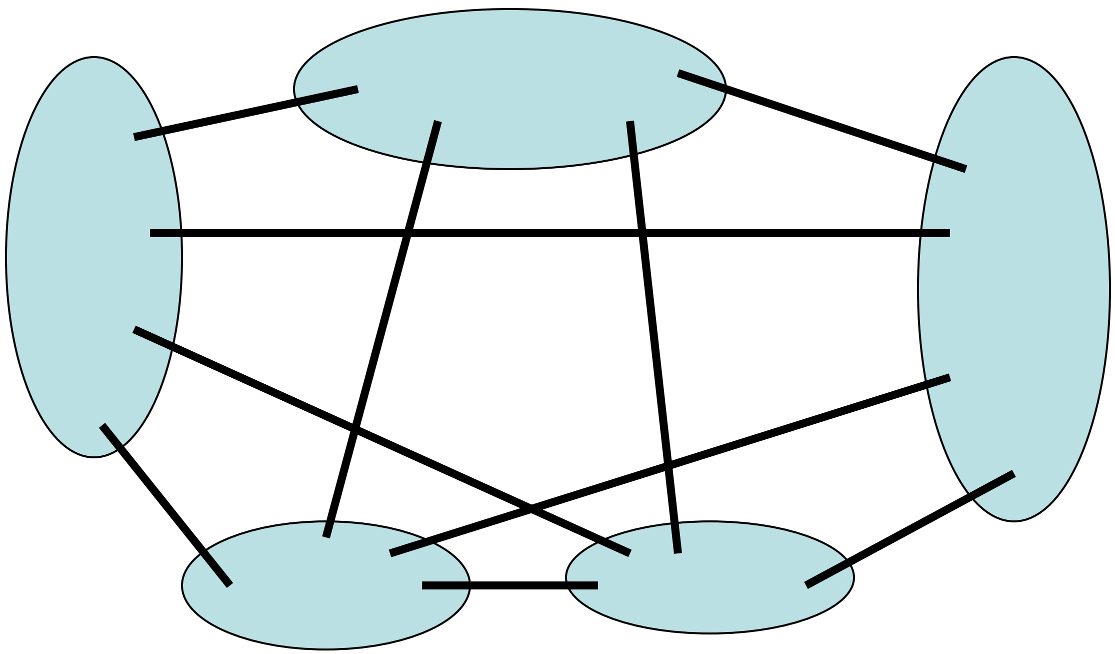
CONFIGURATIONS IN LARGE *t*-CONNECTED GRAPHS

Robin Thomas

School of Mathematics
Georgia Institute of Technology
http://math.gatech.edu/~thomas

joint work with Sergey Norin

- A minor of G is obtained by taking subgraphs and contracting edges.
- Preserves planarity and other properties.
- G has an H minor ($H \leq_m G$) if G has a minor isomorphic to H.
- A K_5 minor:



SUMMARY

THM G is t-connected, big, $G \not\succeq K_t \Rightarrow G \backslash X$ is planar for some $X \subseteq V(G)$ of size $\leq t$-5.

THM G is $(2k+3)$-connected, big $\Rightarrow$ G is k-linked

DEF G is k-linked if for all distinct vertices $s_1, s_2, \ldots, s_k, t_1, t_2, \ldots, t_k$ there exist disjoint paths $P_1, P_2, \ldots, P_k$ such that P_i joins s_i and t_i

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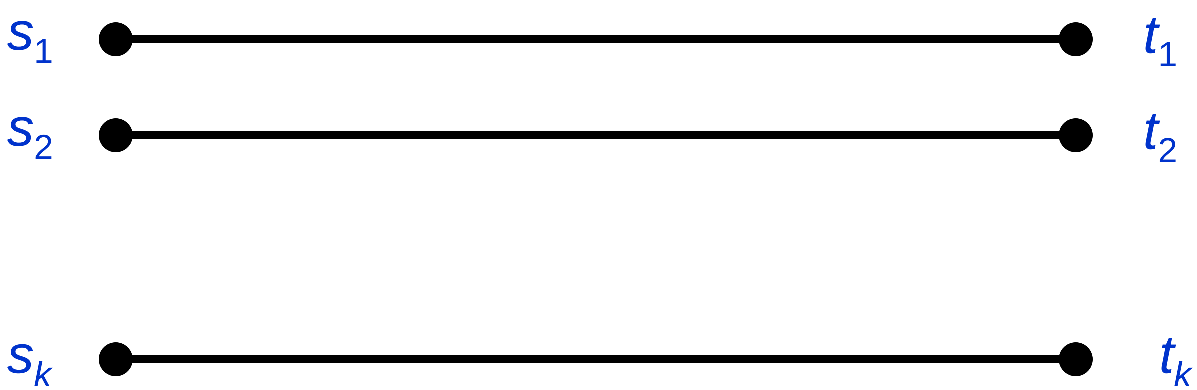

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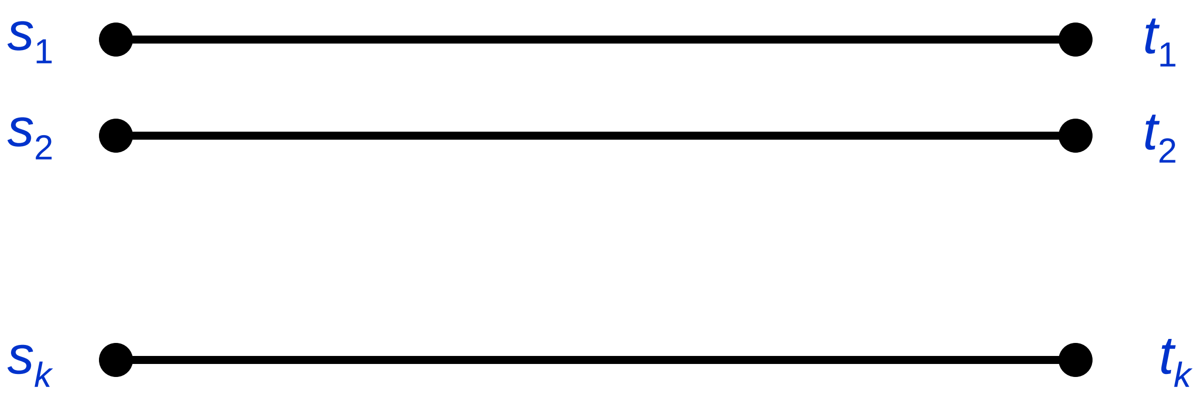

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NOTE $f(k)$=2k+2 would be best possible

NOTE $N \geq 3k$ needed

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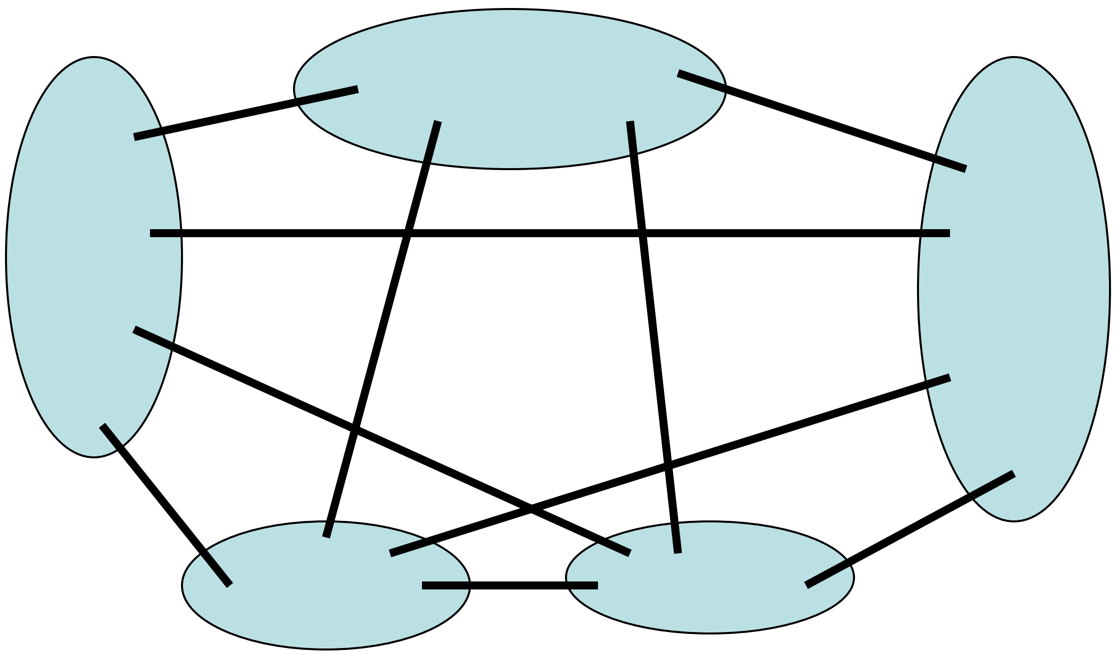
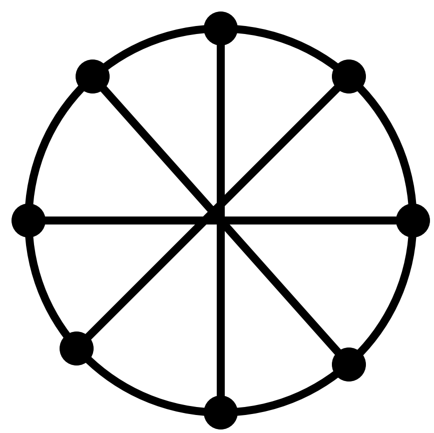


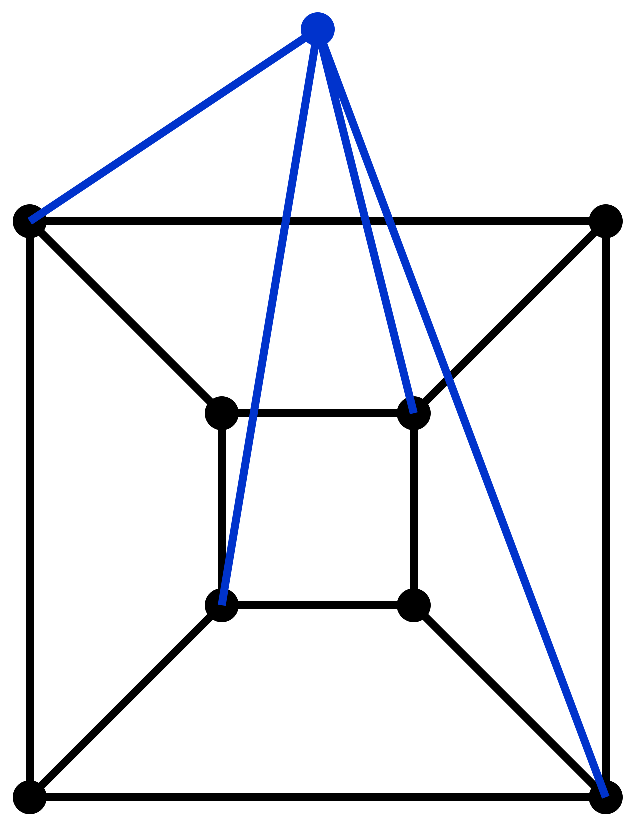
Excluding K_t minors

- $G \not\geq_m K_3$ ⇔ G is a forest (tree-width ≤ 1)
- $G \not\geq_m K_4$ ⇔ G is series-parallel (tree-width ≤ 2)
- $G \not\geq_m K_5$ ⇔ tree-decomposition into planar graphs and V_8 (Wagner 1937)
- $G \not\geq_m K_6$ ⇔ ???



Graphs with no K_6

- apex ($G \backslash v$ planar for some v)



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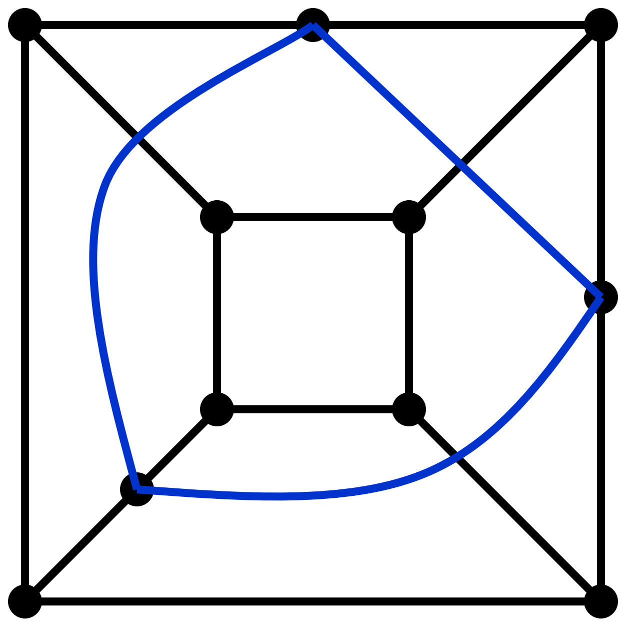
- planar + triangle



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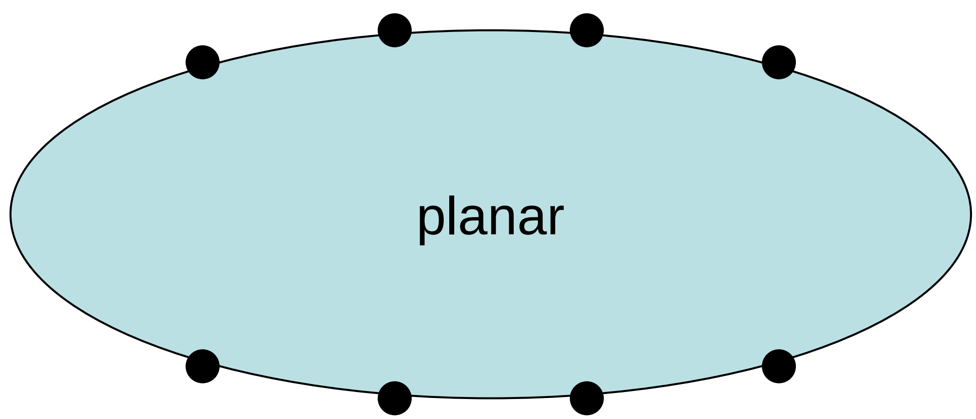

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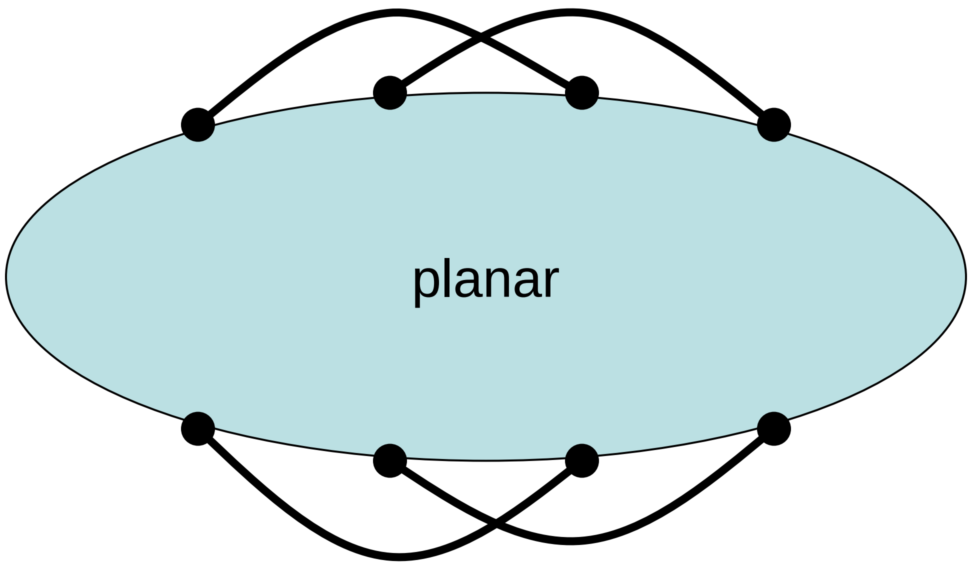

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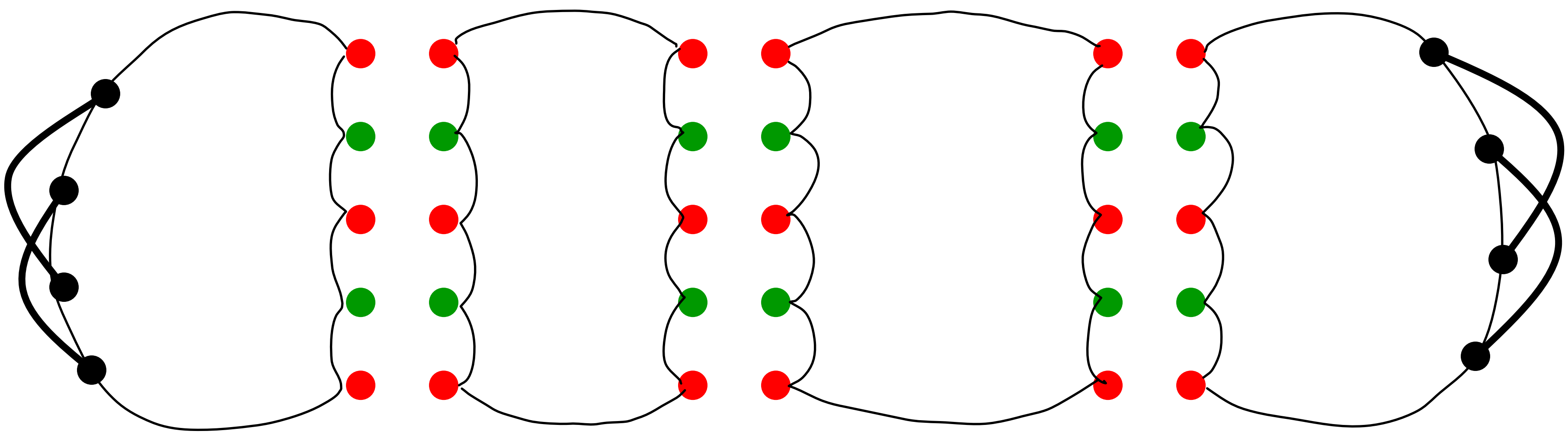

GRAPHS WITH NO K_t MINOR

REMARK

$G \not\geq_m K_t \Rightarrow$ (G + universal vertex) $\not\geq_m K_{t+1}$

REMARK

$G \backslash X$ planar for $X \subseteq V(G)$ of size $\leq t$-5 $\Rightarrow G \not\geq_m K_t$

GRAPHS WITH NO K_t MINOR

THEOREM (Robertson & Seymour)

$G \not\geq_m K_t \Rightarrow G$ has "structure"

Roughly structure means tree-decomposition into pieces that k-almost embed in a surface that does not embed K_t, where $k=k(t)$.

Converse not true, but:

G has "structure" $\Rightarrow G \not\geq_m K_{t'}$ for some $t' >> t$

Our objective is to find a simple iff statement

Extremal results for K_t

- $G \not\geq K_t \Rightarrow |E(G)| \leq (t\text{-}2)n\text{-}(t\text{-}1)(t\text{-}2)/2$ for $t \leq 7$ (Mader)
- $G \not\geq K_8 \not\Rightarrow |E(G)| \leq 6n\text{-}21$, because of $K_{2,2,2,2,2}$
- $G \not\geq K_t \Rightarrow |E(G)| \leq ct(\log t)^{1/2}n$ (Kostochka, Thomason)

CONJ (Seymour, RT) G is $(t\text{-}2)$-connected, big
$G \not\geq K_t \Rightarrow |E(G)| \leq (t\text{-}2)n\text{-}(t\text{-}1)(t\text{-}2)/2$

- $G \not\geq K_8 \Rightarrow |E(G)| \leq 6n\text{-}21$, unless G is a $(K_{2,2,2,2,2},5)$-cockade (Jorgensen)

- $G \not\geq K_9 \Rightarrow |E(G)| \leq 7n\text{-}28$, unless…. (Song, RT)

K_t minors naturally appear in:

Structure theorems:

-series-parallel graphs (Dirac)
-characterization of planarity (Kuratowski)
-linkless embeddings (Robertson, Seymour, RT)
-knotless embeddings (unproven)

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THM (Robertson, Seymour, RT) Every minimal counterexample to Hadwiger for $t=6$ is apex ($G \backslash v$ is planar for some v).

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True for big graphs:

There exists N such that every 6-connected graph $G \not\geq_m K_6$ on $\geq N$ vertices is apex.

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STEPS IN THE PROOF

- Bounded tree-width argument
- Excluded K_t theorem of Robertson & Seymour; reduce to the bounded tree-width case
- Thm of DeVos-Seymour on graphs in a disk

CASE 1 G has bounded tree-width

PROOF Let (T,W) be a tree-decomposition of bounded width. T has a vertex of big degree or a long path.

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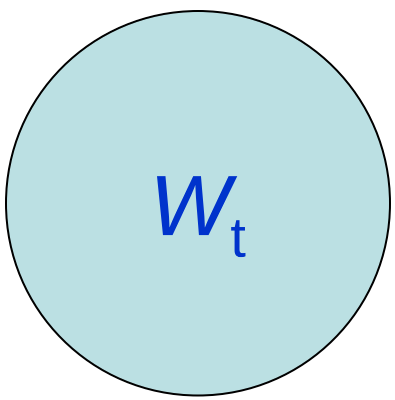

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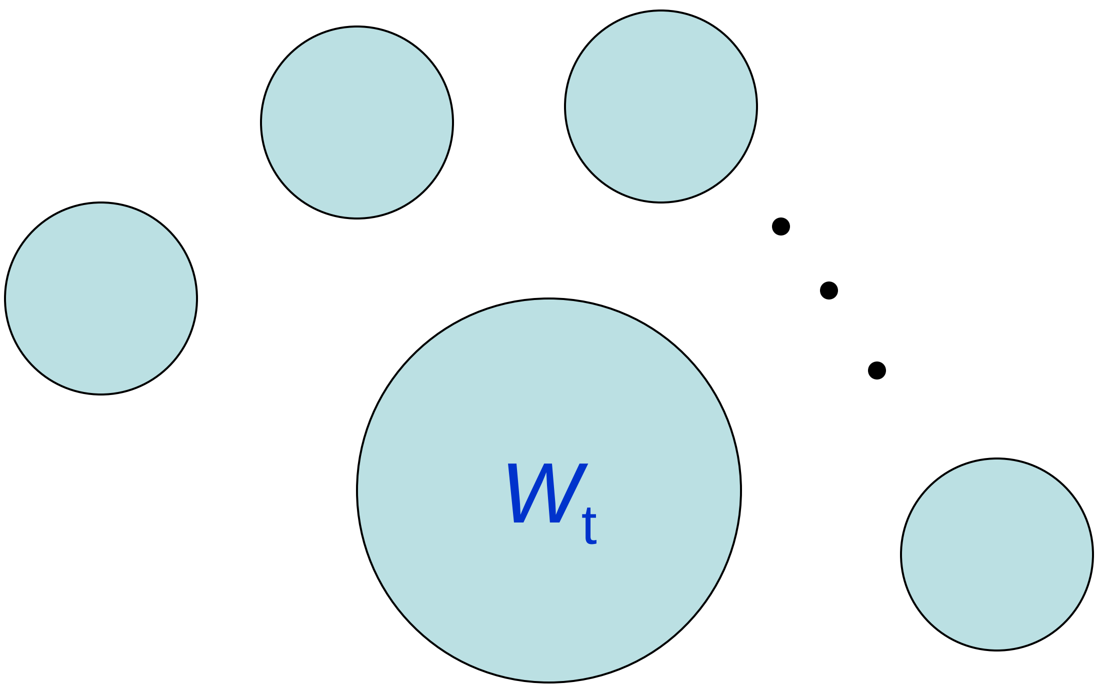

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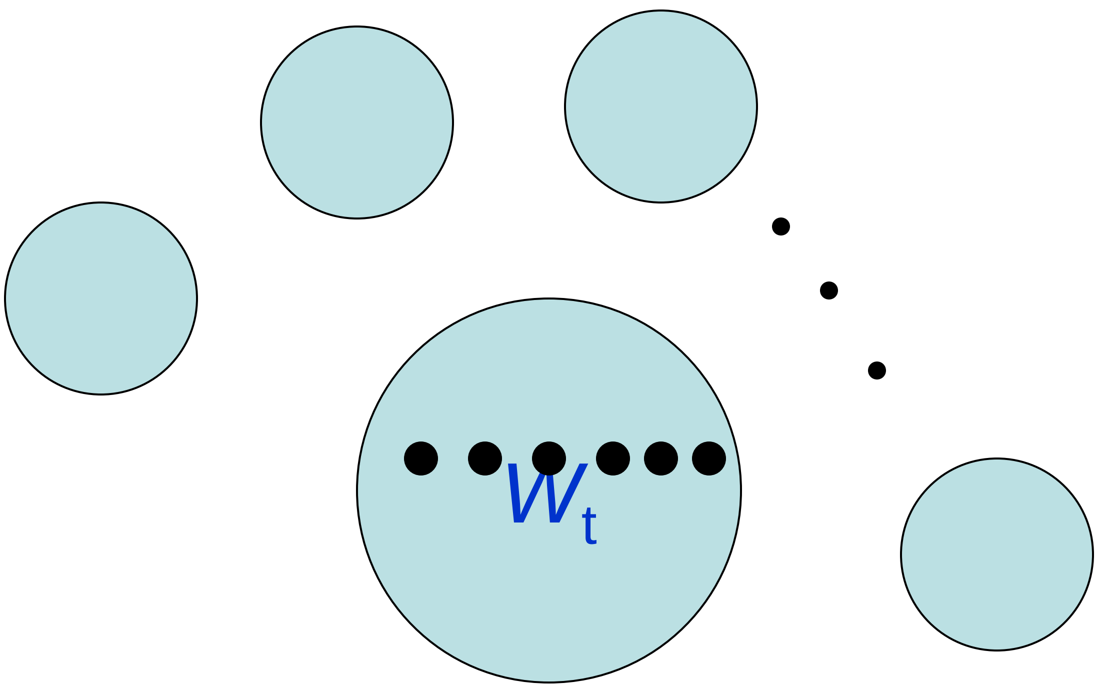

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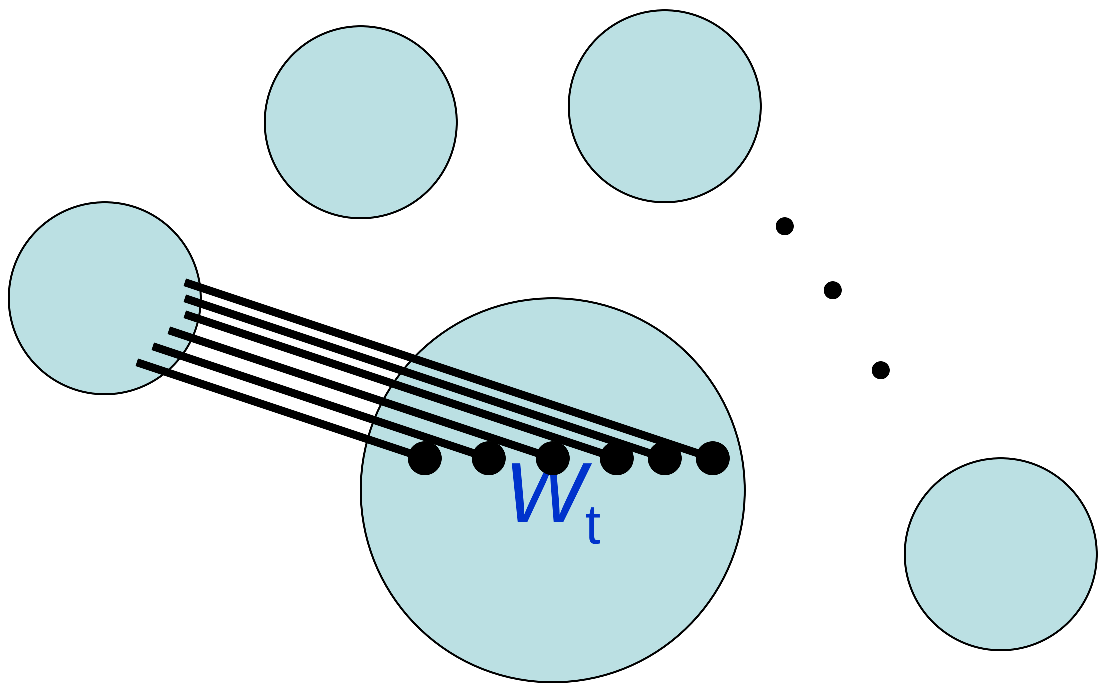

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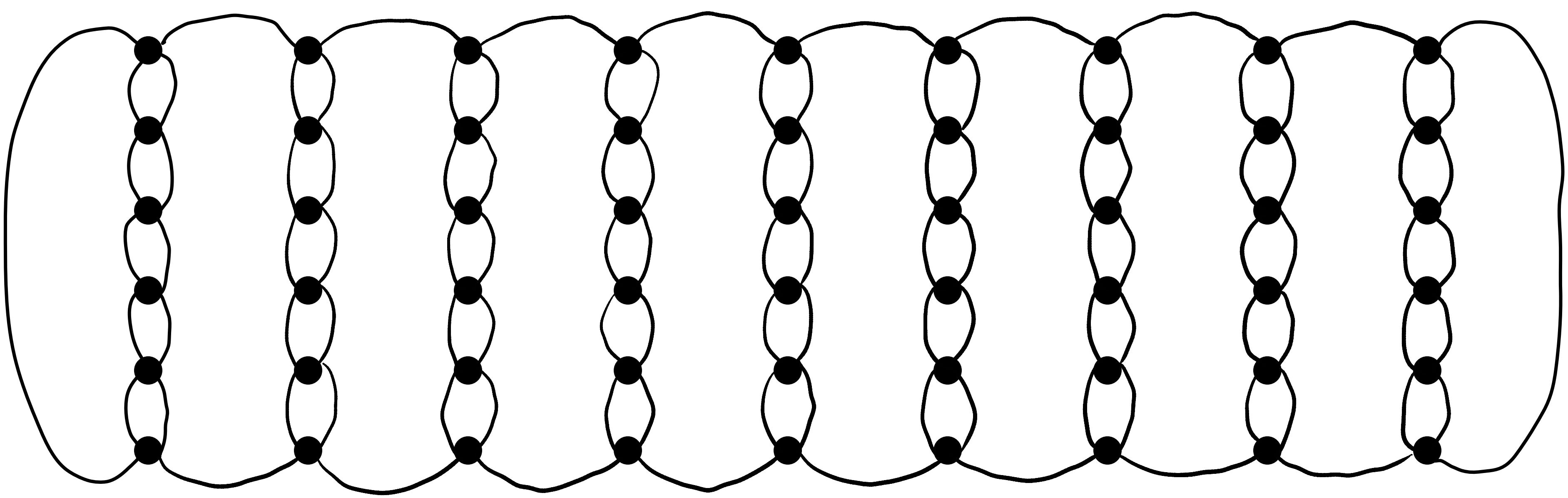

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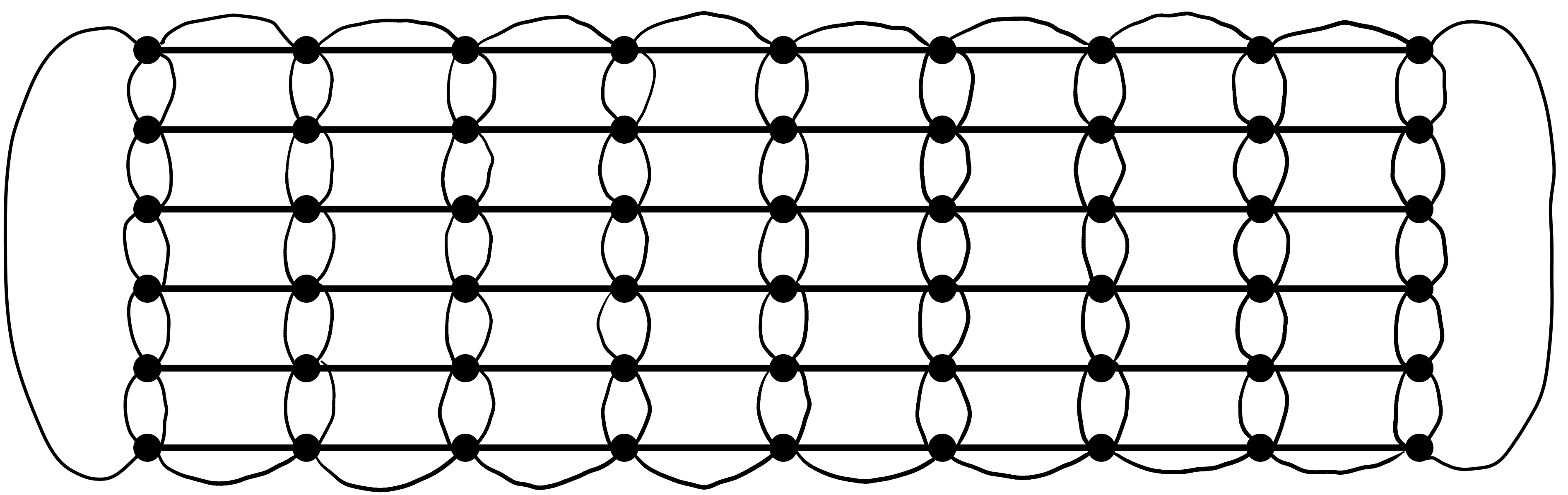

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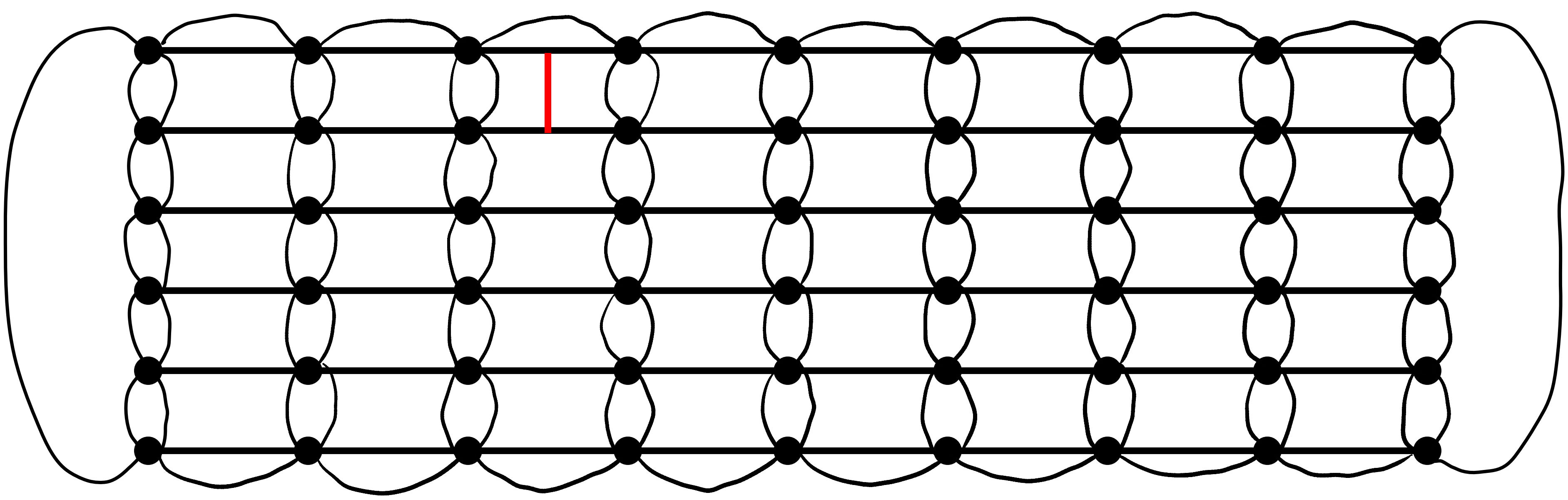

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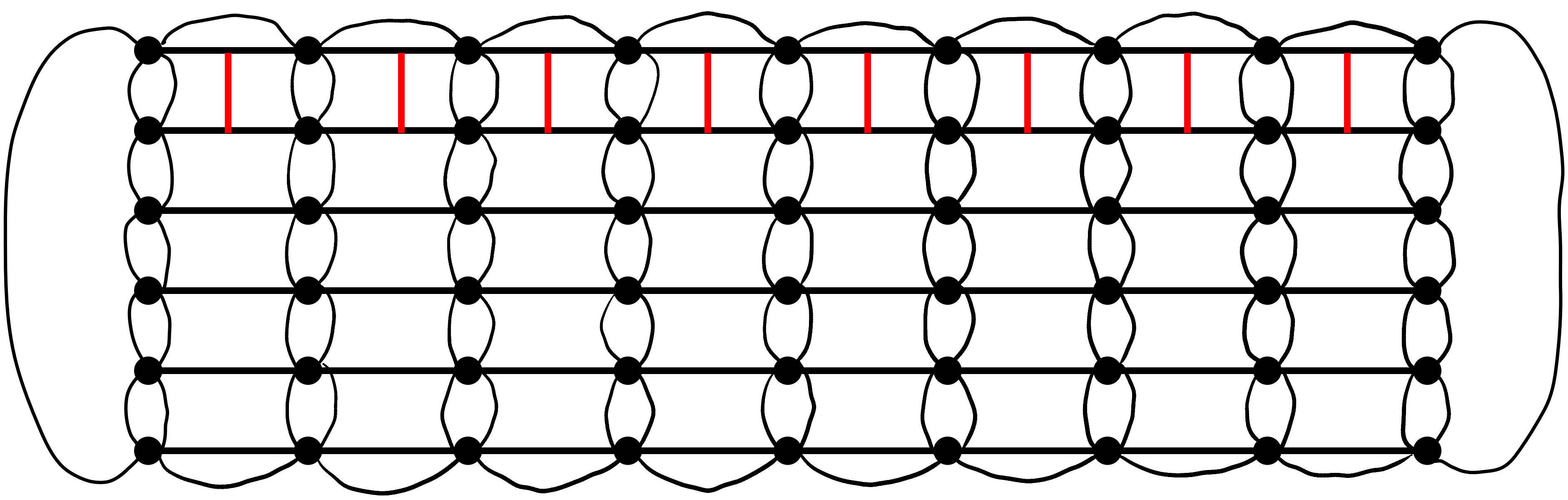

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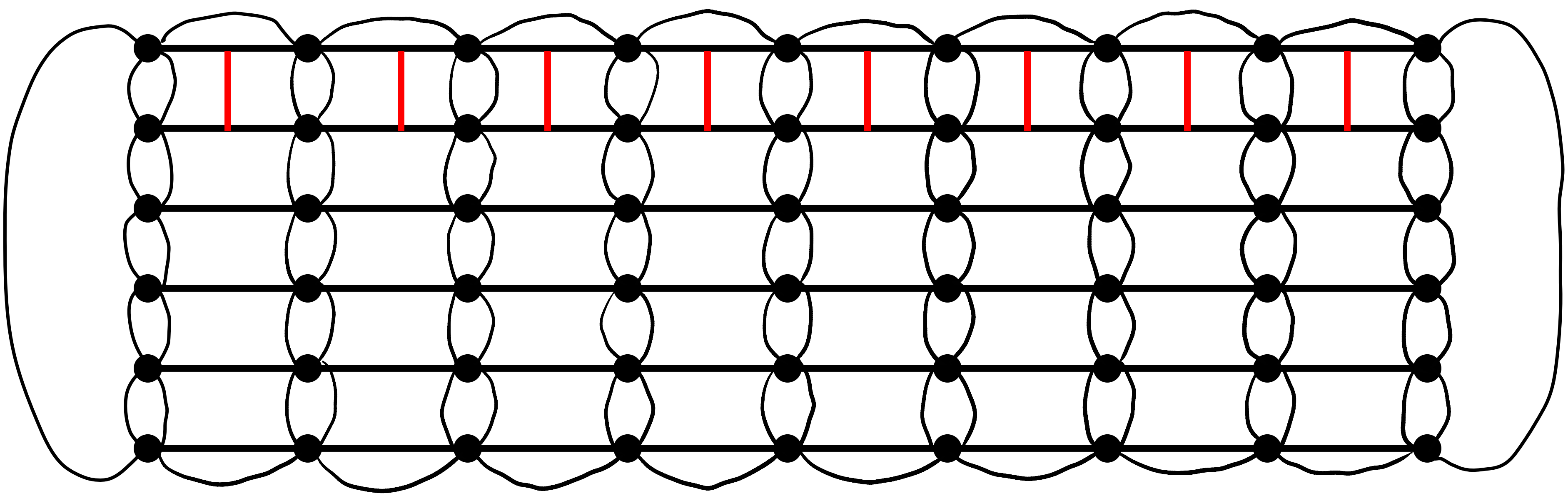
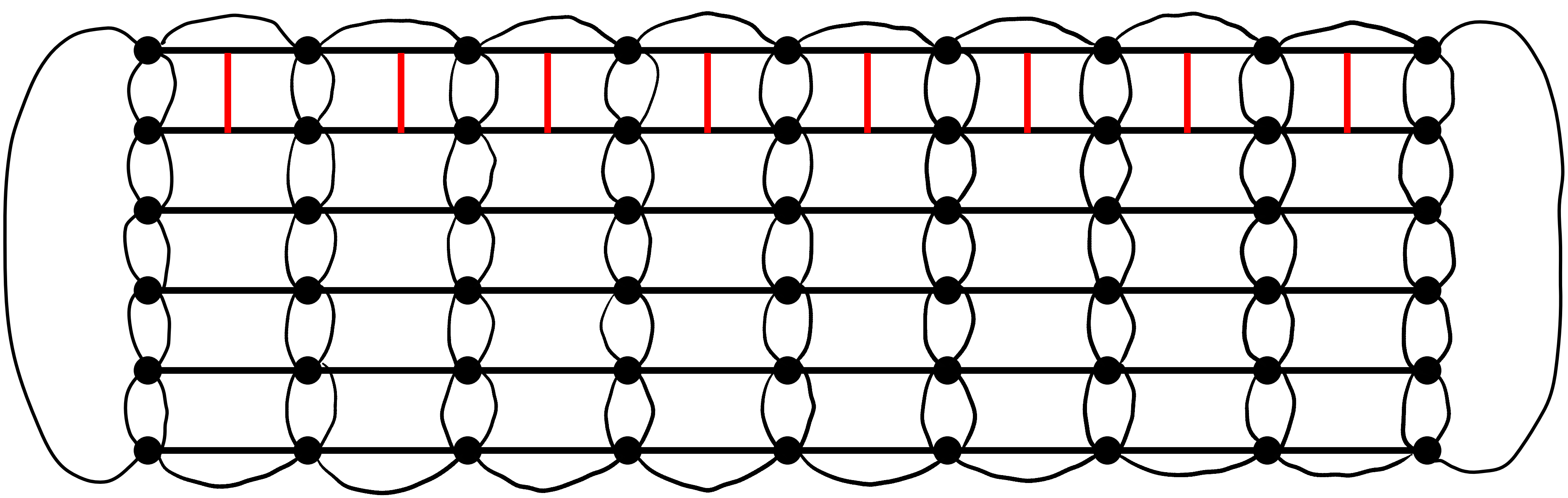
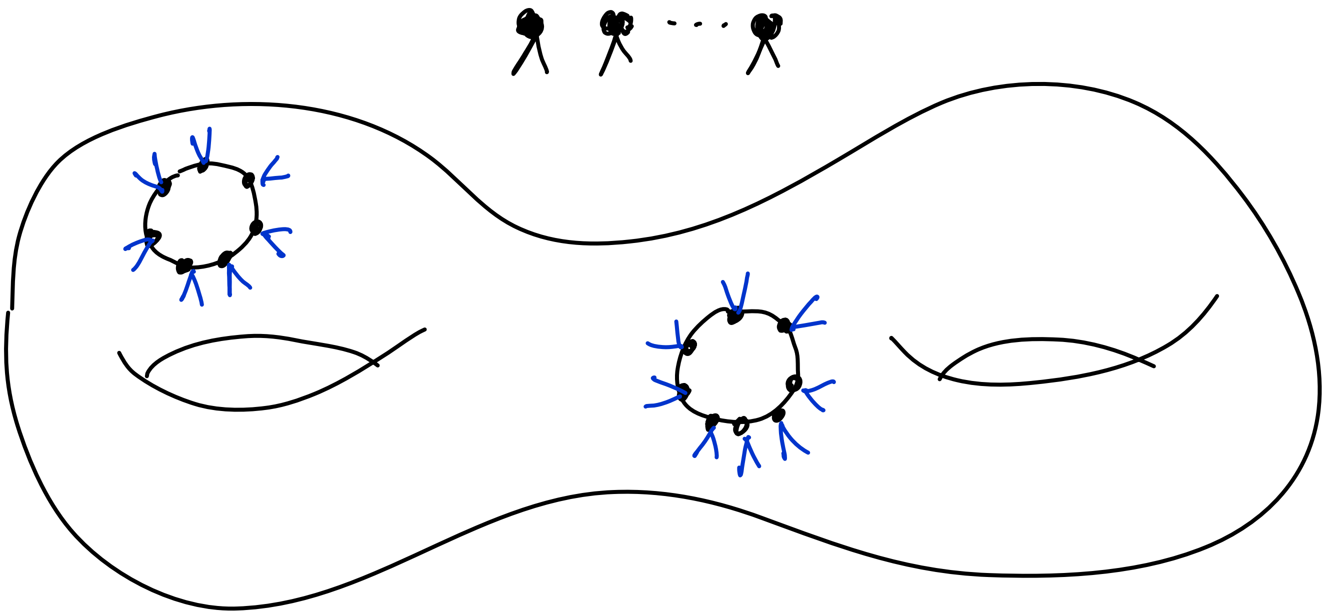


LEMMA Let G be a 2-connected graph on n vertices with a triangle. Then any configuration of n-1 labeled tokens can be moved to any other configuration by repeatedly sliding a token along an edge to an unoccupied vertex.



CASE 2 G has huge tree-width

PROOF By the excluded K_t theorem of Robertson & Seymour we may assume G is k-almost embedded in a surface that does not embed K_t, where $k=k(t)$.



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