Rightarrow$ precoloring extends to a 3-coloring of G

Open problems

Structure of 3-colorable triangle-free toroidal graphs?

Fix a proper minor-closed class F. Can $\chi(G)$ be approximated to within an additive error of 10 in poly-time for $G \in F$?